Constrained Nonlinear Optimisation with Lagrange Multipliers

Attitude Dynamics and Control
Project Report

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Contents

1 Introduction

In this project, given constrained optimisation problem is solved with using Lagrange multipliers and the solution is verified graphically.

$$\text{Minimize } F\left(x_1, x_2\right) = x_1^2 + 320x_1x_2 \tag{1.1}$$

Subject to:

$$\frac{1}{100}\left(x_1 - 60x_2\right) \leq 0 \tag{1.2}$$

$$1 - \frac{1}{3600}x_1\left(x_1 - x_2\right) \leq 0 \tag{1.3}$$

$$x_1, x_2 \geq 0 \tag{1.4}$$

2 Solution

We can define Lagrangian as

$$L = x_1^2 + 320x_1x_2 + \lambda_1\left(\frac{x_1 - 60x_2}{100}\right) + \lambda_2\left[1 - \frac{x_1}{3600}\left(x_1 - x_2\right)\right] + \lambda_3\left(-x_1\right) + \lambda_4\left(-x_2\right) \tag{2.1}$$

Necessary conditions:

$$\frac{\partial L}{\partial x_1} = 2x_1 + 320x_2 + \frac{\lambda_1}{100} - \lambda_2\left(\frac{x_1}{1800} - \frac{x_2}{3600}\right) = 0 \tag{2.2}$$

$$\frac{\partial L}{\partial x_2} = 320x_1 - \frac{3\lambda_1}{5} + \frac{\lambda_2 x_1}{3600} = 0 \tag{2.3}$$

$$\lambda_1\frac{\partial L}{\partial \lambda_1} = \lambda_1\left(\frac{x_1 - 60x_2}{100}\right) = 0 \tag{2.4}$$

$$\lambda_2\frac{\partial L}{\partial \lambda_2} = \lambda_2\left[1 - \frac{x_1}{3600}\left(x_1 - x_2\right)\right] = 0 \tag{2.5}$$

$$\lambda_3\frac{\partial L}{\partial \lambda_3} = -\lambda_3 x_1 = 0 \tag{2.6}$$

$$\lambda_4\frac{\partial L}{\partial \lambda_4} = -\lambda_4 x_2 = 0; \tag{2.7}$$

Now we need to assign active of inactive states to Lagrange multipliers.

- Assume all Lagrange multipliers are inactive: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

$$\frac{\partial L}{\partial x_1} = 2x_1 + 320x_2 = 0 \tag{2.8}$$

$$\frac{\partial L}{\partial x_2} = 320x_2 = 0 \tag{2.9}$$

gives solution $x_1 = x_2 = 0$ and does not satisfy the second constraint.

- Assume $\lambda_1 \geq 0$ (active) and $\lambda_2 = \lambda_3 = \lambda_4 = 0$ (inactive)

$$\frac{\partial L}{\partial x_1} = 2x_1 + 320x_2 + \frac{\lambda_1}{100} = 0 \tag{2.10}$$

$$\frac{\partial L}{\partial x_2} = 320x_1 - \frac{3\lambda_1}{5} = 0 \tag{2.11}$$

$$\lambda_1 \frac{\partial L}{\partial \lambda_1} = \lambda_1 \left(\frac{x_1 - 60x_2}{100} \right) = 0 \tag{2.12}$$

gives solution $x_1 = x_2 = \lambda_1 = 0$ and does not satisfy the second constraint.

- Assume $\lambda_2 \geq 0$ (active) and $\lambda_1 = \lambda_3 = \lambda_4 = 0$ (inactive)

$$\frac{\partial L}{\partial x_1} = 2x_1 + 320x_2 + \lambda_2 \left(\frac{x_1}{1800} - \frac{x_2}{3600} \right) = 0 \tag{2.13}$$

$$\frac{\partial L}{\partial x_2} = 320x_1 + \frac{\lambda_2 x_1}{3600} = 0 \tag{2.14}$$

$$\lambda_2 \frac{\partial L}{\partial \lambda_2} = \lambda_2 \left[1 - \frac{x_1}{3600} (x_1 - x_2) \right] = 0 \tag{2.15}$$

gives solution $x_1 = x_2 = \lambda_2 = 0$ and does not satisfy the second constraint.

- Assume $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ (active) and $\lambda_3 = \lambda_4 = 0$ (inactive)

$$\frac{\partial L}{\partial x_1} = 2x_1 + 320x_2 + \frac{\lambda_1}{100} + \lambda_2 \left(\frac{x_1}{1800} - \frac{x_2}{3600} \right) = 0 \tag{2.16}$$

$$\frac{\partial L}{\partial x_2} = 320x_1 - \frac{3\lambda_1}{5} + \frac{\lambda_2 x_1}{3600} = 0 \tag{2.17}$$

$$\lambda_1 \frac{\partial L}{\partial \lambda_1} = \lambda_1 \left(\frac{x_1 - 60x_2}{100} \right) = 0 \tag{2.18}$$

