[right|thumb|upright=1.35|alt=Graph showing a logarithm curves, which crosses the](/wiki/File:Binary_logarithm_plot_with_ticks.svg" \o "File:Binary logarithm plot with ticks.svg) *[x](/wiki/File:Binary_logarithm_plot_with_ticks.svg" \o "File:Binary logarithm plot with ticks.svg)*[-axis where](/wiki/File:Binary_logarithm_plot_with_ticks.svg" \o "File:Binary logarithm plot with ticks.svg) *[x](/wiki/File:Binary_logarithm_plot_with_ticks.svg" \o "File:Binary logarithm plot with ticks.svg)* [is 1 and extend towards minus infinity along the](/wiki/File:Binary_logarithm_plot_with_ticks.svg" \o "File:Binary logarithm plot with ticks.svg) *[y](/wiki/File:Binary_logarithm_plot_with_ticks.svg" \o "File:Binary logarithm plot with ticks.svg)*[-axis.|The](/wiki/File:Binary_logarithm_plot_with_ticks.svg" \o "File:Binary logarithm plot with ticks.svg) [graph](/wiki/Graph_of_a_function) of the logarithm to base 2 crosses the [*x* axis](/wiki/X_axis) (horizontal axis) at 1 and passes through the points with [coordinates](/wiki/Coordinate) [Template:Nowrap](/wiki/Template:Nowrap), [Template:Nowrap](/wiki/Template:Nowrap), and [Template:Nowrap](/wiki/Template:Nowrap). For example, [Template:Nowrap](/wiki/Template:Nowrap), because [Template:Nowrap](/wiki/Template:Nowrap) The graph gets arbitrarily close to the *y* axis, but [does not meet or intersect it](/wiki/Asymptotic).

[right|thumb|alt=Visualization of how exponents of n can be visualized as a full n-ary tree, and how logarithm relates to exponents using this visualization.|A full 3-ary tree can be used to visualize the exponents of 3 and how the logarithm function relates to them.](/wiki/File:Logarithm_visualization_tree.svg)

In [mathematics](/wiki/Mathematics), the **logarithm** is the [inverse operation](/wiki/Inverse_operation) to [exponentiation](/wiki/Exponentiation). That means the logarithm of a number is the [exponent](/wiki/Exponent) to which another fixed value, the [base](/wiki/Base_(exponentiation)), must be raised to produce that number. In simple cases the logarithm counts repeated multiplication. For example, the base [Template:Math](/wiki/Template:Math) logarithm of [Template:Math](/wiki/Template:Math) is [Template:Math](/wiki/Template:Math), as [Template:Math](/wiki/Template:Math) to the power [Template:Math](/wiki/Template:Math) is [Template:Math](/wiki/Template:Math) ([Template:Math](/wiki/Template:Math)); the multiplication is repeated three times. More generally, exponentiation allows any positive [real number](/wiki/Real_number) to be raised to any real power, always producing a positive result, so the logarithm can be calculated for any two positive real numbers [Template:Math](/wiki/Template:Math) and [Template:Math](/wiki/Template:Math) where [Template:Math](/wiki/Template:Math) is not equal to [Template:Math](/wiki/Template:Math). The logarithm of [Template:Math](/wiki/Template:Math) to *base* [Template:Math](/wiki/Template:Math), denoted [Template:Math](/wiki/Template:Math), is the unique real number [Template:Math](/wiki/Template:Math) such that

[Template:Math](/wiki/Template:Math).

For example, as [Template:Math](/wiki/Template:Math), then:

[Template:Math](/wiki/Template:Math)

The logarithm to base [Template:Math](/wiki/Template:Math) (that is [Template:Math](/wiki/Template:Math)) is called the [common logarithm](/wiki/Common_logarithm) and has many applications in science and engineering. The [natural logarithm](/wiki/Natural_logarithm) has the [number](/wiki/E_(mathematical_constant)) [Template:Nowrap begin](/wiki/Template:Nowrap_begin)[Template:Math](/wiki/Template:Math) ([Template:Math](/wiki/Template:Math)[Template:Nowrap end](/wiki/Template:Nowrap_end)) as its base; its use is widespread in mathematics and [physics](/wiki/Physics), because of its simpler [derivative](/wiki/Derivative). The [binary logarithm](/wiki/Binary_logarithm) uses base [Template:Math](/wiki/Template:Math) (that is [Template:Math](/wiki/Template:Math)) and is commonly used in [computer science](/wiki/Computer_science).

Logarithms were introduced by [John Napier](/wiki/John_Napier) in the early 17th century as a means to simplify calculations. They were rapidly adopted by navigators, scientists, engineers, and others to perform computations more easily, using [slide rules](/wiki/Slide_rule) and [logarithm tables](/wiki/Mathematical_table). Tedious multi-digit multiplication steps can be replaced by table look-ups and simpler addition because of the fact — important in its own right — that the logarithm of a [product](/wiki/Product_(mathematics)) is the [sum](/wiki/Summation) of the logarithms of the factors:

<math> \log\_b(xy) = \log\_b (x) + \log\_b (y), \,</math>

provided that [Template:Math](/wiki/Template:Math), [Template:Math](/wiki/Template:Math) and [Template:Math](/wiki/Template:Math) are all positive and [Template:Math](/wiki/Template:Math). The present-day notion of logarithms comes from [Leonhard Euler](/wiki/Leonhard_Euler), who connected them to the [exponential function](/wiki/Exponential_function) in the 18th century.

[Logarithmic scales](/wiki/Logarithmic_scale) reduce wide-ranging quantities to tiny scopes. For example, the [decibel](/wiki/Decibel) is a [unit](/wiki/Units_of_measurement) [quantifying](/wiki/Level_quantity) signal power log-ratios and amplitude log-ratios (of which [sound pressure](/wiki/Sound_pressure) is a common example). In chemistry, [pH](/wiki/PH) is a logarithmic measure for the [acidity](/wiki/Acid) of an [aqueous solution](/wiki/Aqueous_solution). Logarithms are commonplace in scientific [formulae](/wiki/Formula), and in measurements of the [complexity of algorithms](/wiki/Computational_complexity_theory) and of geometric objects called [fractals](/wiki/Fractal). They describe [musical intervals](/wiki/Interval_(music)), appear in formulas counting [prime numbers](/wiki/Prime_number), inform some models in [psychophysics](/wiki/Psychophysics), and can aid in [forensic accounting](/wiki/Forensic_accounting).

In the same way as the logarithm reverses [exponentiation](/wiki/Exponentiation), the [complex logarithm](/wiki/Complex_logarithm) is the [inverse function](/wiki/Inverse_function) of the exponential function applied to [complex numbers](/wiki/Complex_numbers). The [discrete logarithm](/wiki/Discrete_logarithm) is another variant; it has uses in [public-key cryptography](/wiki/Public-key_cryptography).

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## Motivation and definition[[edit](/index.php?title=(none)&action=edit&section=1)]

The idea of logarithms is to reverse the operation of [exponentiation](/wiki/Exponentiation), that is, raising a number to a power. For example, the third power (or [cube](/wiki/Cube_(algebra))) of 2 is 8, because 8 is the product of three factors of 2:

<math>2^3 = 2 \times 2 \times 2 = 8. \,</math>

It follows that the logarithm of 8 with respect to base 2 is 3, so log2 8 = 3.

### Exponentiation[[edit](/index.php?title=(none)&action=edit&section=2)]

The third power of some number *b* is the product of three factors of *b*. More generally, raising *b* to the [Template:Nowrap](/wiki/Template:Nowrap) power, where *n* is a [natural number](/wiki/Natural_number), is done by multiplying *n* factors of *b*. The [Template:Nowrap](/wiki/Template:Nowrap) power of *b* is written *bn*, so that

<math>b^n = \underbrace{b \times b \times \cdots \times b}\_{n \text{ factors}}.</math>

Exponentiation may be extended to *by*, where *b* is a positive number and the *exponent* *y* is any [real number](/wiki/Real_number). For example, *b*−1 is the [reciprocal](/wiki/Multiplicative_inverse) of *b*, that is, [Template:Nowrap](/wiki/Template:Nowrap). (For further details, including the formula [Template:Nowrap](/wiki/Template:Nowrap), see [exponentiation](/wiki/Exponentiation) or [[1]](#cite_note-1) for an elementary treatise.)

