Taller 2 Señales y Sistemas

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1)

serie de fourier (señal periodica T)

- forma exponencial

$$x(t) = \sum_{k=\infty}^{\infty} c_k e^{jkw_0 t}, \quad c_k = \frac{1}{T}\int_0^T x(t) e^{-jkw_0 t} dt.$$

- forma trigonometrica

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(kw_0 t) + b_k \sin(k w_0 t)].$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(kw_0 t)dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(kw_0 t)dt$$

Transformada de fourier (FT)

Para $x(t)$ de tiempo continuo y no necesariamente periodica

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

DTFT (señal discreta ilimitada)

Para $x[n]$, $n \in \mathbb{Z}$:

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw}) e^{jwn} \, dw$$

DFT (señal discreta finita N)

para $n, k = 0, \dots, N-1$:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \quad , \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

FFT (fast fourier transform)

$$X[k] = \sum_{m=0}^{N/2-1} x[2m] W_{N/2}^{km} + W_N^k \sum_{m=0}^{N/2-1} x[2m+1] W_{N/2}^{km} \quad , \quad W_N = e^{-j\frac{2\pi}{N}}$$

2)

a) $x(t) = e^{-a|t|}$

FT

$$X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt = \int_{-\infty}^{\infty} e^{-a|t|} e^{-jwt}\, dt$$

Para $t \geq 0$

$$\int_0^{\infty} e^{-at} e^{-jwt}\, dt = \int_0^{\infty} e^{-(a+jw)t}\, dt = \frac{1}{a+jw}$$

Para $t < 0$

$$\int_{-\infty}^{0} e^{-a(-t)} e^{-jwt}\, dt = \int_0^{\infty} e^{-au} e^{jwu}\, du = \frac{1}{a-jw}$$

sumo

$$X(w) = \frac{1}{a+jw} + \frac{1}{a-jw}$$

$$= \frac{2a}{a^2+w^2}$$

b) $x(t) = \cos(w_c t)$, $w_c \in \mathbb{R}$

$$X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt$$

$$\cos(w_c t) = \frac{e^{jw_c t} + e^{-jw_c t}}{2}$$

$$X(w) = \int_{-\infty}^{\infty} \frac{e^{jw_c t} + e^{-jw_c t}}{2}\, e^{-jwt}\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(w-w_c)t}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(w+w_c)t}\, dt$$

$$= \frac{1}{2}\left[2\pi\,\delta(w-w_c)\right] + \frac{1}{2}\left[2\pi\,\delta(w+w_c)\right]$$

$$= \pi\left[\delta(w-w_c) + \delta(w+w_c)\right]$$

c) $x(t) = \sin(w_s t), \quad w_s \in \mathbb{R}$

$$X(w) = \int_{-\infty}^{\infty} x(t) e^{-jwt} \, dt$$

$$\text{sen}(w_s t) \quad \text{sen}(w_s t) = \frac{e^{jw_s t} e^{-jw_s t}}{2j}$$

$$X(w) = \frac{1}{2j} \int_{-\infty}^{\infty} e^{jw_s t} e^{-jwt} \, dt - \frac{1}{2j} \int_{-\infty}^{\infty} e^{-jw_s t} e^{-jwt} \, dt$$

$$= \frac{2\pi}{2j} \left[\delta(w - w_s) - \delta(w + w_s) \right]$$

$$= \frac{\pi}{j} \left[\delta(w - w_s) - \delta(w + w_s) \right]$$

$$= \frac{\pi}{j} \left[\delta(w - w_s) - \delta(w + w_s) \right] - j\pi \left[\delta(w + w_s) - \delta(w - w_s) \right]$$

d) $x(t) = f(t)\cos(\omega_c t)$, $f(t) \in \mathbb{R} \to \mathbb{C}$, $\omega_c \in \mathbb{R}$

$\cos(\omega_c t) = \frac{1}{2}(e^{j\omega_c t} + e^{-j\omega_c t}) \Rightarrow x(t) = \frac{1}{2}f(t)e^{j\omega_c t} + \frac{1}{2}f(t)e^{-j\omega_c t}$

