Introduction to Robotics 046212, Spring 2021
Homework 5

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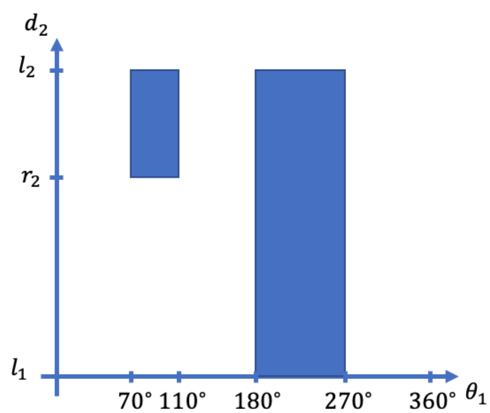
Question 1.

1. The C-space created by the two arches is drawn below. The prismatic joint is limited between l_1 and l_2, and the revolute joint is only limited by $ob1$, which is closer to the origin than the minimum length of the prismatic joint (and therefore prohibits some angles entirely).



2. When d_2 is at $l_1 < d_2 < \frac{l_1+l_2-a\sqrt{2}}{2}$, there are no obstacles. Also, for any point beyond the top and bottom corners of the square, i.e. $d_2 \geq \sqrt{\left(\frac{l_1+l_2}{2}\right)^2 + \left(\frac{a\sqrt{2}}{2}\right)^2} = \frac{1}{2}\sqrt{(l_1+l_2)^2 + 2a^2}$, the obstacle boundaries are vertical lines at $\theta_1 = \pm \arctan\left(\frac{a\sqrt{2}}{l_1+l_2}\right)$. For any point where d_2 is at $\frac{l_1+l_2-a\sqrt{2}}{2} < d_2 < \frac{1}{2}\sqrt{(l_1+l_2)^2+2a^2}$, we have $y_0 = x_0 - \pm\frac{l_1+l_2-a\sqrt{2}}{2}$. Plugging in the transformation to the C-space, which is $x_0 = d_2\cos\theta_1, y_0 = d_2 \sin\theta_1$, we obtain the limits of the obstacle as the two curves defined

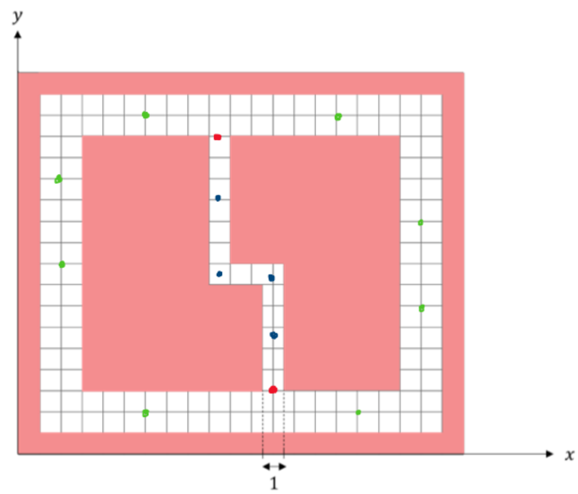
by:

$$d_2 \sin\theta_1 = d_2 \cos\theta_1 \pm \frac{l_1 + l_2 - a\sqrt{2}}{2} \Rightarrow d_2(\sin\theta_1 - \cos\theta_1) = \pm\frac{l_1 + l_2 - a\sqrt{2}}{2}$$

$$\Rightarrow -d_2\sqrt{2}\cos\left(\theta_1 + \frac{\pi}{4}\right) = \pm\frac{l_1 + l_2 - a\sqrt{2}}{2} \Rightarrow \theta_1 = \arccos\left(\mp\frac{l_1 + l_2 - a\sqrt{2}}{2d_2\sqrt{2}}\right) - \frac{\pi}{4}$$

Question 2.

1. Not necessarily, since the lower bound is only a probabilistic one. As a counter-example, consider finding a path connecting the two red points in the configuration space discussed in the tutorial:



Sampling the 4 blue points would yield success. However, sampling the 8 green points will not.