### Definition[[edit](/index.php?title=(none)&action=edit&section=3)]

The *logarithm* of a positive real number *x* with respect to base *b*, a positive real number not equal to 1[Template:Refn](/wiki/Template:Refn), is the exponent by which *b* must be raised to yield *x*. In other words, the logarithm of *x* to base *b* is the solution *y* to the equation[[2]](#cite_note-2): <math>b^y = x. \, </math>

The logarithm is denoted "log*b*(*x*)" (pronounced as "the logarithm of *x* to base *b*" or "the [Template:Nowrap](/wiki/Template:Nowrap) logarithm of *x*"). In the equation *y* = log*b*(*x*), the value *y* is the answer to the question "To what power must *b* be raised, in order to yield *x*?". This question can also be addressed (with a richer answer) for [complex numbers](/wiki/Complex_number), which is done in section ["Complex logarithm"](/wiki/#Complex_logarithm), and this answer is much more extensively investigated in [the page for the complex logarithm](/wiki/Complex_logarithm).

### Examples[[edit](/index.php?title=(none)&action=edit&section=4)]

For example, [Template:Nowrap](/wiki/Template:Nowrap), since [Template:Nowrap](/wiki/Template:Nowrap) [Template:=](/wiki/Template:=) 16. Logarithms can also be negative:

<math>\log\_2 \!\left( \frac{1}{2} \right) = -1,\, </math>

since

<math>2^{-1} = \frac 1 {2^1} = \frac 1 2.</math>

A third example: log10(150) is approximately 2.176, which lies between 2 and 3, just as 150 lies between [Template:Nowrap](/wiki/Template:Nowrap) and [Template:Nowrap](/wiki/Template:Nowrap). Finally, for any base *b*, [Template:Nowrap](/wiki/Template:Nowrap) and [Template:Nowrap](/wiki/Template:Nowrap), since [Template:Nowrap](/wiki/Template:Nowrap) and [Template:Nowrap](/wiki/Template:Nowrap), respectively.

## Logarithmic identities[[edit](/index.php?title=(none)&action=edit&section=5)]

[Template:Main](/wiki/Template:Main)

Several important formulas, sometimes called *logarithmic identities* or *log laws*, relate logarithms to one another.[[3]](#cite_note-3)

### Product, quotient, power and root[[edit](/index.php?title=(none)&action=edit&section=6)]

The logarithm of a product is the sum of the logarithms of the numbers being multiplied; the logarithm of the ratio of two numbers is the difference of the logarithms. The logarithm of the [Template:Nowrap](/wiki/Template:Nowrap) power of a number is *p* times the logarithm of the number itself; the logarithm of a [Template:Nowrap](/wiki/Template:Nowrap) root is the logarithm of the number divided by *p*. The following table lists these identities with examples. Each of the identities can be derived after substitution of the logarithm definitions <math>x = b^{\log\_b(x)}</math> or <math>y = b^{\log\_b(y)}</math> in the left hand sides.

|  |  |  |
| --- | --- | --- |
|  | **Formula** | **Example** |
| product | <math> \log\_b(x y) = \log\_b (x) + \log\_b (y)</math> | <math> \log\_3 (243) = \log\_3(9 \cdot 27) = \log\_3 (9) + \log\_3 (27) = 2 + 3 = 5</math> |
| quotient | <math>\log\_b \!\left(\frac x y \right) = \log\_b (x) - \log\_b (y)</math> | <math> \log\_2 (16) = \log\_2 \!\left ( \frac{64}{4} \right ) = \log\_2 (64) - \log\_2 (4) = 6 - 2 = 4</math> |
| power | <math>\log\_b(x^p) = p \log\_b (x)</math> | <math> \log\_2 (64) = \log\_2 (2^6) = 6 \log\_2 (2) = 6</math> |
| root | <math>\log\_b \sqrt[p]{x} = \frac {\log\_b (x)} p</math> | <math> \log\_{10} \sqrt{1000} = \frac{1}{2}\log\_{10} 1000 = \frac{3}{2} = 1.5 </math> |

### Change of base[[edit](/index.php?title=(none)&action=edit&section=7)]

The logarithm log*b*(*x*) can be computed from the logarithms of *x* and *b* with respect to an arbitrary base *k* using the following formula:

<math> \log\_b(x) = \frac{\log\_k(x)}{\log\_k(b)}.\, </math>

Typical [scientific calculators](/wiki/Scientific_calculators) calculate the logarithms to bases 10 and [*e*](/wiki/E_(mathematical_constant)).[[4]](#cite_note-4) Logarithms with respect to any base *b* can be determined using either of these two logarithms by the previous formula:

<math> \log\_b (x) = \frac{\log\_{10} (x)}{\log\_{10} (b)} = \frac{\log\_{e} (x)}{\log\_{e} (b)}. \,</math>

Given a number *x* and its logarithm log*b*(*x*) to an unknown base *b*, the base is given by:

<math> b = x^\frac{1}{\log\_b(x)}.</math>

## Particular bases[[edit](/index.php?title=(none)&action=edit&section=8)]

Among all choices for the base, three are particularly common. These are *b* = 10, *b* = [*e*](/wiki/E_(mathematical_constant)) (the [irrational](/wiki/Irrational_number) mathematical constant ≈ 2.71828), and *b* = 2. In [mathematical analysis](/wiki/Mathematical_analysis), the logarithm to base *e* is widespread because of its particular analytical properties explained below. On the other hand, [Template:Nowrap](/wiki/Template:Nowrap) logarithms are easy to use for manual calculations in the [decimal](/wiki/Decimal) number system:[[5]](#cite_note-5):<math>\log\_{10}(10 x) = \log\_{10}(10) + \log\_{10}(x) = 1 + \log\_{10}(x).\ </math> Thus, log10(*x*) is related to the number of [decimal digits](/wiki/Decimal_digit) of a positive integer *x*: the number of digits is the smallest [integer](/wiki/Integer) strictly bigger than log10(*x*).[[6]](#cite_note-6) For example, log10(1430) is approximately 3.15. The next integer is 4, which is the number of digits of 1430. Both the natural logarithm and the logarithm to base two are used in [information theory](/wiki/Information_theory), corresponding to the use of [nats](/wiki/Nat_(unit)) or [bits](/wiki/Bit) as the fundamental units of information, respectively.[[7]](#cite_note-7) Binary logarithms are also used in [computer science](/wiki/Computer_science), where the [binary system](/wiki/Binary_numeral_system) is ubiquitous, in [music theory](/wiki/Music_theory), where a pitch ratio of two (the [octave](/wiki/Octave)) is ubiquitous and the [cent](/wiki/Cent_(music)) is the binary logarithm (scaled by 1200) of the ratio between two adjacent equally-tempered pitches, and in [photography](/wiki/Photography) to measure [exposure values](/wiki/Exposure_value).[[8]](#cite_note-8) The following table lists common notations for logarithms to these bases and the fields where they are used. Many disciplines write log(*x*) instead of log*b*(*x*), when the intended base can be determined from the context. The notation *b*log(*x*) also occurs.[[9]](#cite_note-9) The "ISO notation" column lists designations suggested by the [International Organization for Standardization](/wiki/International_Organization_for_Standardization) ([ISO 31-11](/wiki/ISO_31-11)).[[10]](#cite_note-10)

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Base *b*** | **Name for log*b*(*x*)** | **ISO notation** | **Other notations** | **Used in** |
| **2** | [binary logarithm](/wiki/Binary_logarithm) | lb(*x*)[[11]](#cite_note-11) | ld(*x*), log(*x*), lg(*x*),[[12]](#cite_note-12) log2(*x*) | [computer science](/wiki/Computer_science), [information theory](/wiki/Information_theory), [music theory](/wiki/Music_theory), [photography](/wiki/Photography) |
| ***e*** | [natural logarithm](/wiki/Natural_logarithm) | ln(*x*)[Template:Refn](/wiki/Template:Refn) | log(*x*) (in mathematics [[13]](#cite_note-13) and many [programming languages](/wiki/Programming_language)[Template:Refn](/wiki/Template:Refn)) | mathematics, physics, chemistry, [statistics](/wiki/Statistics), [economics](/wiki/Economics), information theory, and some engineering fields |
| **10** | [common logarithm](/wiki/Common_logarithm) | lg(*x*) | log(*x*), log10(*x*) (in engineering, biology, astronomy) | various [engineering](/wiki/Engineering) fields (see [decibel](/wiki/Decibel) and see below),  logarithm [tables](/wiki/Mathematical_table), handheld [calculators](/wiki/Scientific_calculator), [spectroscopy](/wiki/Spectroscopy) |

## History[[edit](/index.php?title=(none)&action=edit&section=9)]

[Template:Main](/wiki/Template:Main) The **history of logarithm** in seventeenth century Europe is the discovery of a new [function](/wiki/Function_(mathematics)) that extended the realm of analysis beyond the scope of algebraic methods. The method of logarithms was publicly propounded by [John Napier](/wiki/John_Napier) in 1614, in a book titled *Mirifici Logarithmorum Canonis Descriptio* (*Description of the Wonderful Rule of Logarithms*).[[14]](#cite_note-14)[[15]](#cite_note-15) Prior to Napier's invention, there had been other techniques of similar scopes, such as the prosthaphaeresis or the use of tables of progressions, extensively developed by [Jost Bürgi](/wiki/Jost_Bürgi) around 1600.[[16]](#cite_note-16)[[17]](#cite_note-17) This theorem states that a [continuous function](/wiki/Continuous_function) that produces two values *m* and *n* also produces any value that lies between *m* and *n*. A function is *continuous* if it does not "jump", that is, if its graph can be drawn without lifting the pen.

