Introduction to Robotics 046212, Spring 2021
Homework 4

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Question 1. We will derive the equations of motion of the system using the Euler-Lagrange equations. We start by writing the Lagrangian $L = T - U$, with the kinetic and potential energy of the system.
The kinetic energy of the system is comprised of the kinetic energies of both masses. The potential of the system includes gravity on the mass m and the potential stored in the spring connected to the mass M.

$$T = \frac{1}{2} M \dot{\vec{x}}^T \dot{\vec{x}} + \frac{1}{2} m \dot{\vec{r}}^T \dot{\vec{r}}$$
$$U = m\vec{g} \cdot \vec{r} + \frac{1}{2} k (x - l_0)^2$$

Where $\vec{x} = (x, 0)^T$ is the position of mass M and $\vec{r} = \begin{bmatrix} x + l\sin\theta \\ l\cos\theta \end{bmatrix}$ is the position of the point mass m. Finding the time derivative of $\vec{r}$, $\dot{\vec{r}}$, and plugging in, we have:

$$L = T - U = \frac{1}{2}\left[M||\dot{x}||^2 + m\left((\dot{x} + l\dot{\theta}\cos\theta)^2 + l^2\dot{\theta}^2\sin^2\theta\right)\right] + mgl\cos\theta - \frac{1}{2}k(x - l_0)^2 =$$
$$= \frac{1}{2}\left[(M + m)\dot{x}^2 + ml^2\dot{\theta}^2 - k(x - l_0)^2\right] + ml(\dot{x}\dot{\theta} + g)\cos\theta$$

We can now calculate the required partial derivatives of the Lagrangian:

$$\frac{\partial L}{\partial \dot{x}} = (M + m)\dot{x} + ml\dot{\theta}\cos\theta \qquad \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} = (M + m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$
$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + ml\dot{x}\cos\theta \qquad \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta} + ml(\ddot{x}\cos\theta - \dot{x}^2\sin\theta)$$
$$\frac{\partial L}{\partial x} = -k(x - l_0)$$
$$\frac{\partial L}{\partial \theta} = -ml(\dot{x}\dot{\theta} + g)\sin\theta$$

Finally, we can calculate the generalized forces:

$$\xi_1 = \vec{F}_1^T \frac{\partial \vec{x}}{\partial x} + \vec{F}_2^T \frac{\partial \vec{r}}{\partial x} = F_1 + F_2$$
$$\xi_2 = \vec{F}_1^T \frac{\partial \vec{x}}{\partial \theta} + \vec{F}_2^T \frac{\partial \vec{r}}{\partial \theta} = F_2 l \cos\theta$$

Plugging it all in, we obtain the equations of motion for the generalized coordinates x and θ:

$$(M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + k(x - l_0) = F_1 + F_2$$
$$ml^2 + ml(\ddot{x}\cos\theta - \dot{x}^2\sin\theta) + ml(\dot{x}\dot{\theta} + g)\sin\theta = F_2 l\cos\theta$$

Question 2. Note: throughout the entire question, d_3 should have included l_2 (which was not mentioned in the original vertion of tutorial 8 question 2, discussing this manipulator). Therefore, we can assume $l_2 = 0$, and then the solution below holds.

1. The matrix form of the Euler-Lagrange equations is:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

We can use the Lagrange equations found in the tutorial to obtain the explicit matrices:

$$\begin{cases} m\ddot{\theta}_1 d_3^2 c_{\theta_2}^2 + 2md_3\dot{d}_3\dot{\theta}_1 c_{\theta_2}^2 - 2m\dot{\theta}_1 d_3^2 c_{\theta_2} s_{\theta_2}\dot{\theta}_2 = \tau_1 \\ md_3^2\ddot{\theta}_2 + 2md_3\dot{d}_3\dot{\theta}_2 + md_3^2 c_{\theta_2} s_{\theta_2}\dot{\theta}_1^2 + mgd_3 c_{\theta_2} = \tau_2 \\ m\ddot{d}_3 - md_3(\dot{\theta}_2^2 + c_{\theta_2}^2\dot{\theta}_1^2) + mgs_{\theta_2} = \tau_3 \end{cases}$$
$$\Downarrow$$
$$\begin{bmatrix} md_3^2 c_{\theta_2}^2 & 0 & 0 \\ 0 & md_3^2 & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{d}_3 \end{bmatrix} + C(q,\dot{q})\dot{q} + \begin{bmatrix} 0 \\ mgd_3 c_{\theta_2} \\ mgs_{\theta_2} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

