Introduction to Robotics 046212, Spring 2021
Homework 3

Orr Krupnik 302629027

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Question 1. We've found the DH parameters for this arm in question 2 of the previous homework assignment:

Link	a_i	α_i	d_i	θ_i
1	$\sqrt{l_1^2 + l_2^2}$	0	d_1	$\arctan\left(\frac{l_2}{l_1}\right)$
2	0	$\frac{\pi}{2}$	l_3	θ_2
3	0	0	d_3	0

We can now use these parameters to find the forward kinematics function for this manipulator:

$$T_3^0(q) = A_1^0(q_1)A_2^1(q_2)A_3^2(q_3) =$$

$$= \begin{bmatrix} c_{\theta_1} & -s_{\theta_1}c_{\alpha_1} & s_{\theta_1}s_{\alpha_1} & a_1c_{\theta_1} \\ s_{\theta_1} & c_{\theta_1}c_{\alpha_1} & -c_{\theta_1}s_{\alpha_1} & a_1s_{\theta_1} \\ 0 & s_{\alpha_1} & c_{\alpha_1} & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\theta_2} & -s_{\theta_2}c_{\alpha_2} & s_{\theta_2}s_{\alpha_2} & a_2c_{\theta_2} \\ s_{\theta_2} & c_{\theta_2}c_{\alpha_2} & -c_{\theta_2}s_{\alpha_2} & a_2s_{\theta_2} \\ 0 & s_{\alpha_2} & c_{\alpha_2} & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c_{\theta_3} & -s_{\theta_3}c_{\alpha_3} & s_{\theta_3}s_{\alpha_3} & a_3c_{\theta_3} \\ s_{\theta_3} & c_{\theta_3}c_{\alpha_3} & -c_{\theta_3}s_{\alpha_3} & a_3s_{\theta_3} \\ 0 & s_{\alpha_3} & c_{\alpha_3} & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 & a_1c_{\theta_1} \\ s_{\theta_1} & c_{\theta_1} & 0 & a_1s_{\theta_1} \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\theta_2} & 0 & s_{\theta_2} & 0 \\ s_{\theta_2} & 0 & -c_{\theta_2} & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 & a_1c_{\theta_1} \\ s_{\theta_1} & c_{\theta_1} & 0 & a_1s_{\theta_1} \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\theta_2} & 0 & s_{\theta_2} & d_3s_{\theta_2} \\ s_{\theta_2} & 0 & -c_{\theta_2} & -d_3c_{\theta_2} \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} c_{\theta_1+\theta_2} & 0 & s_{\theta_1+\theta_2} & d_3s_{\theta_1+\theta_2} + a_1c_{\theta_1} \\ s_{\theta_1+\theta_2} & 0 & -c_{\theta_1+\theta_2} & d_3c_{\theta_1+\theta_2} + a_1s_{\theta_1} \\ 0 & 1 & 0 & l_3 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Our parameter vector q is $(d_1, \theta_2, d_3)^T$. Denoting the position of the end effector $(p_{W_x}, p_{W_y}, p_{W_z})^T$ and plugging in a_1 (and holding off on plugging in θ_1) we have:

$$\begin{aligned}
p_{W_x} &= d_3 \sin(\theta_2 + \theta_1) + \sqrt{l_1^2 + l_2^2}\cos(\theta_1) \\
p_{W_y} &= d_3 \cos(\theta_2 + \theta_1) + \sqrt{l_1^2 + l_2^2}\sin(\theta_1) \\
p_{W_z} &= l_3 + d_1
\end{aligned}$$

We immediately obtain $d_1 = p_{W_x} - l_3$. Additionally, we note that $\cos(\theta_1) = \cos\left(\arctan\left(\frac{l_2}{l_1}\right)\right) = \frac{l_2}{\sqrt{l_1^2+l_2^2}}$ and conversely $\sin(\theta_1) = \frac{l_1}{\sqrt{l_1^2+l_2^2}}$. Plugging this in, moving to the left hand side and adding up the squares of the first two equations, we obtain:

$$\begin{gathered}
p_{W_x} = d_3 \sin(\theta_2 + \theta_1) + l_2 \Leftrightarrow p_{W_x} - l_2 = d_3 \sin(\theta_2 + \theta_1) \\
p_{W_y} = d_3 \cos(\theta_2 + \theta_1) + l_1 \Leftrightarrow p_{W_y} - l_1 = d_3 \cos(\theta_2 + \theta_1) \\
\Downarrow \\
d_3^2 = (p_{W_x} - l_2)^2 + (p_{W_y} - l_1)^2 \Rightarrow d_3 = \sqrt{(p_{W_x} - l_2)^2 + (p_{W_y} - l_1)^2}
\end{gathered}$$