This property can be shown to hold for the function [Template:Nowrap begin](/wiki/Template:Nowrap_begin)*f*(*x*) = *bx*[Template:Nowrap end](/wiki/Template:Nowrap_end). Because *f* takes arbitrarily large and arbitrarily small positive values, any number [Template:Nowrap](/wiki/Template:Nowrap) lies between *f*(*x*0) and *f*(*x*1) for suitable *x*0 and *x*1. Hence, the intermediate value theorem ensures that the equation *f*(*x*) = *y* has a solution. Moreover, there is only one solution to this equation, because the function *f* is [strictly increasing](/wiki/Monotonic_function) (for [Template:Nowrap](/wiki/Template:Nowrap)), or strictly decreasing (for [Template:Nowrap](/wiki/Template:Nowrap)).<ref name=LangIV.2>[Template:Harvard citations](/wiki/Template:Harvard_citations)</ref>

The unique solution *x* is the logarithm of *y* to base *b*, log*b*(*y*). The function that assigns to *y* its logarithm is called *logarithm function* or *logarithmic function* (or just *logarithm*).

The function log*b*(*x*) is essentially characterized by the above product formula

<math>\log\_b(xy) = \log\_b(x) + \log\_b(y).</math>

More precisely, the logarithm to any base [Template:Nowrap](/wiki/Template:Nowrap) is the only [increasing function](/wiki/Increasing_function) *f* from the positive reals to the reals satisfying [Template:Nowrap begin](/wiki/Template:Nowrap_begin)*f*(*b*) = 1[Template:Nowrap end](/wiki/Template:Nowrap_end) and [[28]](#cite_note-28):<math>f(xy)=f(x)+f(y).</math>

### Inverse function[[edit](/index.php?title=(none)&action=edit&section=13)]

[right|thumb|The graph of the logarithm function log*b*(*x*) (blue) is obtained by](/wiki/File:Logarithm_inversefunctiontoexp.svg) [reflecting](/wiki/Reflection_(mathematics)) the graph of the function *bx* (red) at the diagonal line ([Template:Nowrap begin](/wiki/Template:Nowrap_begin)*x* = *y*[Template:Nowrap end](/wiki/Template:Nowrap_end)).|alt=The graphs of two functions. The formula for the logarithm of a power says in particular that for any number *x*,

<math>\log\_b \left (b^x \right) = x \log\_b(b) = x.</math>

In prose, taking the [Template:Nowrap](/wiki/Template:Nowrap) power of *b* and then the [Template:Nowrap](/wiki/Template:Nowrap) logarithm gives back *x*. Conversely, given a positive number *y*, the formula

<math>b^{\log\_b(y)} = y</math>

says that first taking the logarithm and then exponentiating gives back *y*. Thus, the two possible ways of combining (or [composing](/wiki/Composition_(mathematics))) logarithms and exponentiation give back the original number. Therefore, the logarithm to base *b* is the [*inverse function*](/wiki/Inverse_function) of [Template:Nowrap](/wiki/Template:Nowrap).[[29]](#cite_note-29) Inverse functions are closely related to the original functions. Their [graphs](/wiki/Graph_of_a_function) correspond to each other upon exchanging the *x*- and the *y*-coordinates (or upon reflection at the diagonal line *x* = *y*), as shown at the right: a point (*t*, *u* = *bt*) on the graph of *f* yields a point (*u*, *t* = log*bu*) on the graph of the logarithm and vice versa. As a consequence, log*b*(*x*) [diverges to infinity](/wiki/Divergent_sequence) (gets bigger than any given number) if *x* grows to infinity, provided that *b* is greater than one. In that case, log*b*(*x*) is an [increasing function](/wiki/Increasing_function). For [Template:Nowrap](/wiki/Template:Nowrap), log*b*(*x*) tends to minus infinity instead. When *x* approaches zero, log*b*(*x*) goes to minus infinity for [Template:Nowrap](/wiki/Template:Nowrap) (plus infinity for [Template:Nowrap](/wiki/Template:Nowrap), respectively).

### Derivative and antiderivative[[edit](/index.php?title=(none)&action=edit&section=14)]

[right|thumb|220|The graph of the](/wiki/File:Logarithm_derivative.svg) [natural logarithm](/wiki/Natural_logarithm) (green) and its tangent at [Template:Nowrap](/wiki/Template:Nowrap) (black)|alt=A graph of the logarithm function and a line touching it in one point. Analytic properties of functions pass to their inverses.[[30]](#cite_note-30) Thus, as [Template:Nowrap begin](/wiki/Template:Nowrap_begin)*f*(*x*) = *bx*[Template:Nowrap end](/wiki/Template:Nowrap_end) is a continuous and [differentiable function](/wiki/Differentiable_function), so is log*b*(*y*). Roughly, a continuous function is differentiable if its graph has no sharp "corners". Moreover, as the [derivative](/wiki/Derivative) of *f*(*x*) evaluates to ln(*b*)*bx* by the properties of the [exponential function](/wiki/Exponential_function), the [chain rule](/wiki/Chain_rule) implies that the derivative of log*b*(*x*) is given by[[31]](#cite_note-31)[[32]](#cite_note-32): <math>\frac{d}{dx} \log\_b(x) = \frac{1}{x\ln(b)}. </math> That is, the [slope](/wiki/Slope) of the [tangent](/wiki/Tangent) touching the graph of the [Template:Nowrap](/wiki/Template:Nowrap) logarithm at the point [Template:Nowrap](/wiki/Template:Nowrap) equals [Template:Nowrap](/wiki/Template:Nowrap).

The derivative of ln(*x*) is 1/*x*; this implies that ln(*x*) is the unique [antiderivative](/wiki/Antiderivative) of 1/*x* that has the value 0 for *x* =1. This is this very simple formula that motivated to qualify as "natural" the natural logarithm; this is also one of the main reasons of the importance of the constant [*e*](/wiki/E_(mathematical_constant)).

The derivative with a generalised functional argument *f*(*x*) is

<math>\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}.</math>

The quotient at the right hand side is called the [logarithmic derivative](/wiki/Logarithmic_derivative) of *f*. Computing *f'*(*x*) by means of the derivative of ln(*f*(*x*)) is known as [logarithmic differentiation](/wiki/Logarithmic_differentiation).[[33]](#cite_note-33) The antiderivative of the [natural logarithm](/wiki/Natural_logarithm) ln(*x*) is:[[34]](#cite_note-34): <math>\int \ln(x) \,dx = x \ln(x) - x + C.</math> [Related formulas](/wiki/List_of_integrals_of_logarithmic_functions), such as antiderivatives of logarithms to other bases can be derived from this equation using the change of bases.[[35]](#cite_note-35)

### Integral representation of the [[natural logarithm]][[edit](/index.php?title=(none)&action=edit&section=15)]

[right|thumb|The](/wiki/File:Natural_logarithm_integral.svg) [natural logarithm](/wiki/Natural_logarithm) of *t* is the shaded area underneath the graph of the function *f*(*x*) = 1/*x* (reciprocal of *x*).|alt=A hyperbola with part of the area underneath shaded in grey. The [natural logarithm](/wiki/Natural_logarithm) of *t* equals the [integral](/wiki/Integral) of 1/*x* *dx* from 1 to *t*:

<cite id=integral\_naturallog><math>\ln (t) = \int\_1^t \frac{1}{x} \, dx.</math></cite>

In other words, ln(*t*) equals the area between the *x* axis and the graph of the function 1/*x*, ranging from [Template:Nowrap](/wiki/Template:Nowrap) to [Template:Nowrap](/wiki/Template:Nowrap) (figure at the right). This is a consequence of the [fundamental theorem of calculus](/wiki/Fundamental_theorem_of_calculus) and the fact that derivative of ln(*x*) is 1/*x*. The right hand side of this equation can serve as a definition of the [natural logarithm](/wiki/Natural_logarithm). Product and power logarithm formulas can be derived from this definition.[[36]](#cite_note-36) For example, the product formula [Template:Nowrap](/wiki/Template:Nowrap) is deduced as:

<math> \ln(tu) = \int\_1^{tu} \frac{1}{x} \, dx \ \stackrel {(1)} = \int\_1^{t} \frac{1}{x} \, dx + \int\_t^{tu} \frac{1}{x} \, dx \ \stackrel {(2)} = \ln(t) + \int\_1^u \frac{1}{w} \, dw = \ln(t) + \ln(u).</math>

The equality (1) splits the integral into two parts, while the equality (2) is a change of variable ([Template:Nowrap](/wiki/Template:Nowrap)). In the illustration below, the splitting corresponds to dividing the area into the yellow and blue parts. Rescaling the left hand blue area vertically by the factor *t* and shrinking it by the same factor horizontally does not change its size. Moving it appropriately, the area fits the graph of the function [Template:Nowrap](/wiki/Template:Nowrap) again. Therefore, the left hand blue area, which is the integral of *f*(*x*) from *t* to *tu* is the same as the integral from 1 to *u*. This justifies the equality (2) with a more geometric proof.

[thumb|center|500px|A visual proof of the product formula of the natural logarithm|alt=The hyperbola depicted twice. The area underneath is split into different parts.](/wiki/File:Natural_logarithm_product_formula_proven_geometrically.svg)

The power formula [Template:Nowrap](/wiki/Template:Nowrap) may be derived in a similar way:

<math>

\ln(t^r) = \int\_1^{t^r} \frac{1}{x}dx = \int\_1^t \frac{1}{w^r} \left(rw^{r - 1} \, dw\right) = r \int\_1^t \frac{1}{w} \, dw = r \ln(t). </math> The second equality uses a change of variables ([integration by substitution](/wiki/Integration_by_substitution)), [Template:Nowrap](/wiki/Template:Nowrap).

The sum over the reciprocals of natural numbers,

<math>1 + \frac 1 2 + \frac 1 3 + \cdots + \frac 1 n = \sum\_{k=1}^n \frac{1}{k},</math>

is called the [harmonic series](/wiki/Harmonic_series_(mathematics)). It is closely tied to the [natural logarithm](/wiki/Natural_logarithm): as *n* tends to [infinity](/wiki/Infinity), the difference,

<math>\sum\_{k=1}^n \frac{1}{k} - \ln(n),</math>

[converges](/wiki/Limit_of_a_sequence) (i.e., gets arbitrarily close) to a number known as the [Euler–Mascheroni constant](/wiki/Euler–Mascheroni_constant). This relation aids in analyzing the performance of algorithms such as [quicksort](/wiki/Quicksort).[[37]](#cite_note-37) There is also another integral representation of the logarithm that is useful in some situations.

<math> \ln(x) = -\lim\_{\epsilon \to 0} \int\_\epsilon^\infty \frac{dt}{t}\left( e^{-xt} - e^{-t} \right)</math>

This can be verified by showing that it has the same value at [Template:Nowrap](/wiki/Template:Nowrap), and the same derivative.

### Transcendence of the logarithm[[edit](/index.php?title=(none)&action=edit&section=16)]

[Real numbers](/wiki/Real_number) that are not [algebraic](/wiki/Algebraic_number) are called [transcendental](/wiki/Transcendental_number);[[38]](#cite_note-38) for example, [Template:Pi](/wiki/Template:Pi) and [*e*](/wiki/E_(mathematical_constant)) are such numbers, but <math>\sqrt{2-\sqrt 3}</math> is not. [Almost all](/wiki/Almost_all) real numbers are transcendental. The logarithm is an example of a [transcendental function](/wiki/Transcendental_function). The [Gelfond–Schneider theorem](/wiki/Gelfond–Schneider_theorem) asserts that logarithms usually take transcendental, i.e., "difficult" values.[[39]](#cite_note-39)

## Calculation[[edit](/index.php?title=(none)&action=edit&section=17)]

[thumb|The logarithm keys (*lo*g for base-10 and *ln* for base-*e*) on a typical scientific calculator](/wiki/File:Logarithm_keys.jpg) Logarithms are easy to compute in some cases, such as [Template:Nowrap beginlog](/wiki/Template:Nowrap_begin)10(1000) = 3[Template:Nowrap end](/wiki/Template:Nowrap_end). In general, logarithms can be calculated using [power series](/wiki/Power_series) or the [arithmetic–geometric mean](/wiki/Arithmetic–geometric_mean), or be retrieved from a precalculated [logarithm table](/wiki/Logarithm_table) that provides a fixed precision.[[40]](#cite_note-40)[[41]](#cite_note-41)[Newton's method](/wiki/Newton's_method), an iterative method to solve equations approximately, can also be used to calculate the logarithm, because its inverse function, the exponential function, can be computed efficiently.[[42]](#cite_note-42) Using look-up tables, [CORDIC](/wiki/CORDIC)-like methods can be used to compute logarithms if the only available operations are addition and [bit shifts](/wiki/Arithmetic_shift).[[43]](#cite_note-43)[[44]](#cite_note-44) Moreover, the [binary logarithm algorithm](/wiki/Binary_logarithm#Algorithm) calculates lb(*x*) [recursively](/wiki/Recursion) based on repeated squarings of *x*, taking advantage of the relation

<math>\log\_2(x^2) = 2 \log\_2 (x). \,</math>

### Power series[[edit](/index.php?title=(none)&action=edit&section=18)]

Taylor series

[right|thumb|The Taylor series of ln(*z*) centered at *z*](/wiki/File:Taylor_approximation_of_natural_logarithm.gif)[Template:=](/wiki/Template:=) 1. The animation shows the first 10 approximations along with the 99th and 100th. The approximations do not converge beyond a distance of 1 from the center.|alt=An animation showing increasingly good approximations of the logarithm graph. For any real number *z* that satisfies [Template:Nowrap](/wiki/Template:Nowrap), the following formula holds:[Template:Refn](/wiki/Template:Refn)<ref name=AbramowitzStegunp.68>[Template:Harvard citations](/wiki/Template:Harvard_citations)</ref>

<math>

\ln (z) = \frac{(z-1)^1}{1} - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \cdots </math> This is a shorthand for saying that ln(*z*) can be approximated to a more and more accurate value by the following expressions:

<math>

\begin{array}{lllll} (z-1) & & \\ (z-1) & - & \frac{(z-1)^2}{2} & \\ (z-1) & - & \frac{(z-1)^2}{2} & + & \frac{(z-1)^3}{3} \\ \vdots & \end{array} </math> For example, with [Template:Nowrap](/wiki/Template:Nowrap) the third approximation yields 0.4167, which is about 0.011 greater than [Template:Nowrap](/wiki/Template:Nowrap). This [series](/wiki/Series_(mathematics)) approximates ln(*z*) with arbitrary precision, provided the number of summands is large enough. In elementary calculus, ln(*z*) is therefore the [*limit*](/wiki/Limit_(mathematics)) of this series. It is the [Taylor series](/wiki/Taylor_series) of the [natural logarithm](/wiki/Natural_logarithm) at [Template:Nowrap begin](/wiki/Template:Nowrap_begin)*z* = 1[Template:Nowrap end](/wiki/Template:Nowrap_end). The Taylor series of ln *z* provides a particularly useful approximation to ln(1+*z*) when *z* is small, *|z| < 1*, since then

<math>

\ln (1+z) = z - \frac{z^2}{2} +\frac{z^3}{3}\cdots \approx z. </math> For example, with *z* = 0.1 the first-order approximation gives [Template:Math](/wiki/Template:Math), which is less than 5% off the correct value 0.0953.