$F\{f(t)e^{\pm j\omega_c t}\}(\omega) = F(\omega \mp \omega_c)$

$X(\omega) = \frac{1}{2}F(\omega - \omega_c) + \frac{1}{2}F(\omega + \omega_c)$

e) $x(t) = e^{-at^2}$, $a > 0$

$X(\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$

$\int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{\omega^2}{4a}\right)$

f) $x(t) = A\,\mathrm{rect}_d(t)$ con $\mathrm{rect}_d(t) = \begin{cases} 1, & |t| \le \frac{d}{2}, \\ 0, & |t| > \frac{d}{2}. \end{cases}$

$X(\omega) = A\int_{-d/2}^{d/2} e^{-j\omega t}\, dt = A\frac{e^{-j\omega(d/2)} - e^{j\omega(d/2)}}{-j\omega}$

$= A\frac{2\sin\left(\frac{\omega d}{2}\right)}{\omega} = Ad\frac{\sin\left(\frac{\omega d}{2}\right)}{\frac{\omega d}{2}} = Ad\,\mathrm{Sinc}\left(\frac{\omega d}{2}\right)$

$= Ad\,\mathrm{sinc}\left(\frac{\omega d}{2}\right)$, $\mathrm{Sinc}(x) = \frac{\sin x}{x}$

3) a) $F\{e^{-jw_1 t}\cos(w_c t)\}$

$F\{\cos(w_c t)\} = \pi[\delta(w-w_c) + \delta(w+w_c)]$.

$F\{x(t)e^{-jw_1 t}\}(w) = x(w+w_1)$

$F\{e^{-jw_1 t}\cos(w_c t)\} = \pi[\delta(w+w_1-w_c) + \delta(w+w_1+w_c)]$

b) $F\{u(t)\cos^2(w_c t)\}$

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

$u(t)\cos^2(w_c t) = \frac{1}{2}u(t) + \frac{1}{4}u(t)e^{j2w_c t} + \frac{1}{4}u(t)e^{-j2w_c t}$

$F\{u(t)\} = Pv\frac{1}{jw} + \pi\delta(w)$

$F\{u(t)e^{\pm j2w_c t}\} = U(w \mp 2w_c)$

$F\{u(t)\cos^2(w_c t)\} = \frac{1}{2}\left(Pv\frac{1}{jw} + \pi\delta(w)\right) +$

$\frac{1}{4}\left(Pv\frac{1}{j(w-2w_c)} + \pi\delta(w-2w_c)\right) + \frac{1}{4}\left(Pv\frac{1}{j(w+2w_c)} + \pi\delta(w+2w_c)\right)$

c) $F^{-1}\left\{\frac{7}{w^2+6w+45} \cdot \frac{10}{(8+jw/3)^2}\right\}$

Producto en frecuencia <-> convolución en tiempo

$$f(t) = (f_1 * f_2)(t)$$

$$f_1(t) = F^{-1}\left\{\frac{7}{w^2+6w+45}\right\}$$

$$f_2(t) = F^{-1}\left\{\frac{10}{(8+jw/3)^2}\right\}$$

d) $F\{3t^3\}$

$t^n \longleftrightarrow 2\pi j^n \delta^{(n)}(w)$

$n=3$ factor 3 ->

$F\{3t^3\}(w) = 3(2\pi) j^3 \delta^{(3)}(w) = -6\pi j \delta^{(3)}(w)$

e) Señal periodica de periodo T:

$$x(t) = B \sum_{n=-\infty}^{\infty} \Bigg[\underbrace{\frac{1}{a^2+(w-nw_0)^2}}_{\text{Fr de } e^{-a|t|}} + \underbrace{\frac{1}{a+j(w-nw_0)}}_{\text{Fr de } e^{-at}u(t)} \Bigg] \qquad w_0 = \frac{2\pi}{T}$$

$$X(w) = \frac{B}{T} \sum_{n=-\infty}^{\infty} \left[\frac{1}{a^2+(w-nw_0)^2} + \frac{1}{a+j(w-nw_0)} \right]$$