2. Not necessarily. Assume the shorter path selected is of length L', such that $L' = \frac{L}{2}$. However, this path could also have lower clearance: $\rho' = \frac{\rho}{3}$. The new lower bound is:

$$1 - \left\lceil \frac{2\frac{L}{2}}{\frac{\rho}{3}} \right\rceil e^{-\sigma\left(\frac{\rho}{3}\right)^d n} = 1 - \left\lceil \frac{3}{2} \cdot \frac{2L}{\rho} \right\rceil e^{-\sigma\rho^d n \cdot 3^{-d}} \le 1 - \left\lceil \frac{2L}{\rho} \right\rceil e^{-\sigma\rho^d n \cdot 3^{-d}} \le 1 - \left\lceil \frac{2L}{\rho} \right\rceil e^{-\sigma\rho^d n}$$

which is lower than the original one for the longer path.

Question 3.

1. The damping ratio for this type of system is defined as $\zeta = \frac{b}{2\sqrt{km}}$, which in this case gives us $\zeta = \frac{2}{2\sqrt{0.1\cdot 4}} \approx 1.581$. Since $\zeta > 1$, the uncontrolled system is overdamped. The time constant of convergence to the origin (considering $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.1}{4}}$ is:

$$\left(-\zeta\omega + \omega\sqrt{\zeta^2 - 1}\right)^{-1} = \left(-\frac{1}{\sqrt{0.4}}\sqrt{\frac{0.1}{4}} + \sqrt{\frac{0.1}{4}}\sqrt{\left(\frac{1}{\sqrt{0.4}}\right)^2 - 1}\right)^{-1} =$$

$$= \left(-\frac{1}{2\sqrt{0.1}}\frac{\sqrt{0.1}}{2} + \frac{\sqrt{0.1}}{2}\sqrt{\frac{1}{0.4} - 1}\right)^{-1} = \left(-\frac{1}{4} + \frac{\sqrt{0.1}}{2}\sqrt{\frac{6}{4}}\right)^{-1} =$$

$$= \frac{4}{2\sqrt{0.6} - 1} \approx 7.283$$

2. Applying the P controller $f = K_p(x_d - x)$ with $x_d = 0$, we obtain:

$$m\ddot{x} + b\dot{x} + kx = -K_p x$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k + K_p}{m}x = 0 \Rightarrow \omega = \sqrt{\frac{k + K_p}{m}}$$

$$1 = \zeta_P = \frac{b}{2\sqrt{(k + K_p)m}} \Rightarrow K_p = \frac{b^2}{4m} - k = \frac{1}{4} - 0.1 = 0.15$$

Where the last line stems from the requirement to keep the system critically damped.

3. Differentiating the error, we have $f = K_d\dot{x}_e = K_d(\dot{x}_d - \dot{x})$. Plugging in $\dot{x}_d = 0$ we obtain:

$$m\ddot{x} + b\dot{x} + kx = -K_d\dot{x}$$

$$\ddot{x} + \frac{b + K_d}{m}\dot{x} + \frac{k}{m}x = 0$$

$$1 = \zeta_D = \frac{b + K_d}{2m\sqrt{km}} \Rightarrow K_d = 2m\sqrt{km} - b = 2\cdot 4\sqrt{0.1\cdot 4} - 2 \approx 3.06$$

4. The differential equation describing the system with the PD controller $f = K_p(x_d - x) + K_d(\dot{x}_d - \dot{x}) = K_p(1 - x) - K_d\dot{x}$ is:

$$m\ddot{x} + b\dot{x} + kx = K_p(1 - x) - K_d\dot{x}$$

$$\ddot{x} + \frac{b + K_d}{m}\dot{x} + \frac{k + K_p}{m}x = K_p$$

To solve this ODE, we can take $\tilde{x}(t)$ to be the solution of the homogeneous equation:

$$\ddot{\tilde{x}} + \frac{b + K_d}{m}\dot{\tilde{x}} + \frac{k + K_p}{m}\tilde{x} = 0$$

$$\Rightarrow \omega = \sqrt{\frac{k + K_p}{m}} \quad \zeta = \frac{b + K_d}{2\sqrt{m(k + K_p)}} \approx \frac{2 + 3.06}{2\cdot\sqrt{4\cdot(0.1 + 0.15)}} = 2.53 > 1$$

