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Coulomb effects in the Klein-Gordon equation for pions

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We find a simple analytic expression which approximates the relativistic Coulomb scattering of a spinless particle from the Coulomb field of a point source. We discuss the application of our result for various aspects of pion-nucleus scattering. The results are easily extended for spinless heavy ions.

[NUCLEAR REACTIONS Coulomb scattering, Klein-Gordon equation, found analytic expression for $\sigma(\theta)$, application to π scattering.]

Treating the Coulomb interaction with sufficient care for pion-nucleus scattering is difficult for large Z nuclei. Some of the problems have been discussed in Refs. 1 and 2 in connection with the analysis of Coulomb-nuclear interference and transmission experiments in the approximation of omitting the relativistic $V_C{}^2$ potential [see below, Eq. (1)]. In this paper we point out another set of difficulties associated with the $V_C{}^2$ interaction. The purpose of this work is (1) to find a simple analytic form for the relativistic Coulomb amplitudes including the $V_C{}^2$ term and (2) to suggest an efficient method for calculating the nuclear phase shift in the presence of the $V_C{}^2$ interaction.

Although the answers to the questions which we address are in principle known, most recent calculations have not taken account of the $V_C{}^2$ term. For example, several of the most recent standard optical model computer programs[3,4] do not give sufficient attention to the $V_C{}^2$ term. In Ref. 4 the $V_C{}^2$ term is completely absent, and in Ref. 3 the effect is only included in the lowest partial waves. Our work is motivated by the fact that when the $V_C{}^2$ term is important, it is necessary to evaluate the partial wave amplitudes for states of arbitrarily high orbital angular momentum; to do this correctly requires a special treatment, such as the one we propose. For many applications in pion physics the corrections which we are discussing are not important. However, for scattering from heavy nuclei, especially at low energy and for analyzing small angle elastic scattering data to extract the real and imaginary parts of the forward scattering amplitudes, the $V_C{}^2$ term is important. In particular, the analysis procedure of Ref. 1 and Ref. 2 must be modified to take into account correctly the $V_C{}^2$ potential, and to remedy this, point (1) above is especially relevant.

The one-body Klein-Gordon equation for pion-nucleus scattering is taken conventionally as

$$\left[-(\hbar c)^2\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{c^2\hat{L}^2}{r^2}+(m_0c^2)^2\right]\psi(\vec{r}) = [E^2-2EV_C+V_C{}^2-2EV_N]\psi(\vec{r}), \quad (1)$$

where V_C and V_N are the Coulomb and nuclear potentials, respectively, and where the term quadratic in the Coulomb potential arises from the "minimal substitution" prescription[5]

$$E^2 \to (E-V_C)^2 \quad (2)$$

in the Klein-Gordon equation for a free particle.

Pilkuhn[6] has discussed the solution of Eq. (1) for point charged particles and $V_N=0$. In the following we use his notation rather than that used in Refs. 1 and 2. The scattering amplitude $f_{\gamma C}$ is given by

$$f_{\gamma C}(\theta)=\frac{1}{2ik}\sum_L(2L+1)P_L(\cos\theta)(e^{2i\sigma_\gamma}-1), \quad (3a)$$

where

$$e^{2i\sigma_\gamma}=e^{-i\pi(\gamma-L-1/2)}\frac{\Gamma(\gamma+\frac{1}{2}+i\eta)}{\Gamma(\gamma+\frac{1}{2}-i\eta)} \quad (3b)$$

and

$$\gamma^2=(L+\tfrac{1}{2})^2-Z^2\alpha^2, \quad \eta=Z\alpha/\beta \quad (3c)$$

with Z the product of the nuclear and pionic charge, α the fine-structure constant, and β the velocity of the pion. In the limit of small $Z\alpha$ we retrieve the familiar expression for the nonrelativistic Coulomb interaction

$$e^{2i\sigma_L}=\frac{\Gamma(L+1+i\eta)}{\Gamma(L+1-i\eta)}. \quad (4)$$

In this limit the partial wave expansion converges slowly due to the long range $1/r$ behavior of V_C, but an analytic result for the sum in known.[7] We denote the Coulomb scattering amplitude in the absence of $V_C{}^2$ terms by $f_C(\theta)$. For nonzero scattering angles, the following expression is well known:

$$f_C(\theta)=\frac{1}{2ik}\sum_L (2L+1)P_L(\cos\theta)(e^{2i\sigma_L}-1)$$
$$=-\frac{\eta}{2k\sin^2(\frac{1}{2}\theta)}\exp[-2i(\eta\ln\sin\tfrac{1}{2}\theta-\sigma_0)]\,. \quad (5)$$

The long range $1/r^2$ behavior of V_C^2 introduces additional convergence difficulties into the partial wave expansion. To evaluate the corrections due to V_C^2 requires that one sum partial waves to infinite order, and a simple exact result for the sum is not known. However, Pilkuhn has found the leading correction and writes

$$\delta f_{\gamma C}(\theta)\equiv f_{\gamma C}(\theta)-f_C(\theta)$$
$$\simeq -f_C(\theta)\frac{\pi(Z\alpha)^2}{2\eta}e^{2i(\sigma_{-1/2}-\sigma_0)}\sin\tfrac{1}{2}\theta \quad (6)$$

where $f_C(\theta)$ is given in Eq. (5) and $\sigma_{-1/2}$ is evaluated using Eq. (4). Note that the correction term has one fewer power of $\sin\frac{1}{2}\theta$ than the leading term, but is still singular as $1/\theta$ for $\theta\to 0$. There is also a singular phase at small angles, as can be seen in Eq. (5). To get $f_{\gamma C}$ more accurately, the only method of which we are aware is that due to Yennie, Ravenhall, and Wilson.[8] We have adapted this to spinless particles, but our experience has been that numerical inaccuracies are encountered for small angles, where the best accuracy is needed for data analysis[1,2] of certain classes of experiments.