We can discard the negative solution, since d_3 is the extension of a prismatic joint. Finally, we can plug d_3 into either of the first two equations to find θ_2:

$$\begin{aligned}
p_{W_x} &= \sqrt{(p_{W_x} - l_2)^2 + (p_{W_y} - l_1)^2}\sin(\theta_2 + \theta_1) + \sqrt{l_1^2 + l_2^2}\cos(\theta_1) \\
\Rightarrow \sin(\theta_2 + \theta_1) &= \frac{p_{W_x} - l_2}{\sqrt{(p_{W_x} - l_2)^2 + (p_{W_y} - l_1)^2}} \\
\Rightarrow \theta_2 &= \arcsin\left(\frac{p_{W_x} - l_2}{\sqrt{(p_{W_x} - l_2)^2 + (p_{W_y} - l_1)^2}}\right) - \arctan\left(\frac{l_2}{l_1}\right)
\end{aligned}$$

Question 2.

1. With three joints, the geometric Jacobian will be a (6×3) matrix, with each joint contributing a 6-d vector as we've seen in class. We then have:

$$\begin{bmatrix} J_{P_1} \\ J_{O_1} \end{bmatrix} = \begin{bmatrix} z_0 \times (p_3 - p_0) \\ z_0 \end{bmatrix} \quad \begin{bmatrix} J_{P_2} \\ J_{O_2} \end{bmatrix} = \begin{bmatrix} z_1 \times (p_3 - p_1) \\ z_1 \end{bmatrix} \quad \begin{bmatrix} J_{P_3} \\ J_{O_3} \end{bmatrix} = \begin{bmatrix} z_2 \\ 0 \end{bmatrix}$$

   The z vector for the first joint is simply $z_0 = (0, 0, 1)^T$ and it is located at $p_0 = (0, 0, 0)^T$. We note p_2 and p_1 are in the same location; both are at the end of the first link, coinciding with the second joint. The location has a fixed z coordinate, l_1, and has x and y coordinates defined by θ_1: $p_1 = p_2 = (l_2 c_{\theta_1}, l_2 s_{\theta_1}, l_1)^T$. The unit vector for the revolute joint is $z_1 = (s_{\theta_1}, -c_{\theta_1}, 0)^T$, similar to the example in class. The axis of the prismatic joint depends both on θ_1 and θ_2 and can be calculated using the transforms arising from the DH parameters of the arm, or produced from the geometry of the arm: $z_2 = (c_{\theta_1} c_{\theta_2}, s_{\theta_1} c_{\theta_2}, s_{\theta_2})^T$.

The end effector location is at a z coordinate offset from p_2 by $d_3 s_{\theta_2}$, and along the same vector as p_2 in the xy plain, with a radius longer by $d_3 c_{\theta_2}$. Therefore, $p_3 = [(l_2 + d_3 c_{\theta_2})c_{\theta_1}, (l_2 + d_3 c_{\theta_2})s_{\theta_1}, l_1 + d_3 s_{\theta_2}]^T$.
We can now use all of the above to produce the elements of the Jacobian:

$$J_{P_1} = z_0 \times (p_3 - p_0) = (0,0,1)^T \times \begin{bmatrix} (l_2 + d_3 c_{\theta_2})c_{\theta_1} \\ (l_2 + d_3 c_{\theta_2})s_{\theta_1} \\ l_1 + d_3 s_{\theta_2} \end{bmatrix} =$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ (l_2 + d_3 c_{\theta_2})c_{\theta_1} & (l_2 + d_3 c_{\theta_2})s_{\theta_1} & l_1 + d_3 s_{\theta_2} \end{vmatrix} = \begin{bmatrix} -(l_2 + d_3 c_{\theta_2})s_{\theta_1} \\ (l_2 + d_3 c_{\theta_2})c_{\theta_1} \\ 0 \end{bmatrix}$$

$$J_{P_2} = z_1 \times (p_3 - p_1) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ s_{\theta_1} & -c_{\theta_1} & 0 \\ d_3 c_{\theta_2} c_{\theta_1} & d_3 c_{\theta_2} s_{\theta_1} & d_3 s_{\theta_2} \end{vmatrix} = \begin{bmatrix} -d_3 c_{\theta_1} s_{\theta_2} \\ -d_3 s_{\theta_1} s_{\theta_2} \\ d_3 c_{\theta_2} \end{bmatrix}$$