More efficient series

Another series is based on the [area hyperbolic tangent](/wiki/Area_hyperbolic_tangent) function:

<math>

\ln (z) = 2\cdot\operatorname{artanh}\,\frac{z-1}{z+1} = 2 \left ( \frac{z-1}{z+1} + \frac{1}{3}{\left(\frac{z-1}{z+1}\right)}^3 + \frac{1}{5}{\left(\frac{z-1}{z+1}\right)}^5 + \cdots \right ), </math> for any real number *z* > 0.[Template:Refn](/wiki/Template:Refn)[[45]](#cite_note-45) Using the [Sigma notation](/wiki/Sigma_notation), this is also written as

<math>\ln (z) = 2\sum\_{n=0}^\infty\frac{1}{2n+1}\left(\frac{z-1}{z+1}\right)^{2n+1}.</math>

This series can be derived from the above Taylor series. It converges more quickly than the Taylor series, especially if *z* is close to 1. For example, for [Template:Nowrap begin](/wiki/Template:Nowrap_begin)*z* = 1.5[Template:Nowrap end](/wiki/Template:Nowrap_end), the first three terms of the second series approximate ln(1.5) with an error of about [Template:Val](/wiki/Template:Val). The quick convergence for *z* close to 1 can be taken advantage of in the following way: given a low-accuracy approximation [Template:Nowrap](/wiki/Template:Nowrap) and putting

<math>A = \frac z{\exp(y)}, \,</math>

the logarithm of *z* is:

<math>\ln (z)=y+\ln (A). \,</math>

The better the initial approximation *y* is, the closer *A* is to 1, so its logarithm can be calculated efficiently. *A* can be calculated using the [exponential series](/wiki/Exponential_function), which converges quickly provided *y* is not too large. Calculating the logarithm of larger *z* can be reduced to smaller values of *z* by writing [Template:Nowrap](/wiki/Template:Nowrap), so that [Template:Nowrap](/wiki/Template:Nowrap).

A closely related method can be used to compute the logarithm of integers. From the above series, it follows that:

<math>\ln (n+1) = \ln(n) + 2\sum\_{k=0}^\infty\frac{1}{2k+1}\left(\frac{1}{2 n+1}\right)^{2k+1}.</math>

If the logarithm of a large integer [Template:Mvar](/wiki/Template:Mvar) is known, then this series yields a fast converging series for log(*n*+1).

### Arithmetic–geometric mean approximation[[edit](/index.php?title=(none)&action=edit&section=19)]

The [arithmetic–geometric mean](/wiki/Arithmetic–geometric_mean) yields high precision approximations of the [natural logarithm](/wiki/Natural_logarithm). ln(*x*) is approximated to a precision of 2−*p* (or *p* precise bits) by the following formula (due to [Carl Friedrich Gauss](/wiki/Carl_Friedrich_Gauss)):[[46]](#cite_note-46)[[47]](#cite_note-47)

<math>\ln (x) \approx \frac{\pi}{2 M(1,2^{2-m}/x)} - m \ln (2).</math>

Here *M*(x,y) denotes the [arithmetic–geometric mean](/wiki/Arithmetic–geometric_mean) of x and y. It is obtained by repeatedly calculating the average (x+y)/2 ([arithmetic mean](/wiki/Arithmetic_mean)) and sqrt(x\*y) ([geometric mean](/wiki/Geometric_mean)) of x and y then let those two numbers become the next x and y. The two numbers quickly converge to a common limit which is the value of *M*(x,y). *m* is chosen such that

<math>x \,2^m > 2^{p/2}.\, </math>

to insure the required precision. A larger *m* makes the *M*(x,y) calculation take more steps (the initial x and y are farther apart so it takes more steps to converge) but gives more precision. The constants [Template:Pi](/wiki/Template:Pi) and ln(2) can be calculated with quickly converging series.

## Applications[[edit](/index.php?title=(none)&action=edit&section=20)]

[thumb|A](/wiki/File:NautilusCutawayLogarithmicSpiral.jpg) [nautilus](/wiki/Nautilus) displaying a logarithmic spiral|alt=A photograph of a nautilus' shell. Logarithms have many applications inside and outside mathematics. Some of these occurrences are related to the notion of [scale invariance](/wiki/Scale_invariance). For example, each chamber of the shell of a [nautilus](/wiki/Nautilus) is an approximate copy of the next one, scaled by a constant factor. This gives rise to a [logarithmic spiral](/wiki/Logarithmic_spiral).[[48]](#cite_note-48) [Benford's law](/wiki/Benford's_law) on the distribution of leading digits can also be explained by scale invariance.[[49]](#cite_note-49) Logarithms are also linked to [self-similarity](/wiki/Self-similarity). For example, logarithms appear in the analysis of algorithms that solve a problem by dividing it into two similar smaller problems and patching their solutions.[[50]](#cite_note-50) The dimensions of self-similar geometric shapes, that is, shapes whose parts resemble the overall picture are also based on logarithms. [Logarithmic scales](/wiki/Logarithmic_scale) are useful for quantifying the relative change of a value as opposed to its absolute difference. Moreover, because the logarithmic function log(*x*) grows very slowly for large *x*, logarithmic scales are used to compress large-scale scientific data. Logarithms also occur in numerous scientific formulas, such as the [Tsiolkovsky rocket equation](/wiki/Tsiolkovsky_rocket_equation), the [Fenske equation](/wiki/Fenske_equation), or the [Nernst equation](/wiki/Nernst_equation).

### Logarithmic scale[[edit](/index.php?title=(none)&action=edit&section=21)]

[Template:Main](/wiki/Template:Main) [A logarithmic chart depicting the value of one](/wiki/File:Germany_Hyperinflation.svg) [Goldmark](/wiki/German_gold_mark) in [Papiermarks](/wiki/German_Papiermark) during the [German hyperinflation in the 1920s](/wiki/Inflation_in_the_Weimar_Republic)|right|thumb|alt=A graph of the value of one mark over time. The line showing its value is increasing very quickly, even with logarithmic scale. Scientific quantities are often expressed as logarithms of other quantities, using a *logarithmic scale*. For example, the [decibel](/wiki/Decibel) is a [unit of measurement](/wiki/Unit_of_measurement) associated with [logarithmic-scale](/wiki/Logarithmic-scale) [quantities](/wiki/Level_quantity). It is based on the common logarithm of [ratios](/wiki/Ratio)—10 times the common logarithm of a [power](/wiki/Power_(physics)) ratio or 20 times the common logarithm of a [voltage](/wiki/Voltage) ratio. It is used to quantify the loss of voltage levels in transmitting electrical signals,[[51]](#cite_note-51) to describe power levels of sounds in [acoustics](/wiki/Acoustics),[[52]](#cite_note-52) and the [absorbance](/wiki/Absorbance) of light in the fields of [spectrometry](/wiki/Spectrometer) and [optics](/wiki/Optics). The [signal-to-noise ratio](/wiki/Signal-to-noise_ratio) describing the amount of unwanted [noise](/wiki/Noise_(electronic)) in relation to a (meaningful) [signal](/wiki/Signal_(information_theory)) is also measured in decibels.[[53]](#cite_note-53) In a similar vein, the [peak signal-to-noise ratio](/wiki/Peak_signal-to-noise_ratio) is commonly used to assess the quality of sound and [image compression](/wiki/Image_compression) methods using the logarithm.[[54]](#cite_note-54) The strength of an earthquake is measured by taking the common logarithm of the energy emitted at the quake. This is used in the [moment magnitude scale](/wiki/Moment_magnitude_scale) or the [Richter magnitude scale](/wiki/Richter_magnitude_scale). For example, a 5.0 earthquake releases 32 times (101.5) and a 6.0 releases 1000 times (103) the energy of a 4.0.[[55]](#cite_note-55) Another logarithmic scale is [apparent magnitude](/wiki/Apparent_magnitude). It measures the brightness of stars logarithmically.[[56]](#cite_note-56) Yet another example is [pH](/wiki/PH) in [chemistry](/wiki/Chemistry); pH is the negative of the common logarithm of the [activity](/wiki/Activity_(chemistry)) of [hydronium](/wiki/Hydronium) ions (the form [hydrogen](/wiki/Hydrogen) [ions](/wiki/Ion) [Template:Chem](/wiki/Template:Chem) take in water).[[57]](#cite_note-57) The activity of hydronium ions in neutral water is 10−7 [mol·L−1](/wiki/Molar_concentration), hence a pH of 7. Vinegar typically has a pH of about 3. The difference of 4 corresponds to a ratio of 104 of the activity, that is, vinegar's hydronium ion activity is about 10−3 mol·L−1.