The asymptotic solution to Eq. (1) may be expressed in terms of the phase shifts Δ in the various partial wave channels. The complete scattering amplitude is then

$$F(\theta)=\frac{1}{2ik}\sum_L (2L+1)P_L(\cos\theta)(e^{2i\Delta_L}-1)\,. \quad (7)$$

In finding the numerical solution to Eq. (1) one would like to be able to integrate out to a radius R beyond which $V_N=0$ and then match to a Couomb wave function. For large $Z\alpha$ this can present some difficulties, which we discuss below. For now we assume that Δ_L is known. The convergence difficulties mentioned above for purely Coulombic interactions are also present in Eq. (7), and we now want to present a convenient and accurate method of finding $F(\theta)$.

Let us first consider the following identity, valid for any set of phase shifts $\bar\sigma_L$:

$$F(\theta)=\bar f_{\gamma C}(\theta)+\frac{1}{2ik}\sum_L (2L+1)e^{2i\bar\sigma_L}(e^{2i(\Delta_L-\bar\sigma_L)}-1)\times P_L(\cos\theta)\,, \quad (8a)$$

where

$$\bar f_{\gamma C}(\theta)=\frac{1}{2ik}\sum_L (2L+1)P_L(\cos\theta)(e^{2i\bar\sigma_L}-1)\,. \quad (8b)$$

It should be clear from Eq. (8a) that Eq. (7) may be summed conveniently provided $\bar\sigma_L$ satisfies the following conditions:

(a) $\Delta_L\to\bar\sigma_L$ rapidly for $L\gtrsim kR$ (implying $\bar\sigma_L\to\sigma_{\gamma L}$ rapidly, also).

(b) A convenient analytic expression exists for $\bar f_{\gamma C}$ [Eq. (8b)].

Our main result is an extension of the result in Eq. (6) found by Pilkuhn.[6] We obtain our result by writing the following expansion, which is justified in the appendices:

$$e^{2i\sigma_\gamma}=\frac{\Gamma(L+1+i\eta)}{\Gamma(L+1-i\eta)}+\sum_{m=1}^{\infty}g_m\frac{\Gamma(L+1+i\eta-m/2)}{\Gamma(L+1-i\eta+m/2)}\,. \quad (9)$$

Pilkuhn showed that

$$g_1=i\pi(Z\alpha)^2/2\,. \quad (10)$$

We have derived expressions for the next two coefficients

$$g_2=-\tfrac{1}{8}\pi^2(Z\alpha)^4-i\eta(Z\alpha)^2\,, \quad (11a)$$

$$g_3=\frac{\pi}{2}(Z\alpha)^4\left\{\eta+\frac{i}{4}\left[1-\frac{(\pi Z\alpha)^2}{6}\right]\right\}-\frac{i\eta}{2}g_1(i\eta-1/2) \quad (11b)$$

(see Appendix A). We shall argue below that with g_1, g_2, and g_3 we are able to calculate $\bar f_{\gamma C}$ with sufficient accuracy to satisfy condition (a) above for pion physics.

Condition (b) is satisfied if $e^{2i\sigma_\gamma}$ is given as the expansion in Eq. (9), and the result is

$$f_{\gamma C}=f_C(\theta)+f_C(\theta)\sum_{m=1}^{\infty}A_m(\sin\tfrac{1}{2}\theta)^m\,, \quad (12a)$$

where

$$A_m=\frac{-g_m e^{2i(\sigma_{-m/2}-\sigma_0)}}{i\eta(-i\eta+m/2-1)(-i\eta+m/2-2)\dots(i\eta-m/2+1)}\,. \quad (12b)$$

We derive this result in Appendix B.

If we retain only three terms in Eq. (9), then we have the following definition of $\bar{\sigma}_L$:

$$\bar{\sigma}_L \equiv \frac{1}{2i}\ln\left(e^{2i\sigma_L} + \sum_{m=1}^{3} g_m \frac{\Gamma(L+1+i\eta-m/2)}{\Gamma(L+1-i\eta+m/2)}\right), \quad (13a)$$

giving

$$\bar{f}_{\gamma C}(\theta) = f_C(\theta)[1 + A_1 \sin\tfrac{1}{2}\theta + A_2(\sin\tfrac{1}{2}\theta)^2 + A_3(\sin\tfrac{1}{2}\theta)^3], \quad (13b)$$

and

$$A_1 = \frac{-g_1 e^{2i(\sigma_{-1/2}-\sigma_0)}}{i\eta},$$
$$A_2 = \frac{-g_2 e^{2i(\sigma_{-1}-\sigma_0)}}{\eta^2}, \quad (13c)$$
$$A_3 = \frac{g_3 e^{2i(\sigma_{-3/2}-\sigma_0)}}{i\eta(\eta^2+\frac{1}{4})}$$

From Eqs. (5) and (13b) one sees that it is sufficient to truncate the series at $m=2$ in order to obtain a sum $f_{\gamma C}$ which contains all of the terms which do not go to zero as $\theta \to 0$. In Table I we compare this definition of $\bar{\sigma}_L$ to σ_γ. This shows the results for a calculation on Pb ($Z=82$) for $T = 225$ MeV. We find that $\bar{\sigma}_L = \sigma_\gamma$ to one part in 10^6 for $L \geq 12$ and better than 1% for $L=1$. Thus conditions (a) and (b) are satisfied, and we have accomplished our goal of writing a convenient expression, Eqs. (8) and (13), for $F(\theta)$ defined in Eq. (7).