By definition, we have $c_{ijk} = \frac{1}{2}\left(\frac{\partial d_{ki}(q)}{\partial(q_j)} + \frac{\partial d_{kj}(q)}{\partial(q_i)} - \frac{\partial d_{ij}(q)}{\partial(q_k)}\right)$. In our case, this gives us:

$$c_{121} = c_{211} = -md_3^2 c_{\theta_2} s_{\theta_2}$$
$$c_{112} = md_3^2 c_{\theta_2} s_{\theta_2}$$
$$c_{223} = -md_3$$
$$c_{232} = c_{322} = md_3$$
$$c_{113} = -md_3 c_{\theta_2}^2$$
$$c_{131} = c_{311} = md_3 c_{\theta_2}^2$$

Therefore:

$$D(q) = \begin{bmatrix} md_3^2 c_{\theta_2}^2 & 0 & 0 \\ 0 & md_3^2 & 0 \\ 0 & 0 & m \end{bmatrix}$$
$$C(q,\dot{q}) = \begin{bmatrix} -md_3^2 c_{\theta_2} s_{\theta_2} \dot{\theta}_2 + md_3 c_{\theta_2}^2 \dot{d}_3 & -md_3^2 c_{\theta_2} s_{\theta_2} \dot{\theta}_1 & md_3 c_{\theta_2}^2 \dot{\theta}_1 \\ md_3^2 c_{\theta_2} s_{\theta_2} \dot{\theta}_1 & md_3 \dot{d}_3 & md_3 \dot{\theta}_2 \\ -md_3 c_{\theta_2}^2 \dot{\theta}_1 & -md_3 \dot{\theta}_2 & 0 \end{bmatrix}$$
$$G(q) = \begin{bmatrix} 0 \\ mgd_3 c_{\theta_2} \\ mgs_{\theta_2} \end{bmatrix}$$

2. The inertia matrix of the manipulator, $D(q)$, is defined as:

$$D(q) = \sum_{i=1}^{n} (m_i J_{P_i}^T J_{P_i} + J_{O_i}^T R_i^T I_i R_i J_{O_i})$$

In the previous section, the only mass in our manipulator was a point mass, and therefore the inertia matrix had no rotational element. To add the rotational element required in this section, we need to find J_{O_3}, R_3 and I_3, the latter being the tensor of inertia of the end-effector in a reference frame attached to its center of mass.
The tensor of inertia of a radius R ball and uniform density ρ can be calculated as follows:

$$I_{xx} = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} \rho(y^2+z^2)dxdydz = ... = \frac{2}{5}mR^2$$
$$I_{xy} = -\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} \rho xydxdydz = ... = 0$$

and from symmetry, we have $I_3 = \frac{2}{5}mR^2 I$[1]. The element added to $D(q)$ is therefore:
$J_{O_3}^T R_3^T I_3 R_3 J_{O_3} = \frac{2}{5}mR^2 J_{O_3}^T R_3^T I R_3 J_{O_3} = \frac{2}{5}mR^2 J_{O_3}^T J_{O_3}$
Using the DH convention, we have $z_0 = [0,0,1]^T$, $z_1 = [s_{\theta_1}, -c_{\theta_1}, 0]^T$ and $z_2 = [c_{\theta_1}c_{\theta_2}, s_{\theta_1}c_{\theta_2}, s_{\theta_2}]^T$, therefore $J_{O_3} = \begin{bmatrix} 0 & s_{\theta_1} & 0 \\ 0 & -c_{\theta_1} & 0 \\ 1 & 0 & 0 \end{bmatrix}$. This gives us the new $D(q)$ (noting that the position of the mass is now $d_3 + R$):

$$D(q) = \begin{bmatrix} m(d_3+R)^2 c_{\theta_2}^2 & 0 & 0 \\ 0 & m(d_3+R)^2 & 0 \\ 0 & 0 & m \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ s_{\theta_1} & -c_{\theta_1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & s_{\theta_1} & 0 \\ 0 & -c_{\theta_1} & 0 \\ 1 & 0 & 0 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 + m(d_3+R)^2 c_{\theta_2}^2 & 0 & 0 \\ 0 & 1 + m(d_3+R)^2 & 0 \\ 0 & 0 & m \end{bmatrix}$$

[1] Source: https://scienceworld.wolfram.com/physics/MomentofInertiaSphere.html

The final Lagrangian will have $d_3 + R$ instead of d_3 in $C(q, \dot{q})$ and $G(q)$ as well. The rest remains unchanged.

3. A gravity field in the direction of $\hat{y}_0$ would affect the potential energy, which would depend on the coordinate of the end-effector mass along the y_0 axis: $d_3 s_{\theta_1} c_{\theta_2}$. The potential energy is then $U = -mgd_3 s_{\theta_1} c_{\theta_2}$. Taking the gradient of the Lagrangian w.r.t. q (recall the potential energy is subtracted), we have:

$$G(q) = \begin{bmatrix} mgd_3 c_{\theta_1} c_{\theta_2} \\ -mgd_3 s_{\theta_1} s_{\theta_2} \\ mgs_{\theta_1} c_{\theta_2} \end{bmatrix}$$