

# 086761 - Homework 1

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# 1

## 1.1

$$f_x(x) = \frac{1}{\sqrt{|2\pi\Sigma_x|}} \cdot e^{-\frac{1}{2}(x-\mu_x)^T \Sigma_x^{-1}(x-\mu_x)} \quad (1.1)$$

## 1.2

$\mu_y, \Sigma_y$  can be calculated using expectation properties (and do not depend on  $y$  being Gaussian):

$$\mu_y = \mathbb{E}Ax + b = A\mathbb{E}x + b = \boxed{A\mu_x + b} \quad (1.2)$$

$$\Sigma_y = \mathbb{E} \left[ (Ax + b - \mu_y)(Ax + b - \mu_y)^T \right] = \mathbb{E} \left[ A(x - \mu_x)(x - \mu_x)^T A^T \right] = \quad (1.3)$$

$$A\mathbb{E} \left[ (x - \mu_x)(x - \mu_x)^T \right] A^T = \boxed{A\Sigma_x A^T} \quad (1.4)$$

Next, we show that  $y$  is Gaussian.

Note: In retrospective, an easier approach than the below is to consider that any projection of  $Ax$  is ultimately a projection of  $x$  and so is, by definition, Gaussian (and hence so is  $Ax + b$ ). Another alternative approach is to show that the characteristic function of  $Ax + b$  has the same form as that of a Gaussian density. The direct approach below gets complicated (but still valid) when density of  $Ax + b$  is degenerate.

We proceed to show that  $y$  is Gaussian. In the case where  $A$  is invertible this can be directly obtained from random vector transformation formula (1-to-1 transformation). Since  $y = Ax + b$  the Jacobian matrix is  $A$  and since the mapping is one-to-one ( $A$  is invertible) according to the transformation theorem:

$$f_Y(y) = \frac{1}{|A|} f_X(A^{-1}(y - b)) \quad (1.5)$$

ignoring the constant normalization factors (since they don't depend on  $x$  and only guarantee the distributions' summing to 1) we calculate the exponent in the right hand side:

$$-\frac{1}{2} \left( A^{-1}(y - b) - \mu_x \right)^T \Sigma_x^{-1} \left( A^{-1}(y - b) - \mu_x \right) = \quad (1.6)$$

$$-\frac{1}{2} (y - b - A\mu_x)^T A^{-T} \Sigma_x^{-1} A^{-1} (y - b - A\mu_x) = \quad (1.7)$$

$$-\frac{1}{2} (y - (A\mu_x + b))^T \left( A\Sigma_x A^T \right)^{-1} (y - (A\mu_x + b)), \quad (1.8)$$

which exactly gives a Gaussian distribution (in particular, with the mean and variance we calculated beforehand).

We now deal with the case where  $A \in \mathbb{R}^{m \times n}$  is not invertible - either rectangular or square with zero determinant. Consider the SVD decomposition of  $A$ :

$$A = U\Sigma V^T \quad (1.9)$$

Here,  $U$  and  $V$  are square invertible matrices of respective sizes  $m \times m$  and  $n \times n$ . Note that  $V^T = V^*$  as  $A$  is real.  $\Sigma$  is a rectangular matrix the same size as  $A$  with the main diagonal containing the singular values (and zero elsewhere). We can now write:

$$y = U\Sigma V^T x + b \quad (1.10)$$

Here,  $V^T x$  is Gaussian, since  $V$  is invertible. Without loss of generality denote

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \quad V^T x = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad (1.11)$$

where  $\Sigma_1$  is (square) diagonal, with positive entries (and in particular, is invertible), and has the same number of columns as the length of vector  $v_1$ . We have then that

$$\Sigma V^T x = \begin{pmatrix} \Sigma_1 v_1 \\ 0 \end{pmatrix}, \quad (1.12)$$

where  $\Sigma_1 v_1$  is Gaussian since  $v_1$  is and  $\Sigma_1$  is invertible, and hence  $\Sigma V^T x = U^T(y - b)$  is (possibly degenerate) Gaussian (and hence, so is  $y$ ).