A careful look at Table I shows that for $L=0$ σ_γ has a nonzero imaginary part. The reason for this can be seen in Eq. (3c); i.e., for $Z\alpha > \frac{1}{2}$, γ is purely imaginary. Among other things, this means that the expansion in Eq. (9) will not converge for $L=0$. In general, there will be some critical angular momentum L_C for which

$$(L+\tfrac{1}{2})^2 < (Z\alpha)^2 \quad \text{for all} \quad L \leq L_C. \quad (14)$$

TABLE I. Comparison between σ_γ [Eq. (3b)] and $\bar{\sigma}_L$ [Eq. (13a)] for 225-MeV pions on Pb.

L	$\mathrm{Re}\sigma_\gamma$	$\mathrm{Im}\sigma_\gamma$	$\mathrm{Re}\bar{\sigma}_L$	$\mathrm{Im}\bar{\sigma}_L$
0	0.003 27	−0.003 11	0.000 86	−0.003 17
1	0.435 7	0	0.438 1	-7×10^{-3}
2	0.700 2	0	0.701 5	-4×10^{-4}
3	0.888 7	0	0.889 1	1×10^{-5}
4	1.034 8	0	1.034 9	2×10^{-5}
5	1.154 0	0	1.154 1	2×10^{-5}
6	1.254 7	0	1.254 8	1×10^{-5}
7	1.341 9	0	1.341 9	7×10^{-6}
8	1.418 7	0	1.418 7	4×10^{-6}

For example, in the heavy ion collision, Pb + Pb,

$$L_C = (82)^2\alpha - \tfrac{1}{2} \approx 49. \quad (15)$$

For pions $L=0$ is the only anomalous partial wave which can exist for actual nuclei because Eq. (14) requires $Z=205$ in order for $L_C=1$, and larger Z for $L_C>1$. Becuase Eq. (8) is an *identity*, valid for any set $\bar{\sigma}_L$, the fact that Eq. (9) may not converge for small L in no way affects the utility of the definition in Eq. (13a).

One corollary of our work is that the exact Coulomb scattering amplitude is given by the following expression:

$$f_{\gamma C}(\theta) = \bar{f}_{\gamma C}(\theta) + \frac{1}{2ik}\sum_L (2L+1)e^{2i\sigma_L}(e^{2i(\sigma_\gamma-\bar{\sigma}_L)} - 1) \times P_l(\cos\theta), \quad (16)$$

where $\bar{f}_C$ and $\bar{\sigma}_L$ are again given by Eq. (13). Because $\sigma_\gamma \approx \bar{\sigma}_L$ for $L \gtrsim L_C$, the summation extends over just a few values of L in practice.

In order to calculate $F(\theta)$ according to Eq. (8), one clearly needs to know the phase shifts Δ_L. We have already remarked that for large $Z\alpha$ one may encounter some numerical difficulties. For practical reasons, one wants to be able to integrate Eq. (1) only out to the distance R beyond which the nuclear potential is zero, i.e., evaluate the phase shift Δ_L relative to the relativistic Coulomb wave function. Otherwise, to evaluate Δ_L relative to the nonrelativistic Coulomb wave function, one would have to integrate out to a much larger radius where V_C^2 is negligible. We point out that Δ_L may be obtained accurately and simply by matching to linear combinations of relativistic Coulomb wave functions, i.e., solutions of Eq. (1) with V_C the point Coulomb potential and V_N absent. Solutions of this equation are the same as nonrelativistic Coulomb functions evaluated for a nonintegral angular momentum L',

$$L' = \gamma - \tfrac{1}{2} = [(L+\tfrac{1}{2})^2 - Z^2\alpha^2]^{1/2} - \tfrac{1}{2}. \quad (17)$$

The relativistic Coulomb functions are obtained in this fashion using the technique of Barrett[9] with some minor modifications. Since the phase shift δ_L is then the phase shift relative to the point relativistic Coulomb wave function, to obtain Δ_L one must add σ_γ, i.e., $\Delta_L = \delta_L + \sigma_\gamma$. Note that δ_L is not equal to the quantity $\Delta_L - \bar{\sigma}_L$, which is needed for the evaluation of Eq. (8a).

A technical point arises in this context also when $\gamma^2 < 1$ in Eq. (3c). In this case the phase shift has an imaginary part; the loss of flux corresponds to the trapping of a classical particle by an attractive $1/r^2$ potential. Physically, this is impossible for pions because of the nuclear charge form factor cuts off the $1/r^2$ short range behavior; in order to obtain a sensible result ($\Delta_0 =$ real) one must match the solution of Eq. (1) to the proper relativistic Coulomb wave functions; i.e., those with complex phase shifts corresponding to a *point* relativistic Coulomb interaction. We have found that the method in Ref. 9 is especially well suited to calculating with nonintegral complex values of L. The matching to the relativistic Coulomb wave function of complex L will produce a complex $\delta_L = \Delta_L - \sigma_\gamma$ because the Coulomb wave functions are now complex. (Some care must be taken to do all the boundary matching using complex arithmetic in the computer programs.) However, $\Delta_L = \delta_L + \sigma_\gamma$ is guaranteed to be real.

Two other methods for evaluating the amplitude F are often used. One is to set V_C^2 to zero in Eq. (1) (i.e., use just the V_C term). The other is to lump the V_C^2 contribution into the "nuclear part" of F and to treat it as a short range interaction. These methods are used, respectively, in the popular programs PIPIT[4] and PIRK.[3] The value of f_N is then found by solving the scattering equation, but matching to Coulomb wave functions with integral L and by summing a limited number of partial waves. The arguments we have given show that these procedures are incorrect for large $Z\alpha$, and we want to compare these approximate methods to the exact method proposed above.

The figure displays the errors that can be made in calculating the Coulomb scattering cross section for scattering of π^+ from Pb at $T_\pi = 100$ MeV using the two approximate calculational methods. The form factor is taken to be a Woods-Saxon of half radius 6.5 fm and diffuseness 0.53 fm. The dashed curve is nonrelativistic Coulomb scattering. The dot-dashed curve includes the V_C^2 term as a short range interaction using a matching radius of 10 fm and 20 angular momentum values. These are to be compared to the exact solution which is plotted as the solid curve. Over most of the angular distribution, the deviations from the exact result are larger than modern experimental uncertainties.

