[Template:Pp-protect](/wiki/Template:Pp-protect" \o "Template:Pp-protect) [Template:Redirect](/wiki/Template:Redirect) [Template:Use dmy dates](/wiki/Template:Use_dmy_dates) [Template:Pi box](/wiki/Template:Pi_box) The number [**Template:Pi**](/wiki/Template:Pi) is a [mathematical constant](/wiki/Mathematical_constant), the [ratio](/wiki/Ratio) of a [circle's](/wiki/Circle) [circumference](/wiki/Circumference) to its [diameter](/wiki/Diameter), commonly approximated as 3.14159. It has been represented by the Greek letter "[Template:Pi](/wiki/Template:Pi)" since the mid-18th century, though it is also sometimes spelled out as "**pi**" ([Template:IPAc-en](/wiki/Template:IPAc-en)).

Being an [irrational number](/wiki/Irrational_number), [Template:Pi](/wiki/Template:Pi) cannot be expressed exactly as a [fraction](/wiki/Common_fraction) (equivalently, its [decimal representation](/wiki/Decimal_representation) never ends and never [settles into a permanent repeating pattern](/wiki/Repeating_decimal)). Still, fractions such as 22/7 and other rational numbers are commonly used to [approximate](/wiki/Approximations_of_π) [Template:Pi](/wiki/Template:Pi). The digits appear to be randomly distributed; however, to date, no proof of this has been discovered. Also, [Template:Pi](/wiki/Template:Pi) is a [transcendental number](/wiki/Transcendental_number) – a number that is not the [root](/wiki/Root_of_a_polynomial) of any non-zero [polynomial](/wiki/Polynomial) having [rational](/wiki/Rational_number) [coefficients](/wiki/Coefficient). This transcendence of [Template:Pi](/wiki/Template:Pi) implies that it is impossible to solve the ancient challenge of [squaring the circle](/wiki/Squaring_the_circle) with a [compass and straightedge](/wiki/Compass-and-straightedge_construction).

Ancient civilizations needed the value of [Template:Pi](/wiki/Template:Pi) to be computed accurately for practical reasons. It was calculated to seven digits, using geometrical techniques, in [Chinese mathematics](/wiki/History_of_mathematics#Chinese_mathematics) and to about five in [Indian mathematics](/wiki/History_of_mathematics#Indian_mathematics) in the 5th century AD. The historically first exact formula for [Template:Pi](/wiki/Template:Pi), based on [infinite series](/wiki/Series_(mathematics)), was not available until a millennium later, when in the 14th century the [Madhava–Leibniz series](/wiki/Leibniz_formula_for_pi) was discovered in Indian mathematics.[[1]](#cite_note-1)[[2]](#cite_note-2) In the 20th and 21st centuries, mathematicians and [computer scientists](/wiki/Computer_science) discovered new approaches that, when combined with increasing computational power, extended the decimal representation of [Template:Pi](/wiki/Template:Pi) to, as of 2015, over 13.3 trillion (1013) digits.[[3]](#cite_note-3) Practically all scientific applications require no more than a few hundred digits of [Template:Pi](/wiki/Template:Pi)[Template:Discuss](/wiki/Template:Discuss), and many substantially fewer, so the primary motivation for these computations is the human desire to break records.[[4]](#cite_note-4)[[5]](#cite_note-5) However, the extensive calculations involved have been used to test [supercomputers](/wiki/Supercomputer) and high-precision multiplication [algorithms](/wiki/Algorithms).

Because its definition relates to the circle, [Template:Pi](/wiki/Template:Pi) is found in many formulae in [trigonometry](/wiki/Trigonometry) and [geometry](/wiki/Geometry), especially those concerning circles, ellipses or spheres. Because of its special role as an [eigenvalue](/wiki/Eigenvalue), [Template:Pi](/wiki/Template:Pi) appears in areas of mathematics and the sciences having little to do with the geometry of circles, such as [number theory](/wiki/Number_theory) and [statistics](/wiki/Statistics). It is also found in [cosmology](/wiki/Cosmology), [thermodynamics](/wiki/Thermodynamics), [mechanics](/wiki/Mechanics) and [electromagnetism](/wiki/Electromagnetism). The ubiquity of [Template:Pi](/wiki/Template:Pi) makes it one of the most widely known mathematical constants both inside and outside the scientific community: Several books devoted to it have been published, the number is celebrated on [Pi Day](/wiki/Pi_Day) and record-setting calculations of the digits of [Template:Pi](/wiki/Template:Pi) often result in news headlines. Attempts to memorize the value of [Template:Pi](/wiki/Template:Pi) with increasing precision have led to records of over 70,000 digits.

[Template:TOC limit](/wiki/Template:TOC_limit)

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## Fundamentals[[edit](/index.php?title=(none)&action=edit&section=1)]

### Name[[edit](/index.php?title=(none)&action=edit&section=2)]

The symbol used by mathematicians to represent the ratio of a circle's circumference to its diameter is the lowercase [Greek letter](/wiki/Pi_(letter)) [Template:Pi](/wiki/Template:Pi), sometimes spelled out as *pi,* and derived from the first letter of the Greek word *perimetros,* meaning circumference.[[6]](#cite_note-6) In English, [Template:Pi](/wiki/Template:Pi) is [pronounced as "pie"](/wiki/English_pronunciation_of_Greek_letters) ( [Template:IPAc-en](/wiki/Template:IPAc-en), [Template:Respell](/wiki/Template:Respell)).[[7]](#cite_note-7) In mathematical use, the lowercase letter [Template:Pi](/wiki/Template:Pi) (or π in [sans-serif](/wiki/Sans-serif) font) is distinguished from its capital counterpart [Template:PI](/wiki/Template:PI), which denotes a [product of a sequence](/wiki/Product_of_a_sequence).

The choice of the symbol [Template:Pi](/wiki/Template:Pi) is discussed in the section [*Adoption of the symbol*](/wiki/#Adoption_of_the_symbol_π) [*Template:Pi*](/wiki/Template:Pi).

### Definition[[edit](/index.php?title=(none)&action=edit&section=3)]

[alt=A diagram of a circle, with the width labeled as diameter, and the perimeter labeled as circumference|thumb|right|The circumference of a circle is slightly more than three times as long as its diameter. The exact ratio is called](/wiki/File:Pi_eq_C_over_d.svg) [Template:Pi](/wiki/Template:Pi). [Template:Pi](/wiki/Template:Pi) is commonly defined as the [ratio](/wiki/Ratio) of a [circle's](/wiki/Circle) [circumference](/wiki/Circumference) [Template:Math](/wiki/Template:Math) to its [diameter](/wiki/Diameter) [Template:Math](/wiki/Template:Math):[[8]](#cite_note-8):<math> \pi = \frac{C}{d}</math> The ratio [Template:Math](/wiki/Template:Math) is constant, regardless of the circle's size. For example, if a circle has twice the diameter of another circle it will also have twice the circumference, preserving the ratio [Template:Math](/wiki/Template:Math). This definition of [Template:Pi](/wiki/Template:Pi) implicitly makes use of [flat (Euclidean) geometry](/wiki/Euclidean_geometry); although the notion of a circle can be extended to any [curved (non-Euclidean) geometry](/wiki/Non-Euclidean_geometry), these new circles will no longer satisfy the formula [Template:Math](/wiki/Template:Math).[[8]](#cite_note-8) Here, the circumference of a circle is the [arc length](/wiki/Arc_length) around the perimeter of the circle, a quantity which can be formally defined independently of geometry using [limits](/wiki/Limit_(mathematics)), a concept in [calculus](/wiki/Calculus).[[9]](#cite_note-9) For example, one may compute directly the arc length of the top half of the unit circle given in [Cartesian coordinates](/wiki/Cartesian_coordinates) by [Template:Math](/wiki/Template:Math), as the [integral](/wiki/Integral):[[10]](#cite_note-10):<math>\pi = \int\_{-1}^1 \frac{dx}{\sqrt{1-x^2}}.</math> An integral such as this was adopted as the definition of [Template:Pi](/wiki/Template:Pi) by [Karl Weierstrass](/wiki/Karl_Weierstrass), who defined it directly as an integral in 1841.[[11]](#cite_note-11) Definitions of [Template:Pi](/wiki/Template:Pi) such as these that rely on a notion of circumference, and hence implicitly on concepts of the [integral calculus](/wiki/Integral_calculus), are no longer common in the literature. [Template:Harvtxt](/wiki/Template:Harvtxt) explains that this is because in many modern treatments of calculus, [differential calculus](/wiki/Differential_calculus) typically precedes integral calculus in the university curriculum, so it is desirable to have a definition of [Template:Pi](/wiki/Template:Pi) that does not rely on the latter. One such definition, due to [Richard Baltzer](/wiki/Richard_Baltzer),[[12]](#cite_note-12) and popularized by [Edmund Landau](/wiki/Edmund_Landau),[[13]](#cite_note-13) is the following: [Template:Pi](/wiki/Template:Pi) is twice the smallest positive number at which the [cosine function](/wiki/Cosine) equals 0.[[8]](#cite_note-8)[[10]](#cite_note-10)[[14]](#cite_note-14) The cosine can be defined independently of geometry as a [power series](/wiki/Power_series),[[15]](#cite_note-15) or as the solution of a [differential equation](/wiki/Differential_equation).[[14]](#cite_note-14) In a similar spirit, [Template:Pi](/wiki/Template:Pi) can be defined instead using properties of the [complex exponential](/wiki/Complex_exponential), [Template:Math](/wiki/Template:Math), of a [complex](/wiki/Complex_number) variable [Template:Math](/wiki/Template:Math). Like the cosine, the complex exponential can be defined in one of several ways. The set of complex numbers at which [Template:Math](/wiki/Template:Math) is equal to one is then an (imaginary) arithmetic progression of the form:

<math>\{\dots,-2\pi i, 0, 2\pi i, 4\pi i,\dots\} = \{2\pi ki| k\in\mathbb Z\}</math>

and there is a unique positive real number [Template:Pi](/wiki/Template:Pi) with this property.[[10]](#cite_note-10)[[16]](#cite_note-16)A more abstract variation on the same idea, making use of sophisticated mathematical concepts of [topology](/wiki/Topology) and [algebra](/wiki/Algebra), is the following theorem:[[17]](#cite_note-17) there is a unique [continuous](/wiki/Continuous_function) [isomorphism](/wiki/Isomorphism) from the [group](/wiki/Group_(mathematics)) **R**/**Z** of real numbers under addition [modulo](/wiki/Quotient_group) integers (the [circle group](/wiki/Circle_group)) onto the multiplicative group of [complex numbers](/wiki/Complex_numbers) of [absolute value](/wiki/Absolute_value) one. The number [Template:Pi](/wiki/Template:Pi) is then defined as half the magnitude of the derivative of this homomorphism.[[18]](#cite_note-18) A circle encloses the largest area that can be attained within a given perimeter. Thus the number [Template:Pi](/wiki/Template:Pi) is also characterized as the best constant in the [isoperimetric inequality](/wiki/Isoperimetric_inequality) (times one-fourth). There are many other, closely related, ways in which [Template:Pi](/wiki/Template:Pi) appears as an [eigenvalue](/wiki/Eigenvalue) of some geometrical or physical process; see [below](/wiki/#Spectral_characterizations).

### Irrationality and normality[[edit](/index.php?title=(none)&action=edit&section=4)]

[Template:Pi](/wiki/Template:Pi) is an [irrational number](/wiki/Irrational_number), meaning that it cannot be written as the [ratio of two integers](/wiki/Rational_number) (fractions such as [Template:Math](/wiki/Template:Math) are commonly used to approximate [Template:Pi](/wiki/Template:Pi); no [common fraction](/wiki/Common_fraction) (ratio of whole numbers) can be its exact value).[[19]](#cite_note-19) Because [Template:Pi](/wiki/Template:Pi) is irrational, it has an infinite number of digits in its [decimal representation](/wiki/Decimal_representation), and it does not settle into an infinitely [repeating pattern](/wiki/Repeating_decimal) of digits. There are several [proofs that](/wiki/Proof_that_π_is_irrational) [Template:Pi](/wiki/Template:Pi) is irrational; they generally require calculus and rely on the [reductio ad absurdum](/wiki/Reductio_ad_absurdum) technique. The degree to which [Template:Pi](/wiki/Template:Pi) can be approximated by [rational numbers](/wiki/Rational_number) (called the [irrationality measure](/wiki/Irrationality_measure)) is not precisely known; estimates have established that the irrationality measure is larger than the measure of [Template:Math](/wiki/Template:Math) or [Template:Math](/wiki/Template:Math) but smaller than the measure of [Liouville numbers](/wiki/Liouville_number).[[20]](#cite_note-20) The digits of [Template:Pi](/wiki/Template:Pi) have no apparent pattern and have passed tests for [statistical randomness](/wiki/Statistical_randomness), including tests for [normality](/wiki/Normal_number); a number of infinite length is called normal when all possible sequences of digits (of any given length) appear equally often.[[21]](#cite_note-21) The conjecture that [Template:Pi](/wiki/Template:Pi) is [normal](/wiki/Normal_number) has not been proven or disproven.[[21]](#cite_note-21) Since the advent of computers, a large number of digits of [Template:Pi](/wiki/Template:Pi) have been available on which to perform statistical analysis. [Yasumasa Kanada](/wiki/Yasumasa_Kanada) has performed detailed statistical analyses on the decimal digits of [Template:Pi](/wiki/Template:Pi) and found them consistent with normality; for example, the frequency of the ten digits 0 to 9 were subjected to [statistical significance tests](/wiki/Statistical_significance_test), and no evidence of a pattern was found.[[22]](#cite_note-22) Any random sequence of digits contains arbitrarily long subsequences that appear non-random, by the [infinite monkey theorem](/wiki/Infinite_monkey_theorem). Thus, because the sequence of [Template:Pi's](/wiki/Template:Pi) digits passes statistical tests for randomness, it contains some sequences of digits that may appear non-random, such as a [sequence of six consecutive 9s](/wiki/Six_nines_in_pi) that begins at the 762nd decimal place of the decimal representation of [Template:Pi](/wiki/Template:Pi).[[23]](#cite_note-23)

