Bundle Adjustment

$$p(x,l|z) = \frac{p(x,l)\,p(z|x,l)}{p(z)} \propto \underbrace{p(x,l)}_{\text{prior}}\,\underbrace{p(z|x,l)}_{\text{Meas. likelihood}}$$

$$p(X,L|Z) \propto \prod_{i}^{N}\prod_{j\in\mathcal{M}_i} p(z_{i,j}|x_i,l_j) \qquad \begin{cases} z_{i,j} = \pi(x_i,l_j)+v \longrightarrow p(z_{i,j}|x_i,l_j) \\ p(z|x,l) = \dfrac{1}{\sqrt{|2\pi\Sigma_v|}}\exp\left(-\dfrac{1}{2}\|z-\pi(x,l)\|^2_{\Sigma_v}\right)\end{cases}$$

$$X^\star, L^\star = \arg\max_{X,L} p(X,L|Z)$$

Equivalent to minimizing:

$$J_{BA}(X,L) \doteq \sum_{i=1}^{N}\sum_{j\in\mathcal{M}_i}\|z_{i,j}-\pi(x_i,l_j)\|^2_{\Sigma}$$

Solve: Gauss-Newton: (1) Linearize (Taylor 1st)

$$J(\bar X+\Delta X,\bar L+\Delta L)\approx \sum_{i=1}^{N}\sum_{j\in\mathcal{M}_i}\|z_{i,j}-\pi(\bar x_i,\bar l_j)-\nabla_{x_i}\pi\cdot\Delta x_i-\nabla_{l_j}\pi\cdot\Delta l_j\|^2_{\Sigma}$$

$$J(\bar\Theta+\Delta\Theta)\approx \sum_{i=1}^{N}\sum_{j\in\mathcal{M}_i}\|z_{i,j}-\pi(\bar x_i,\bar l_j)-A_{i,j}\Delta\Theta\|^2_{\Sigma} = \sum_{i=1}^{N}\sum_{j\in\mathcal{M}_i}\|A_{i,j}\Delta\Theta-b_{i,j}\|^2_{\Sigma} \qquad \Theta\doteq\{X,L\}$$

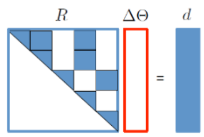
$$J(\bar\Theta+\Delta\Theta)\approx \sum_{i=1}^{N}\sum_{j\in\mathcal{M}_i}\Big\|\underbrace{\Sigma^{-1/2}A_{i,j}}_{\doteq\mathcal{A}_{i,j}}\Delta\Theta-\underbrace{\Sigma^{-1/2}b_{i,j}}_{\doteq\breve b_{i,j}}\Big\|^2 \rightarrow J(\bar\Theta+\Delta\Theta)\approx\|\mathcal{A}\Delta\Theta-\breve b\|^2 \qquad \begin{cases}\mathcal{A}\doteq\begin{pmatrix}\vdots\\ \mathcal{A}_{i,j}\\ \vdots\end{pmatrix}\\ \breve b\doteq\begin{pmatrix}\vdots\\ \breve b_{i,j}\\ \vdots\end{pmatrix}\end{cases}$$

(2) Calculate optimal increment ΔΘ, update, repeat $\bar\Theta\leftarrow\bar\Theta+\Delta\Theta$

Calculating the increment: (a) for dense matrixes: $\Delta\Theta=(\mathcal{A}^T\mathcal{A})^{-1}\mathcal{A}^T\breve b$

(b) sparse matrixes – QR factorization $Q^T\mathcal{A}=\begin{bmatrix}R\\0\end{bmatrix}$ $Q^T\breve b\doteq\begin{bmatrix}d\\e\end{bmatrix}$

$$\|\mathcal{A}\Delta\Theta-\breve b\|_2^2=\|R\Delta\Theta-d\|_2^2+\|e\|_2^2$$



- obtain least squares solution via back-substitution

Smoothing and Mapping (SAM)

$$x^\star_{0:k},L^\star_k=\arg\max_{x_{0:k},L_k}p(x_{0:k},L_k|u_{0:k-1},z_{0:k})$$

- full joint pdf :

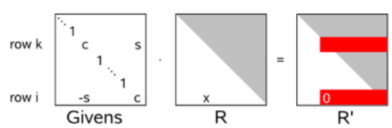
$$p(x_{0:k},L_k|u_{0:k-1},z_{0:k})=\eta p(x_0)\prod_i\left[p(x_i|x_{i-1},u_{i-1})\prod_{j\in\mathcal{M}_i}p(z_{i,j}|x_i,l_j)\right]$$