[Semilog](/wiki/Semi-log_plot) (log-linear) graphs use the logarithmic scale concept for visualization: one axis, typically the vertical one, is scaled logarithmically. For example, the chart at the right compresses the steep increase from 1 million to 1 trillion to the same space (on the vertical axis) as the increase from 1 to 1 million. In such graphs, [exponential functions](/wiki/Exponential_function) of the form [Template:Nowrap begin](/wiki/Template:Nowrap_begin)*f*(*x*) = *a* · *bx*[Template:Nowrap end](/wiki/Template:Nowrap_end) appear as straight lines with [slope](/wiki/Slope) equal to the logarithm of *b*. [Log-log](/wiki/Log-log_plot) graphs scale both axes logarithmically, which causes functions of the form [Template:Nowrap begin](/wiki/Template:Nowrap_begin)*f*(*x*) = *a* · *xk*[Template:Nowrap end](/wiki/Template:Nowrap_end) to be depicted as straight lines with slope equal to the exponent *k*. This is applied in visualizing and analyzing [power laws](/wiki/Power_law).[[58]](#cite_note-58)

### Psychology[[edit](/index.php?title=(none)&action=edit&section=22)]

Logarithms occur in several laws describing [human perception](/wiki/Human_perception):[[59]](#cite_note-59)[[60]](#cite_note-60)[Hick's law](/wiki/Hick's_law) proposes a logarithmic relation between the time individuals take to choose an alternative and the number of choices they have.[[61]](#cite_note-61) [Fitts's law](/wiki/Fitts's_law) predicts that the time required to rapidly move to a target area is a logarithmic function of the distance to and the size of the target.[[62]](#cite_note-62) In [psychophysics](/wiki/Psychophysics), the [Weber–Fechner law](/wiki/Weber–Fechner_law) proposes a logarithmic relationship between [stimulus](/wiki/Stimulus_(psychology)) and [sensation](/wiki/Sensation_(psychology)) such as the actual vs. the perceived weight of an item a person is carrying.[[63]](#cite_note-63) (This "law", however, is less precise than more recent models, such as the [Stevens' power law](/wiki/Stevens'_power_law).[[64]](#cite_note-64))

Psychological studies found that individuals with little mathematics education tend to estimate quantities logarithmically, that is, they position a number on an unmarked line according to its logarithm, so that 10 is positioned as close to 100 as 100 is to 1000. Increasing education shifts this to a linear estimate (positioning 1000 10x as far away) in some circumstances, while logarithms are used when the numbers to be plotted are difficult to plot linearly.[[65]](#cite_note-65)[[66]](#cite_note-66)

### Probability theory and statistics[[edit](/index.php?title=(none)&action=edit&section=23)]

[thumb|right|alt=Three asymmetric PDF curves|Three](/wiki/File:PDF-log_normal_distributions.svg) [probability density functions](/wiki/Probability_density_function) (PDF) of random variables with log-normal distributions. The location parameter [Template:Math](/wiki/Template:Math), which is zero for all three of the PDFs shown, is the mean of the logarithm of the random variable, not the mean of the variable itself. [Distribution of first digits (in %, red bars) in the](/wiki/File:Benfords_law_illustrated_by_world's_countries_population.png) [population of the 237 countries](/wiki/List_of_countries_by_population) of the world. Black dots indicate the distribution predicted by Benford's law.|thumb|right|alt=A bar chart and a superimposed second chart. The two differ slightly, but both decrease in a similar fashion. Logarithms arise in [probability theory](/wiki/Probability_theory): the [law of large numbers](/wiki/Law_of_large_numbers) dictates that, for a [fair coin](/wiki/Fair_coin), as the number of coin-tosses increases to infinity, the observed proportion of heads [approaches one-half](/wiki/Binomial_distribution#Symmetric_binomial_distribution_(p_=_0.5)). The fluctuations of this proportion about one-half are described by the [law of the iterated logarithm](/wiki/Law_of_the_iterated_logarithm).[[67]](#cite_note-67) Logarithms also occur in [log-normal distributions](/wiki/Log-normal_distribution). When the logarithm of a [random variable](/wiki/Random_variable) has a [normal distribution](/wiki/Normal_distribution), the variable is said to have a log-normal distribution.[[68]](#cite_note-68) Log-normal distributions are encountered in many fields, wherever a variable is formed as the product of many independent positive random variables, for example in the study of turbulence.[[69]](#cite_note-69) Logarithms are used for [maximum-likelihood estimation](/wiki/Maximum-likelihood_estimation) of parametric [statistical models](/wiki/Statistical_model). For such a model, the [likelihood function](/wiki/Likelihood_function) depends on at least one [parameter](/wiki/Parametric_model) that must be estimated. A maximum of the likelihood function occurs at the same parameter-value as a maximum of the logarithm of the likelihood (the "*log likelihood*"), because the logarithm is an increasing function. The log-likelihood is easier to maximize, especially for the multiplied likelihoods for [independent](/wiki/Independence_(probability)) random variables.[[70]](#cite_note-70) [Benford's law](/wiki/Benford's_law) describes the occurrence of digits in many [data sets](/wiki/Data_set), such as heights of buildings. According to Benford's law, the probability that the first decimal-digit of an item in the data sample is *d* (from 1 to 9) equals log10(*d* + 1) − log10(*d*), *regardless* of the unit of measurement.[[71]](#cite_note-71) Thus, about 30% of the data can be expected to have 1 as first digit, 18% start with 2, etc. Auditors examine deviations from Benford's law to detect fraudulent accounting.[[72]](#cite_note-72)

### Computational complexity[[edit](/index.php?title=(none)&action=edit&section=24)]

[Analysis of algorithms](/wiki/Analysis_of_algorithms) is a branch of [computer science](/wiki/Computer_science) that studies the [performance](/wiki/Time_complexity) of [algorithms](/wiki/Algorithm) (computer programs solving a certain problem).<ref name=Wegener>[Template:Citation](/wiki/Template:Citation), pages 1-2</ref> Logarithms are valuable for describing algorithms that [divide a problem](/wiki/Divide_and_conquer_algorithm) into smaller ones, and join the solutions of the subproblems.[[73]](#cite_note-73) For example, to find a number in a sorted list, the [binary search algorithm](/wiki/Binary_search_algorithm) checks the middle entry and proceeds with the half before or after the middle entry if the number is still not found. This algorithm requires, on average, log2(*N*) comparisons, where *N* is the list's length.[[74]](#cite_note-74) Similarly, the [merge sort](/wiki/Merge_sort) algorithm sorts an unsorted list by dividing the list into halves and sorting these first before merging the results. Merge sort algorithms typically require a time [approximately proportional to](/wiki/Big_O_notation) [Template:Nowrap](/wiki/Template:Nowrap).[[75]](#cite_note-75) The base of the logarithm is not specified here, because the result only changes by a constant factor when another base is used. A constant factor is usually disregarded in the analysis of algorithms under the standard [uniform cost model](/wiki/Uniform_cost_model).<ref name=Wegener20>[Template:Citation](/wiki/Template:Citation)</ref>

A function *f*(*x*) is said to [grow logarithmically](/wiki/Logarithmic_growth) if *f*(*x*) is (exactly or approximately) proportional to the logarithm of *x*. (Biological descriptions of organism growth, however, use this term for an exponential function.[[76]](#cite_note-76)) For example, any [natural number](/wiki/Natural_number) *N* can be represented in [binary form](/wiki/Binary_numeral_system) in no more than [Template:Nowrap](/wiki/Template:Nowrap) [bits](/wiki/Bit). In other words, the amount of [memory](/wiki/Memory_(computing)) needed to store *N* grows logarithmically with *N*.

### Entropy and chaos[[edit](/index.php?title=(none)&action=edit&section=25)]

[right|thumb|](/wiki/File:Chaotic_Bunimovich_stadium.png)[Billiards](/wiki/Dynamical_billiards) on an oval [billiard table](/wiki/Billiard_table). Two particles, starting at the center with an angle differing by one degree, take paths that diverge chaotically because of [reflections](/wiki/Reflection_(physics)) at the boundary.|alt=An oval shape with the trajectories of two particles.

[Entropy](/wiki/Entropy) is broadly a measure of the disorder of some system. In [statistical thermodynamics](/wiki/Statistical_thermodynamics), the entropy *S* of some physical system is defined as

<math> S = - k \sum\_i p\_i \ln(p\_i).\, </math>

The sum is over all possible states *i* of the system in question, such as the positions of gas particles in a container. Moreover, *pi* is the probability that the state *i* is attained and *k* is the [Boltzmann constant](/wiki/Boltzmann_constant). Similarly, [entropy in information theory](/wiki/Entropy_(information_theory)) measures the quantity of information. If a message recipient may expect any one of *N* possible messages with equal likelihood, then the amount of information conveyed by any one such message is quantified as log2(*N*) bits.[[77]](#cite_note-77) [Lyapunov exponents](/wiki/Lyapunov_exponent) use logarithms to gauge the degree of chaoticity of a [dynamical system](/wiki/Dynamical_system). For example, for a particle moving on an oval billiard table, even small changes of the initial conditions result in very different paths of the particle. Such systems are [chaotic](/wiki/Chaos_theory) in a [deterministic](/wiki/Deterministic_system) way, because small measurement errors of the initial state predictably lead to largely different final states.[[78]](#cite_note-78) At least one Lyapunov exponent of a deterministically chaotic system is positive.