### Transcendence[[edit](/index.php?title=(none)&action=edit&section=5)]

[thumb|alt=A diagram of a square and circle, both with identical area; the length of the side of the square is the square root of pi|Because](/wiki/File:Squaring_the_circle.svg) [Template:Pi](/wiki/Template:Pi) is a [transcendental number](/wiki/Transcendental_number), [squaring the circle](/wiki/Squaring_the_circle) is not possible in a finite number of steps using the classical tools of [compass and straightedge](/wiki/Compass-and-straightedge_construction). In addition to being irrational, more strongly [Template:Pi](/wiki/Template:Pi) is a [transcendental number](/wiki/Transcendental_number), which means that it is not the [solution](/wiki/Root_of_a_function) of any non-constant [polynomial](/wiki/Polynomial) with [rational](/wiki/Rational_number) coefficients, such as [Template:Math](/wiki/Template:Math).[[24]](#cite_note-24)[[25]](#cite_note-25) The transcendence of [Template:Pi](/wiki/Template:Pi) has two important consequences: First, [Template:Pi](/wiki/Template:Pi) cannot be expressed using any finite combination of rational numbers and square roots or [*n*-th roots](/wiki/N-th_root) such as [Template:Math](/wiki/Template:Math) or [Template:Math](/wiki/Template:Math). Second, since no transcendental number can be [constructed](/wiki/Constructible_number) with [compass and straightedge](/wiki/Compass-and-straightedge_construction), it is not possible to "[square the circle](/wiki/Squaring_the_circle)". In other words, it is impossible to construct, using compass and straightedge alone, a square whose area is equal to the area of a given circle.[[26]](#cite_note-26) Squaring a circle was one of the important geometry problems of the [classical antiquity](/wiki/Classical_antiquity).[[27]](#cite_note-27) Amateur mathematicians in modern times have sometimes attempted to square the circle and sometimes claim success despite the fact that it is impossible.[[28]](#cite_note-28)

### Continued fractions[[edit](/index.php?title=(none)&action=edit&section=6)]

[thumb|alt=A photograph of the Greek letter pi, created as a large stone mosaic embedded in the ground.|The constant](/wiki/File:Matheon2.jpg) [Template:Pi](/wiki/Template:Pi) is represented in this [mosaic](/wiki/Mosaic) outside the Mathematics Building at the [Technical University of Berlin](/wiki/Technical_University_of_Berlin). Like all irrational numbers, [Template:Pi](/wiki/Template:Pi) cannot be represented as a [common fraction](/wiki/Common_fraction) (also known as a [simple](/wiki/Simple_fractions) or [vulgar fraction](/wiki/Vulgar_fraction)), by the very definition of "irrational". But every irrational number, including [Template:Pi](/wiki/Template:Pi), can be represented by an infinite series of nested fractions, called a [continued fraction](/wiki/Continued_fraction):

<math>

\pi=3+\textstyle \frac{1}{7+\textstyle \frac{1}{15+\textstyle \frac{1}{1+\textstyle \frac{1}{292+\textstyle \frac{1}{1+\textstyle \frac{1}{1+\textstyle \frac{1}{1+\ddots}}}}}}}</math> [Template:OEIS2C](/wiki/Template:OEIS2C)

Truncating the continued fraction at any point yields a rational approximation for [Template:Pi](/wiki/Template:Pi); the first four of these are 3, 22/7, 333/106, and 355/113. These numbers are among the most well-known and widely used historical approximations of the constant. Each approximation generated in this way is a best rational approximation; that is, each is closer to [Template:Pi](/wiki/Template:Pi) than any other fraction with the same or a smaller denominator.[[29]](#cite_note-29) Because [Template:Pi](/wiki/Template:Pi) is known to be transcendental, it is by definition not algebraic and so cannot be a [quadratic irrational](/wiki/Quadratic_irrational). Therefore, [Template:Pi](/wiki/Template:Pi) cannot have a [periodic continued fraction](/wiki/Periodic_continued_fraction). Although the simple continued fraction for [Template:Pi](/wiki/Template:Pi) (shown above) also does not exhibit any other obvious pattern,[[30]](#cite_note-30) mathematicians have discovered several [generalized continued fractions](/wiki/Generalized_continued_fraction) that do, such as:[[31]](#cite_note-31):<math>\pi=\textstyle \cfrac{4}{1+\textstyle \frac{1^2}{2+\textstyle \frac{3^2}{2+\textstyle \frac{5^2}{2+\textstyle \frac{7^2}{2+\textstyle \frac{9^2}{2+\ddots}}}}}} =3+\textstyle \frac{1^2}{6+\textstyle \frac{3^2}{6+\textstyle \frac{5^2}{6+\textstyle \frac{7^2}{6+\textstyle \frac{9^2}{6+\ddots}}}}} =\textstyle \cfrac{4}{1+\textstyle \frac{1^2}{3+\textstyle \frac{2^2}{5+\textstyle \frac{3^2}{7+\textstyle \frac{4^2}{9+\ddots}}}}}</math>

### Approximate value[[edit](/index.php?title=(none)&action=edit&section=7)]

Some [approximations of *pi*](/wiki/Approximations_of_π) include:

* **Integers**: [3](/wiki/3_(number))
* **Fractions**: Approximate fractions include (in order of increasing accuracy) [Template:Sfrac](/wiki/Template:Sfrac), [Template:Sfrac](/wiki/Template:Sfrac), [Template:Sfrac](/wiki/Template:Sfrac), [Template:Sfrac](/wiki/Template:Sfrac), [Template:Sfrac](/wiki/Template:Sfrac), and [Template:Sfrac](/wiki/Template:Sfrac).[[29]](#cite_note-29) (List is selected terms from [Template:OEIS2C](/wiki/Template:OEIS2C) and [Template:OEIS2C](/wiki/Template:OEIS2C).)
* **Decimal**: The first 50 decimal digits are [Template:Gaps](/wiki/Template:Gaps)[[32]](#cite_note-32) [Template:OEIS2C](/wiki/Template:OEIS2C)
* [**Binary**](/wiki/Binary_numeral_system): The [base](/wiki/Radix) 2 approximation to 48 digits is [Template:Gaps](/wiki/Template:Gaps)
* [**Hexadecimal**](/wiki/Hexadecimal): The [base](/wiki/Radix) 16 approximation to 20 digits is [Template:Gaps](/wiki/Template:Gaps)[[33]](#cite_note-33)\* [**Sexagesimal**](/wiki/Sexagesimal): A base 60 approximation to five sexagesimal digits is 3;8,29,44,0,47[[34]](#cite_note-34)

### Complex numbers and Euler's identity[[edit](/index.php?title=(none)&action=edit&section=8)]

[thumb|alt=A diagram of a unit circle centered at the origin in the complex plane, including a ray from the center of the circle to its edge, with the triangle legs labeled with sine and cosine functions.|The association between imaginary powers of the number](/wiki/File:Euler's_formula.svg) [Template:Math](/wiki/Template:Math) and [points](/wiki/Point_(geometry)) on the [unit circle](/wiki/Unit_circle) centered at the [origin](/wiki/Origin_(mathematics)) in the [complex plane](/wiki/Complex_plane) given by [Euler's formula](/wiki/Euler's_formula).

Any [complex number](/wiki/Complex_number), say [Template:Math](/wiki/Template:Math), can be expressed using a pair of [real numbers](/wiki/Real_number). In the [polar coordinate system](/wiki/Polar_coordinate_system#Complex_numbers), one number (radius or *r*) is used to represent [Template:Math's](/wiki/Template:Math) distance from the [origin](/wiki/Origin_(mathematics)) of the [complex plane](/wiki/Complex_plane) and the other (angle or [Template:Math](/wiki/Template:Math)) to represent a counter-clockwise [rotation](/wiki/Rotation) from the positive real line as follows:[[35]](#cite_note-35):<math>z = r\cdot(\cos\varphi + i\sin\varphi),</math> where [Template:Math](/wiki/Template:Math) is the [imaginary unit](/wiki/Imaginary_unit) satisfying [Template:Math](/wiki/Template:Math) = −1. The frequent appearance of [Template:Pi](/wiki/Template:Pi) in [complex analysis](/wiki/Complex_analysis) can be related to the behavior of the [exponential function](/wiki/Exponential_function) of a complex variable, described by [Euler's formula](/wiki/Euler's_formula):[[36]](#cite_note-36)

<math>e^{i\varphi} = \cos \varphi + i\sin \varphi,</math>

where [the constant](/wiki/E_(mathematical_constant)) [Template:Math](/wiki/Template:Math) is the base of the [natural logarithm](/wiki/Natural_logarithm). This formula establishes a correspondence between imaginary powers of [Template:Math](/wiki/Template:Math) and points on the [unit circle](/wiki/Unit_circle) centered at the origin of the complex plane. Setting [Template:Math](/wiki/Template:Math) = [Template:Pi](/wiki/Template:Pi) in Euler's formula results in [Euler's identity](/wiki/Euler's_identity), celebrated by mathematicians because it contains the five most important mathematical constants:[[36]](#cite_note-36)[[37]](#cite_note-37):<math>e^{i \pi} + 1 = 0.</math>

There are [Template:Math](/wiki/Template:Math) different [complex numbers](/wiki/Complex_number) [Template:Math](/wiki/Template:Math) satisfying [Template:Math](/wiki/Template:Math), and these are called the "[Template:Math](/wiki/Template:Math)-th [roots of unity](/wiki/Root_of_unity)".[[38]](#cite_note-38) They are given by this formula:

<math>e^{2 \pi i k/n} \qquad (k = 0, 1, 2, \dots, n - 1).</math>

### Spectral characterizations[[edit](/index.php?title=(none)&action=edit&section=9)]

[thumb|right|The](/wiki/File:Harmonic_partials_on_strings.svg) [overtones](/wiki/Overtone) of a vibrating string are [eigenfunctions](/wiki/Eigenfunction) of the second derivative, and form a [harmonic progression](/wiki/Harmonic_series_(music)). The associated eigenvalues form the [arithmetic progression](/wiki/Arithmetic_progression) of integer multiples of [Template:Pi](/wiki/Template:Pi). Many [of the appearances](/wiki/#Uses) of [Template:Pi](/wiki/Template:Pi) in the formulas of mathematics and the sciences have to do with its close relationship with geometry. However, [Template:Pi](/wiki/Template:Pi) also appears in many natural situations having apparently nothing to do with geometry.

In many applications it plays a distinguished role as an [eigenvalue](/wiki/Eigenvalue). For example, an idealized [vibrating string](/wiki/Vibrating_string) can be modelled as the graph of a function [Template:Math](/wiki/Template:Math) on the unit interval [Template:Math](/wiki/Template:Math), with [fixed ends](/wiki/Boundary_conditions) [Template:Math](/wiki/Template:Math). The modes of vibration of the string are solutions of the [differential equation](/wiki/Differential_equation) [Template:Math](/wiki/Template:Math). Here [Template:Math](/wiki/Template:Math) is an associated eigenvalue, which is constrained by [Sturm–Liouville theory](/wiki/Sturm–Liouville_theory) to take on only certain specific values. The value [Template:Math](/wiki/Template:Math) is one such eigenvalue, as the function [Template:Math](/wiki/Template:Math) satisfies the boundary conditions and the differential equation with [Template:Math](/wiki/Template:Math).[[39]](#cite_note-39) [thumb|right|The](/wiki/File:Sir_William_Thompson_illustration_of_Carthage.png) [ancient city of Carthage](/wiki/Ancient_Carthage) was the solution to an isoperimetric problem, according to a legend recounted by [Lord Kelvin](/wiki/Lord_Kelvin) [Template:Harv](/wiki/Template:Harv): those lands bordering the sea that [Queen Dido](/wiki/Dido) could enclose on all other sides within a single given oxhide, cut into strips. The value [Template:Pi](/wiki/Template:Pi) is in fact the *least* such eigenvalue, and is associated with the [fundamental mode](/wiki/Fundamental_mode) of vibration of the string. One way to obtain this is by estimating the [energy](/wiki/Energy). The energy satisfies an inequality, [Wirtinger's inequality for functions](/wiki/Wirtinger's_inequality_for_functions),[[40]](#cite_note-40) which states that if a function [Template:Math](/wiki/Template:Math) is given such that [Template:Math](/wiki/Template:Math) and [Template:Math](/wiki/Template:Math) and [Template:Math](/wiki/Template:Math) are both [square integrable](/wiki/Square_integrable), then the inequality holds:

<math>\pi^2\int\_0^1|f(x)|^2\,dx\le \int\_0^1|f'(x)|^2\,dx,</math>

and the case of equality holds precisely when [Template:Math](/wiki/Template:Math) is a multiple of [Template:Math](/wiki/Template:Math). So [Template:Pi](/wiki/Template:Pi) appears as an optimal constant in Wirtinger's inequality, and from this it follows that it is the smallest such eigenvalue (by [Rayleigh quotient](/wiki/Rayleigh_quotient) methods).

The number [Template:Pi](/wiki/Template:Pi) serves a similar role in higher-dimensional analysis, appearing as eigenvalues for other similar kinds of problems. As mentioned [above](/wiki/#Definition), it can be characterized via its role as the best constant in the [isoperimetric inequality](/wiki/Isoperimetric_inequality): the area [Template:Mvar](/wiki/Template:Mvar) enclosed by a plane [Jordan curve](/wiki/Jordan_curve) of perimeter [Template:Mvar](/wiki/Template:Mvar) satisfies the inequality

<math>4\pi A\le P^2,</math>

and equality is clearly achieved for the circle, since in that case [Template:Math](/wiki/Template:Math) and [Template:Math](/wiki/Template:Math).[[41]](#cite_note-41) [thumb|right|An animation of a](/wiki/File:Animation_of_Heisenberg_geodesic.gif) [geodesic in the Heisenberg group](/wiki/Heisenberg_group#As_a_sub-Riemannian_manifold), showing the close connection between the Heisenberg group, isoperimetry, and the constant [Template:Pi](/wiki/Template:Pi). The cumulative height of the geodesic is equal to the area of the shaded portion of the unit circle, while the arc length (in the [Carnot–Carathéodory metric](/wiki/Carnot–Carathéodory_metric)) is equal to the circumference. Ultimately as a consequence of the isoperimetric inequality, the constant [Template:Pi](/wiki/Template:Pi) is associated with best constants of the [Poincaré inequality](/wiki/Poincaré_inequality).[[42]](#cite_note-42) As a special case, [Template:Pi](/wiki/Template:Pi) appears as the optimal smallest [eigenvalue](/wiki/Eigenvalue) of the [Dirichlet energy](/wiki/Dirichlet_energy), in dimensions 1 and 2, which thus characterizes the role of [Template:Pi](/wiki/Template:Pi) in many physical phenomena as well, for example those of classical [potential theory](/wiki/Potential_theory).[[43]](#cite_note-43)[[44]](#cite_note-44)[[45]](#cite_note-45) The one-dimensional case is just Wirtinger's inequality.