- MAP, sol: least sq. problem

$$x^\star_{0:k},L^\star_k=\arg\min_{x_{0:k},L_k}\left\{\|x_0-\hat x_0\|^2_{\Sigma_0}+\sum_i\left[\|x_i-f(x_{i-1},u_{i-1})\|^2_{\Sigma_w}+\sum_{j\in\mathcal{M}_i}\|z_{i,j}-h(x_i,l_j)\|^2_{\Sigma_v}\right]\right\}$$

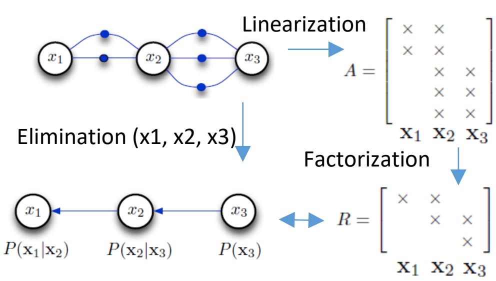
$\Theta\doteq\{x_{0:k},L_k\}$ - solve as in BA

iSAM : using the Givens rotations



- identify nodes involved in new factors
- find path in previous bayes net from previous root to involved nodes
- re-eliminate nodes on found paths

BBN (Bayesian Belief network) **and FG**



Epipolar Geometry, constrains

-epipolar constraint: $\hat x'\,[t]_\times R\hat x=0$ $\begin{cases}\hat x\doteq K^{-1}x\\ x=K\begin{bmatrix}R & t\end{bmatrix}X\end{cases}$

Normalized image coords camera motion

-Essential matrix: $E=[t]_\times R\in\mathbb{R}^{3\times3}$ $\hat x'^TE\hat x=0$ $x'^TFx=0$

-Fundamental Matrix: $E=K'^TFK$ - singularity constraint: $\det(F)=0$

(1) estimate essential matrix
(2) recover camera motion

Recovering F : for n image correspondences ($\hat x'\leftrightarrow\hat x$) :

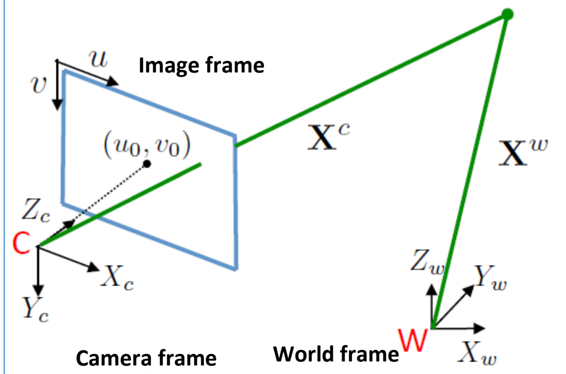
$$\mathbf{Af}=\begin{bmatrix}x_1'x_1 & x_1'y_1 & x_1' & y_1'x_1 & y_1'y_1 & y_1' & x_1 & y_1 & 1\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\\ x_n'x_n & x_n'y_n & x_n' & y_n'x_n & y_n'y_n & y_n' & x_n & y_n & 1\end{bmatrix}\mathbf{f}=\mathbf{0}$$

Motion estimation:
(a) calc E from F
(b) $t^TE=0$
(c) Extract t^T via SVD
(d) Extract R
(e) 4 possible solutions

Triangulation Methods (M known, find 3D point)

- Linear approach: -solve with Least Squares $A\begin{pmatrix}X\\Y\\Z\end{pmatrix}=b$

$$0=(um_{31}-m_{11})X+(um_{32}-m_{12})Y+(um_{33}-m_{13})Z+(um_{34}-m_{14})$$
$$0=(vm_{31}-m_{21})X+(vm_{32}-m_{22})Y+(vm_{33}-m_{23})Z+(vm_{34}-m_{24})$$



-Projection matrix: $M=K[R\,|\,t]$

$$\begin{pmatrix}\tilde u\\ \tilde v\\ \tilde w\end{pmatrix}=\begin{bmatrix}f_x & s & u_0\\ & f_y & v_0\\ & & 1\end{bmatrix}\overbrace{\begin{bmatrix}R & | & t\end{bmatrix}}^{=(x)}\overbrace{\begin{pmatrix}X_w\\Y_w\\Z_w\\1\end{pmatrix}}^{=(l)}$$