$$\lambda_2 \frac{\partial L}{\partial \lambda_2} = \lambda_2 \left[1 - \frac{x_1}{3600} (x_1 - x_2) \right] = 0 \tag{2.19}$$

gives solution as:

$$x_1 = \begin{bmatrix} 0 \\ -60.506338090753289 \\ 60.506338090753289 \end{bmatrix} \tag{2.20}$$

$$x_2 = \begin{bmatrix} 0 \\ -1.008438968179221 \\ 1.008438968179221 \end{bmatrix} \tag{2.21}$$

$$\lambda_2 = \begin{bmatrix} 0 \\ -3.291955004598611 \\ 3.291955004598611 \end{bmatrix} 10^4 \tag{2.22}$$

$$\lambda_1 = \begin{bmatrix} 0 \\ 2.318644067796610 \\ 2.318644067796610 \end{bmatrix} 10^4 \tag{2.23}$$

Among these results, $x_1 = 60.506338090753289$, $x_1 = 1.008438968179221$ satisfy the constraints and gives result as $F = 2.318644067796610 \times 10^4$

By asolving the problem also graphically, we obtain the figure below as:

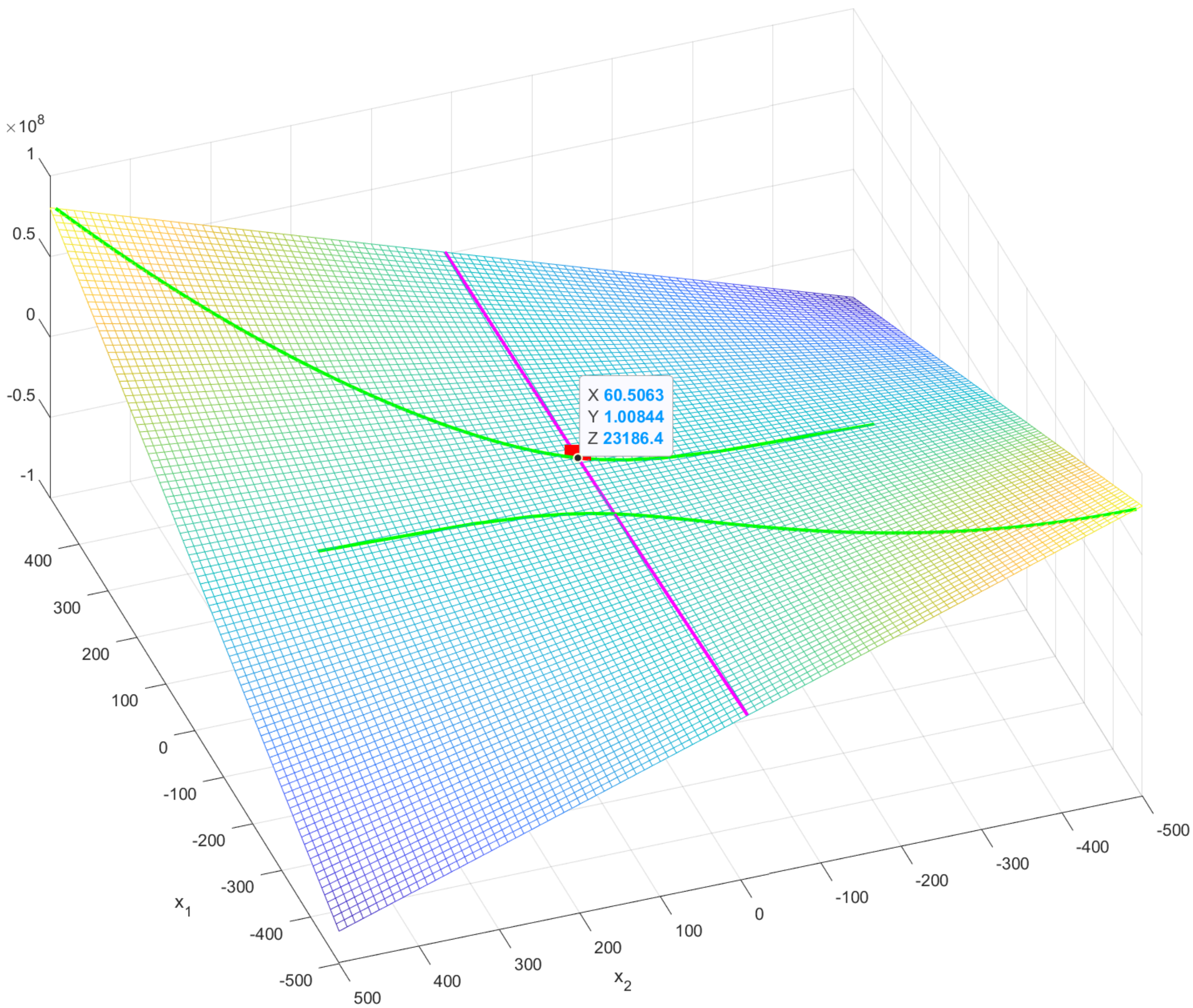

Figure 1: Graphical solution of problem with fmincon function from MATLAB

And from figure[1], we can see that $F = 2.31864 \times 10^4$ which means our result of minimum point obtained manually with using lagrange multipliers is correct.