We can now obtain the full Jacobian:

$$J(q) = \begin{bmatrix} J_{P_1} & J_{P_2} & J_{P_3} \\ J_{O_1} & J_{O_2} & J_{O_3} \end{bmatrix} = \begin{bmatrix} -(l_2 + d_3 c_{\theta_2})s_{\theta_1} & -d_3 c_{\theta_1} s_{\theta_2} & c_{\theta_1} c_{\theta_2} \\ (l_2 + d_3 c_{\theta_2})c_{\theta_1} & -d_3 s_{\theta_1} s_{\theta_2} & s_{\theta_1} c_{\theta_2} \\ 0 & d_3 c_{\theta_2} & s_{\theta_2} \\ 0 & s_{\theta_1} & 0 \\ 0 & -c_{\theta_1} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

2. To find the directions of reduced manipulability, we need to find the singularities of the position Jacobian, J_P, by calculating its determinant:

$$\det(J_P) = \begin{vmatrix} -(l_2 + d_3 c_{\theta_2})s_{\theta_1} & -d_3 c_{\theta_1} s_{\theta_2} & c_{\theta_1} c_{\theta_2} \\ (l_2 + d_3 c_{\theta_2})c_{\theta_1} & -d_3 s_{\theta_1} s_{\theta_2} & s_{\theta_1} c_{\theta_2} \\ 0 & d_3 c_{\theta_2} & s_{\theta_2} \end{vmatrix} =$$

$$= -(l_2 + d_3 c_{\theta_2})s_{\theta_1}[-d_3 s_{\theta_1} s^2_{\theta_2} - c^2_{\theta_2} d_3 s_{\theta_1}] + d_3 c_{\theta_1} s_{\theta_2}(l_2 + d_3 c_{\theta_2})c_{\theta_1} s_{\theta_2} +$$
$$+ c_{\theta_1} c_{\theta_2}(l_2 + d_3 c_{\theta_2})c_{\theta_1} d_3 c_{\theta_2} =$$
$$= d_3(l_2 + d_3 c_{\theta_2})[s^2_{\theta_1} + c^2_{\theta_1} s^2_{\theta_2} + c^2_{\theta_1} c^2_{\theta_2}] = d_3(l_2 + d_3 c_{\theta_2})$$

The determinant is 0 in two cases: $d_3 = 0$ or $l_2 + d_3 s_{\theta_2} = 0$ (with the latter only possible if the maximum extension for d_3 is at least l_2).
For each of these configurations, we can find the directions of reduced manipulability by finding the eigenvectors matching the 0 eigenvalues of J_P^T.

- For $d_3 = 0$ (assuming $s_{\theta_2} \neq 0$ and $c_{\theta_1} \neq 0$):

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = J_P^T v = \begin{bmatrix} -l_2 s_{\theta_1} & l_2 c_{\theta_1} & 0 \\ 0 & 0 & 0 \\ c_{\theta_1} c_{\theta_2} & s_{\theta_1} c_{\theta_2} & s_{\theta_2} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \Rightarrow \begin{cases} l_2 s_{\theta_1} v_x = l_2 c_{\theta_1} v_y \Rightarrow v_y = \frac{s_{\theta_1}}{c_{\theta_1}} v_x \\ c_{\theta_1} c_{\theta_2} v_x + s_{\theta_1} c_{\theta_2} v_y + s_{\theta_2} v_z = 0 \end{cases}$$

$$\Rightarrow v_z = -\frac{c_{\theta_2}}{s_{\theta_2}} \left(c_{\theta_1} + \frac{s^2_{\theta_1}}{c_{\theta_1}} \right) v_x = -\frac{c_{\theta_2}}{s_{\theta_2} c_{\theta_1}} v_x$$

We can choose $v_x = c_{\theta_1}$ and the direction of reduced manipulability matching this singularity is $v = \left[c_{\theta_1}, s_{\theta_1}, -\frac{c_{\theta_2}}{s_{\theta_2}}\right]^T$. This singularity happens when the prismatic joint is not extended at all, which means rotations of joint 2 will not change the position of the end effector.

