Rolling Mean and Variance of a Discrete Time Series

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Abstract

In this brief report, I am going to derive an expression for the rolling mean and variance of a discrete time series, which is useful for online applications, where limited computational resources don't allow the computation of the mean every time a new data point is received. This approach considers that the rolling mean and variance are computed with one incoming data point on every update, but can be extended to consider multiple data points at once.

1 Windowed formulation for a window of duration $N-1$

For a time window $[1, N]$, the mean and variance respectively become:

$$\mu_{1:N} = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{1}$$

$$\sigma^2_{1:N} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2 \tag{2}$$

2 Derivation of Rolling Mean

For the time window $[2, N+1]$, the mean becomes:

$$\begin{aligned}\mu_{2:N+1} &= \frac{1}{N}\sum_{i=2}^{N+1} x_i & (3)\\ &= \frac{1}{N}\sum_{i=1}^{N}\left(x_i + x_{N+1} - x_1 right\right) & (4)\\ &= \mu_{1:N} + \frac{1}{N}\left(x_{N+1} - x_1\right) & (5)\end{aligned}$$

Thus, we can express the windowed mean for a time window $[2, N+1]$, as a function of the mean for window $[1, N]$ and old data point x_1 and incoming data point x_{N+1}.

3 Derivation of Rolling Variance

For the time window $[1, N]$, the variance becomes:

$$\sigma_{1:N} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_{1:N})^2 \tag{6}$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left(x_i^2 - 2x_i\mu_{1:N} + \mu_{1:N}^2\right) \tag{7}$$

$$= \frac{1}{N}\sum_{i=1}^{N}x_i^2 - \mu_{1:N}^2 \tag{8}$$

Similarly, for the window $[2, N+1]$:

$$\sigma_{2:N+1} = \frac{1}{N}\sum_{i=2}^{N+1}x_i^2 - \mu_{2:N+1}^2 \tag{9}$$

Let $S = \sum_{i=1}^{N} x_i^2$. It follows that:

$$\sum_{i=2}^{N+1}x_i^2 = \sum_{i=1}^{N}x_i^2 - x_1^2 + x_{N+1}^2 \tag{10}$$

Thus,

$$\sigma_{2:N+1}^2 = \frac{1}{N}\sum_{i=2}^{N+1}x_i^2 - \mu_{2:N+1}^2 \tag{11}$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left(x_i^2 - x_1^2 + x_{N+1}^2\right) - \mu_{2:N+1}^2 \tag{12}$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left(x_i^2 - x_1^2 + x_{N+1}^2\right) - \mu_{1:N}^1 + \left(-x_1^2 + x_{N+1}^2\right) \tag{13}$$

Now we have an equation for the variance for the second time window $[2, N+1]$, with the precomputed terms from the first time window $[1, N]$, the old point x_1, the old mean $\mu_{1:N}$, and the new point x_N. This equation is useful if we want to compute the variance of a time series on-the-fly, without having to recompute the mean of the entire window every time we receive an incoming point.