The constant [Template:Pi](/wiki/Template:Pi) also appears as a critical spectral parameter in the [Fourier transform](/wiki/Fourier_transform). This is the [integral transform](/wiki/Integral_transform), that takes a complex-valued integrable function [Template:Math](/wiki/Template:Math) on the real line to the function defined as:

<math>\hat{f}(\xi) = \int\_{-\infty}^\infty f(x) e^{-2\pi i x\xi}\,dx.</math>

There are several different conventions for the Fourier transform, all of which involve a factor of [Template:Pi](/wiki/Template:Pi) that is placed *somewhere*. The appearance of [Template:Pi](/wiki/Template:Pi) is essential in these formulas, as there is there is no possibility to remove [Template:Pi](/wiki/Template:Pi) altogether from the Fourier transform and its inverse transform. The definition given above is the most canonical however, because it describes the unique unitary operator on [Template:Math](/wiki/Template:Math) that is also an algebra homomorphism of [Template:Math](/wiki/Template:Math) to [Template:Math](/wiki/Template:Math).[[46]](#cite_note-46) The [Heisenberg uncertainty principle](/wiki/Heisenberg_uncertainty_principle) also contains the number [Template:Pi](/wiki/Template:Pi). The uncertainty principle gives a sharp lower bound on the extent to which it is possible to localize a function both in space and in frequency: with our conventions for the Fourier transform,

<math>\int\_{-\infty}^\infty x^2|f(x)|^2\,dx\ \int\_{-\infty}^\infty \xi^2|\hat{f}(\xi)|^2\,d\xi\ge \left(\frac{1}{4\pi}\int\_{-\infty}^\infty |f(x)|^2\,dx\right)^2.</math>

The physical consequence, about the uncertainty in simultaneous position and momentum observations of a [quantum mechanical](/wiki/Quantum_mechanical) system, is [discussed below](/wiki/#Describing_physical_phenomena). The appearance of [Template:Pi](/wiki/Template:Pi) in the formulae of Fourier analysis is ultimately a consequence of the [Stone–von Neumann theorem](/wiki/Stone–von_Neumann_theorem), asserting the uniqueness of the [Schrödinger representation](/wiki/Schrödinger_representation) of the [Heisenberg group](/wiki/Heisenberg_group).[[47]](#cite_note-47)

### Gaussian integrals[[edit](/index.php?title=(none)&action=edit&section=10)]

[thumb|right|A graph of the](/wiki/File:E%5e(-x%5e2).svg) [Gaussian function](/wiki/Gaussian_function) [Template:Math](/wiki/Template:Math). The colored region between the function and the *x*-axis has area [Template:Math](/wiki/Template:Math).

The fields of [probability](/wiki/Probability) and [statistics](/wiki/Statistics) frequently use the [normal distribution](/wiki/Normal_distribution) as a simple model for complex phenomena; for example, scientists generally assume that the observational error in most experiments follows a normal distribution.[[48]](#cite_note-48) The [Gaussian function](/wiki/Gaussian_function), which is the [probability density function](/wiki/Probability_density_function) of the normal distribution with [mean](/wiki/Mean) [Template:Math](/wiki/Template:Math) and [standard deviation](/wiki/Standard_deviation) [Template:Math](/wiki/Template:Math), naturally contains [Template:Pi](/wiki/Template:Pi):[[49]](#cite_note-49)

<math>f(x) = {1 \over \sigma\sqrt{2\pi} }\,e^{-(x-\mu )^2/(2\sigma^2)}</math>

For this to be a probability density, the area under the graph of [Template:Math](/wiki/Template:Math) needs to be equal to one. This follows from a [change of variables](/wiki/Integration_by_substitution) in the [Gaussian integral](/wiki/Gaussian_integral):[[49]](#cite_note-49)

<math>\int\_{-\infty}^\infty e^{-u^2} \, du=\sqrt{\pi}</math>

which says that the area under the basic [Bell curve](/wiki/Bell_curve) in the figure is equal to the square root of [Template:Pi](/wiki/Template:Pi).

[thumb|right|](/wiki/File:Random_walk_simulation.png)[Template:Pi](/wiki/Template:Pi) [can be computed](/wiki/Arcsine_laws_(Wiener_process)) from the distribution of zeros of a one-dimensional [Wiener process](/wiki/Wiener_process) The [central limit theorem](/wiki/Central_limit_theorem) explains the central role of normal distributions, and thus of [Template:Pi](/wiki/Template:Pi), in probability and statistics. This theorem is ultimately connected with the [spectral characterization](/wiki/#Spectral_characterizations) of [Template:Pi](/wiki/Template:Pi) as the eigenvalue associated with the Heisenberg uncertainty principle, and the fact that equality holds in the uncertainty principle only for the Gaussian function.[[50]](#cite_note-50) Equivalently, [Template:Pi](/wiki/Template:Pi) is the unique constant making the Gaussian normal distribution [Template:Math](/wiki/Template:Math) equal to its own Fourier transform.[[51]](#cite_note-51) Indeed, according to [Template:Harvtxt](/wiki/Template:Harvtxt), the "whole business" of establishing the fundamental theorems Fourier analysis reduces to the Gaussian integral.

## History[[edit](/index.php?title=(none)&action=edit&section=11)]

[Template:Main](/wiki/Template:Main) [Template:See also](/wiki/Template:See_also)

### Antiquity[[edit](/index.php?title=(none)&action=edit&section=12)]

The best known approximations to [Template:Pi](/wiki/Template:Pi) dating [before the Common Era](/wiki/1st_millennium_BC) were accurate to two decimal places; this was improved upon in [Chinese mathematics](/wiki/Chinese_mathematics) in particular by the mid first millennium, to an accuracy of seven decimal places. After this, no further progress was made until the late medieval period.

Some Egyptologists[[52]](#cite_note-52) have claimed that the [ancient Egyptians](/wiki/Ancient_Egypt) used an approximation of [Template:Pi](/wiki/Template:Pi) as [Template:Sfrac](/wiki/Template:Sfrac) from as early as the [Old Kingdom](/wiki/Old_Kingdom).[[53]](#cite_note-53)This claim has met with skepticism.[[54]](#cite_note-54)[[55]](#cite_note-55)[[56]](#cite_note-56)[[57]](#cite_note-57) The earliest written approximations of [Template:Pi](/wiki/Template:Pi) are found in [Egypt](/wiki/Ancient_Egypt) and [Babylon](/wiki/Babylon), both within one percent of the true value. In Babylon, a [clay tablet](/wiki/Clay_tablet) dated 1900–1600 BC has a geometrical statement that, by implication, treats [Template:Pi](/wiki/Template:Pi) as [Template:Sfrac](/wiki/Template:Sfrac) = 3.125.[[58]](#cite_note-58) In Egypt, the [Rhind Papyrus](/wiki/Rhind_Papyrus), dated around 1650 BC but copied from a document dated to 1850 BC, has a formula for the area of a circle that treats [Template:Pi](/wiki/Template:Pi) as ([Template:Sfrac](/wiki/Template:Sfrac))2 ≈ 3.1605.[[58]](#cite_note-58) Astronomical calculations in the [*Shatapatha Brahmana*](/wiki/Shatapatha_Brahmana) (ca. 4th century BC) use a fractional approximation of [Template:Sfrac](/wiki/Template:Sfrac) ≈ 3.139 (an accuracy of 9×10−4).[[59]](#cite_note-59) Other Indian sources by about 150 BC treat [Template:Pi](/wiki/Template:Pi) as [Template:Math](/wiki/Template:Math) ≈ 3.1622.[[60]](#cite_note-60)

### Polygon approximation era[[edit](/index.php?title=(none)&action=edit&section=13)]

[350px|right|thumb|alt=diagram of a hexagon and pentagon circumscribed outside a circle|](/wiki/File:Archimedes_pi.svg)[Template:Pi](/wiki/Template:Pi) can be estimated by computing the perimeters of circumscribed and inscribed polygons. The first recorded algorithm for rigorously calculating the value of [Template:Pi](/wiki/Template:Pi) was a geometrical approach using polygons, devised around 250 BC by the Greek mathematician [Archimedes](/wiki/Archimedes).[[61]](#cite_note-61) This polygonal algorithm dominated for over 1,000 years, and as a result [Template:Pi](/wiki/Template:Pi) is sometimes referred to as "Archimedes' constant".[[62]](#cite_note-62) Archimedes computed upper and lower bounds of [Template:Pi](/wiki/Template:Pi) by drawing a regular hexagon inside and outside a circle, and successively doubling the number of sides until he reached a 96-sided regular polygon. By calculating the perimeters of these polygons, he proved that [Template:Math](/wiki/Template:Math) (that is [Template:Math](/wiki/Template:Math)).[[63]](#cite_note-63) Archimedes' upper bound of [Template:Math](/wiki/Template:Math) may have led to a widespread popular belief that [Template:Pi](/wiki/Template:Pi) is equal to [Template:Math](/wiki/Template:Math).[[64]](#cite_note-64) Around 150 AD, Greek-Roman scientist [Ptolemy](/wiki/Ptolemy), in his [*Almagest*](/wiki/Almagest), gave a value for [Template:Pi](/wiki/Template:Pi) of 3.1416, which he may have obtained from Archimedes or from [Apollonius of Perga](/wiki/Apollonius_of_Perga).[[65]](#cite_note-65) Mathematicians using polygonal algorithms reached 39 digits of [Template:Pi](/wiki/Template:Pi) in 1630, a record only broken in 1699 when infinite series were used to reach 71 digits.[[66]](#cite_note-66)[thumb|upright|alt=A painting of a man studying|](/wiki/File:Domenico-Fetti_Archimedes_1620.jpg)[Archimedes](/wiki/Archimedes) developed the polygonal approach to approximating [Template:Pi](/wiki/Template:Pi).

In [ancient China](/wiki/Ancient_China), values for [Template:Pi](/wiki/Template:Pi) included 3.1547 (around 1 AD), [Template:Math](/wiki/Template:Math) (100 AD, approximately 3.1623), and [Template:Math](/wiki/Template:Math) (3rd century, approximately 3.1556).[[67]](#cite_note-67) Around 265 AD, the [Wei Kingdom](/wiki/Wei_Kingdom) mathematician [Liu Hui](/wiki/Liu_Hui) created a [polygon-based iterative algorithm](/wiki/Liu_Hui's_π_algorithm) and used it with a 3,072-sided polygon to obtain a value of [Template:Pi](/wiki/Template:Pi) of 3.1416.[[68]](#cite_note-68)[[69]](#cite_note-69) Liu later invented a faster method of calculating [Template:Pi](/wiki/Template:Pi) and obtained a value of 3.14 with a 96-sided polygon, by taking advantage of the fact that the differences in area of successive polygons form a geometric series with a factor of 4.[[68]](#cite_note-68) The Chinese mathematician [Zu Chongzhi](/wiki/Zu_Chongzhi), around 480 AD, calculated that [Template:Math](/wiki/Template:Math) (a fraction that goes by the name [*Milü*](/wiki/Milü) in Chinese), using [Liu Hui's algorithm](/wiki/Liu_Hui's_π_algorithm) applied to a 12,288-sided polygon. With a correct value for its seven first decimal digits, this value of 3.141592920... remained the most accurate approximation of [Template:Pi](/wiki/Template:Pi) available for the next 800 years.[[70]](#cite_note-70) The Indian astronomer [Aryabhata](/wiki/Aryabhata) used a value of 3.1416 in his [*Āryabhaṭīya*](/wiki/Āryabhaṭīya) (499 AD).[[71]](#cite_note-71) [Fibonacci](/wiki/Fibonacci) in c. 1220 computed 3.1418 using a polygonal method, independent of Archimedes.[[72]](#cite_note-72) Italian author [Dante](/wiki/Dante) apparently employed the value [Template:Math](/wiki/Template:Math).[[72]](#cite_note-72) The Persian astronomer [Jamshīd al-Kāshī](/wiki/Jamshīd_al-Kāshī) produced 9 sexagesimal digits, roughly the equivalent of 16 decimal digits, in 1424 using a polygon with 3×228 sides,[[73]](#cite_note-73)[[74]](#cite_note-74) which stood as the world record for about 180 years.[[75]](#cite_note-75) French mathematician [François Viète](/wiki/François_Viète) in 1579 achieved 9 digits with a polygon of 3×217 sides.[[75]](#cite_note-75) Flemish mathematician [Adriaan van Roomen](/wiki/Adriaan_van_Roomen) arrived at 15 decimal places in 1593.[[75]](#cite_note-75) In 1596, Dutch mathematician [Ludolph van Ceulen](/wiki/Ludolph_van_Ceulen) reached 20 digits, a record he later increased to 35 digits (as a result, [Template:Pi](/wiki/Template:Pi) was called the "Ludolphian number" in Germany until the early 20th century).[[76]](#cite_note-76) Dutch scientist [Willebrord Snellius](/wiki/Willebrord_Snellius) reached 34 digits in 1621,[[77]](#cite_note-77) and Austrian astronomer [Christoph Grienberger](/wiki/Christoph_Grienberger) arrived at 38 digits in 1630 using 1040 sides,[[78]](#cite_note-78) which remains the most accurate approximation manually achieved using polygonal algorithms.[[77]](#cite_note-77)