- For $l_2 + d_3 c_{\theta_2} = 0 \Leftrightarrow c_{\theta_2} = -l_2/d_3$, assuming w.l.o.g. $d_3 \neq 0$:

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = J_P^T v = \begin{bmatrix}0 & 0 & 0\\ -d_3 c_{\theta_1} s_{\theta_2} & -d_3 s_{\theta_1} s_{\theta_2} & -l_2\\ c_{\theta_1} c_{\theta_2} & s_{\theta_1} c_{\theta_2} & s_{\theta_2}\end{bmatrix}\begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix}$$

$$\Rightarrow \begin{cases}-d_3 s_{\theta_2}(c_{\theta_1} v_x + s_{\theta_1} v_y) = l_2 v_z\\ -c_{\theta_2}(c_{\theta_1} v_x + s_{\theta_1} v_y) = s_{\theta_2} v_z\end{cases}$$

$$\Rightarrow (c_{\theta_1} v_x + s_{\theta_1} v_y)\frac{s_{\theta_2}}{c_{\theta_2}} = -(c_{\theta_1} v_x + s_{\theta_1} v_y)\frac{c_{\theta_2}}{s_{\theta_2}} \Rightarrow c_{\theta_1} v_x + s_{\theta_1} v_y = 0$$

Which gives us $v_y = -\frac{c_{\theta_1}}{s_{\theta_1}} v_x$. Plugging back into one of the previous equations, we also obtain $v_z = 0$. We can select $v_x = s_{\theta_1}$, and the direction of reduced manipulability is then $v = [s_{\theta_1}, -c_{\theta_1}, 0]^T$. This singularity matches the case when the end effector is directly above joint 1, and the direction we found matches the vector in the $x_0 y_0$ plain orthogonal to l_2, which makes sense - moving joint 1 in this configuration will not yield any movement of the end effector.

3. The generalized forces vector operating on the end effector consists of gravity alone, and is therefore $\gamma_e = [0, 0, -Mg, 0, 0, 0]^T$. Using the geometric jacobian we found in the previous section, we can find the forces operating on the joints (neglecting, for now, the masses of the links):

$$\tau = J^T(q)\gamma_e = \begin{bmatrix}-(l_2 + d_3 c_{\theta_2}) s_{\theta_1} & (l_2 + d_3 c_{\theta_2}) c_{\theta_1} & 0 & 0 & 0 & 1\\ -d_3 c_{\theta_1} s_{\theta_2} & -d_3 s_{\theta_1} s_{\theta_2} & d_3 c_{\theta_2} & s_{\theta_1} & -c_{\theta_1} & 0\\ c_{\theta_1} c_{\theta_2} & s_{\theta_1} c_{\theta_2} & s_{\theta_2} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\-Mg\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\-Mg d_3 c_{\theta_2}\\-Mg s_{\theta_2}\end{bmatrix}$$

For the prismatic joint, This allows for configurations with no force required to hold an equilibrium when $s_{\theta_2} = 0$. These are configurations where the prismatic joint is parallel to the $x_0 y_0$ plain, which means that the mass of the object does not exert any force on the joint.
No torque is needed in joint 1 in any configuration. For joint 2, no torque is required to hold equilibrium when $c_{\theta_2} = 0$, which means the end effector is right above or right below joint 2; or when $d_3 = 0$, which means the mass held in the end effector is at the same location as joint 2. In either case, the direction of the prismatic joint is aligned with the gravity force, and therefore no torque is required to keep an equilibrium.

Question 3.

