Introduction to Robotics 046212, Spring 2021
Homework 1

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Question 1.

A. Rotating the blue gimbal first corresponds to a rotation around the fixed x axis. Since the green gimbal hasn't moved in the process, the next rotation (of the green gimbal) is around the fixed y axis (assuming the orientation of the gimbals matches the drawn reference frame). Third, rotating the purple gimbal (which hasn't moved yet) is again a rotation around the fixed-frame x axis. Therefore, the matching Euler parametrization is XYX**, all in the fixed frame**.

B. Rotating the purple gimbal first is a rotation around the fixed x axis. Then, rotating the green gimbal is a rotation around the y' axis in the current frame. Finally, rotating the blue gimbal is a rotation around the new x'' axis in the current frame. All in all, the parametrization is XYX**, all w.r.t. the current frame** (for the first rotation, the fixed frame is also the current one).

C. We can use the observation from sec. A, that all rotations are in the fixed frame when rotating in the order blue-green-purple. Therefore, the only change necessary to match the Roll-Pitch-Yaw parametrization is to fix the purple gimbal axis parallel to the z axis, and leave the other two as they are.

Question 2.

A. A possible set of rotations is a $-90°$ rotation around the current (fixed-frame) x axis, and then a $45°$ rotation around the new y axis (which corresponds to a $-45°$ rotation around the fixed-frame z axis).

B. We can compose the elementary rotations around x and y to obtain the rotation matrix (postmultiplying since we are using rotations w.r.t current frame):

$$R = R_x(-90°)R_y(45°) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(-90°) & -\sin(-90°) \\ 0 & \sin(-90°) & \cos(-90°) \end{pmatrix} \begin{pmatrix} \cos 45° & 0 & \sin 45° \\ 0 & 1 & 0 \\ -\sin 45° & 0 & \cos 45° \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 \end{pmatrix}$$

Finally, we can obtain the axis and angle parametrization using the formula from the tutorial:

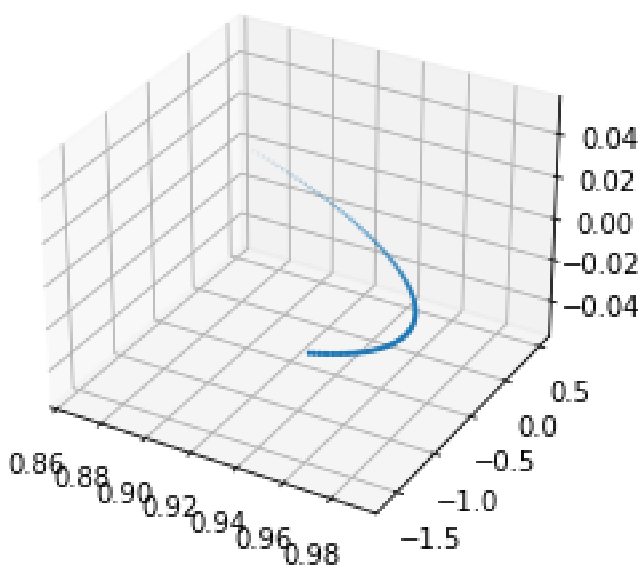
$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r33 - 1}{2}\right) = \cos^{-1}\left(\frac{\frac{\sqrt{2}}{2} - 1}{2}\right) = \cos^{-1}\left(\frac{\sqrt{2} - 2}{4}\right) \approx 98.42°$$

$$r = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = \frac{1}{2\sin(98.42°)}\begin{bmatrix} -1 - \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \approx 0.505 \begin{bmatrix} -1 - \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

C. To find the point's new pose, we need to apply the rotation matrix to its original pose:

$$p' = Rp = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 \end{pmatrix} (\cos(\alpha), \sin(\alpha), 1)^T = \begin{bmatrix} \frac{\sqrt{2}}{2}(1 + \cos\alpha) \\ \frac{\sqrt{2}}{2}(1 - \cos\alpha) \\ -\sin\alpha \end{bmatrix}$$

D. We can use the angle and axis parametrization as a single-parameter rotation to draw the trajectory of the given point. Below is a drawing of the 3D trajectory of the point:



The following code was used to draw the trajectory:

```
import numpy as np
import matplotlib.pyplot as plt

alpha = np.radians(30)
point = np.array((np.cos(alpha), np.sin(alpha), 1))

angle = 98.42
theta = np.radians(angle)

r = [-1.0 - np.sqrt(2.0) / 2.0, np.sqrt(2.0) / 2.0, -np.sqrt(2.0) /
    2.0] / (2 * np.sin(theta))

def rotate(x, t):
    c = np.cos(t)
    s = np.sin(t)

    rotation = np.array([
        [r[0] * r[0] * (1 - c) + c, r[0] * r[1] *(1 - c) + r[2] * s,
    r[0] * r[2] * (1-c) + r[1] * s],
        [r[0] * r[1] * (1 - c) + r[2] * s, r[1] * r[1] * (1 - c) + c,
     r[1] * r[2] * (1 - c) + r[0] * s],
        [r[0] * r[2] * (1 - c) + r[1] * s, r[1] * r[2] * (1 - c) + r
    [0] * s, r[2] * r[2] * (1 - c) + s]
    ])

    return np.dot(rotation, x)

xyz = []
for t in np.linspace(0.0, theta, num=100):
    xyz.append(rotate(point, t))

points = np.stack(xyz)

fig = plt.figure()
ax = fig.add_subplot(projection='3d')
plt.scatter(points[:, 0], points[:, 1], points[:, 2])
```

Question 3.

A. Since the columns of a rotation matrix are the axes of the new frame of reference projected on the old one, they are orthonormal vectors, which makes the rotation matrix an orthogonal matrix. As such (and as seen in class), we have $RR^T = I \Leftrightarrow R^T = R^{-1}$. Taking the determinant of both sides of this equation, we obtain $\det(R) = \det(R^T) = \det(R^{-1}) = \frac{1}{\det(R)}$. The only real numbers which are their own inverse are ± 1, so it follows directly that $|\det(R)| = 1$.

B. We can use the axis-angle parametrization of a rotation matrix to construct $f(\theta)$:

$$R(r, \theta) = \begin{bmatrix} r_x^2(1-\cos\theta)+\cos\theta & r_xr_y(1-\cos\theta)-r_z\sin\theta & r_xr_z(1-\cos\theta)+r_y\sin\theta \\ r_xr_y(1-\cos\theta)+r_z\sin\theta & r_y^2(1-\cos\theta)+\cos\theta & r_yr_z(1-\cos\theta)-r_x\sin\theta \\ r_xr_z(1-\cos\theta)-r_y\sin\theta & r_yr_z(1-\cos\theta)+r_x\sin\theta & r_z^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

Obtaining a fixed r from R using the formula in the tutorial, we can define $f(\theta) = R(r,\theta)$ (in case $\sin\theta = 0$, the formula from the tutorial won't work, and we can instead solve $Rr = r$ for r). For a fixed r, f is continuous in θ since each element of the matrix is an affine composition of sin and cos functions, which are continuous in θ.

C. For any square matrix, $\det(\cdot)$ can be calculated as a linear combination of the determinants of its minors. This can be expanded all the way down to 2×2 matrices, for which the determinant is defined as $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, which is a continuous function of the elements. Therefore, building back up to a determinant of an $n \times n$ matrix, each step is a linear combination of continuous functions, which is in turn continuous.

D. Since $|\det(R)| = 1$ for any R, the only values $\det(R)$ could have are 1 and -1. I is the identity rotation matrix, and we have $\det(I) = \det(f(0)) = 1$. Since $\det(R) = \det(f(\nu))$ and both $\det(\cdot)$ and $f(\theta)$ are continuous function, it follows that $\det(R) \neq -1$, as the discontinuity moving from 1 to -1 is impossible.

Question 4.

A. The translation from B to A is 3 along x_A. The rotation from B to A is simply a π rotation around the z axis. Therefore:

$$A_B^A = \begin{bmatrix} R_B^A & o_B^A \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the same manner, the translation from C to B is 2 along z_B. The rotation is 90° around y_B, and then 150° around the new x'_B. We can compose the two elementary matrices to obtain the rotation:

$$R_C^B = \begin{bmatrix} \cos(90^\circ) & 0 & \sin(90^\circ) \\ 0 & 1 & 0 \\ -\sin(90^\circ) & 0 & \cos(90^\circ) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(150^\circ) & -\sin(150^\circ) \\ 0 & \sin(150^\circ) & \cos(150^\circ) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix}$$

Plugging in, we obtain:

$$A_C^B = \begin{bmatrix} R_C^B & o_C^B \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, we have:

$$A_C^A = A_B^A A_C^B = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 3 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

B. Since $CD = AB = 3$, we have $FC = 6$. Therefore, $FD = FC \cos 30^\circ = 3\sqrt{3}$. P is located next to the centroid, which means its coordinates in frame A are $P^A = (-4, \frac{3\sqrt{3}}{2}, 1)^T$. We can now use the inverse transform to express P in frame C:

$$\tilde{P}^C = A_A^C \tilde{P}^A = \begin{bmatrix} (R_C^A)^T & -(R_C^A)^T o_C^A \\ 0^T & 1 \end{bmatrix} \tilde{P}^A = \begin{bmatrix} 0 & 0 & -1 & 2 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & \frac{3}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{3\sqrt{3}}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ \frac{3\sqrt{3}}{2} \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ -\frac{11\sqrt{3}}{4} \\ 1 \end{bmatrix}$$