### Infinite series[[edit](/index.php?title=(none)&action=edit&section=14)]

[Template:Comparison pi infinite series.svg](/wiki/Template:Comparison_pi_infinite_series.svg) The calculation of [Template:Pi](/wiki/Template:Pi) was revolutionized by the development of [infinite series](/wiki/Infinite_series) techniques in the 16th and 17th centuries. An infinite series is the sum of the terms of an infinite [sequence](/wiki/Sequence_(mathematics)).[[79]](#cite_note-79) Infinite series allowed mathematicians to compute [Template:Pi](/wiki/Template:Pi) with much greater precision than [Archimedes](/wiki/Archimedes) and others who used geometrical techniques.[[79]](#cite_note-79) Although infinite series were exploited for [Template:Pi](/wiki/Template:Pi) most notably by European mathematicians such as [James Gregory](/wiki/James_Gregory_(mathematician)) and [Gottfried Wilhelm Leibniz](/wiki/Gottfried_Wilhelm_Leibniz), the approach was first discovered in [India](/wiki/India) sometime between 1400 and 1500 AD.[[80]](#cite_note-80) The first written description of an infinite series that could be used to compute [Template:Pi](/wiki/Template:Pi) was laid out in Sanskrit verse by Indian astronomer [Nilakantha Somayaji](/wiki/Nilakantha_Somayaji) in his [*Tantrasamgraha*](/wiki/Tantrasamgraha), around 1500 AD.[[81]](#cite_note-81) The series are presented without proof, but proofs are presented in a later Indian work, [*Yuktibhāṣā*](/wiki/Yuktibhāṣā), from around 1530 AD. Nilakantha attributes the series to an earlier Indian mathematician, [Madhava of Sangamagrama](/wiki/Madhava_of_Sangamagrama), who lived c. 1350 – c. 1425.[[81]](#cite_note-81) Several infinite series are described, including series for sine, tangent, and cosine, which are now referred to as the [Madhava series](/wiki/Madhava_series) or [Gregory–Leibniz series](/wiki/Leibniz_formula_for_π).[[81]](#cite_note-81) Madhava used infinite series to estimate [Template:Pi](/wiki/Template:Pi) to 11 digits around 1400, but that value was improved on around 1430 by the Persian mathematician [Jamshīd al-Kāshī](/wiki/Jamshīd_al-Kāshī), using a polygonal algorithm.[[82]](#cite_note-82)[thumb|upright|alt=A formal portrait of a man, with long hair|](/wiki/File:GodfreyKneller-IsaacNewton-1689.jpg)[Isaac Newton](/wiki/Isaac_Newton) used [infinite series](/wiki/Infinite_series) to compute [Template:Pi](/wiki/Template:Pi) to 15 digits, later writing "I am ashamed to tell you to how many figures I carried these computations".[[83]](#cite_note-83)

The [first infinite sequence discovered in Europe](/wiki/Viète's_formula) was an [infinite product](/wiki/Infinite_product) (rather than an [infinite sum](/wiki/Infinite_sum), which are more typically used in [Template:Pi](/wiki/Template:Pi) calculations) found by French mathematician [François Viète](/wiki/François_Viète) in 1593:[[84]](#cite_note-84)

<math> \frac2\pi = \frac{\sqrt2}2 \cdot \frac{\sqrt{2+\sqrt2}}2 \cdot \frac{\sqrt{2+\sqrt{2+\sqrt2}}}2 \cdots</math> [Template:OEIS2C](/wiki/Template:OEIS2C)

The [second infinite sequence found in Europe](/wiki/Wallis_product), by [John Wallis](/wiki/John_Wallis) in 1655, was also an infinite product:[[84]](#cite_note-84):<math> \frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots </math> The discovery of [calculus](/wiki/Calculus), by English scientist [Isaac Newton](/wiki/Isaac_Newton) and German mathematician [Gottfried Wilhelm Leibniz](/wiki/Gottfried_Wilhelm_Leibniz) in the 1660s, led to the development of many infinite series for approximating [Template:Pi](/wiki/Template:Pi). Newton himself used an arcsin series to compute a 15 digit approximation of [Template:Pi](/wiki/Template:Pi) in 1665 or 1666, later writing "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."[[83]](#cite_note-83) In Europe, Madhava's formula was rediscovered by Scottish mathematician [James Gregory](/wiki/James_Gregory_(mathematician)) in 1671, and by Leibniz in 1674:[[85]](#cite_note-85)[[86]](#cite_note-86)

<math>

\arctan z = z - \frac {z^3} {3} +\frac {z^5} {5} -\frac {z^7} {7} +\cdots </math>

This formula, the Gregory–Leibniz series, equals [Template:Math](/wiki/Template:Math) when evaluated with [Template:Math](/wiki/Template:Math) = 1.[[86]](#cite_note-86) In 1699, English mathematician [Abraham Sharp](/wiki/Abraham_Sharp) used the Gregory–Leibniz series for <math>z=\frac{1}{\sqrt{3}}</math> to compute [Template:Pi](/wiki/Template:Pi) to 71 digits, breaking the previous record of 39 digits, which was set with a polygonal algorithm.[[87]](#cite_note-87) The Gregory–Leibniz for <math>z=1</math> series is simple, but [converges](/wiki/Convergent_series) very slowly (that is, approaches the answer gradually), so it is not used in modern [Template:Pi](/wiki/Template:Pi) calculations.[[88]](#cite_note-88) In 1706 [John Machin](/wiki/John_Machin) used the Gregory–Leibniz series to produce an algorithm that converged much faster:[[89]](#cite_note-89):<math> \frac{\pi}{4} = 4 \, \arctan \frac{1}{5} - \arctan \frac{1}{239}</math> Machin reached 100 digits of [Template:Pi](/wiki/Template:Pi) with this formula.[[90]](#cite_note-90) Other mathematicians created variants, now known as [Machin-like formulae](/wiki/Machin-like_formula), that were used to set several successive records for calculating digits of [Template:Pi](/wiki/Template:Pi).[[90]](#cite_note-90) Machin-like formulae remained the best-known method for calculating [Template:Pi](/wiki/Template:Pi) well into the age of computers, and were used to set records for 250 years, culminating in a 620-digit approximation in 1946 by Daniel Ferguson – the best approximation achieved without the aid of a calculating device.[[91]](#cite_note-91) A record was set by the calculating prodigy [Zacharias Dase](/wiki/Zacharias_Dase), who in 1844 employed a Machin-like formula to calculate 200 decimals of [Template:Pi](/wiki/Template:Pi) in his head at the behest of German mathematician [Carl Friedrich Gauss](/wiki/Carl_Friedrich_Gauss).[[92]](#cite_note-92) British mathematician [William Shanks](/wiki/William_Shanks) famously took 15 years to calculate [Template:Pi](/wiki/Template:Pi) to 707 digits, but made a mistake in the 528th digit, rendering all subsequent digits incorrect.[[92]](#cite_note-92)

#### Rate of convergence[[edit](/index.php?title=(none)&action=edit&section=15)]

Some infinite series for [Template:Pi](/wiki/Template:Pi) [converge](/wiki/Convergent_series) faster than others. Given the choice of two infinite series for [Template:Pi](/wiki/Template:Pi), mathematicians will generally use the one that converges more rapidly because faster convergence reduces the amount of computation needed to calculate [Template:Pi](/wiki/Template:Pi) to any given accuracy.[[93]](#cite_note-93) A simple infinite series for [Template:Pi](/wiki/Template:Pi) is the [Gregory–Leibniz series](/wiki/Leibniz_formula_for_π):[[94]](#cite_note-94):<math> \pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \cdots</math> As individual terms of this infinite series are added to the sum, the total gradually gets closer to [Template:Pi](/wiki/Template:Pi), and – with a sufficient number of terms – can get as close to [Template:Pi](/wiki/Template:Pi) as desired. It converges quite slowly, though – after 500,000 terms, it produces only five correct decimal digits of [Template:Pi](/wiki/Template:Pi).[[95]](#cite_note-95) An infinite series for [Template:Pi](/wiki/Template:Pi) (published by Nilakantha in the 15th century) that converges more rapidly than the Gregory–Leibniz series is:[[96]](#cite_note-96)

<math> \pi = 3 + \frac{4}{2\times3\times4} - \frac{4}{4\times5\times6} + \frac{4}{6\times7\times8} - \frac{4}{8\times9\times10} + \cdots</math>

The following table compares the convergence rates of these two series:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Infinite series for** [**Template:Pi**](/wiki/Template:Pi) | **After 1st term** | **After 2nd term** | **After 3rd term** | **After 4th term** | **After 5th term** | **Converges to:** |
| <math>\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} \cdots.</math> | 4.0000 | 2.6666... | 3.4666... | 2.8952... | 3.3396... | rowspan=2| [Template:Pi](/wiki/Template:Pi) = 3.1415... |
| <math>\pi = [Template:3](/wiki/Template:3) + \frac[Template:4](/wiki/Template:4){2\times3\times4} - \frac[Template:4](/wiki/Template:4){4\times5\times6} + \frac[Template:4](/wiki/Template:4){6\times7\times8} \cdots. </math> | 3.0000 | 3.1666... | 3.1333... | 3.1452... | 3.1396... |  |

After five terms, the sum of the Gregory–Leibniz series is within 0.2 of the correct value of [Template:Pi](/wiki/Template:Pi), whereas the sum of Nilakantha's series is within 0.002 of the correct value of [Template:Pi](/wiki/Template:Pi). Nilakantha's series converges faster and is more useful for computing digits of [Template:Pi](/wiki/Template:Pi). Series that converge even faster include [Machin's series](/wiki/Machin-like_formula) and [Chudnovsky's series](/wiki/Chudnovsky_algorithm), the latter producing 14 correct decimal digits per term.[[93]](#cite_note-93)

### Irrationality and transcendence[[edit](/index.php?title=(none)&action=edit&section=16)]

[Template:See also](/wiki/Template:See_also) Not all mathematical advances relating to [Template:Pi](/wiki/Template:Pi) were aimed at increasing the accuracy of approximations. When Euler solved the [Basel problem](/wiki/Basel_problem) in 1735, finding the exact value of the sum of the reciprocal squares, he established a connection between [Template:Pi](/wiki/Template:Pi) and the [prime numbers](/wiki/Prime_number) that later contributed to the development and study of the [Riemann zeta function](/wiki/Riemann_zeta_function):[[97]](#cite_note-97)

<math> \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots</math>

Swiss scientist [Johann Heinrich Lambert](/wiki/Johann_Heinrich_Lambert) in 1761 proved that [Template:Pi](/wiki/Template:Pi) is [irrational](/wiki/Irrational_number), meaning it is not equal to the quotient of any two whole numbers.[[19]](#cite_note-19) [Lambert's proof](/wiki/Proof_that_π_is_irrational) exploited a continued-fraction representation of the tangent function.[[98]](#cite_note-98) French mathematician [Adrien-Marie Legendre](/wiki/Adrien-Marie_Legendre) proved in 1794 that [Template:Pi](/wiki/Template:Pi)2 is also irrational. In 1882, German mathematician [Ferdinand von Lindemann](/wiki/Ferdinand_von_Lindemann) proved that [Template:Pi](/wiki/Template:Pi) is [transcendental](/wiki/Transcendental_number), confirming a conjecture made by both [Legendre](/wiki/Adrien-Marie_Legendre) and Euler.[[99]](#cite_note-99)[[100]](#cite_note-100) Hardy and Wright states that "the proofs were afterwards modified and simplified by Hilbert, Hurwitz, and other writers".[[101]](#cite_note-101)

### Adoption of the symbol {{pi}}[[edit](/index.php?title=(none)&action=edit&section=17)]

[thumb|upright|](/wiki/File:Leonhard_Euler.jpg)[Leonhard Euler](/wiki/Leonhard_Euler) popularized the use of the Greek letter [Template:Pi](/wiki/Template:Pi) in works he published in 1736 and 1748. The earliest known use of the Greek letter [Template:Pi](/wiki/Template:Pi) to represent the ratio of a circle's circumference to its diameter was by Welsh mathematician [William Jones](/wiki/William_Jones_(mathematician)) in his 1706 work *Synopsis Palmariorum Matheseos; or, a New Introduction to the Mathematics*.[[102]](#cite_note-102) The Greek letter first appears there in the phrase "1/2 Periphery ([Template:Pi](/wiki/Template:Pi))" in the discussion of a circle with radius one. Jones may have chosen [Template:Pi](/wiki/Template:Pi) because it was the first letter in the Greek spelling of the word *periphery*.[[103]](#cite_note-103) However, he writes that his equations for [Template:Pi](/wiki/Template:Pi) are from the "ready pen of the truly ingenious Mr. John Machin", leading to speculation that [Machin](/wiki/John_Machin) may have employed the Greek letter before Jones.[[104]](#cite_note-104) It had indeed been used earlier for geometric concepts.[[104]](#cite_note-104) [William Oughtred](/wiki/William_Oughtred) used [Template:Pi](/wiki/Template:Pi) and δ, the Greek letter equivalents of p and d, to express ratios of periphery and diameter in the 1647 and later editions of *Clavis Mathematicae*.

After Jones introduced the Greek letter in 1706, it was not adopted by other mathematicians until [Euler](/wiki/Euler) started using it, beginning with his 1736 work [*Mechanica*](/wiki/Mechanica). Before then, mathematicians sometimes used letters such as *c* or *p* instead.[[104]](#cite_note-104) Because Euler corresponded heavily with other mathematicians in Europe, the use of the Greek letter spread rapidly.[[104]](#cite_note-104) In 1748, Euler used [Template:Pi](/wiki/Template:Pi) in his widely read work [*Introductio in analysin infinitorum*](/wiki/Introductio_in_analysin_infinitorum) (he wrote: "for the sake of brevity we will write this number as [Template:Pi](/wiki/Template:Pi); thus [Template:Pi](/wiki/Template:Pi) is equal to half the circumference of a circle of radius 1") and the practice was universally adopted thereafter in the [Western world](/wiki/Western_world).[[104]](#cite_note-104)

## Modern quest for more digits[[edit](/index.php?title=(none)&action=edit&section=18)]

### Computer era and iterative algorithms[[edit](/index.php?title=(none)&action=edit&section=19)]

[thumb|upright|alt=Formal photo of a balding man wearing a suit|](/wiki/File:JohnvonNeumann-LosAlamos.gif)[John von Neumann](/wiki/John_von_Neumann) was part of the team that first used a digital computer, [ENIAC](/wiki/ENIAC), to compute [Template:Pi](/wiki/Template:Pi). [Template:Quote box](/wiki/Template:Quote_box)

The development of computers in the mid-20th century again revolutionized the hunt for digits of [Template:Pi](/wiki/Template:Pi). American mathematicians [John Wrench](/wiki/John_Wrench) and Levi Smith reached 1,120 digits in 1949 using a desk calculator.[[105]](#cite_note-105) Using an [inverse tangent](/wiki/Inverse_tangent) (arctan) infinite series, a team led by George Reitwiesner and [John von Neumann](/wiki/John_von_Neumann) that same year achieved 2,037 digits with a calculation that took 70 hours of computer time on the [ENIAC](/wiki/ENIAC) computer.[[106]](#cite_note-106) The record, always relying on an arctan series, was broken repeatedly (7,480 digits in 1957; 10,000 digits in 1958; 100,000 digits in 1961) until 1 million digits were reached in 1973.[[107]](#cite_note-107) Two additional developments around 1980 once again accelerated the ability to compute [Template:Pi](/wiki/Template:Pi). First, the discovery of new [iterative algorithms](/wiki/Iterative_algorithm) for computing [Template:Pi](/wiki/Template:Pi), which were much faster than the infinite series; and second, the invention of [fast multiplication algorithms](/wiki/Multiplication_algorithm#Fast_multiplication_algorithms_for_large_inputs) that could multiply large numbers very rapidly.[[108]](#cite_note-108) Such algorithms are particularly important in modern [Template:Pi](/wiki/Template:Pi) computations, because most of the computer's time is devoted to multiplication.[[109]](#cite_note-109) They include the [Karatsuba algorithm](/wiki/Karatsuba_algorithm), [Toom–Cook multiplication](/wiki/Toom–Cook_multiplication), and [Fourier transform-based methods](/wiki/FFT_multiplication#Fourier_transform_methods).[[110]](#cite_note-110) The iterative algorithms were independently published in 1975–1976 by American physicist [Eugene Salamin](/wiki/Eugene_Salamin_(mathematician)) and Australian scientist [Richard Brent](/wiki/Richard_Brent_(scientist)).[[111]](#cite_note-111) These avoid reliance on infinite series. An iterative algorithm repeats a specific calculation, each iteration using the outputs from prior steps as its inputs, and produces a result in each step that converges to the desired value. The approach was actually invented over 160 years earlier by [Carl Friedrich Gauss](/wiki/Carl_Friedrich_Gauss), in what is now termed the [arithmetic–geometric mean method](/wiki/AGM_method) (AGM method) or [Gauss–Legendre algorithm](/wiki/Gauss–Legendre_algorithm).[[111]](#cite_note-111) As modified by Salamin and Brent, it is also referred to as the Brent–Salamin algorithm.