## 2

### 2.1

$$p(x | z) = \frac{p(z | x)p(x)}{p(z)} = \frac{p(z | x)p(x)}{\int_x p(z | x)p(x) \cdot dx} \propto p(z | x) \cdot p(x) \quad (2.1)$$

### 2.2

We are interested in the MAP estimate:

$$x_{MAP}^* \doteq \arg \max_x p(x | z) = \arg \max_x p(z | x) \cdot p(x) = \arg \max_x \log p(z | x) + \log p(x) = \quad (2.2)$$

$$\arg \min_x ||z - Hx||_R^2 + ||x - x_0||_{\Sigma_0}^2 \quad (2.3)$$

Develop the latter:

$$||z - Hx||_R^2 + ||x - x_0||_{\Sigma_0}^2 = ||R^{-1/2}(Hx - z)||^2 + ||\Sigma_0^{-1/2}(x - x_0)||^2 = \quad (2.4)$$

$$||\begin{pmatrix} R^{-1/2}(Hx - z) \\ \Sigma_0^{-1/2}(x - x_0) \end{pmatrix}||^2 = ||\begin{pmatrix} R^{-1/2}H \\ \Sigma_0^{-1/2} \end{pmatrix} x - \begin{pmatrix} R^{-1/2}z \\ \Sigma_0^{-1/2}x_0 \end{pmatrix}||^2, \quad (2.5)$$

from where  $x_{MAP}^*$  can be obtained as the least-squares solution using the pseudoinverse  $A^\dagger b = (A^T A)^{-1} A^T b$ , or equivalently proceed directly, noting that the function is convex and has a unique extremum which is the minimum. Develop the above into:

$$= x^T (H^T R^{-1} H + \Sigma_0^{-1}) x - 2 (z^T R^{-1} H + x_0^T \Sigma_0^{-1}) x + (z^T R^{-1} z + x_0^T \Sigma_0^{-1} x_0) \quad (2.6)$$

find the zero of the gradient:

$$\nabla(\cdot) = 2 \left( H^T R^{-1} H + \Sigma_0^{-1} \right) x^* - 2 \left( \Sigma_0^{-1} x_0 + H^T R^{-1} z \right) = 0 \quad (2.7)$$

$$\implies \boxed{x_{MAP}^* = \left( H^T R^{-1} H + \Sigma_0^{-1} \right)^{-1} \left( \Sigma_0^{-1} x_0 + H^T R^{-1} z \right)} \quad (2.8)$$

The associated covariance is

$$\boxed{\Sigma = \left( H^T R^{-1} H + \Sigma_0^{-1} \right)^{-1}}, \quad (2.9)$$

as can be seen from the quadratic term in Eq.(2.6).

## 3 Code

```
In [2]: import numpy as np
        from numpy.linalg import norm
        from math import sin, cos, pi, asin, atan2, degrees
        from tabulate import tabulate
```

```
np.set_printoptions(suppress=True,precision=11)
```

### 3.1 Rotations

```
In [19]: def wedge(v):
        return np.matrix([[0, -v[2], v[1]],
                           [v[2], 0, -v[0]],
                           [-v[1], v[0], 0]]);

        def get_rotation_rodriguez(v, theta=None):
            if theta is None:
                theta = norm(v);
            n = v/theta;
            else:
                n = v;

            cM = wedge(n);
            return np.identity(3) + sin(theta)*cM + (1-cos(theta))*cM*cM
```

#### 3.1.1

```
In [14]: def rotx(phi):
        v = phi*np.array([1, 0, 0]);
        return get_rotation_rodriguez(v)

        def roty(theta):
            v = theta*np.array([0, 1, 0]);
            return get_rotation_rodriguez(v)
```

```

def rotz(psi):
    v = psi*np.array([0, 0, 1]);
    return get_rotation_rodriguez(v)

def angles2rot(phi, theta, psi):
    return rotz(psi)*roty(theta)*rotx(phi)

```

### 3.1.2

```
In [16]: print('R= \n' + str(angles2rot(phi=pi/4, theta=pi/5, psi=pi/7)))
```

```

R=
[[ 0.72889912554  0.06766479742  0.68126906578]
 [ 0.35101931853  0.81741496597 -0.45674742629]
 [-0.58778525229  0.57206140282  0.57206140282]]

```

### 3.1.3

Development here is based on [1].

We first write the rotation matrix as the product of individual axis rotation matrices:

$$\begin{pmatrix} \cos(\psi) \cos(\theta) & \cos(\psi) \sin(\phi) \sin(\theta) - \cos(\phi) \sin(\psi) & \sin(\phi) \sin(\psi) + \cos(\phi) \cos(\psi) \sin(\theta) \\ \cos(\theta) \sin(\psi) & \cos(\phi) \cos(\psi) + \sin(\phi) \sin(\psi) \sin(\theta) & \cos(\phi) \sin(\psi) \sin(\theta) - \cos(\psi) \sin(\phi) \\ -\sin(\theta) & \cos(\theta) \sin(\phi) & \cos(\phi) \cos(\theta) \end{pmatrix} \quad (3.1)$$