### Fractals[[edit](/index.php?title=(none)&action=edit&section=26)]

[The Sierpinski triangle (at the right) is constructed by repeatedly replacing](/wiki/File:Sierpinski_dimension.svg) [equilateral triangles](/wiki/Equilateral_triangle) by three smaller ones.|right|thumb|400px|alt=Parts of a triangle are removed in an iterated way.

Logarithms occur in definitions of the [dimension](/wiki/Fractal_dimension) of [fractals](/wiki/Fractal).[[79]](#cite_note-79) Fractals are geometric objects that are [self-similar](/wiki/Self-similarity): small parts reproduce, at least roughly, the entire global structure. The [Sierpinski triangle](/wiki/Sierpinski_triangle) (pictured) can be covered by three copies of itself, each having sides half the original length. This makes the [Hausdorff dimension](/wiki/Hausdorff_dimension) of this structure [Template:Nowrap beginln](/wiki/Template:Nowrap_begin)(3)/ln(2) ≈ 1.58[Template:Nowrap end](/wiki/Template:Nowrap_end). Another logarithm-based notion of dimension is obtained by [counting the number of boxes](/wiki/Box-counting_dimension) needed to cover the fractal in question.

### Music[[edit](/index.php?title=(none)&action=edit&section=27)]

[Template:Multiple image](/wiki/Template:Multiple_image)

Logarithms are related to musical tones and [intervals](/wiki/Interval_(music)). In [equal temperament](/wiki/Equal_temperament), the frequency ratio depends only on the interval between two tones, not on the specific frequency, or [pitch](/wiki/Pitch_(music)), of the individual tones. For example, the [note *A*](/wiki/A_(musical_note)) has a frequency of 440 [Hz](/wiki/Hertz) and [*B-flat*](/wiki/B♭_(musical_note)) has a frequency of 466 Hz. The interval between *A* and *B-flat* is a [semitone](/wiki/Semitone), as is the one between *B-flat* and [*B*](/wiki/B_(musical_note)) (frequency 493 Hz). Accordingly, the frequency ratios agree:

<math>\frac{466}{440} \approx \frac{493}{466} \approx 1.059 \approx \sqrt[12]2.</math>

Therefore, logarithms can be used to describe the intervals: an interval is measured in semitones by taking the [Template:Nowrap](/wiki/Template:Nowrap) logarithm of the [frequency](/wiki/Frequency) ratio, while the [Template:Nowrap](/wiki/Template:Nowrap) logarithm of the frequency ratio expresses the interval in [cents](/wiki/Cent_(music)), hundredths of a semitone. The latter is used for finer encoding, as it is needed for non-equal temperaments.[[80]](#cite_note-80)

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Interval** (the two tones are played at the same time) | [1/12 tone](/wiki/72_tone_equal_temperament) [Template:Audio](/wiki/Template:Audio) | [Semitone](/wiki/Semitone) [Template:Audio](/wiki/Template:Audio) | [Just major third](/wiki/Just_major_third) [Template:Audio](/wiki/Template:Audio) | [Major third](/wiki/Major_third) [Template:Audio](/wiki/Template:Audio) | [Tritone](/wiki/Tritone) [Template:Audio](/wiki/Template:Audio) | [Octave](/wiki/Octave) [Template:Audio](/wiki/Template:Audio) |
| **Frequency ratio** *r* | <math>2^{\frac 1 {72}} \approx 1.0097</math> | <math>2^{\frac 1 {12}} \approx 1.0595</math> | <math>\tfrac 5 4 = 1.25</math> | <math>\begin{align} 2^{\frac 4 {12}} & = \sqrt[3] 2 \\ & \approx 1.2599 \end{align} </math> | <math>\begin{align} 2^{\frac 6 {12}} & = \sqrt 2 \\ & \approx 1.4142 \end{align} </math> | <math> 2^{\frac {12} {12}} = 2 </math> |
| **Corresponding number of semitones** <math>\log\_{\sqrt[12] 2}(r) = 12 \log\_2 (r)</math> | <math>\tfrac 1 6 \,</math> | <math>1 \,</math> | <math>\approx 3.8631 \,</math> | <math>4 \,</math> | <math>6 \,</math> | <math>12 \,</math> |
| **Corresponding number of cents** <math>\log\_{\sqrt[1200] 2}(r) = 1200 \log\_2 (r)</math> | <math>16 \tfrac 2 3 \,</math> | <math>100 \,</math> | <math>\approx 386.31 \,</math> | <math>400 \,</math> | <math>600 \,</math> | <math>1200 \,</math> |

### Number theory[[edit](/index.php?title=(none)&action=edit&section=28)]

[Natural logarithms](/wiki/Natural_logarithm) are closely linked to [counting prime numbers](/wiki/Prime-counting_function) (2, 3, 5, 7, 11, ...), an important topic in [number theory](/wiki/Number_theory). For any [integer](/wiki/Integer) *x*, the quantity of [prime numbers](/wiki/Prime_number) less than or equal to *x* is denoted [Template:Pi](/wiki/Template:Pi)(*x*). The [prime number theorem](/wiki/Prime_number_theorem) asserts that [Template:Pi](/wiki/Template:Pi)(*x*) is approximately given by

<math>\frac{x}{\ln(x)},</math>

in the sense that the ratio of [Template:Pi](/wiki/Template:Pi)(*x*) and that fraction approaches 1 when *x* tends to infinity.[[81]](#cite_note-81) As a consequence, the probability that a randomly chosen number between 1 and *x* is prime is inversely [proportional](/wiki/Proportionality_(mathematics)) to the number of decimal digits of *x*. A far better estimate of [Template:Pi](/wiki/Template:Pi)(*x*) is given by the [offset logarithmic integral](/wiki/Logarithmic_integral_function) function Li(*x*), defined by

<math> \mathrm{Li}(x) = \int\_2^x \frac1{\ln(t)} \,dt. </math>

The [Riemann hypothesis](/wiki/Riemann_hypothesis), one of the oldest open mathematical [conjectures](/wiki/Conjecture), can be stated in terms of comparing [Template:Pi](/wiki/Template:Pi)(*x*) and Li(*x*).[[82]](#cite_note-82) The [Erdős–Kac theorem](/wiki/Erdős–Kac_theorem) describing the number of distinct [prime factors](/wiki/Prime_factor) also involves the [natural logarithm](/wiki/Natural_logarithm).

The logarithm of *n* [factorial](/wiki/Factorial), [Template:Nowrap begin](/wiki/Template:Nowrap_begin)*n*! = 1 · 2 · ... · *n*[Template:Nowrap end](/wiki/Template:Nowrap_end), is given by

<math> \ln (n!) = \ln (1) + \ln (2) + \cdots + \ln (n). \,</math>

This can be used to obtain [Stirling's formula](/wiki/Stirling's_formula), an approximation of *n*! for large *n*.[[83]](#cite_note-83)

## Generalizations[[edit](/index.php?title=(none)&action=edit&section=29)]

### Complex logarithm[[edit](/index.php?title=(none)&action=edit&section=30)]

[Template:Main](/wiki/Template:Main) [thumb|right|Polar form of](/wiki/File:Complex_number_illustration_multiple_arguments.svg) [Template:Nowrap](/wiki/Template:Nowrap). Both φ and φ' are arguments of *z*.|alt=An illustration of the polar form: a point is described by an arrow or equivalently by its length and angle to the *x* axis.