1. The constraints for our trajectory are:

$$\begin{aligned}
&x(0) = x_0 \quad x(T) = x_f \quad y(0) = y_0 \quad y(T) = y_f \\
&\dot{x}(0) = \dot{y}(0) = \ddot{x}(0) = \ddot{y}(0) = 0 \\
&\dot{x}(T) = \dot{y}(T) = \ddot{x}(T) = \ddot{y}(T) = 0
\end{aligned}$$

With twelve constraints, we can have two 5-deg polynomials (for x and y):

$$\begin{aligned}
x(t) &= \sum_{i=0}^{5} a_i t^i \quad \dot{x}(t) = \sum_{i=0}^{4} (i+1) a_{i+1} t^i \quad \ddot{x}(t) = \sum_{i=0}^{3} (i+1)(i+2) a_{i+2} t^i \\
y(t) &= \sum_{i=0}^{5} b_i t^i \quad \dot{y}(t) = \sum_{i=0}^{4} (i+1) b_{i+1} t^i \quad \ddot{y}(t) = \sum_{i=0}^{3} (i+1)(i+2) b_{i+2} t^i
\end{aligned}$$

Plugging in our constraints:

$$\begin{aligned}
x(0) = x_0 &\Rightarrow a_0 = x_0 \\
y(0) = y_0 &\Rightarrow b_0 = y_0 \\
\dot{x}(0) = 0 &\Rightarrow a_1 = 0 \\
\dot{y}(0) = 0 &\Rightarrow b_1 = 0 \\
\ddot{x}(0) = 0 &\Rightarrow a_2 = 0 \\
\ddot{y}(0) = 0 &\Rightarrow b_2 = 0 \\
x(T) = x_f &\Rightarrow a_5 T^5 + a_4 T^4 + a_3 T^3 + x_0 = x_f \\
y(T) = y_f &\Rightarrow b_5 T^5 + b_4 T^4 + b_3 T^3 + y_0 = y_f \\
\dot{x}(T) = 0 &\Rightarrow 5a_5 T^4 + 4a_4 T^3 + 3a_3 T^2 = 0 \\
\dot{y}(T) = 0 &\Rightarrow 5b_5 T^4 + 4b_4 T^3 + 3b_3 T^2 = 0 \\
\ddot{x}(T) = 0 &\Rightarrow 20a_5 T^3 + 12a_4 T^2 + 6a_3 T = 0 \\
\ddot{y}(T) = 0 &\Rightarrow 20b_5 T^3 + 12b_4 T^2 + 6b_3 T = 0
\end{aligned}$$

We now have six linear equations for the six remaining coefficients, which we can separate into two similar linear systems of three equations each:

$$\begin{bmatrix} T^5 & T^4 & T^3 \\ 5T^4 & 4T^3 & 3T^2 \\ 20T^3 & 12T^2 & 6T \end{bmatrix} \begin{bmatrix} a_5 \\ a_4 \\ a_3 \end{bmatrix} = \begin{bmatrix} x_f - x_0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} T^5 & T^4 & T^3 \\ 5T^4 & 4T^3 & 3T^2 \\ 20T^3 & 12T^2 & 6T \end{bmatrix} \begin{bmatrix} b_5 \\ b_4 \\ b_3 \end{bmatrix} = \begin{bmatrix} y_f - y_0 \\ 0 \\ 0 \end{bmatrix}$$

The solution of which is:

$$\begin{aligned}
a_5 &= \frac{6}{T^5}(x_f - x_0) \quad b_5 = \frac{6}{T^5}(y_f - y_0) \\
a_4 &= -\frac{15}{T^4}(x_f - x_0) \quad b_4 = -\frac{15}{T^4}(y_f - y_0) \\
a_3 &= \frac{10}{T^3}(x_f - x_0) \quad b_3 = \frac{10}{T^3}(y_f - y_0)
\end{aligned}$$

Plugging it all in, we have:

$$x(t) = \frac{6}{T^5}(x_f - x_0)t^5 - \frac{15}{T^4}(x_f - x_0)t^4 + \frac{10}{T^3}(x_f - x_0)t^3 + x_0$$
$$y(t) = \frac{6}{T^5}(y_f - y_0)t^5 - \frac{15}{T^4}(y_f - y_0)t^4 + \frac{10}{T^3}(y_f - y_0)t^3 + y_0$$