Note that $\sigma/\sigma_R < 1$ almost everywhere in the figure. This means that Coulomb scattering and nuclear scattering are comparable in size over the *entire* angular region plotted. Thus, in this example, inaccuracies in the Coulomb amplitude are significant at both large and small angles. In the more customary cases, where Coulomb scattering dominates at small angles, the V_C^2 correction will be more important compared to the nuclear amplitude than indicated by our example. A small percentage deviation at small angles is, of course, a large discrepancy because the Rutherford cross section σ_R is very large. It should be clear that summing a finite number of

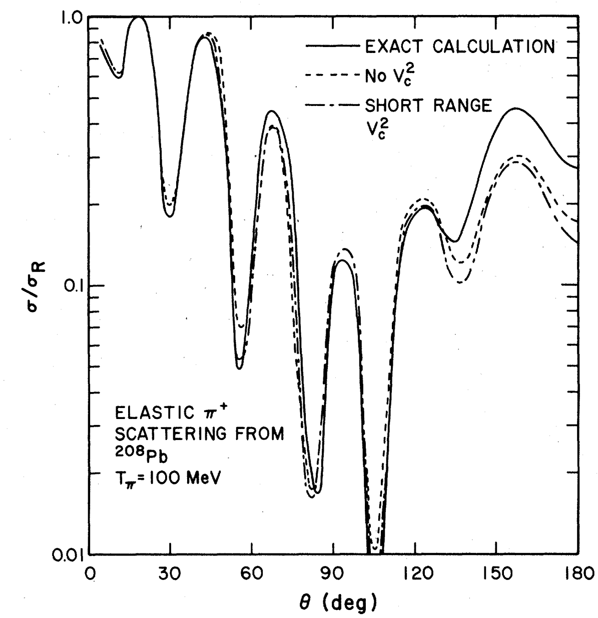

FIG. 1. The angular distributions divided by the Rutherford cross section are plotted for 100 MeV pions on ^{208}Pb. The solid curve is the exact calculation described in the text; the dot-dashed curve uses a short-range V_C^2 approximation; and the dashed curve contains no V_C^2 term at all. Whenever they are not plotted, the dot-dashed curve overlaps the solid curve, and the dashed curve overlaps the dot-dashed curve.

partial waves cannot build up the singular correction at small angles; hence the "short ranged V_C^2" approximation which is commonly employed (e.g., Ref. 3) cannot reproduce the correct Coulomb scattering. We see in the figure that even at large angles only a small part of the V_C^2 correction to $f_C(\theta)$ is included by summing twenty partial waves.

In this paper we have shown how to easily sum the relativistic Coulomb scattering amplitude for point bosons. Additionally, the incorporation of the V_C^2 interaction into optical model programs has been shown to be both important and straightforward.

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APPENDIX A

In this appendix we want to describe how we calculated the coefficients g_m in Eq. (9). We begin by expanding $e^{2i\sigma_\gamma}$ as a series in $(L+\frac{1}{2})^{-m}$ using Eq. (3b). We find

$$e^{2i\sigma_\gamma} = e^{2i\sigma_L} \sum_m \frac{B_m}{(L+\frac{1}{2})^m}, \tag{A1}$$

where the first four coefficients are

$$B_0=1, \quad B_1=\tfrac{1}{2} i\pi Z^2\alpha^2,$$
$$B_2=-\tfrac{1}{8}\pi^2 Z^4\alpha^4 - i\eta Z^2\alpha^2, \tag{A2}$$
$$B_3=\frac{\pi}{2}(Z\alpha)^4\left[\eta+\frac{i}{4}\left(1-\frac{\pi^2Z^2\alpha^2}{6}\right)\right].$$

We many now combine the factors of $(L+\frac{1}{2})^{-m}$ with the Γ functions in the expression for $e^{2i\sigma_L}$ given in Eq. (4). This gives

$$e^{2i\sigma_\gamma}=\sum_{m=0} C_m \frac{\Gamma(L+1+i\eta-m)}{\Gamma(L+1-i\eta)}. \tag{A3}$$

The first four coefficients are

$$C_0=1, \quad C_1=B_1, \quad C_2=B_2+(i\eta-\tfrac{1}{2})B_1,$$
$$C_3=B_3+2(i\eta-1)B_2+(i\eta-\tfrac{1}{2})(i\eta-\tfrac{3}{2})B_1. \tag{A4}$$

We next express $\Gamma(L+1+i\eta-m)/\Gamma(L+1-i\eta)$ as an expansion in terms of $\Gamma(L+1+i\eta-n/2)/\Gamma(L+1-i\eta+n/2)$,

$$\frac{\Gamma(L+1+i\eta-m)}{\Gamma(L+1-i\eta)}=\sum_{n=0} E_n^m \frac{\Gamma[L+1+i\eta-(m+n)/2]}{\Gamma[L+1-i\eta+(m+n)/2]}$$
$$=\sum_{n=m} E_{n-m}^m \frac{\Gamma(L+1+i\eta-n/2)}{\Gamma(L+1-i\eta+n/2)}. \tag{A5}$$

The first three coefficients in this expansion are

$$E_0^m=1, \quad E_1^m=\tfrac{1}{2}m(m-2i\eta),$$
$$E_2^m=\tfrac{1}{8}m^2(m-2i\eta)(1+m-2i\eta). \tag{A6}$$

Combining Eqs. (A3) and (A5) gives, after some simple algebra,

$$e^{2i\sigma_\gamma}=\frac{\Gamma(L+1+i\eta)}{\Gamma(L+1-i\eta)}+\sum_{m=1} g_m \frac{\Gamma(L+1+i\eta-m/2)}{\Gamma(L+1-i\eta+m/2)}, \tag{A7}$$

where

$$g_1=B_1, \quad g_2=B_2, \quad g_3=B_3-B_1(i\eta-\tfrac{1}{2})\tfrac{1}{2}i\eta. \tag{A8}$$

This gives the results in Eqs. (11a) and (11b).