The iterative algorithms were widely used after 1980 because they are faster than infinite series algorithms: whereas infinite series typically increase the number of correct digits additively in successive terms, iterative algorithms generally *multiply* the number of correct digits at each step. For example, the Brent-Salamin algorithm doubles the number of digits in each iteration. In 1984, the Canadian brothers [John](/wiki/Jonathan_Borwein) and [Peter Borwein](/wiki/Peter_Borwein) produced an iterative algorithm that quadruples the number of digits in each step; and in 1987, one that increases the number of digits five times in each step.[[112]](#cite_note-112) Iterative methods were used by Japanese mathematician [Yasumasa Kanada](/wiki/Yasumasa_Kanada) to set several records for computing [Template:Pi](/wiki/Template:Pi) between 1995 and 2002.[[113]](#cite_note-113) This rapid convergence comes at a price: the iterative algorithms require significantly more memory than infinite series.[[113]](#cite_note-113)

### Motivations for computing {{pi}}[[edit](/index.php?title=(none)&action=edit&section=20)]

[thumb|400px|right|As mathematicians discovered new algorithms, and computers became available, the number of known decimal digits of](/wiki/File:Record_pi_approximations.svg) [Template:Pi](/wiki/Template:Pi) increased dramatically. Note that the vertical scale is [logarithmic](/wiki/Logarithm). For most numerical calculations involving [Template:Pi](/wiki/Template:Pi), a handful of digits provide sufficient precision. According to Jörg Arndt and Christoph Haenel, thirty-nine digits are sufficient to perform most [cosmological](/wiki/Cosmological) calculations, because that is the accuracy necessary to calculate the circumference of the [observable universe](/wiki/Observable_universe) with a precision of one atom.[[114]](#cite_note-114) Despite this, people have worked strenuously to compute [Template:Pi](/wiki/Template:Pi) to thousands and millions of digits.[[115]](#cite_note-115) This effort may be partly ascribed to the human compulsion to break records, and such achievements with [Template:Pi](/wiki/Template:Pi) often make headlines around the world.[[116]](#cite_note-116)[[117]](#cite_note-117) They also have practical benefits, such as testing [supercomputers](/wiki/Supercomputer), testing numerical analysis algorithms (including [high-precision multiplication algorithms](/wiki/Multiplication_algorithm#Fast_multiplication_algorithms_for_large_inputs)); and within pure mathematics itself, providing data for evaluating the randomness of the digits of [Template:Pi](/wiki/Template:Pi).[[118]](#cite_note-118)

### Rapidly convergent series[[edit](/index.php?title=(none)&action=edit&section=21)]

[thumb|upright|alt=Photo portrait of a man|](/wiki/File:Srinivasa_Ramanujan_-_OPC_-_1.jpg) [Srinivasa Ramanujan](/wiki/Srinivasa_Ramanujan), working in isolation in India, produced many innovative series for computing [Template:Pi](/wiki/Template:Pi). Modern [Template:Pi](/wiki/Template:Pi) calculators do not use iterative algorithms exclusively. New infinite series were discovered in the 1980s and 1990s that are as fast as iterative algorithms, yet are simpler and less memory intensive.[[113]](#cite_note-113) The fast iterative algorithms were anticipated in 1914, when the Indian mathematician [Srinivasa Ramanujan](/wiki/Srinivasa_Ramanujan) published dozens of innovative new formulae for [Template:Pi](/wiki/Template:Pi), remarkable for their elegance, mathematical depth, and rapid convergence.[[119]](#cite_note-119) One of his formulae, based on [modular equations](/wiki/Modular_equation), is

<math>\frac{1}{\pi} = \frac{2 \sqrt 2}{9801} \sum\_{k=0}^\infty \frac{(4k)!(1103+26390k)}{k!^4(396^{4k})}.</math>

This series converges much more rapidly than most arctan series, including Machin's formula.[[120]](#cite_note-120) [Bill Gosper](/wiki/Bill_Gosper) was the first to use it for advances in the calculation of [Template:Pi](/wiki/Template:Pi), setting a record of 17 million digits in 1985.[[121]](#cite_note-121) Ramanujan's formulae anticipated the modern algorithms developed by the Borwein brothers and the [Chudnovsky brothers](/wiki/Chudnovsky_brothers).[[122]](#cite_note-122) The [Chudnovsky formula](/wiki/Chudnovsky_algorithm) developed in 1987 is

<math>\frac{1}{\pi} = \frac{12}{640320^{3/2}} \sum\_{k=0}^\infty \frac{(6k)! (13591409 + 545140134k)}{(3k)!(k!)^3 (-640320)^{3k}}.</math>

It produces about 14 digits of [Template:Pi](/wiki/Template:Pi) per term,[[123]](#cite_note-123) and has been used for several record-setting [Template:Pi](/wiki/Template:Pi) calculations, including the first to surpass 1 billion (109) digits in 1989 by the Chudnovsky brothers, 2.7 trillion (2.7×1012) digits by [Fabrice Bellard](/wiki/Fabrice_Bellard) in 2009, and 10 trillion (1013) digits in 2011 by Alexander Yee and Shigeru Kondo.[[124]](#cite_note-124)[[125]](#cite_note-125) For similar formulas, see also the [Ramanujan–Sato series](/wiki/Ramanujan–Sato_series).

In 2006, Canadian mathematician [Simon Plouffe](/wiki/Simon_Plouffe) used the PSLQ [integer relation algorithm](/wiki/Integer_relation_algorithm)[[126]](#cite_note-126) to generate several new formulas for [Template:Pi](/wiki/Template:Pi), conforming to the following template:

<math>\pi^k = \sum\_{n=1}^\infty \frac{1}{n^k} \left(\frac{a}{q^n-1} + \frac{b}{q^{2n}-1} + \frac{c}{q^{4n}-1}\right),</math>

where [Template:Math](/wiki/Template:Math) is [Template:Math](/wiki/Template:Math)[Template:Pi](/wiki/Template:Pi) (Gelfond's constant), [Template:Math](/wiki/Template:Math) is an [odd number](/wiki/Odd_number), and [Template:Math](/wiki/Template:Math) are certain rational numbers that Plouffe computed.[[127]](#cite_note-127)

### Monte Carlo methods[[edit](/index.php?title=(none)&action=edit&section=22)]

[Template:Multiple image](/wiki/Template:Multiple_image) [Monte Carlo methods](/wiki/Monte_Carlo_methods), which evaluate the results of multiple random trials, can be used to create approximations of [Template:Pi](/wiki/Template:Pi).[[128]](#cite_note-128) [Buffon's needle](/wiki/Buffon's_needle) is one such technique: If a needle of length [Template:Math](/wiki/Template:Math) is dropped [Template:Math](/wiki/Template:Math) times on a surface on which parallel lines are drawn [Template:Math](/wiki/Template:Math) units apart, and if [Template:Math](/wiki/Template:Math) of those times it comes to rest crossing a line ([Template:Math](/wiki/Template:Math) > 0), then one may approximate [Template:Pi](/wiki/Template:Pi) based on the counts:[[129]](#cite_note-129)

<math>\pi \approx \frac{2n\ell}{xt}</math>

Another Monte Carlo method for computing [Template:Pi](/wiki/Template:Pi) is to draw a circle inscribed in a square, and randomly place dots in the square. The ratio of dots inside the circle to the total number of dots will approximately equal [Template:Math](/wiki/Template:Math).[[130]](#cite_note-130) [thumb|right|Five random walks with 200 steps. The sample mean of](/wiki/File:Five_random_walks.png) [Template:Math](/wiki/Template:Math) is [Template:Math](/wiki/Template:Math), and so [Template:Math](/wiki/Template:Math) is within [Template:Math](/wiki/Template:Math) of [Template:Pi](/wiki/Template:Pi) Another way to calculate [Template:Pi](/wiki/Template:Pi) using probability is to start with a [random walk](/wiki/Random_walk), generated by a sequence of (fair) coin tosses: independent [random variables](/wiki/Random_variable) [Template:Math](/wiki/Template:Math) such that [Template:Math](/wiki/Template:Math)} with equal probabilities. The associated random walk is

<math>W\_n = \sum\_{k=1}^n X\_k</math>

so that, for each *n*, [Template:Math](/wiki/Template:Math) is drawn from a standard [binomial distribution](/wiki/Binomial_distribution). As *n* varies [Template:Math](/wiki/Template:Math) defines a (discrete) [stochastic process](/wiki/Stochastic_process). Then [Template:Pi](/wiki/Template:Pi) can be calculated by[[131]](#cite_note-131):<math>\pi = \lim\_{n\to\infty} \frac{2n}{E[|W\_n|]^2}.</math> This Monte Carlo method is independent of any relation to circles, and is a consequence of the [central limit theorem](/wiki/Central_limit_theorem), discussed [above](/wiki/#Gaussian_integrals).

These Monte Carlo methods for approximating [Template:Pi](/wiki/Template:Pi) are very slow compared to other methods, and do not provide any information on the exact number of digits that are obtained. Thus they are never used to approximate [Template:Pi](/wiki/Template:Pi) when speed or accuracy is desired.[[132]](#cite_note-132)

### Spigot algorithms[[edit](/index.php?title=(none)&action=edit&section=23)]

Two algorithms were discovered in 1995 that opened up new avenues of research into [Template:Pi](/wiki/Template:Pi). They are called [spigot algorithms](/wiki/Spigot_algorithm) because, like water dripping from a [spigot](/wiki/Tap_(valve)), they produce single digits of [Template:Pi](/wiki/Template:Pi) that are not reused after they are calculated.[[133]](#cite_note-133)[[134]](#cite_note-134) This is in contrast to infinite series or iterative algorithms, which retain and use all intermediate digits until the final result is produced.[[133]](#cite_note-133) American mathematicians [Stan Wagon](/wiki/Stan_Wagon) and Stanley Rabinowitz produced a simple spigot algorithm in 1995.[[134]](#cite_note-134)[[135]](#cite_note-135)[[136]](#cite_note-136) Its speed is comparable to arctan algorithms, but not as fast as iterative algorithms.[[135]](#cite_note-135) Another spigot algorithm, the [BBP](/wiki/Bailey–Borwein–Plouffe_formula) [digit extraction algorithm](/wiki/Digit_extraction_algorithm), was discovered in 1995 by Simon Plouffe:[[137]](#cite_note-137)[[138]](#cite_note-138):<math> \pi = \sum\_{k=0}^\infty \frac{1}{16^k} \left( \frac{4}{8k + 1} - \frac{2}{8k + 4} - \frac{1}{8k + 5} - \frac{1}{8k + 6}\right)</math> This formula, unlike others before it, can produce any individual [hexadecimal](/wiki/Hexadecimal) digit of [Template:Pi](/wiki/Template:Pi) without calculating all the preceding digits.[[137]](#cite_note-137) Individual binary digits may be extracted from individual hexadecimal digits, and [octal](/wiki/Octal) digits can be extracted from one or two hexadecimal digits. Variations of the algorithm have been discovered, but no digit extraction algorithm has yet been found that rapidly produces decimal digits.[[139]](#cite_note-139) An important application of digit extraction algorithms is to validate new claims of record [Template:Pi](/wiki/Template:Pi) computations: After a new record is claimed, the decimal result is converted to hexadecimal, and then a digit extraction algorithm is used to calculate several random hexadecimal digits near the end; if they match, this provides a measure of confidence that the entire computation is correct.[[125]](#cite_note-125) Between 1998 and 2000, the [distributed computing](/wiki/Distributed_computing) project [PiHex](/wiki/PiHex) used [Bellard's formula](/wiki/Bellard's_formula) (a modification of the BBP algorithm) to compute the quadrillionth (1015th) bit of [Template:Pi](/wiki/Template:Pi), which turned out to be 0.[[140]](#cite_note-140) In September 2010, a [Yahoo!](/wiki/Yahoo!) employee used the company's [Hadoop](/wiki/Apache_Hadoop) application on one thousand computers over a 23-day period to compute 256 [bits](/wiki/Bit) of [Template:Pi](/wiki/Template:Pi) at the two-quadrillionth (2×1015th) bit, which also happens to be zero.[[141]](#cite_note-141)

## Use[[edit](/index.php?title=(none)&action=edit&section=24)]

Because [Template:Pi](/wiki/Template:Pi) is closely related to the circle, it is found in [many formulae](/wiki/List_of_formulae_involving_π) from the fields of geometry and trigonometry, particularly those concerning circles, spheres, or ellipses. Other branches of science, such as statistics, physics, Fourier analysis, and number theory, also include [Template:Pi](/wiki/Template:Pi) in some of their important formulae.

### Geometry and trigonometry[[edit](/index.php?title=(none)&action=edit&section=25)]

[thumb|alt=A diagram of a circle with a square coving the circle's upper right quadrant.|right|The area of the circle equals](/wiki/File:Circle_Area.svg) [Template:Pi](/wiki/Template:Pi) times the shaded area. [Template:Pi](/wiki/Template:Pi) appears in formulae for areas and volumes of geometrical shapes based on circles, such as [ellipses](/wiki/Ellipse), [spheres](/wiki/Sphere), [cones](/wiki/Cone_(geometry)), and [tori](/wiki/Torus). Below are some of the more common formulae that involve [Template:Pi](/wiki/Template:Pi).[[142]](#cite_note-142)\* The circumference of a circle with radius [Template:Math](/wiki/Template:Math) is [Template:Math](/wiki/Template:Math).