Given rotation matrix  $R$ , we will want to extract angles by demanding per-element equality. We treat separately the case where  $\cos(\theta) = 0$ . For the case  $\theta = \frac{\pi}{2}$  we obtain the matrix:

$$\begin{pmatrix} 0 & \cos(\psi) \sin(\phi) - \cos(\phi) \sin(\psi) & \sin(\phi) \sin(\psi) + \cos(\phi) \cos(\psi) \\ 0 & \cos(\phi) \cos(\psi) + \sin(\phi) \sin(\psi) & \cos(\phi) \sin(\psi) - \cos(\psi) \sin(\phi) \\ -1 & 0 & 0 \end{pmatrix} \quad (3.2)$$

$$= \begin{pmatrix} 0 & \sin(\phi - \psi) & \cos(\phi - \psi) \\ 0 & \cos(\phi - \psi) & -\sin(\phi - \psi) \\ -1 & 0 & 0 \end{pmatrix} \quad (3.3)$$

For the case  $\theta = -\frac{\pi}{2}$ :

$$\begin{pmatrix} 0 & -\cos(\psi) \sin(\phi) - \cos(\phi) \sin(\psi) & \sin(\phi) \sin(\psi) - \cos(\phi) \cos(\psi) \\ 0 & \cos(\phi) \cos(\psi) - \sin(\phi) \sin(\psi) & -\cos(\phi) \sin(\psi) - \cos(\psi) \sin(\phi) \\ 1 & 0 & 0 \end{pmatrix} \quad (3.4)$$

$$= \begin{pmatrix} 0 & -\sin(\phi + \psi) & -\cos(\phi + \psi) \\ 0 & \cos(\phi + \psi) & -\sin(\phi + \psi) \\ 1 & 0 & 0 \end{pmatrix} \quad (3.5)$$

From here we proceed extracting the angles (see code and [1]).

```
In [17]: def rot2angles(M):
    sin_theta = -M[2, 0];
    theta = asin(sin_theta);

    if sin_theta==1:
        psi = 0; # arbitrary
        phi = atan2(M[0,1],M[0,2]);
    elif sin_theta==-1:
        psi = 0; # arbitrary
        phi = atan2(-M[0,1],-M[0,2]);
    else:
        # another valid solution: pi-theta.
        cos_theta = cos(theta);
        phi = atan2(M[2,1]/cos_theta, M[2, 2]/cos_theta);
        psi = atan2(M[1,0]/cos_theta, M[0, 0]/cos_theta);

    return phi, theta, psi
```

### 3.1.4

```
In [22]: phi, theta, psi = (degrees(x) for x in rot2angles(
np.mat(' 0.813797681 -0.440969611 0.378522306; \
        0.46984631  0.882564119 0.0180283112; \
        -0.342020143 0.163175911 0.925416578'))))
print('phi=' + str(phi), 'theta=' + str(theta), 'psi=' + str(psi))

phi=9.999999994217536 theta=19.999999980143038 psi=29.99999998990158
```

## 3.2 3D Rigid Transformation

$$l^C = R_G^C l^G + t_{C \rightarrow G}^C = R_G^C (l^G + t_{C \rightarrow G}^G) \quad (3.6)$$

```
In [23]: trans = np.mat([-451.2459, 257.0322, 400])
rot = np.mat('0.5363 -0.8440 0; 0.8440 0.5363 0; 0 0 1');

pt = np.mat([450, 400, 50])

Tgl = rot.dot((pt + trans).T)
print(Tgl)

[[-555.20335297]
 [ 351.31482926]
 [ 450.         ]]
```

### 3.3 Pose Composition

Commanded:

$$x_{k+1}^c = x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.7)$$

$$R_{k+1}^c = R_k^c \quad (3.8)$$

$${}_{k+1}^k T^{(commanded)} = \begin{pmatrix} I & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \quad (3.9)$$

Actual:

$$x_{k+1}^a = x_k + \begin{pmatrix} 1.01 \\ 0 \end{pmatrix} \quad (3.10)$$

$$R_{k+1}^a = R(\theta = 1^\circ) \cdot R_k^c \quad (3.11)$$

$${}_{k+1}^k T^{(actual)} = \begin{pmatrix} R(\theta^\circ) & \begin{pmatrix} 1.01 \\ 0 \end{pmatrix} \end{pmatrix} \quad (3.12)$$

```
In [122]: %matplotlib inline
```

```
import matplotlib.pyplot as plt
from copy import deepcopy
```

```
def rotation2(theta):
    # rotates by -theta
    return np.mat([[cos(theta), sin(theta)],
                  [-sin(theta), cos(theta)]]);
```

```
def rot2angle(R):
    return atan2(R[0, 1], R[1, 1]);
```

```
Tc = np.concatenate((np.identity(2), np.mat([[1],[0]])), axis=1);
```

```
dtheta = pi/180;
```

```
Ta = np.concatenate((rotation2(-dtheta), np.mat([[1.01],[0]])), axis=1);
```

```
# Initial pose
```

```
origin = np.mat([[0,0]]).T;
```

```
orientation = np.mat([[1, 0]]).T;
```

```
Ti = np.concatenate((np.identity(2), np.mat([[0],[0]])), axis=1);
```

```
# Lists hold nominal and actual pose history
```

```
T1 = [{"T" : Ti, "origin" : origin, "orientation" : orientation}];
```

```
T2 = deepcopy(T1);
```

```
for ii in range(10):
```

```
    # commanded
```

```
    Tk = Tc.dot(np.vstack((T1[-1]['T'], [0, 0, 1])));
```

```
    nominal_origin = Tk[:,-1];
```

```
    Rk = Tk[0:2,0:2];
```

```
    nominal_orientation = Rk.dot(orientation);
```

```
    T1.append({"T" : Tk, "origin" : nominal_origin, "orientation" : nominal_orientation})
```

```

# actual
Tk = Ta.dot(np.vstack((T2[-1]['T'], [0, 0, 1])));
actual_origin = Tk[:,-1];
Rk = Tk[0:2,0:2];
actual_orientation = Rk.dot(orientation);
T2.append({"T" : Tk, "origin" : actual_origin, "orientation" : actual_orientation})

def disp_track(locations, orientations, color='b'):
x = [o[0].item(0) for o in locations]
y = [o[1].item(0) for o in locations]
u = [o[0].item(0) for o in orientations]
v = [o[1].item(0) for o in orientations]
plt.plot(x, y, '-.', color=color)
plt.quiver(x, y, u, v, color='k',zorder=10)

plt.figure()

Pc = [P['origin'] for P in T1]
Uc = [P['orientation'] for P in T1]
disp_track(Pc, Uc)

Pa = [P['origin'] for P in T2]
Ua = [P['orientation'] for P in T2]
disp_track(Pa, Ua, 'r')

plt.grid(True)
plt.axis('equal');
plt.legend(['commanded/nominal', 'actual']);
plt.title('Nominal vs. actual robot poses');

print('Location difference: ' + str(norm(Pc[-1]-Pa[-1])) + ' m')
print('Orientation difference: ' + str(degrees(acos(Ua[-1].T.dot(Uc[-1])))) + u'\N{DEGREE S

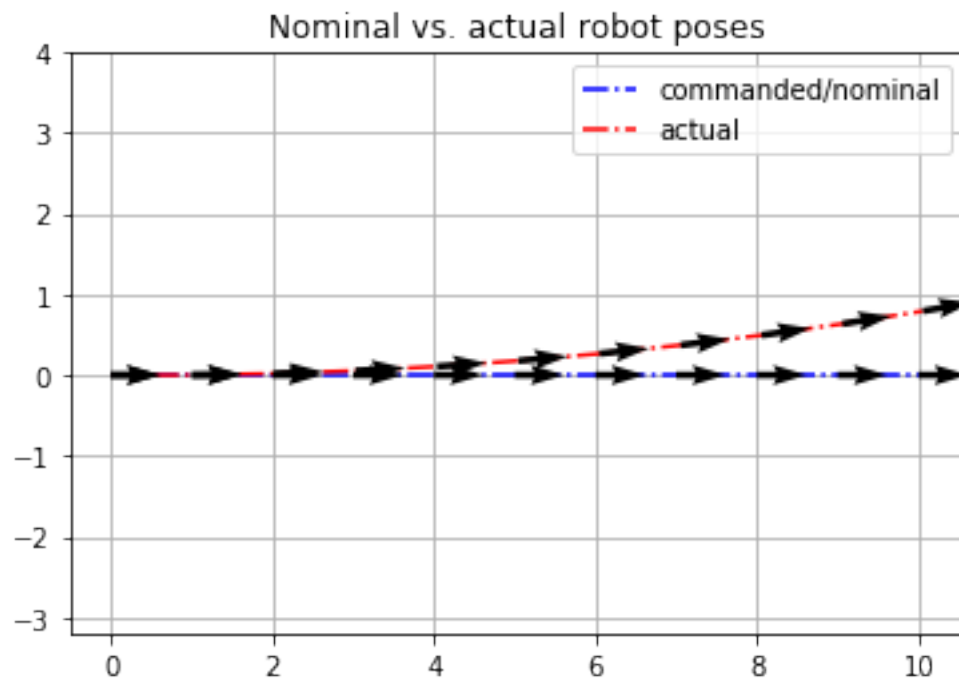
```

```

Location difference: 0.793435624902 m
Orientation difference: 9.999999999999902°

```





## Bibliography

- [1] Gregory G Slabaugh. Computing euler angles from a rotation matrix. *Retrieved on August, 6(2000):39–63*, 1999.