This procedure used to obtain the coefficients is straightforward and may be used to obtain g_m for $m>3$. However, the algebra involved in obtaining the B, C, and E coefficients is tedious, and we next give some details of these steps.

B Coefficients

Begin by expanding $e^{2i\sigma_\gamma}$ as a power series about the point $\gamma=L+\frac{1}{2}$,

$$e^{2i\sigma_\gamma}=\sum_{m=0}\frac{\xi^m}{m!}\frac{d^m}{d\xi^m}e^{2i\sigma_\gamma}\Big|_{\xi=0},$$

where $\xi=\gamma-(L+\frac{1}{2})$. (AB1)

The quantity γ [Eq. (3c)] may be expanded as a power series in $(L+\frac{1}{2})^{-1}$ for sufficiently large L. We find

$$\xi=-\frac{1}{2}\frac{Z^2\alpha^2}{L+\frac{1}{2}}-\frac{1}{8}\frac{Z^4\alpha^4}{(L+\frac{1}{2})^3}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^5\right), \tag{AB2}$$

$$\xi^2=\frac{1}{4}\frac{Z^4\alpha^4}{(L+\frac{1}{2})^2}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^4\right), \tag{AB3}$$

$$\xi^3=-\frac{1}{8}\frac{Z^6\alpha^6}{(L+\frac{1}{2})^3}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^5\right). \tag{AB4}$$

To find the B coefficients to $O(1/(L+\frac{1}{2})^3)$ we need

$$\frac{d}{d\xi}e^{2i\sigma_\gamma}\Big|_{\xi=0}$$

to $O(1/(L+\frac{1}{2})^2)$ for the $m=1$ term in Eq. (AB1), to $O(1/(L+\frac{1}{2})^1)$ for the $m=2$ term and to $O(1/(L+\frac{1}{2})^0)$ for the $m=3$ term. Using the definition in Eq. (3b) we find

$$\frac{d}{d\xi}e^{2i\sigma_\gamma}\Big|_{\xi=0}=e^{2i\sigma_L}\left[-i\pi+\frac{d}{dL}\frac{\Gamma(L+1+i\eta)}{\Gamma(L+1-i\eta)}\right]. \tag{AB5}$$

But

$$\frac{d}{dL}\ln\frac{\Gamma(L+1+i\eta)}{\Gamma(L+1-i\eta)}=\sum_{m=1}\left(\frac{\Delta^m-\Delta^{*m}}{m!}\right)\frac{d^{m+1}}{dL^{m+1}}\ln\Gamma(L+\tfrac{1}{2}),$$
$$\Delta=\tfrac{1}{2}+i\eta. \tag{AB6}$$

For large L the derivatives of $\ln\Gamma(L+\frac{1}{2})$ may be determined from the asymptotic expansion for the Γ function[10]

$$\ln\Gamma(z)=(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln 2\pi+\frac{1}{12z}+O((1/z^3)). \tag{AB7}$$

We find from Eqs. (AB7) and (AB6)

$$\frac{d}{d\xi}e^{2i\sigma_\gamma}\Big|_{\xi=0}=e^{2i\sigma_L}\left[-i\pi+\frac{2i\eta}{L+\frac{1}{2}}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^3\right)\right]. \tag{AB8}$$

Higher order derivatives are similarly worked out, and we find

$$\frac{d^2}{d\xi^2}e^{2i\sigma_\gamma}\Big|_{\xi=0}=e^{2i\sigma_L}\left[-\pi^2+\frac{4\pi\eta}{L+\frac{1}{2}}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^2\right)\right] \tag{AB9}$$

and

$$\frac{d^3}{d\xi^3}e^{2i\sigma_\gamma}\Big|_{\xi=0}=e^{2i\sigma_L}[i\pi^3]. \tag{AB10}$$

Inserting Eqs. (AB8)–(AB10) and Eqs. (AB2)–(AB4) in Eq. (AB1) and rearranging gives the results in Eqs. (A1) and (A2).

C Coefficients

Consider first

$$\frac{\Gamma(L+1+i\eta)}{(L+\frac{1}{2})\Gamma(L+1-i\eta)}. \tag{AC1}$$

Using the recursion relation

$$\Gamma(1+z)=z\Gamma(z), \tag{AC2}$$

we have

$$\frac{\Gamma(L+1+i\eta)}{(L+\frac{1}{2})\Gamma(L+1-i\eta)}=\frac{L+i\eta}{L+\frac{1}{2}}\frac{\Gamma(L+i\eta)}{\Gamma(L+1-i\eta)}=\frac{\Gamma(L+i\eta)}{\Gamma(L+1-i\eta)}+\frac{i\eta-\frac{1}{2}}{L+\frac{1}{2}}\times\frac{\Gamma(L+i\eta)}{\Gamma(L+1-i\eta)}. \tag{AC3}$$

Using Eq. (AC2) once more this becomes

$$\frac{\Gamma(L+i\eta)}{\Gamma(L+1-i\eta)}+(i\eta-\tfrac{1}{2})\frac{L-1+i\eta}{L+\frac{1}{2}}\frac{\Gamma(L-1+i\eta)}{\Gamma(L+1-i\eta)}. \tag{AC4}$$