* The [area of a circle](/wiki/Area_of_a_disk) with radius [Template:Math](/wiki/Template:Math) is [Template:Math](/wiki/Template:Math).
* The volume of a sphere with radius [Template:Math](/wiki/Template:Math) is [Template:Math](/wiki/Template:Math).
* The surface area of a sphere with radius [Template:Math](/wiki/Template:Math) is [Template:Math](/wiki/Template:Math).

The formulae above are special cases of the volume of the [*n*-dimensional ball](/wiki/N-ball) and the surface area of its boundary, the [(*n*−1)-dimensional sphere](/wiki/N-sphere), given [below](/wiki/#Gamma_function_and_Stirling's_approximation).

[Definite integrals](/wiki/Integral) that describe circumference, area, or volume of shapes generated by circles typically have values that involve [Template:Pi](/wiki/Template:Pi). For example, an integral that specifies half the area of a circle of radius one is given by:[[143]](#cite_note-143)

<math>\int\_{-1}^1 \sqrt{1-x^2}\,dx = \frac{\pi}{2}.</math>

In that integral the function [Template:Math](/wiki/Template:Math) represents the top half of a circle (the [square root](/wiki/Square_root) is a consequence of the [Pythagorean theorem](/wiki/Pythagorean_theorem)), and the integral [Template:Math](/wiki/Template:Math) computes the area between that half of a circle and the [Template:Math](/wiki/Template:Math) axis. [thumb|340px|alt=Diagram showing graphs of functions|](/wiki/File:Sine_cosine_one_period.svg)[Sine](/wiki/Sine) and [cosine](/wiki/Cosine) functions repeat with period 2[Template:Pi](/wiki/Template:Pi). The [trigonometric functions](/wiki/Trigonometric_function) rely on angles, and mathematicians generally use radians as units of measurement. [Template:Pi](/wiki/Template:Pi) plays an important role in angles measured in [radians](/wiki/Radian), which are defined so that a complete circle spans an angle of 2[Template:Pi](/wiki/Template:Pi) radians.[[144]](#cite_note-144) The angle measure of 180° is equal to [Template:Pi](/wiki/Template:Pi) radians, and 1° = [Template:Pi](/wiki/Template:Pi)/180 radians.[[144]](#cite_note-144) Common trigonometric functions have periods that are multiples of [Template:Pi](/wiki/Template:Pi); for example, sine and cosine have period 2[Template:Pi](/wiki/Template:Pi),[[145]](#cite_note-145) so for any angle [Template:Math](/wiki/Template:Math) and any integer [Template:Math](/wiki/Template:Math),

<math> \sin\theta = \sin\left(\theta + 2\pi k \right) \text{ and } \cos\theta = \cos\left(\theta + 2\pi k \right).</math>[[145]](#cite_note-145)

### Topology[[edit](/index.php?title=(none)&action=edit&section=26)]

[thumb|right|](/wiki/File:Uniform_tiling_73-t2.png)[Uniformization](/wiki/Uniformization_theorem) of the [Klein quartic](/wiki/Klein_quartic), a surface of [genus](/wiki/Genus_(mathematics)) three and Euler characteristic −4, as a quotient of the [hyperbolic plane](/wiki/Poincaré_disk_model) by the [symmetry group](/wiki/Symmetry_group) [PSL(2,7)](/wiki/PSL(2,7)) of the [Fano plane](/wiki/Fano_plane). The hyperbolic area of a fundamental domain is [Template:Math](/wiki/Template:Math), by Gauss–Bonnet. The constant [Template:Pi](/wiki/Template:Pi) appears in the [Gauss–Bonnet formula](/wiki/Gauss–Bonnet_formula) which relates the [differential geometry of surfaces](/wiki/Differential_geometry_of_surfaces) to their [topology](/wiki/Topology). Specifically, if a [compact](/wiki/Compact_space) surface [Template:Math](/wiki/Template:Math) has [Gauss curvature](/wiki/Gauss_curvature) *K*, then

<math>\int\_\Sigma K\,dA = 2\pi \chi(\Sigma)</math>

where [Template:Math](/wiki/Template:Math) is the [Euler characteristic](/wiki/Euler_characteristic), which is an integer.[[146]](#cite_note-146) An example is the surface area of a sphere *S* of curvature 1 (so that its [radius of curvature](/wiki/Radius_of_curvature), which coincides with its radius, is also 1.) The Euler characteristic of a sphere can be computed from its [homology groups](/wiki/Homology_group), and is found to be equal to two. Thus we have

<math>A(S) = \int\_S 1\,dA = 2\pi\cdot 2 = 4\pi</math>

reproducing the formula for the surface area of a sphere of radius 1.

The constant appears in many other integral formulae in topology, in particular those involving [characteristic classes](/wiki/Characteristic_class) via the [Chern–Weil homomorphism](/wiki/Chern–Weil_homomorphism).[[147]](#cite_note-147)

### Vector calculus[[edit](/index.php?title=(none)&action=edit&section=27)]

[thumb|right|The techniques of vector calculus can be understood in terms of decompositions into](/wiki/File:YL10M5sph.png) [spherical harmonics](/wiki/Spherical_harmonic) (shown) [Vector calculus](/wiki/Vector_calculus) is a branch of calculus that is concerned with the properties of [vector fields](/wiki/Vector_field), and has many physical applications such as to [electricity and magnetism](/wiki/Electricity_and_magnetism). The [Newtonian potential](/wiki/Newtonian_potential) for a point source [Template:Mvar](/wiki/Template:Mvar) situated at the origin of a three dimensional Cartesian coordinate system is[[148]](#cite_note-148):<math>V(\mathbf{x}) = -\frac{k Q}{|\mathbf{x}|}</math> which represents the [potential energy](/wiki/Potential_energy) of a unit mass (or charge) placed a distance [Template:Math](/wiki/Template:Math) from the source, and [Template:Mvar](/wiki/Template:Mvar) is a dimensional constant. The field, denoted here by [Template:Math](/wiki/Template:Math), which may be the (Newtonian) [gravitational field](/wiki/Gravitational_field) or the (Coulomb) [electric field](/wiki/Electric_field), is the negative [gradient](/wiki/Gradient) of the potential:

<math>\mathbf{E} = -\nabla V.</math>

Special cases include [Coulomb's law](/wiki/Coulomb's_law) and [Newton's law of universal gravitation](/wiki/Newton's_law_of_universal_gravitation). [Gauss' law](/wiki/Gauss'_law) states that the outward [flux](/wiki/Flux) of the field through any smooth, simple, closed, orientable surface [Template:Mvar](/wiki/Template:Mvar) containing the origin is equal to [Template:Math](/wiki/Template:Math):

[Template:Oiint](/wiki/Template:Oiint)

It is standard to absorb this factor of [Template:Math](/wiki/Template:Math) into the constant [Template:Mvar](/wiki/Template:Mvar), but this argument shows why it must appear *somewhere*. Furthermore, [Template:Math](/wiki/Template:Math) is the surface area of the unit sphere, but we have not assumed that [Template:Mvar](/wiki/Template:Mvar) is the sphere. However, as a consequence of the [divergence theorem](/wiki/Divergence_theorem), because the region away from the origin is vacuum (source-free) it is only the [homology class](/wiki/Homology_class) of the surface [Template:Mvar](/wiki/Template:Mvar) in [Template:Math](/wiki/Template:Math) that matters in computing the integral, so it can be replaced by any convenient surface in the same homology class, in particular a sphere, where spherical coordinates can be used to calculate the integral.

A consequence of the Gauss law is that the negative [Laplacian](/wiki/Laplacian) of the potential [Template:Mvar](/wiki/Template:Mvar) is equal to [Template:Math](/wiki/Template:Math) times the [Dirac delta function](/wiki/Dirac_delta_function):

<math>\Delta V(\mathbf x) = -4\pi k Q\delta(\mathbf x).</math>

More general distributions of matter (or charge) are obtained from this by [convolution](/wiki/Convolution), giving the [Poisson equation](/wiki/Poisson_equation)

<math>\Delta V(\mathbf x) = -4\pi k \rho(\mathbf x)</math>

where [Template:Math](/wiki/Template:Math) is the distribution function.

[thumb|right|Einstein's equation states that the curvature of space-time is produced by the matter-energy content.](/wiki/File:Spacetime_lattice_analogy.svg) The constant [Template:Pi](/wiki/Template:Pi) also plays an analogous role in four-dimensional potentials associated with [Einstein's equations](/wiki/Einstein's_equations), a fundamental formula which forms the basis of the [general theory of relativity](/wiki/General_theory_of_relativity) and describes the [fundamental interaction](/wiki/Fundamental_interaction) of [gravitation](/wiki/Gravitation) as a result of [spacetime](/wiki/Spacetime) being [curved](/wiki/Curvature) by [matter](/wiki/Matter) and [energy](/wiki/Energy):[[149]](#cite_note-149):<math> R\_{\mu\nu} - \frac{1}{2} R g\_{\mu\nu} + \Lambda g\_{\mu\nu} = \frac{8 \pi G}{c^4} T\_{\mu\nu},</math> where [Template:Math](/wiki/Template:Math) is the [Ricci curvature tensor](/wiki/Ricci_curvature_tensor), [Template:Mvar](/wiki/Template:Mvar) is the [scalar curvature](/wiki/Scalar_curvature), [Template:Math](/wiki/Template:Math) is the [metric tensor](/wiki/Metric_tensor_(general_relativity)), [Template:Math](/wiki/Template:Math) is the [cosmological constant](/wiki/Cosmological_constant), [Template:Mvar](/wiki/Template:Mvar) is [Newton's gravitational constant](/wiki/Gravitational_constant), [Template:Mvar](/wiki/Template:Mvar) is the [speed of light](/wiki/Speed_of_light) in vacuum, and [Template:Math](/wiki/Template:Math) is the [stress–energy tensor](/wiki/Stress–energy_tensor). The left-hand side of Einstein's equation is a non-linear analog of the Laplacian of the metric tensor (and reduces to that in the weak field limit), and the right hand side is the analog of the distribution function, times [Template:Math](/wiki/Template:Math).

### Cauchy's integral formula[[edit](/index.php?title=(none)&action=edit&section=28)]

[thumb|right|Complex analytic functions can be visualized as a collection of streamlines and equipotentials, systems of curves intersecting at right angles. Here illustrated is the complex logarithm of the Gamma function.](/wiki/File:Factorial05.jpg) One of the key tools in [complex analysis](/wiki/Complex_analysis) is [contour integration](/wiki/Contour_integration) of a function over a positively oriented ([rectifiable](/wiki/Rectifiable_curve)) [Jordan curve](/wiki/Jordan_curve) [Template:Math](/wiki/Template:Math). A form of [Cauchy's integral formula](/wiki/Cauchy's_integral_formula) states that if a point [Template:Math](/wiki/Template:Math) is interior to [Template:Math](/wiki/Template:Math), then[[150]](#cite_note-150)

<math>\oint\_\gamma \frac{dz}{z-z\_0} = 2\pi i.</math>

Although the curve [Template:Math](/wiki/Template:Math) is not a circle, and hence does not have any obvious connection to the constant [Template:Pi](/wiki/Template:Pi), a standard proof of this result uses [Morera's theorem](/wiki/Morera's_theorem), which implies that the integral is invariant under [homotopy](/wiki/Homotopy) of the curve, so that it can be deformed to a circle and then integrated explicitly in polar coordinates. More generally, it is true that if a rectifiable closed curve [Template:Math](/wiki/Template:Math) does not contain [Template:Math](/wiki/Template:Math), then the above integral is [Template:Math](/wiki/Template:Math) times the [winding number](/wiki/Winding_number) of the curve.

The general form of Cauchy's integral formula establishes the relationship between the values of a [complex analytic function](/wiki/Complex_analytic_function) [Template:Math](/wiki/Template:Math) on the Jordan curve [Template:Math](/wiki/Template:Math) and the value of [Template:Math](/wiki/Template:Math) at any interior point [Template:Math](/wiki/Template:Math) of [Template:Math](/wiki/Template:Math):[[151]](#cite_note-151)[[152]](#cite_note-152):<math>\oint\_\gamma { f(z) \over z-z\_0 }\,dz = 2\pi i f (z\_{0})</math> provided [Template:Math](/wiki/Template:Math) is analytic in the region enclosed by [Template:Math](/wiki/Template:Math) and extends continuously to [Template:Math](/wiki/Template:Math). Cauchy's integral formula is a special case of the [residue theorem](/wiki/Residue_theorem), that if [Template:Math](/wiki/Template:Math) is a [meromorphic function](/wiki/Meromorphic_function) the region enclosed by [Template:Math](/wiki/Template:Math) and is continuous in a neighborhood of [Template:Math](/wiki/Template:Math), then

<math>\oint\_\gamma g(z)\, dz =2\pi i \sum \operatorname{Res}( g, a\_k ) </math>

where the sum is of the [residues](/wiki/Residue_(mathematics)) at the [poles](/wiki/Pole_(complex_analysis)) of [Template:Math](/wiki/Template:Math).

### The gamma function and Stirling's approximation[[edit](/index.php?title=(none)&action=edit&section=29)]

[thumb|right|The](/wiki/File:Hopf_Fibration.png) [Hopf fibration](/wiki/Hopf_fibration) of the 3-sphere, by [Villarceaux circles](/wiki/Villarceaux_circles), over the [complex projective line](/wiki/Complex_projective_line) with its [Fubini–Study metric](/wiki/Fubini–Study_metric) (three parallels are shown). The identity [Template:Math](/wiki/Template:Math) [is a consequence](/wiki/Fubini–Study_metric#The_n_=_1_case). The factorial function [Template:Math](/wiki/Template:Math) is the product of all of the positive integers through [Template:Math](/wiki/Template:Math). The [gamma function](/wiki/Gamma_function) extends the concept of [factorial](/wiki/Factorial) (normally defined only for non-negative integers) to all complex numbers, except the negative real integers. When the gamma function is evaluated at half-integers, the result contains [Template:Pi](/wiki/Template:Pi); for example <math> \Gamma(1/2) = \sqrt{\pi} </math> and <math>\Gamma(5/2) = \frac {3 \sqrt{\pi}} {4} </math>.[[153]](#cite_note-153) The gamma function is defined by its [Weierstrass product](/wiki/Weierstrass_product) development:[[154]](#cite_note-154):<math>\Gamma(z) = e^{-\gamma z}\prod\_{n=1}^\infty \frac{e^{z/n}}{1+z/n}</math> where [Template:Math](/wiki/Template:Math) is the [Euler–Mascheroni constant](/wiki/Euler–Mascheroni_constant). Evaluated at [Template:Math](/wiki/Template:Math) and squared, the equation [Template:Math](/wiki/Template:Math) reduces to the Wallis product formula. The gamma function is also connected to the [Riemann zeta function](/wiki/Riemann_zeta_function) and identities for the [functional determinant](/wiki/Functional_determinant), in which the constant [Template:Pi](/wiki/Template:Pi) [plays an important role](/wiki/#Number_theory_and_Riemann_zeta_function).