2. To find the end effector's velocity and acceleration, we simply need to differentiate the trajectory:

$$v(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} t^2(x_f - x_0)\frac{30}{T^3}\left[\frac{1}{T^2}t^2 - \frac{2}{T}t + 1\right] \\ t^2(y_f - y_0)\frac{30}{T^3}\left[\frac{1}{T^2}t^2 - \frac{2}{T}t + 1\right] \end{bmatrix} = t^2\frac{30}{T^3}\left(\frac{t}{T} - 1\right)^2 \begin{bmatrix} x_f - x_0 \\ y_f - y_0 \end{bmatrix}$$
$$a(t) = \begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} t(x_f - x_0)\frac{60}{T^3}\left[\frac{2}{T^2}t^2 - \frac{3}{T}t + 1\right] \\ t(y_f - y_0)\frac{60}{T^3}\left[\frac{2}{T^2}t^2 - \frac{3}{T}t + 1\right] \end{bmatrix} = t\frac{60}{T^3}\left(\frac{2}{T^2}t^2 - \frac{3}{T}t + 1\right) \begin{bmatrix} x_f - x_0 \\ y_f - y_0 \end{bmatrix}$$

3. To find $\theta_1(t)$ and $d_2(t)$ we can look at the position of the end effector in polar coordinates (we drop the (t) notation for convenience):

$$x = r\cos\alpha \quad y = r\sin\alpha$$
$$r = \sqrt{(l_1 + d_2)^2 + l_2^2} \quad \alpha = \theta_1 + \arctan\left(\frac{l_2}{l_1 + d_2}\right)$$
$$r^2 = l_1^2 + 2d_2 l_1 + d_2^2 + l_2^2 \Rightarrow d_2^2 + 2l_1 d_2 + l_1^2 + l_2^2 - r^2 = 0$$
$$\Rightarrow d_2 = \frac{-2l_1 \pm \sqrt{4l_1^2 - 4(l_1^2 + l_2^2 - r^2}}{2} = -l_1 + \sqrt{r^2 - l_2^2}$$

Where the second solution for d_2 can be discarded as it may be negative (assuming d_2 is limited to positive values). Plugging in $r^2 = x^2 + y^2$ we obtain:

$$d_2 = -l_1 + \sqrt{x^2 + y^2 - l_2^2}$$
$$\theta_1 = \alpha - \arctan\left(\frac{l_2}{l_1 + d_2}\right) = \arctan\left(\frac{y}{x}\right) - \arctan\left(\frac{l_2}{\sqrt{x^2 + y^2 - l_2^2}}\right)$$

And we can plug in $x(t), y(t)$ we found earlier to obtain the full trajectory.

4. Since we are required to keep a trajectory along a straight line where $y = 1$ and $l_1 > 1$, there will be certain positions of joint 1 where the straight line constraint cannot be satisfied (when d_2 is constrained to positive values).
To overcome the problem, we can set up a midway point which is feasible with the joint constraints, and is reachable in a straight line from both the start and finish points (such as $(x_m, y_m) = (0, \sqrt{l_1^2 + l_2^2})$). This would essentially break up the trajectory into two segments.

Question 4. We require the transformation $T(\phi_e)$ which satisfies $\dot{\phi}_e = T(\phi_e)^{-1}\omega_e$. We note that the columns of $T(\phi_e)$ are the axes of rotation. In our case:

For the first X rotation: $\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \dot{\phi}$

For the Y rotation: $\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \dot{\nu} = \begin{bmatrix} 0 \\ c_\phi \\ s_\phi \end{bmatrix} \dot{\nu}$

For the second X rotation: $\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix} \begin{bmatrix} c_\nu & 0 & s_\nu \\ 0 & 1 & 0 \\ -s_\nu & 0 & c_\nu \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \dot{\psi} = \begin{bmatrix} c_\nu \\ s_\phi s_\nu \\ -c_\phi s_\nu \end{bmatrix} \dot{\psi}$

All in all, we have:

$$\omega_e = \begin{bmatrix} 1 & 0 & c_\nu \\ 0 & c_\phi & s_\phi s_\nu \\ 0 & s_\phi & -c_\phi s_\nu \end{bmatrix} \dot{\phi}_e$$

$$\dot{\phi}_e = \begin{bmatrix} 1 & 0 & c_\nu \\ 0 & c_\phi & s_\phi s_\nu \\ 0 & s_\phi & -c_\phi s_\nu \end{bmatrix}^{-1} \omega_e = \begin{bmatrix} 1 & s_\phi \frac{c_\nu}{s_\nu} & -c_\phi \frac{c_\nu}{s_\nu} \\ 0 & -c_\phi & -s_\phi \\ 0 & -s_\phi \frac{1}{s_\nu} & c_\phi \frac{1}{s_\nu} \end{bmatrix} \omega_e$$

Which is singular when $s_\nu = 0$.