Since this system is overdamped, the solution is $\tilde{x}(t) = c_1 e^{(-\zeta\omega+\omega\sqrt{\zeta^2-1})t} + c_2 e^{(-\zeta\omega-\omega\sqrt{\zeta^2-1})t}$. Using the fact that $\tilde{x}$ solves the homogeneous ODE, we can verify that the solution to the non-homogeneous equation is $x(t) = \tilde{x}(t) + c_3$, as follows:

$$\ddot{x} + \frac{b+K_d}{m}\dot{x} + \frac{k+K_p}{m}x = K_p$$
$$\underbrace{\ddot{\tilde{x}} + \frac{b+K_d}{m}\dot{\tilde{x}} + \frac{k+K_p}{m}\tilde{x}}_{=0} + \frac{k+K_p}{m}c_3 = K_p$$
$$\Rightarrow c_3 = \frac{mK_p}{k+K_p} \Rightarrow x(t) = c_1 e^{(-\zeta\omega+\omega\sqrt{\zeta^2-1})t} + c_2 e^{(-\zeta\omega-\omega\sqrt{\zeta^2-1})t} + \frac{mK_p}{k+K_p}$$

As $t \to \infty$, we have $x(t) \to \frac{mK_p}{k+K_p}$ (the system is stable since $\zeta\omega > 0$ and $\omega^2 > 0$). Therefore, the steady state error is $\lim_{t\to\infty} x_e(t) = \lim_{t\to\infty}(x_d - x(t)) = 1 - \frac{mK_p}{k+K_p}$. The steady state control force can be found by plugging this in:

$$\lim_{t\to\infty} f = \lim_{t\to\infty}(K_p x_e(t) + K_d \dot{x}_e(t)) = K_p \lim_{t\to\infty} x_e(t) - K_d \lim_{t\to\infty} \dot{x}(t) =$$
$$= K_p\left(1 - \frac{mK_p}{k+K_p}\right)$$

5. A PID controller is of the form $f(t) = K_p x_e(t) + K_d \dot{x}_e(t) + K_i \int_0^t x_e(\tau)d\tau$. Plugging into the equation of motion (and dropping the (t) notation for simplicity), we have:

$$m\ddot{x} + b\dot{x} + kx = K_p x_e + K_d \dot{x}_e + K_i \int_0^t x_e d\tau$$

Since $x_e = x_d - x$ and $\ddot{x}_d = \dot{x}_d = 0$, we have $\dot{x}_e = -\dot{x}, \ddot{x}_e = -\ddot{x}$. Therefore:

$$-m\ddot{x}_e - b\dot{x}_e + k(x_d - x_e) = K_p x_e + K_d \dot{x}_e + K_i \int_0^t x_e d\tau$$
$$m\ddot{x}_e + (b+K_d)\dot{x}_e + (k+K_p)x_e + K_i \int_0^t x_e d\tau = kx_d$$

Taking the time derivative of this equation (and assuming x_d is constant), we have:

$$m\dddot{x}_e + (b+K_d)\ddot{x}_e + (k+K_p)\dot{x}_e + K_i x_e = 0$$

To find the stability conditions, we can use the Routh-Hurwitz method:

m	$k+K_p$	0
$b+K_d$	K_i	0
$\frac{(b+K_d)K_p - mK_i}{b+K_d}$	0	0
K_i	0	0

Since $m > 0$, for the system to be stable the entire left column of the table should be positive. Therefore, the conditions on K_p, K_d and K_i for the system to be stable are:

$$K_i > 0, \quad K_d > -b, \quad (b + K_d)K_p > mK_i$$

To show that a zero steady-state error is possible, we select K_p, K_d and K_i (in accordance with the conditions) to be: $K_p = 11.9, K_d = 10, K_i = 4$. Plugging in and normalizing, we obtain the following ODE:

$$\begin{aligned}
&\dddot{x}_e + \frac{b + K_d}{m}\ddot{x}_e + \frac{k + K_p}{m}\dot{x}_e + \frac{K_i}{m}x_e = 0 \\
&\dddot{x}_e + 3\ddot{x}_e + 3\dot{x}_e + x_e = 0
\end{aligned}$$

An easy-to-guess solution to this ODE is $x_e(t) = Ce^{-t}$, which clearly produces zero steady-state error as $t \to \infty$.