The gamma function is used to calculate the volume [Template:Math](/wiki/Template:Math) of the [*n*-dimensional ball](/wiki/N-ball) of radius *r* in Euclidean *n*-dimensional space, and the surface area [Template:Math](/wiki/Template:Math) of its boundary, the [(*n*−1)-dimensional sphere](/wiki/N-sphere):[[155]](#cite_note-155)

<math>V\_n(r) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}r^n</math>

<math>S\_{n-1}(r) = \frac{n\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}r^{n-1}</math>

Further, it follows from the [functional equation](/wiki/Functional_equation) that

<math>2\pi r = \frac{S\_{n+1}(r)}{V\_n(r)}.</math>

The gamma function can be used to create a simple approximation to the factorial function [Template:Math](/wiki/Template:Math) for large [Template:Math](/wiki/Template:Math): <math> n! \sim \sqrt{2 \pi n} \left(\frac{n}{e}\right)^n</math> which is known as [Stirling's approximation](/wiki/Stirling's_approximation).[[156]](#cite_note-156) Equivalently,

<math>\pi = \lim\_{n\to\infty} \frac{e^{2n}n!^2}{2 n^{2n+1}}.</math>

As a geometrical application of Stirling's approximation, let [Template:Math](/wiki/Template:Math) denote the [standard simplex](/wiki/Simplex) in *n*-dimensional Euclidean space, and [Template:Math](/wiki/Template:Math) denote the simplex having all of its sides scaled up by a factor of [Template:Math](/wiki/Template:Math). Then

<math>\operatorname{Vol}((n+1)\Delta\_n) = \frac{(n+1)^n}{n!} \sim \frac{e^{n+1}}{\sqrt{2\pi n}}.</math>

[Ehrhart's volume conjecture](/wiki/Ehrhart's_volume_conjecture) is that this is the (optimal) upper bound on the volume of a [convex body](/wiki/Convex_body) containing only one [lattice point](/wiki/Lattice_point).[[157]](#cite_note-157)

### Number theory and Riemann zeta function[[edit](/index.php?title=(none)&action=edit&section=30)]

[thumb|right|Each prime has an associated](/wiki/File:Prüfer.png) [Prüfer group](/wiki/Prüfer_group), which are arithmetic localizations of the circle. The [L-functions](/wiki/L-function) of analytic number theory are also localized in each prime *p*. [thumb|right|Solution of the Basel problem using the](/wiki/File:ModularGroup-FundamentalDomain-01.png) [Weil conjecture](/wiki/Weil_conjecture_on_Tamagawa_numbers): the value of <math>\zeta(2)</math> is the [hyperbolic](/wiki/Poincaré_half-plane_model) area of a fundamental domain of the [modular group](/wiki/Modular_group), times <math>2\pi</math> The [Riemann zeta function](/wiki/Riemann_zeta_function) [Template:Math](/wiki/Template:Math) is used in many areas of mathematics. When evaluated at [Template:Math](/wiki/Template:Math) it can be written as

<math> \zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots</math>

Finding a [simple solution](/wiki/Closed-form_expression) for this infinite series was a famous problem in mathematics called the [Basel problem](/wiki/Basel_problem). [Leonhard Euler](/wiki/Leonhard_Euler) solved it in 1735 when he showed it was equal to [Template:Math](/wiki/Template:Math).[[97]](#cite_note-97) Euler's result leads to the [number theory](/wiki/Number_theory) result that the probability of two random numbers being [relatively prime](/wiki/Relatively_prime) (that is, having no shared factors) is equal to [Template:Math](/wiki/Template:Math).[[158]](#cite_note-158)[[159]](#cite_note-159) This probability is based on the observation that the probability that any number is [divisible](/wiki/Divisible) by a prime [Template:Math](/wiki/Template:Math) is [Template:Math](/wiki/Template:Math) (for example, every 7th integer is divisible by 7.) Hence the probability that two numbers are both divisible by this prime is [Template:Math](/wiki/Template:Math), and the probability that at least one of them is not is [Template:Math](/wiki/Template:Math). For distinct primes, these divisibility events are mutually independent; so the probability that two numbers are relatively prime is given by a product over all primes:[[160]](#cite_note-160): <math>\begin{align} \prod\_p^\infty \left(1-\frac{1}{p^2}\right) &= \left( \prod\_p^\infty \frac{1}{1-p^{-2}} \right)^{-1}\\ &= \frac{1}{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots }\\ &= \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 61\%. \end{align}</math> This probability can be used in conjunction with a [random number generator](/wiki/Random_number_generator) to approximate [Template:Pi](/wiki/Template:Pi) using a Monte Carlo approach.[[161]](#cite_note-161) The solution to the Basel problem implies that the geometrically derived quantity [Template:Pi](/wiki/Template:Pi) is connected in a deep way to the distribution of prime numbers. This is a special case of [Weil's conjecture on Tamagawa numbers](/wiki/Weil's_conjecture_on_Tamagawa_numbers), which asserts the equality of similar such infinite products of *arithmetic* quantities, localized at each prime *p*, and a *geometrical* quantity: the reciprocal of the volume of a certain [locally symmetric space](/wiki/Locally_symmetric_space). In the case of the Basel problem, it is the [hyperbolic 3-manifold](/wiki/Hyperbolic_3-manifold) [Template:Math](/wiki/Template:Math).[[162]](#cite_note-162) The zeta function also satisfies Riemann's functional equation, which involves [Template:Pi](/wiki/Template:Pi) as well as the gamma function:

<math>

\zeta(s) = 2^s\pi^{s-1}\ \sin\left(\frac{\pi s}{2}\right)\ \Gamma(1-s)\ \zeta(1-s)\!.</math>

Furthermore, the derivative of the zeta function satisfies

<math>\exp(-\zeta'(0)) = \sqrt{2\pi}.</math>

A consequence is that [Template:Pi](/wiki/Template:Pi) can be obtained from the [functional determinant](/wiki/Functional_determinant) of the [harmonic oscillator](/wiki/Harmonic_oscillator). This functional determinant can be computed via a product expansion, and is equivalent to the Wallis product formula.[[163]](#cite_note-163) The calculation can be recast in [quantum mechanics](/wiki/Quantum_mechanics), specifically the [variational approach](/wiki/Calculus_of_variations) to the [spectrum of the hydrogen atom](/wiki/Bohr_model).[[164]](#cite_note-164)

### Fourier series[[edit](/index.php?title=(none)&action=edit&section=31)]

[thumb|right|](/wiki/File:2-adic_integers_with_dual_colorings.svg)[Template:Pi](/wiki/Template:Pi) appears in characters of [p-adic numbers](/wiki/P-adic_numbers) (shown), which are elements of a [Prüfer group](/wiki/Prüfer_group). [Tate's thesis](/wiki/Tate's_thesis) makes heavy use of this machinery.[[165]](#cite_note-165) The constant [Template:Pi](/wiki/Template:Pi) also appears naturally in [Fourier series](/wiki/Fourier_series) of [periodic functions](/wiki/Periodic_function). Periodic functions are functions on the group [Template:Math](/wiki/Template:Math) of fractional parts of real numbers. The Fourier decomposition shows that a complex-valued function [Template:Math](/wiki/Template:Math) on [Template:Math](/wiki/Template:Math) can be written as an infinite linear superposition of [unitary characters](/wiki/Unitary_character) of [Template:Math](/wiki/Template:Math). That is, continuous [group homomorphisms](/wiki/Group_homomorphism) from [Template:Math](/wiki/Template:Math) to the [circle group](/wiki/Circle_group) [Template:Math](/wiki/Template:Math) of unit modulus complex numbers. It is a theorem that every character of [Template:Math](/wiki/Template:Math) is one of the complex exponentials <math>e\_n(x)= e^{2\pi i n x}</math>.

There is a unique character on [Template:Math](/wiki/Template:Math), up to complex conjugation, that is a group isomorphism. Using the [Haar measure](/wiki/Haar_measure) on the circle group, the constant [Template:Pi](/wiki/Template:Pi) is half the magnitude of the [Radon–Nikodym derivative](/wiki/Radon–Nikodym_derivative) of this character. The other characters have derivatives whose magnitudes are positive integral multiples of 2[Template:Pi](/wiki/Template:Pi).[[18]](#cite_note-18) As a result, the constant [Template:Pi](/wiki/Template:Pi) is the unique number such that the group **T**, equipped with its Haar measure, is [Pontrjagin dual](/wiki/Pontrjagin_dual) to the [lattice](/wiki/Lattice_(group)) of integral multiples of 2[Template:Pi](/wiki/Template:Pi).[[166]](#cite_note-166) This is a version of the one-dimensional [Poisson summation formula](/wiki/Poisson_summation_formula).

### Modular forms and theta functions[[edit](/index.php?title=(none)&action=edit&section=32)]

[thumb|right|Theta functions transform under the](/wiki/File:Lattice_with_tau.svg) [lattice](/wiki/Lattice_(group)) of periods of an elliptic curve. The constant [Template:Pi](/wiki/Template:Pi) is connected in a deep way with the theory of [modular forms](/wiki/Modular_form) and [theta functions](/wiki/Theta_function). For example, the [Chudnovsky algorithm](/wiki/Chudnovsky_algorithm) involves in an essential way the [j-invariant](/wiki/J-invariant) of an [elliptic curve](/wiki/Elliptic_curve).

[Modular forms](/wiki/Modular_form) are [holomorphic functions](/wiki/Holomorphic_function) in the [upper half plane](/wiki/Upper_half_plane) characterized by their transformation properties under the [modular group](/wiki/Modular_group) <math>SL\_2(\mathbb Z)</math> (or its various subgroups), a lattice in the group <math>SL\_2(\mathbb R)</math>. An example is the [Jacobi theta function](/wiki/Jacobi_theta_function)

<math>\theta(z,\tau) = \sum\_{n=-\infty}^\infty e^{2\pi i nz + i\pi n^2\tau}</math>

which is a kind of modular form called a [Jacobi form](/wiki/Jacobi_form).[[167]](#cite_note-167) This is sometimes written in terms of the [nome](/wiki/Nome_(mathematics)) <math>q=e^{\pi i \tau}</math>.

The constant [Template:Pi](/wiki/Template:Pi) is the unique constant making the Jacobi theta function an [automorphic form](/wiki/Automorphic_form), which means that it transforms in a specific way. Certain identities hold for all automorphic forms. An example is

<math>\theta(z+\tau,\tau) = e^{-\pi i\tau -2\pi i z}\theta(z,\tau),</math>

which implies that [Template:Math](/wiki/Template:Math) transforms as a representation under the discrete [Heisenberg group](/wiki/Heisenberg_group). General modular forms and other [theta functions](/wiki/Theta_function) also involve [Template:Pi](/wiki/Template:Pi), once again because of the [Stone–von Neumann theorem](/wiki/Stone–von_Neumann_theorem).[[167]](#cite_note-167)

### Cauchy distribution and potential theory[[edit](/index.php?title=(none)&action=edit&section=33)]

[thumb|right|The](/wiki/File:Witch_of_Agnesi,_construction.svg) [Witch of Agnesi](/wiki/Witch_of_Agnesi), named for [Maria Agnesi](/wiki/Maria_Gaetana_Agnesi) (1718–1799), is a geometrical construction of the graph of the Cauchy distribution. The [Cauchy distribution](/wiki/Cauchy_distribution)

<math>g(x)=\frac{1}{\pi}\cdot\frac{1}{x^2+1}</math>

is a [probability density function](/wiki/Probability_density_function). The total probability is equal to one, owing to the integral:

<math>\int\_{-\infty }^{\infty } \frac{1}{x^2+1} \, dx = \pi.</math>

The [Shannon entropy](/wiki/Shannon_entropy) of the Cauchy distribution is equal to [Template:Math](/wiki/Template:Math), which also involves [Template:Pi](/wiki/Template:Pi).

[thumb|right|The Cauchy distribution governs the passage of](/wiki/File:Brownianmotion_beads_in_water_spim_video.gif) [Brownian particles](/wiki/Brownian_motion) through a membrane. The Cauchy distribution plays an important role in [potential theory](/wiki/Potential_theory) because it is the simplest [Furstenberg measure](/wiki/Furstenberg_boundary), the classical [Poisson kernel](/wiki/Poisson_kernel) associated with a [Brownian motion](/wiki/Brownian_motion) in a half-plane.[[168]](#cite_note-168) [Conjugate harmonic functions](/wiki/Conjugate_harmonic_function) and so also the [Hilbert transform](/wiki/Hilbert_transform) are associated with the asymptotics of the Poisson kernel. The Hilbert transform *H* is the integral transform given by the [Cauchy principal value](/wiki/Cauchy_principal_value) of the [singular integral](/wiki/Singular_integral)

<math>Hf(t) = \frac{1}{\pi}\int\_{-\infty}^\infty \frac{f(x)}{x-t}.</math>

The constant [Template:Pi](/wiki/Template:Pi) is the unique (positive) normalizing factor such that *H* defines a [linear complex structure](/wiki/Linear_complex_structure) on the Hilbert space of square-integrable real-valued functions on the real line.[[169]](#cite_note-169) The Hilbert transform, like the Fourier transform, can be characterized purely in terms of its transformation properties on the Hilbert space [Template:Math](/wiki/Template:Math): up to a normalization factor, it is the unique bounded linear operator that commutes with positive dilations and anticommutes with all reflections of the real line.[[170]](#cite_note-170) The constant [Template:Pi](/wiki/Template:Pi) is the unique normalizing factor that makes this transformation unitary.