The [complex numbers](/wiki/Complex_number) *a* solving the equation

<math>e^a=z.\,</math>

are called *complex logarithms*. Here, *z* is a complex number. A complex number is commonly represented as [Template:Nowrap begin](/wiki/Template:Nowrap_begin)*z = x + iy*[Template:Nowrap end](/wiki/Template:Nowrap_end), where *x* and *y* are real numbers and *i* is the [imaginary unit](/wiki/Imaginary_unit). Such a number can be visualized by a point in the [complex plane](/wiki/Complex_plane), as shown at the right. The [polar form](/wiki/Polar_form) encodes a non-zero complex number *z* by its [absolute value](/wiki/Absolute_value), that is, the distance *r* to the [origin](/wiki/Origin_(mathematics)), and an angle between the *x* axis and the line passing through the origin and *z*. This angle is called the [argument](/wiki/Argument_(complex_analysis)) of *z*. The absolute value *r* of *z* is

<math>r=\sqrt{x^2+y^2}. \,</math>

The argument is not uniquely specified by *z*: both [Template:Math](/wiki/Template:Math) and [Template:Math'](/wiki/Template:Math) = [Template:Math](/wiki/Template:Math) + 2[Template:Pi](/wiki/Template:Pi) are arguments of *z* because adding 2[Template:Pi](/wiki/Template:Pi) [radians](/wiki/Radian) or 360 degrees[Template:Refn](/wiki/Template:Refn) to φ corresponds to "winding" around the origin counter-clock-wise by a [turn](/wiki/Turn_(geometry)). The resulting complex number is again *z*, as illustrated at the right. However, exactly one argument φ satisfies [Template:Nowrap](/wiki/Template:Nowrap) and [Template:Nowrap](/wiki/Template:Nowrap). It is called the *principal argument*, denoted Arg(*z*), with a capital *A*.[[84]](#cite_note-84) (An alternative normalization is [Template:Nowrap](/wiki/Template:Nowrap).[[85]](#cite_note-85))

[right|thumb|The principal branch of the complex logarithm, Log(*z*). The black point at](/wiki/File:Complex_log.jpg) [Template:Nowrap](/wiki/Template:Nowrap) corresponds to absolute value zero and brighter (more [saturated](/wiki/Saturation_(color_theory))) colors refer to bigger absolute values. The [hue](/wiki/Hue) of the color encodes the argument of Log(*z*).|alt=A density plot. In the middle there is a black point, at the negative axis the hue jumps sharply and evolves smoothly otherwise. Using [trigonometric functions](/wiki/Trigonometric_functions) [sine](/wiki/Sine) and [cosine](/wiki/Cosine), or the [complex exponential](/wiki/Complex_exponential), respectively, *r* and φ are such that the following identities hold:[[86]](#cite_note-86)

<math>\begin{array}{lll}z& = & r \left (\cos \varphi + i \sin \varphi\right) \\

& = & r e^{i \varphi}. \end{array} \, </math>

This implies that the [Template:Nowrap](/wiki/Template:Nowrap) power of *e* equals *z*, where

<math>a = \ln (r) + i ( \varphi + 2 n \pi ), \,</math>

φ is the principal argument Arg(*z*) and *n* is an arbitrary integer. Any such *a* is called a complex logarithm of *z*. There are infinitely many of them, in contrast to the uniquely defined real logarithm. If [Template:Nowrap begin](/wiki/Template:Nowrap_begin)*n* = 0[Template:Nowrap end](/wiki/Template:Nowrap_end), *a* is called the *principal value* of the logarithm, denoted Log(*z*). The principal argument of any positive real number *x* is 0; hence Log(*x*) is a real number and equals the real (natural) logarithm. However, the above formulas for logarithms of products and powers [do *not* generalize](/wiki/Exponentiation#Failure_of_power_and_logarithm_identities) to the principal value of the complex logarithm.[[87]](#cite_note-87) The illustration at the right depicts Log(*z*). The discontinuity, that is, the jump in the hue at the negative part of the *x*- or real axis, is caused by the jump of the principal argument there. This locus is called a [branch cut](/wiki/Branch_cut). This behavior can only be circumvented by dropping the range restriction on φ. Then the argument of *z* and, consequently, its logarithm become [multi-valued functions](/wiki/Multi-valued_function).

### Inverses of other exponential functions[[edit](/index.php?title=(none)&action=edit&section=31)]

Exponentiation occurs in many areas of mathematics and its inverse function is often referred to as the logarithm. For example, the [logarithm of a matrix](/wiki/Logarithm_of_a_matrix) is the (multi-valued) inverse function of the [matrix exponential](/wiki/Matrix_exponential).[[88]](#cite_note-88) Another example is the [*p*-adic logarithm](/wiki/P-adic_logarithm_function), the inverse function of the [*p*-adic exponential](/wiki/P-adic_exponential_function). Both are defined via Taylor series analogous to the real case.[[89]](#cite_note-89) In the context of [differential geometry](/wiki/Differential_geometry), the [exponential map](/wiki/Exponential_map_(Riemannian_geometry)) maps the [tangent space](/wiki/Tangent_space) at a point of a [manifold](/wiki/Differentiable_manifold) to a [neighborhood](/wiki/Neighborhood_(mathematics)) of that point. Its inverse is also called the logarithmic (or log) map.[[90]](#cite_note-90) In the context of [finite groups](/wiki/Finite_groups) exponentiation is given by repeatedly multiplying one group element *b* with itself. The [discrete logarithm](/wiki/Discrete_logarithm) is the integer *n* solving the equation

<math>b^n = x,\,</math>

where *x* is an element of the group. Carrying out the exponentiation can be done efficiently, but the discrete logarithm is believed to be very hard to calculate in some groups. This asymmetry has important applications in [public key cryptography](/wiki/Public_key_cryptography), such as for example in the [Diffie–Hellman key exchange](/wiki/Diffie–Hellman_key_exchange), a routine that allows secure exchanges of [cryptographic](/wiki/Cryptography) keys over unsecured information channels.[[91]](#cite_note-91) [Zech's logarithm](/wiki/Zech's_logarithm) is related to the discrete logarithm in the multiplicative group of non-zero elements of a [finite field](/wiki/Finite_field).[[92]](#cite_note-92) [Template:AnchorFurther](/wiki/Template:Anchor) logarithm-like inverse functions include the *double logarithm* ln(ln(*x*)), the [*super- or hyper-4-logarithm*](/wiki/Super-logarithm) (a slight variation of which is called [iterated logarithm](/wiki/Iterated_logarithm) in computer science), the [Lambert W function](/wiki/Lambert_W_function), and the [logit](/wiki/Logit). They are the inverse functions of the [double exponential function](/wiki/Double_exponential_function), [tetration](/wiki/Tetration), of [Template:Nowrap](/wiki/Template:Nowrap),[[93]](#cite_note-93) and of the [logistic function](/wiki/Logistic_function), respectively.[[94]](#cite_note-94)

### Related concepts[[edit](/index.php?title=(none)&action=edit&section=32)]

From the perspective of [group theory](/wiki/Group_theory), the identity [Template:Nowrap](/wiki/Template:Nowrap) expresses a [group isomorphism](/wiki/Group_isomorphism) between positive [reals](/wiki/Real_number) under multiplication and reals under addition. Logarithmic functions are the only continuous isomorphisms between these groups.[[95]](#cite_note-95) By means of that isomorphism, the [Haar measure](/wiki/Haar_measure) ([Lebesgue measure](/wiki/Lebesgue_measure)) *dx* on the reals corresponds to the Haar measure *dx*/*x* on the positive reals.[[96]](#cite_note-96) In [complex analysis](/wiki/Complex_analysis) and [algebraic geometry](/wiki/Algebraic_geometry), [differential forms](/wiki/Differential_form) of the form [Template:Nowrap begin](/wiki/Template:Nowrap_begin)*df*/*f* [Template:Nowrap end](/wiki/Template:Nowrap_end) are known as forms with logarithmic [poles](/wiki/Pole_(complex_analysis)).[[97]](#cite_note-97) The [polylogarithm](/wiki/Polylogarithm) is the function defined by

<math>

\operatorname{Li}\_s(z) = \sum\_{k=1}^\infty {z^k \over k^s}. </math> It is related to the [natural logarithm](/wiki/Natural_logarithm) by [Template:Nowrap beginLi](/wiki/Template:Nowrap_begin)1(*z*) = −ln(1 − *z*)[Template:Nowrap end](/wiki/Template:Nowrap_end). Moreover, Li*s*(1) equals the [Riemann zeta function](/wiki/Riemann_zeta_function) ζ(*s*).[[98]](#cite_note-98)

## See also[[edit](/index.php?title=(none)&action=edit&section=33)]

* [Cologarithm](/wiki/Cologarithm)
* [Exponential function](/wiki/Exponential_function)
* [Decimal exponent](/wiki/Decimal_exponent) (dex)
* [Index of logarithm articles](/wiki/Index_of_logarithm_articles)

## Notes[[edit](/index.php?title=(none)&action=edit&section=34)]

[Template:Reflist](/wiki/Template:Reflist)

## References[[edit](/index.php?title=(none)&action=edit&section=35)]

[Template:Reflist](/wiki/Template:Reflist)

## External links[[edit](/index.php?title=(none)&action=edit&section=36)]

* [Template:Commons category inline](/wiki/Template:Commons_category_inline)
* [Template:Wiktionary-inline](/wiki/Template:Wiktionary-inline)
* [Khan Academy: Logarithms, free online micro lectures](http://wayback.archive.org/web/20121218200616/http://www.khanacademy.org/math/algebra/logarithms-tutorial)
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