Writing

$$\frac{L-1+i\eta}{L+\frac{1}{2}}=1+\frac{i\eta-\frac{3}{2}}{L+\frac{1}{2}} \tag{AC5}$$

and using (AC3) once more on the last term in Eq. (AC4) gives

$$\frac{\Gamma(L+1+i\eta)}{(L+\frac{1}{2})\Gamma(L+1-i\eta)}=\frac{\Gamma(L+i\eta)}{\Gamma(L+1-i\eta)}+(i\eta-\tfrac{1}{2})\frac{\Gamma(L-1+i\eta)}{\Gamma(L+1-i\eta)}+(i\eta-\tfrac{1}{2})(i\eta-\tfrac{3}{2})\frac{\Gamma(L-2+i\eta)}{\Gamma(L+1-i\eta)}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^4\right). \tag{AC6}$$

Similarly, we find

$$\frac{\Gamma(L+1+i\eta)}{(L+\frac{1}{2})^2\Gamma(L+1+i\eta)}=\frac{\Gamma(L-1+i\eta)}{\Gamma(L+1-i\eta)}+2(i\eta-1)\frac{\Gamma(L-2+i\eta)}{\Gamma(L+1-i\eta)}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^4\right), \tag{AC7}$$

$$\frac{\Gamma(L+1+i\eta)}{(L+\frac{1}{2})^3\Gamma(L+1-i\eta)}=\frac{\Gamma(L-2+i\eta)}{\Gamma(L+1-i\eta)}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^4\right). \tag{AC8}$$

Combining terms proportional to $\Gamma(L+1+i\eta-m)/\Gamma(L+1-i\eta)$ for a given m gives the results in Eq. (A4).

E Coefficients

Begin by assuming that the coefficients F_n^M are known in the expansion

$$\frac{\Gamma(L+1-M+i\eta)}{\Gamma(L+1-i\eta)}=\frac{\Gamma(L+1-M/2+i\eta)}{\Gamma(L+1+M/2-i\eta)}\left(1+\frac{F_1^M}{L+\frac{1}{2}}+\frac{F_2^M}{(L+\frac{1}{2})^2}+\cdots\right)+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^{M+3}\right),$$
$$F_1^M=\frac{M}{2}(M-2i\eta),\quad F_2^M=\frac{M^2}{8}(M-2i\eta)(1+M-2i\eta). \tag{AE1}$$

This will be proved subsequently. The point is that the first term in Eq. (AE1) is of the proper form, i.e., the form of a term on the right-hand side of Eq. (A5). In order to cast subsequent terms into the same form it is necessary to do more algebra. The idea is to apply Eq. (AE1) iteratively to cast successively more terms into the proper form. In order to do this use relation (AC2) to write

$$\frac{\Gamma(L+1-M/2+i\eta)}{(L+\frac{1}{2})\Gamma(L+1+M/2-i\eta)}=\frac{\Gamma(L+1-1-M/2+i\eta)}{\Gamma(L+1+M/2-i\eta)}+(i\eta-\tfrac{1}{2}-M/2)\frac{\Gamma(L+1-M/2-2+i\eta)}{\Gamma(L+1+M/2-i\eta)}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^{M+3}\right) \tag{AE2}$$

and

$$\frac{\Gamma(L+1-M/2+i\eta)}{(L+\frac{1}{2})^2\Gamma(L+1+M/2-i\eta)}=\frac{\Gamma(L+1-M/2-2+i\eta)}{\Gamma(L+1+M/2-i\eta)}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^{M+3}\right), \tag{AE3}$$

so that Eq. (AE1) becomes

$$\frac{\Gamma(L+1-M+i\eta)}{\Gamma(L+1-i\eta)}=\frac{\Gamma(L+1-M/2+i\eta)}{\Gamma(L+1+M/2-i\eta)}+F_1^M\frac{\Gamma(L+1-1-M/2+i\eta)}{\Gamma(L+1+M/2-i\eta)}+[F_2^M+(i\eta-\tfrac{1}{2}-M/2)F_1^M]\frac{\Gamma(L+1-M/2-2+i\eta)}{\Gamma(L+1+M/2-i\eta)}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^{M+3}\right). \tag{AE4}$$

But now define

$$i\eta'=-M/2+i\eta, \tag{AE5}$$

and apply Eq. (AE1) to the second and third terms on the right-hand side of Eq. (AE4)

$$\frac{\Gamma(L+1-1+i\eta')}{\Gamma(L+1-i\eta')}=\frac{\Gamma(L+1-\frac{1}{2}+i\eta')}{\Gamma(L+1+\frac{1}{2}-i\eta')}\left(1+\frac{F_1^1(i\eta')}{L+\frac{1}{2}}\right)+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^{M+3}\right), \tag{AE6}$$

$$\frac{\Gamma(L+1-2+i\eta')}{\Gamma(L+1-i\eta')}=\frac{\Gamma(L+1-1+i\eta')}{\Gamma(L+1+1-i\eta')}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^{M+3}\right), \tag{AE7}$$

to get

$$\frac{\Gamma(L+1-M+i\eta)}{\Gamma(L+1-i\eta)}=\frac{\Gamma(L+1-M/2+i\eta)}{\Gamma(L+1+M/2-i\eta)}+F_1^M\frac{\Gamma(L+1-\frac{1}{2}+i\eta')}{\Gamma(L+1+\frac{1}{2}-i\eta')}+\frac{F_1^MF_1^1(i\eta')}{L+\frac{1}{2}}\frac{\Gamma(L+1-\frac{1}{2}+i\eta')}{\Gamma(L+1+\frac{1}{2}-i\eta')}$$
$$+[F_2^M+(i\eta-\tfrac{1}{2}-M/2)F_1^M]\frac{\Gamma(L+1-1+i\eta')}{\Gamma(L+1+1-i\eta')}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^{M+3}\right). \tag{AE8}$$