### Complex dynamics[[edit](/index.php?title=(none)&action=edit&section=34)]

[alt=An complex black shape on a blue background.|thumb|](/wiki/File:Mandel_zoom_00_mandelbrot_set.jpg)[Template:Pi](/wiki/Template:Pi) can be computed from the [Mandelbrot set](/wiki/Mandelbrot_set), by counting the number of iterations required before point (−0.75, ε) diverges. An occurrence of [Template:Pi](/wiki/Template:Pi) in the [Mandelbrot set](/wiki/Mandelbrot_set) [fractal](/wiki/Fractal) was discovered by David Boll in 1991.[[171]](#cite_note-171) He examined the behavior of the Mandelbrot set near the "neck" at (−0.75, 0). If points with coordinates (−0.75, ε) are considered, as ε tends to zero, the number of iterations until divergence for the point multiplied by ε converges to [Template:Pi](/wiki/Template:Pi). The point (0.25, ε) at the cusp of the large "valley" on the right side of the Mandelbrot set behaves similarly: the number of iterations until divergence multiplied by the square root of ε tends to [Template:Pi](/wiki/Template:Pi).[[171]](#cite_note-171)[[172]](#cite_note-172)

## Outside mathematics[[edit](/index.php?title=(none)&action=edit&section=35)]

### Describing physical phenomena[[edit](/index.php?title=(none)&action=edit&section=36)]

Although not a [physical constant](/wiki/Physical_constant), [Template:Pi](/wiki/Template:Pi) appears routinely in equations describing fundamental principles of the universe, often because of [Template:Pi's](/wiki/Template:Pi) relationship to the circle and to [spherical coordinate systems](/wiki/Spherical_coordinate_system). A simple formula from the field of [classical mechanics](/wiki/Classical_mechanics) gives the approximate period [Template:Math](/wiki/Template:Math) of a simple [pendulum](/wiki/Pendulum) of length [Template:Math](/wiki/Template:Math), swinging with a small amplitude ([Template:Math](/wiki/Template:Math) is the [earth's gravitational acceleration](/wiki/Gravity_of_Earth)):[[173]](#cite_note-173):<math>T \approx 2\pi \sqrt\frac{L}{g}.</math> One of the key formulae of [quantum mechanics](/wiki/Quantum_mechanics) is [Heisenberg's uncertainty principle](/wiki/Heisenberg's_uncertainty_principle), which shows that the uncertainty in the measurement of a particle's position (Δ[Template:Math](/wiki/Template:Math)) and [momentum](/wiki/Momentum) (Δ[Template:Math](/wiki/Template:Math)) cannot both be arbitrarily small at the same time (where [Template:Math](/wiki/Template:Math) is [Planck's constant](/wiki/Planck's_constant)):[[174]](#cite_note-174):<math> \Delta x\, \Delta p \ge \frac{h}{4\pi}.</math>

The fact that [Template:Pi](/wiki/Template:Pi) is approximately equal to 3 plays a role in the relatively long lifetime of [orthopositronium](/wiki/Orthopositronium). The inverse lifetime to lowest order in the [fine-structure constant](/wiki/Fine-structure_constant) [Template:Math](/wiki/Template:Math) is[[175]](#cite_note-175):<math>\frac{1}{\tau} = 2\frac{\pi^2 - 9}{9\pi}m\alpha^{6},</math> where [Template:Math](/wiki/Template:Math) is the mass of the electron.

[Template:Pi](/wiki/Template:Pi) is present in some structural engineering formulae, such as the [buckling](/wiki/Buckling) formula derived by Euler, which gives the maximum axial load [Template:Math](/wiki/Template:Math) that a long, slender column of length [Template:Math](/wiki/Template:Math), [modulus of elasticity](/wiki/Modulus_of_elasticity) [Template:Math](/wiki/Template:Math), and [area moment of inertia](/wiki/Area_moment_of_inertia) [Template:Math](/wiki/Template:Math) can carry without buckling:[[176]](#cite_note-176):<math>F =\frac{\pi^2EI}{L^2}.</math>

The field of [fluid dynamics](/wiki/Fluid_dynamics) contains [Template:Pi](/wiki/Template:Pi) in [Stokes' law](/wiki/Stokes'_law), which approximates the [frictional force](/wiki/Drag_force) [Template:Math](/wiki/Template:Math) exerted on small, [spherical](/wiki/Sphere) objects of radius [Template:Math](/wiki/Template:Math), moving with velocity [Template:Math](/wiki/Template:Math) in a [fluid](/wiki/Fluid) with [dynamic viscosity](/wiki/Dynamic_viscosity) [Template:Math](/wiki/Template:Math):[[177]](#cite_note-177):<math>F =6 \, \pi \, \eta \, R \, v .</math>

Under ideal conditions (uniform gentle slope on an homogeneously erodible substrate), the [sinuosity](/wiki/Sinuosity) of a [meandering](/wiki/Meander) river approaches [Template:Pi](/wiki/Template:Pi). The sinuosity is the ratio between the actual length and the straight-line distance from source to mouth. Faster currents along the outside edges of a river's bends cause more erosion than along the inside edges, thus pushing the bends even farther out, and increasing the overall loopiness of the river. However, that loopiness eventually causes the river to double back on itself in places and "short-circuit", creating an [ox-bow lake](/wiki/Ox-bow_lake) in the process. The balance between these two opposing factors leads to an average ratio of [Template:Pi](/wiki/Template:Pi) between the actual length and the direct distance between source and mouth.[[178]](#cite_note-178)[[179]](#cite_note-179)

### Memorizing digits[[edit](/index.php?title=(none)&action=edit&section=37)]

[Template:Main](/wiki/Template:Main) [Piphilology](/wiki/Piphilology) is the practice of memorizing large numbers of digits of [Template:Pi](/wiki/Template:Pi),[[180]](#cite_note-180) and world-records are kept by the [*Guinness World Records*](/wiki/Guinness_World_Records). The record for memorizing digits of [Template:Pi](/wiki/Template:Pi), certified by Guinness World Records, is 70,000 digits, recited in India by Rajveer Meena in 9 hours and 27 minutes on 21 March 2015.[[181]](#cite_note-181) In 2006, [Akira Haraguchi](/wiki/Akira_Haraguchi), a retired Japanese engineer, claimed to have recited 100,000 decimal places, but the claim was not verified by Guinness World Records.[[182]](#cite_note-182) One common technique is to memorize a story or poem in which the word lengths represent the digits of [Template:Pi](/wiki/Template:Pi): The first word has three letters, the second word has one, the third has four, the fourth has one, the fifth has five, and so on. An early example of a memorization aid, originally devised by English scientist [James Jeans](/wiki/James_Hopwood_Jeans), is "How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics."[[180]](#cite_note-180) When a poem is used, it is sometimes referred to as a *piem*. Poems for memorizing [Template:Pi](/wiki/Template:Pi) have been composed in several languages in addition to English.[[180]](#cite_note-180) Record-setting [Template:Pi](/wiki/Template:Pi) memorizers typically do not rely on poems, but instead use methods such as remembering number patterns and the [method of loci](/wiki/Method_of_loci).[[183]](#cite_note-183) A few authors have used the digits of [Template:Pi](/wiki/Template:Pi) to establish a new form of [constrained writing](/wiki/Constrained_writing), where the word lengths are required to represent the digits of [Template:Pi](/wiki/Template:Pi). The [*Cadaeic Cadenza*](/wiki/Cadaeic_Cadenza) contains the first 3835 digits of [Template:Pi](/wiki/Template:Pi) in this manner,[[184]](#cite_note-184) and the full-length book *Not a Wake* contains 10,000 words, each representing one digit of [Template:Pi](/wiki/Template:Pi).<ref name=KeithNAW>[Template:Cite book](/wiki/Template:Cite_book)</ref>

### In popular culture[[edit](/index.php?title=(none)&action=edit&section=38)]

[thumb|right|alt=Pi Pie at Delft University|A pi pie. The circular shape of](/wiki/File:Pi_pie2.jpg) [pie](/wiki/Pie) makes it a frequent subject of pi [puns](/wiki/Pun).|bottom Perhaps because of the simplicity of its definition and its ubiquitous presence in formulae, [Template:Pi](/wiki/Template:Pi) has been represented in popular culture more than other mathematical constructs.[[185]](#cite_note-185) In the 2008 [Open University](/wiki/Open_University) and [BBC](/wiki/BBC) documentary co-production, [*The Story of Maths*](/wiki/The_Story_of_Maths), aired in October 2008 on [BBC Four](/wiki/BBC_Four), British mathematician [Marcus du Sautoy](/wiki/Marcus_du_Sautoy) shows a [visualization](/wiki/Information_graphics) of the - historically first exact - [formula for calculating](/wiki/Madhava_of_Sangamagrama#The_value_of_π_(pi)) [Template:Pi](/wiki/Template:Pi) when visiting India and exploring its contributions to trigonometry.[[186]](#cite_note-186) In the [Palais de la Découverte](/wiki/Palais_de_la_Découverte) (a science museum in Paris) there is a circular room known as the *pi room*. On its wall are inscribed 707 digits of [Template:Pi](/wiki/Template:Pi). The digits are large wooden characters attached to the dome-like ceiling. The digits were based on an 1853 calculation by English mathematician [William Shanks](/wiki/William_Shanks), which included an error beginning at the 528th digit. The error was detected in 1946 and corrected in 1949.[[187]](#cite_note-187) In [Carl Sagan's](/wiki/Carl_Sagan) novel [*Contact*](/wiki/Contact_(novel)) it is suggested that the creator of the universe buried a message deep within the digits of [Template:Pi](/wiki/Template:Pi).[[188]](#cite_note-188) The digits of [Template:Pi](/wiki/Template:Pi) have also been incorporated into the lyrics of the song "Pi" from the album [*Aerial*](/wiki/Aerial_(album)) by [Kate Bush](/wiki/Kate_Bush).[[189]](#cite_note-189) In the United States, [Pi Day](/wiki/Pi_Day) falls on 14 March (written 3/14 in the US style), and is popular among students.[[190]](#cite_note-190) [Template:Pi](/wiki/Template:Pi) and its digital representation are often used by self-described "math [geeks](/wiki/Geek)" for [inside jokes](/wiki/Inside_joke) among mathematically and technologically minded groups. Several college [cheers](/wiki/Cheering) at the [Massachusetts Institute of Technology](/wiki/Massachusetts_Institute_of_Technology) include "3.14159".[[191]](#cite_note-191) Pi Day in 2015 was particularly significant because the date and time 3/14/15 9:26:53 reflected many more digits of pi.[[192]](#cite_note-192) During the 2011 auction for [Nortel's](/wiki/Nortel) portfolio of valuable technology patents, [Google](/wiki/Google) made a series of unusually specific bids based on mathematical and scientific constants, including [Template:Pi](/wiki/Template:Pi).[[193]](#cite_note-193) [Template:Anchor](/wiki/Template:Anchor) In 1958 [Albert Eagle](/wiki/Albert_Eagle) proposed replacing [Template:Pi](/wiki/Template:Pi) by [Template:Tau](/wiki/Template:Tau) = [Template:Pi](/wiki/Template:Pi)/2 to simplify formulas.<ref name=AE1958>[Template:Cite book](/wiki/Template:Cite_book)</ref> However, no other authors are known to use [Template:Tau](/wiki/Template:Tau) in this way. Some people use a different value, [Template:Tau](/wiki/Template:Tau) = 6.283185... = 2[Template:Pi](/wiki/Template:Pi),[[194]](#cite_note-194) arguing that [Template:Tau](/wiki/Template:Tau), as the number of radians in one [turn](/wiki/Turn_(geometry)) or as the ratio of a circle's circumference to its radius rather than its diameter, is more natural than [Template:Pi](/wiki/Template:Pi) and simplifies many formulas.[[195]](#cite_note-195)[[196]](#cite_note-196) Celebrations of this number, because it approximately equals 6.28, by making 28 June "Tau Day" and eating "twice the pie",[[197]](#cite_note-197) have been reported in the media. However, this use of [Template:Math](/wiki/Template:Math) has not made its way into mainstream mathematics.[[198]](#cite_note-198) In 1897, an amateur American mathematician attempted to persuade the [Indiana legislature](/wiki/Indiana_General_Assembly) to pass the [Indiana Pi Bill](/wiki/Indiana_Pi_Bill), which described a method to [square the circle](/wiki/Squaring_the_circle) and contained text that implied various incorrect values for [Template:Pi](/wiki/Template:Pi), including 3.2. The bill is notorious as an attempt to establish a value of scientific constant by legislative fiat. The bill was passed by the Indiana House of Representatives, but rejected by the Senate.[[199]](#cite_note-199)

## Notes[[edit](/index.php?title=(none)&action=edit&section=39)]

**Footnotes** [Template:Reflist](/wiki/Template:Reflist)

**References** [Template:Refbegin](/wiki/Template:Refbegin)

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* [Template:Cite book](/wiki/Template:Cite_book), English translation by Stephen Wilson.
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## Further reading[[edit](/index.php?title=(none)&action=edit&section=40)]

[Template:Refbegin](/wiki/Template:Refbegin)

* [Template:Cite book](/wiki/Template:Cite_book)
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## External links[[edit](/index.php?title=(none)&action=edit&section=41)]

[Template:Commons category](/wiki/Template:Commons_category)

* [Template:Dmoz](/wiki/Template:Dmoz)
* ["Pi"](http://mathworld.wolfram.com/Pi.html) at Wolfram Mathworld
* [Representations of Pi](http://www.wolframalpha.com/input/?i=Representations+of+Pi) at [Wolfram Alpha](/wiki/Wolfram_Alpha)
* [Pi Search Engine](http://www.subidiom.com/pi): 2 billion searchable digits of [Template:Pi](/wiki/Template:Pi), [Template:Math](/wiki/Template:Math), and [Template:Math](/wiki/Template:Math)
* [Template:Cite web](/wiki/Template:Cite_web)
* [Template:Cite web](/wiki/Template:Cite_web)

[Template:Featured article](/wiki/Template:Featured_article)

* Demonstration by Lambert (1761) of irrationality of [Template:Pi](/wiki/Template:Pi), [online](https://www.bibnum.education.fr/mathematiques/theorie-des-nombres/lambert-et-l-irrationalite-de-p-1761) and analyzed [*BibNum*](https://www.bibnum.education.fr/sites/default/files/24-lambert-analysis.pdf) (PDF).

[Template:Irrational number](/wiki/Template:Irrational_number)

[Template:Authority control](/wiki/Template:Authority_control)

[Category:Pi](/wiki/Category:Pi) [Category:Complex analysis](/wiki/Category:Complex_analysis) [Category:Mathematical series](/wiki/Category:Mathematical_series) [Category:Real transcendental numbers](/wiki/Category:Real_transcendental_numbers)