-Calibration Matrix (intrinsic) (K) 5DOF -Rot/translation matrix (extrinsic) 6DOF

$$\mathbf{X}^c=R\mathbf{X}^w+t=\begin{bmatrix}R & t\end{bmatrix}\begin{pmatrix}X_w\\Y_w\\Z_w\\1\end{pmatrix}$$

$$R\equiv R^c_w=R_{w\to c}\qquad t\equiv t^c_{c\to w}$$

-Projection: $\pi(x,l)\doteq K\begin{bmatrix}R & t\end{bmatrix}l$

-reprojection error: $\underbrace{z}_{\text{-observation}}-\underbrace{\pi(x,l)}_{\text{-predicted measurement}}$

Feature Matching: detection, description, matching

-detection: (1) Harris -> not scale invariant

Summing up squared differences (SSD):

$$E(u,v)=\sum_{(x,y)\in W}[u\ v]\underbrace{\begin{bmatrix}I_x^2 & I_xI_y\\ I_yI_x & I_y^2\end{bmatrix}}_{H}\begin{bmatrix}u\\v\end{bmatrix}$$

Big eigenvalues of H (big f) -> corners $f=\dfrac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$

(2) SIFT – scale invariant feature transform

-calculate DoGs (difference of Gaussians)

-Feature: Image coordinates (x,y), scale (σ), orientation (θ)

Descriptor: 128 element vector. Match :min Euclidean distance

Deal with outliers: RANSAC

Camera Extrinsic & Intrinsic calibration

-input: (1) n 2D/3D correspondences (2) known 3D points

-11 DOF. Procedure: (1) calculate M (2) Decompose -> K, R, t

$$\begin{pmatrix}\tilde u\\ \tilde v\\ \tilde w\end{pmatrix}=\underbrace{K\begin{bmatrix}R & | & t\end{bmatrix}}_{M\in\mathbb{R}^{3\times4}}\begin{pmatrix}X\\Y\\Z\\1\end{pmatrix}\qquad M=\begin{bmatrix}m_{11}&m_{12}&m_{13}&m_{14}\\ m_{21}&m_{22}&m_{23}&m_{24}\\ m_{31}&m_{32}&m_{33}&m_{34}\end{bmatrix}$$

-Step (1) : calc M : $\begin{cases}u=\dfrac{\tilde u}{\tilde w}=\dfrac{m_{11}X+m_{12}Y+m_{13}Z+m_{14}}{m_{31}X+m_{32}Y+m_{33}Z+m_{34}}\\ v=\dfrac{\tilde v}{\tilde w}=\dfrac{m_{21}X+m_{22}Y+m_{23}Z+m_{24}}{m_{31}X+m_{32}Y+m_{33}Z+m_{34}}\end{cases}$

$$\begin{cases}(m_{31}X+m_{32}Y+m_{33}Z+m_{34})\,u=m_{11}X+m_{12}Y+m_{13}Z+m_{14}\\ (m_{31}X+m_{32}Y+m_{33}Z+m_{34})\,v=m_{21}X+m_{22}Y+m_{23}Z+m_{24}\end{cases}$$

$$\begin{bmatrix}X & Y & Z & 1 & 0 & 0 & 0 & 0 & -uX & -uY & -uZ & -u\\ 0 & 0 & 0 & 0 & X & Y & Z & 1 & -vX & -vY & -vZ & -v\end{bmatrix}\mathbf{m}=\mathbf{0}$$

$A\mathbf{m}=\mathbf{0}$ - solve via:
- DLT (direct Linear transform algorithm
- reality – SVD (least singular value vector)

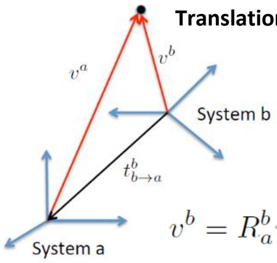
$$\mathbf{m}=\begin{pmatrix}m_{11}\\m_{12}\\m_{13}\\ \vdots\\ m_{34}\end{pmatrix}\in\mathbb{R}^{12}$$

-Step (2) : decompose M : - K : upper triangular, R : orthogonal

(a) QR factorization

(b) calculate t: $t=K^{-1}\begin{pmatrix}m_{14}\\m_{24}\\m_{34}\end{pmatrix}$

Translations, rotations



$$v^b = R^b_a v^a + t^b_{b\to a}$$

R^b_a : rotation from a to b

$t^b_{b\to a}$: translation from b to a, expressed in system b (origin of system a relative to system b)

$$\begin{pmatrix} v^b \\ 1 \end{pmatrix} = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} \begin{pmatrix} v^a \\ 1 \end{pmatrix}$$