3 Appendix A: MATLAB Codes

3.1 Appendix A-1: Code for solving partial derivatives of Lagrange equation

```
clear; clc; close all;
syms x1 x2 lambda1 lambda2 lambda3 lambda4
%case 1: all multipliers are inactive
fprintf('Case 1: All multipliers are inactive\n');
f1 = 2*x1 +  320*x2 == 0;
f2 = 320*x2 == 0;
solve(f1,f2)

%case 2: lambda1 is active and lambda2, lambda3 and lambda4 are inactive
fprintf('\nCase 2: lambda1 is active and lambda2, lambda3 and lambda4 ...
    are inactive\n');
f3 = 2*x1 + 320*x2 + lambda1/100 == 0;
f4 = 320*x1 - ((3*lambda1)/5) == 0;
f5 = lambda1*((x1-(60*x2))/100) == 0;
solve(f3,f4,f5)

%case 3: lambda2 is active and lambda1, lambda3 and lambda4 are inactive
fprintf('\nCase 3: lambda2 is active and lambda1, lambda3 and lambda4 ...
    are inactive\n');
f6 = 2*x1 + 320*x2 + lambda2*((x1/1800)-(x2/3600)) == 0;
f7 = 320*x1 - ((lambda2*x1)/3600) == 0;
f8 = lambda2*(1-((x1/3600)*(x1-x2))) == 0;
solve(f6,f7,f8)

%case 4: lambda1 and lambda2 are active and lambda3 and lambda4 are
%inactive
fprintf('\nCase 4: lambda1 and lambda2 are active and lambda3 and ...
    lambda4 are inactive\n');
fA = 2*x1 + 320*x2 + lambda1/100 - lambda2*(x1/1800 - x2/3600) == 0;
fB = 320*x1 - (3*lambda1)/5 + (lambda2*x1)/3600 == 0;
fC = lambda1*(x1/100 - (3*x2)/5) == 0;
fD = lambda2*(1 - (x1*(x1 - x2))/3600) == 0;
[lambda1,lambda2,x1,x2] = solve(fA,fB,fC,fD);
x1 = double(x1)
x2 = double(x2)
lambda1 = double(lambda1)
```

```
lambda2 = double(lambda2)

%for i = 1:1:3
%    if (x1(i) > 0) & (x2(i)>0) & (((x1(i)-(60*x2(i)))/100) < 0) & ...
    ((1-((x1(i)*(x1(i)-x2(i)))/3600)) < 0)
%        xx1 = x1(i);
%        xx2 = x2(i);
%    end
%end

fprintf('\nWith results which satisfy the constraints, minimum of the ...
    function is:\n');
F = x1(3)^2 + 320*x1(3)*x2(3)
```

3.2 Appendix A-2: Code for solving the problem graphically

```
clear;clc; close all;
figure(1),clf
xx=[-500:10:500]';
for  ii=1:length(xx)
for  jj=1:length(xx)
FF(ii,jj)=  xx(ii)^2+320*xx(ii)*xx(jj);
end
end
hh=mesh(xx,xx,FF);%
hold  on;%

plot3(xx,xx/60,xx.^2+(320/60)*xx.^2,"m-","LineWidth",2);  %first constraint
plot3(xx,xx-(3600./xx),xx.^2+320.*xx.*(xx-(3600./xx)),"g-","LineWidth",2); ...
    %second constraint

xlabel("x_1");  ylabel("x_2");  %
hold  off;  %axis([-3  3  -3  3  0  20])%
hh=get(gcf,"children");%
%set(hh,"View",[-109  74],"CameraPosition",[-26.5555  13.5307  151.881]);%

xx=fmincon("exampleF",[0;0],[],[],[],[],[],[],"exampleC");
hold  on
plot3(xx(1),xx(2),xx(1)^2+320*xx(1)*xx(2),"rs","MarkerSize",20,"MarkerFace","r")
xx(1)^2+320*xx(1)*xx(2)

```

```
%[x,fval,exitflag,output,lambda,grad,hessian] = ...
    fmincon("exampleF",[0;0],[],[],[],[],[],[],"exampleC");

function  [c,ceq]=exampleC(X)
c1 = (1/100)*(X(1)-(60*X(2)));
c2 = 1-(X(1)*(1/3600)*(X(1)-X(2)));
c3 = -X(1);
c4 = -X(2);
c=[c1;c2;c3;c4];
ceq=0;
return
end

function  F = exampleF(X)
F=X(1)^2+320*X(1)*X(2);
return
end
```

4 References

[1] Kirk D.E., *Optimal Control Theory*, Dover Publications, 1998.

[2] Venkataraman P., *Applied Optimization with MATLAB Programming*, John Wiley & Sons ,2022.

[3] Xue D., *Solving Optimization Problems with MATLAB*, De Gruyter, 2020.

[4] Başpınar B., *Istanbul Technical University UCK439E - Introduction to Optimal Control Lecture Notes*, 2022.