Applying relation (AC2) and Eq. (AE1) to the third term on the right-hand side of Eq. (AE8) gives

$$\frac{1}{L+\frac{1}{2}}\frac{\Gamma(L+1-\frac{1}{2}+i\eta')}{\Gamma(L+1+\frac{1}{2}-i\eta')}=\frac{\Gamma(L+1-\frac{1}{2}+(i\eta'-\frac{1}{2}))}{\Gamma(L+1+\frac{1}{2}-(i\eta'-\frac{1}{2}))}+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^{M+3}\right). \tag{AE9}$$

Substituting Eq. (AE9) into Eq. (AE8), using definition (AE5), and combining terms, we get

$$\frac{\Gamma(L+1-M+i\eta)}{\Gamma(L+1-i\eta)}=\frac{\Gamma(L+1+i\eta-M/2)}{\Gamma(L+1-i\eta+M/2)}+E_1^M\frac{\Gamma(L+1+i\eta-M/2-\frac{1}{2})}{\Gamma(L+1-i\eta+M/2+\frac{1}{2})}+E_2^M\frac{\Gamma(L+1+i\eta-M/2-1)}{\Gamma(L+1-i\eta+M/2+1)}$$
$$+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^3\right), \tag{AE10}$$

where

$$E_1^M=F_1^M=\frac{M}{2}(M-2i\eta), \tag{AE11}$$

$$E_2^M=F_1^MF_1^1(i\eta-M/2)+F_2^M+(i\eta-\tfrac{1}{2}-M/2)F_1^M=\frac{M^2}{8}(M-2i\eta)(1+M-2i\eta), \tag{AE12}$$

where we have used the values of F_1^M and F_2^M given in Eq. (AE1). The algorithm explained here may be extended to obtain E_n^M for $n>2$.

F Coefficients

We want to derive expressions for the F coefficients in Eq. (AE1). Begin by writing

$$\frac{\Gamma(L+1-M+i\eta)}{\Gamma(L+1-i\eta)}=\frac{\Gamma(L+1-M/2+i\eta)}{\Gamma(L+1+M/2-i\eta)}\frac{\Gamma(z_1+\phi)}{\Gamma(z_1)}\frac{\Gamma(z_2)}{\Gamma(z_2+\phi)}, \tag{AF1}$$

where

$$z_1=L+1-(M/2-i\eta),\quad z_2=L+1+(M/2-i\eta),\quad \phi=-M/2. \tag{AF2}$$

But, as we shall show below,

$$\frac{\Gamma(z+\phi)}{\Gamma(z)}=z^\phi\exp\left[\sum_{n=1}\frac{D_n(\phi)}{z^n}\right], \tag{AF3}$$

where the first three D coefficients are

$$D_1(\phi)=\frac{\phi}{2}(\phi-1),\quad D_2(\phi)=-\frac{\phi}{12}(\phi-1)(2\phi-1),\quad D_3(\phi)=\frac{\phi^2}{24}(\phi^2-4\phi+2). \tag{AF4}$$

Therefore,

$$\frac{\Gamma(L+1-M+i\eta)}{\Gamma(L+1-i\eta)}=\frac{\Gamma(L+1-M/2+i\eta)}{\Gamma(L+1+M/2-i\eta)}\left(\frac{z_2}{z_1}\right)^{M/2}\exp\left\{\sum_{n=1}D_n\left(-\frac{M}{2}\right)\left(\frac{1}{z_1{}^n}-\frac{1}{z_2{}^n}\right)\right\}. \tag{AF5}$$

But we want to expand only through order $(1/(L+\frac{1}{2})^{M+3})$, and therefore we write

$$\frac{\Gamma(L+1-M+i\eta)}{\Gamma(L+1-i\eta)}=\frac{(L+1-M/2+i\eta)}{(L+1+M/2-i\eta)}\left(\frac{1+[(M+1)/2-i\eta]/(L+\frac{1}{2})}{1+[(1-M)/2+i\eta]/(L+\frac{1}{2})}\right)^{M/2}\left(1+D_1(-M/2)\frac{(M-2i\eta)}{(L+\frac{1}{2})^2}\right)$$
$$+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^{M+3}\right) \tag{AF6}$$
$$=\frac{\Gamma(L+1-M/2+i\eta)}{\Gamma(L+1+M/2-i\eta)}\left(1+\frac{M-2i\eta}{L+\frac{1}{2}}-\frac{(M-2i\eta)[(1-M)/2+i\eta]}{(L+\frac{1}{2})^2}\right)^{M/2}\left(1+D_1\left(-\frac{M}{2}\right)\frac{(M-2i\eta)}{(L+\frac{1}{2})^2}\right)$$
$$+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^{M+3}\right) \tag{AF7}$$
$$=\frac{\Gamma(L+1-M/2+i\eta)}{\Gamma(L+1+M/2-i\eta)}\left(1+\frac{F_1^M}{L+\frac{1}{2}}+\frac{F_2^M}{(L+\frac{1}{2})^2}\right)+O\left(\left(\frac{1}{L+\frac{1}{2}}\right)^{M+3}\right) \tag{AF8}$$

where

$$F_1^M=\frac{M}{2}(M-2i\eta),\quad F_2^M=\frac{M^2}{8}(M-2i\eta)(1+M-2i\eta). \tag{AF9}$$

D Coefficients

Here we want to show how to obtain the coefficients in the expansion of Eq. (AF3). Consider the expansion of $\ln[\Gamma(z+\phi)/\Gamma(z)]$:

$$\ln\Gamma(z+\phi)=\ln\Gamma(z)+\phi\frac{d}{dz}\ln\Gamma(z)+\frac{1}{2}\phi^2\frac{d^2}{dz^2}\ln\Gamma(z)+\cdots \tag{AD1}$$

or

$$\ln\frac{\Gamma(z+\phi)}{\Gamma(z)}=\phi\frac{d}{dz}\ln\Gamma(z)+\frac{\phi^2}{2}\frac{d^2}{dz^2}\ln\Gamma(z)+\cdots . \tag{AD2}$$

But the derivatives may be determined easily from the expansion in Eq. (AB7), giving

$$\frac{d}{dz}\ln\Gamma(z)=\ln z-\frac{1}{2z}-\frac{1}{12z^2}+O\left(\left(\frac{1}{z}\right)^4\right), \tag{AD3}$$

$$\frac{d^2}{dz^2}\ln\Gamma(z)=\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+O\left(\left(\frac{1}{z}\right)^5\right), \tag{AD4}$$

$$\frac{d^3}{dz^3}\ln\Gamma(z)=-\frac{1}{z^2}-\frac{1}{z^3}+O\left(\left(\frac{1}{z}\right)^4\right), \tag{AD5}$$

$$\frac{d^4}{dz^4}\ln\Gamma(z)=\frac{2}{z^3}+O\left(\left(\frac{1}{z}\right)^4\right). \tag{AD6}$$

Combining Eqs. (AD2)–(AD6) we find

$$\ln\frac{\Gamma(z+\phi)}{\Gamma(z)}=\phi\ln z+\frac{1}{z}\left(-\frac{\phi}{2}+\frac{\phi^2}{2}\right)+\frac{1}{z^2}\left(-\frac{\phi}{12}+\frac{\phi^2}{4}-\frac{\phi^3}{6}\right)+\frac{1}{z^3}\left(\frac{\phi^2}{12}-\frac{\phi^3}{6}+\frac{\phi^4}{24}\right)+O\left(\left(\frac{1}{z}\right)^4\right). \tag{AD7}$$

Exponentiating both sides of Eq. (AD7) gives the form in Eq. (AF3) and the coefficients in Eq. (AF4) may be read directly from Eq. (AD7).

APPENDIX B

Here we want to assume that Eq. (9) is a valid expansion and prove that the amplitude with these phase shifts is the one given in Eq. (12). Using Eqs. (5) and (4) we see

$$\frac{1}{2ik}\sum_L(2L+1)P_L(\cos\theta)\left[\frac{\Gamma(L+1+i\eta)}{\Gamma(L+1-i\eta)}-1\right]$$
$$=\frac{i(i\eta)e^{2i\sigma_0(i\eta)}}{2k\sin^2(\frac{1}{2}\theta)}(\sin\tfrac{1}{2}\theta)^{-2i\eta}. \tag{B1}$$

This expression is valid for real or complex values of η. It is clear from Eq. (9) that each term in the expansion of $e^{2i\sigma_\gamma}$ will give rise to a sum similar to Eq. (B1), i.e.,

$$\frac{1}{2ik}\sum(2L+1)P_L(\cos\theta)\times\left[\frac{\Gamma(L+1+i\eta-m/2)}{\Gamma(L+1-i\eta+m/2)}-1\right]. \qquad \text{(B2)}$$

We may use Eq. (B1) to evaluate this, by replacing $i\eta \rightarrow i\eta - m/2$. Thus

$$\text{Sum(B2)} = \frac{i(i\eta - m/2)}{2k\sin^2(\frac{1}{2}\theta)} e^{2i\sigma_0(i\eta - m/2)} (\sin\tfrac{1}{2}\theta)^{-2(i\eta - m/2)}$$
$$= f_C(\theta)\left(\frac{i\eta - m/2}{i\eta}\right)e^{2i\Delta\sigma_0(m)}(\sin\tfrac{1}{2}\theta)^m, \qquad \text{(B3)}$$

where

$$\Delta\sigma_0(m) \equiv \sigma_0(i\eta - m/2) - \sigma_0(i\eta). \qquad \text{(B4)}$$

Thus, the relativistic Coulomb amplitude is

$$f_{\gamma C}(\theta) = \frac{1}{2ik}\sum_L (2L+1)P_L(\cos\theta)(e^{2i\sigma_\gamma}-1)$$
$$= f_C(\theta) + f_C(\theta)\sum_m\left(\frac{i\eta - m/2}{i\eta}\right)e^{2i\Delta\sigma_0(m)}\times(\sin\tfrac{1}{2}\theta)^m. \qquad \text{(B5)}$$

The term with -1 gives a $\delta(\cos\theta - 1)$ and is not written explicitly. It is convenient to rewrite $e^{2i\Delta\sigma_0(m)}$. Consider the definition Eq. (4),

$$e^{2i\sigma_0(i\eta - m/2)} = \frac{\Gamma(1+i\eta-m/2)}{\Gamma(1-i\eta+m/2)}. \qquad \text{(B6)}$$

But

$$\Gamma(1-i\eta+m/2) = (-i\eta+m/2)(-i\eta+m/2-1)\cdots\times(-i\eta-m/2+1)\times\Gamma(-i\eta-m/2+1), \qquad \text{(B7)}$$

where there are m factors in front of the Γ function on the right-hand side of Eq. (B7). Thus, Eq. (B6) reads

$$e^{2i\sigma_0(i\eta-m/2)} = \frac{e^{2i\sigma_{-m/2}(i\eta)}}{(-i\eta+m/2)(-i\eta+m/2-1)\cdots(-i\eta-m/2+1)}, \qquad \text{(B8)}$$

where

$$\frac{\Gamma(1+i\eta-m/2)}{\Gamma(1-i\eta-m/2)} \equiv e^{2i\sigma_{-n/2}(i\eta)} \qquad \text{(B9)}$$

is a pure phase. Combining Eq. (B9) with Eq. (B5) we get the desired result

$$f_{\gamma C}(\theta) = f_C(\theta) + f_C(\theta)\sum_m A_m(\sin\tfrac{1}{2}\theta)^m \qquad \text{(B10)}$$

with

$$A_m = \frac{-g_m \exp[2i(\sigma_{-m/2}-\sigma_0)]}{i\eta(-i\eta+m/2-1)(-i\eta+m/2-2)\cdots(-i\eta-m/2+1)} \qquad \text{(B11)}$$

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