4x4 matrix

$$T = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_z(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_1 = \{R_1, t_1\} \quad x_2 = \{R_2, t_2\} \longrightarrow x \doteq x_1 \oplus x_2 = \{R_1 R_2, R_1 t_2 + t_1\}$$

$${}^G_1T \doteq \begin{bmatrix} R^G_1 & t^G_{G\to 1} \\ 0^T & 1 \end{bmatrix} \quad {}^1_2T \doteq \begin{bmatrix} R^1_2 & t^1_{1\to 2} \\ 0^T & 1 \end{bmatrix}$$

$${}^G_2T = {}^G_1T\,{}^1_2T = \begin{bmatrix} R^G_1 R^1_2 & R^G_1 t^1_{1\to 2} + t^G_{G\to 1} \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R^G_2 & t^G_{G\to 2} \\ 0^T & 1 \end{bmatrix}$$

IMU measurements

$$\omega_m(t) = \omega(t) + \underline{b_g(t)} + n_g(t)$$

$$a_m(t) = R^I_G\left(a^G(t) - g^G\right) + \underline{b_a(t)} + n_a$$

Position: $\dot{p}^G_I(t) = v^G_I(t)$

Velocity: $\dot{v}^G_I(t) = a^G(t)$

Orientation: $\dot{R}^G_I(t) = R^G_I(t)\,\Omega(\omega)$

$$\Omega(\omega) = [\omega^I]_\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Bayesian Inference

- x, y independent: $p(x|y) = p(x)$
- x, y dependent: $p(x|y) = \dfrac{p(x,y)}{p(y)}$
- Conditional independence: $p(x,y|z) = p(x|z)\,p(y|z)$

(does not mean x, y are independent)

- Marginalization:

$$p(x) = \int p(x,y)\,dy = \int p(x|y)\,p(y)\,dy$$

- Expectation (linear):

$$\mathbb{E}[X] = \int x p(x)\,dx$$

$$\mathbb{E}[aX+b] = a\mathbb{E}[X] + b$$

- Covariance (scalar case):

$$Cov[X] = \mathbb{E}[X - \mathbb{E}[X]]^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

- Covariance (multivariate case): $X \in \mathbb{R}^n$

$$Cov[X] = \mathbb{E}\left[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^T\right]$$

- Bayes and Chain Rules

$$p(x|y) = \frac{p(y|x)\,p(x)}{p(y)} = \frac{p(y|x)\,p(x)}{\int p(y|x')\,p(x')\,dx'}$$

$$p(x,y) = p(x|y)\,p(y)$$

Multivariable Gaussian (Covariance): $x \sim N(x;\mu,\Sigma)$

$$p(x) = \frac{1}{\sqrt{\det(2\pi\Sigma)}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

Mahalanobis Norm

$$\|x-\mu\|^2_\Sigma \doteq (x-\mu)^T\Sigma^{-1}(x-\mu)$$

Information Form: $x \sim N^{-1}(x;\eta,\Lambda)$

Information matrix, vector: $\Lambda \doteq \Sigma^{-1} \quad \eta \doteq \Lambda\mu$

$$p(x) = N^{-1}(\eta,\Lambda) = \frac{\exp\left(-\frac{1}{2}\eta^T\Lambda^{-1}\eta\right)}{\sqrt{\det(2\pi\Lambda^{-1})}}\exp\left(-\frac{1}{2}x^T\Lambda x + \eta^T x\right)$$

$$p(x,y) = \mathcal{N}\left(\begin{bmatrix}\mu_x \\ \mu_y\end{bmatrix}, \begin{bmatrix}\Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix}\right) = \mathcal{N}^{-1}\left(\begin{bmatrix}\eta_x \\ \eta_y\end{bmatrix}, \begin{bmatrix}I_{xx} & I_{xy} \\ I_{yx} & I_{yy}\end{bmatrix}\right)$$

	Marginalization	**Conditioning**
	$p(x) = \int p(x,y)\,dy \doteq \mathcal{N}(\mu,\Sigma) \doteq \mathcal{N}^{-1}(\eta, I)$	$p(x\|y) = \frac{p(x,y)}{p(y)} \doteq \mathcal{N}(\mu',\Sigma') \doteq \mathcal{N}^{-1}(\eta', I')$
Covariance form	$\mu = \mu_x$, $\Sigma = \Sigma_{xx}$	$\mu' = \mu_x + \Sigma_{xy}\Sigma^{-1}_{yy}(y-\mu_y)$, $\Sigma' = \Sigma_{xx} - \Sigma_{xy}\Sigma^{-1}_{yy}\Sigma_{yx}$
Information form	$\eta = \eta_x - I_{xy}I^{-1}_{yy}\eta_y$, $I = I_{xx} - I_{xy}I^{-1}_{yy}I_{yx}$	$\eta' = \eta_x - I_{xy}y$, $I' = I_{xx}$

$$p(x|z,a) = \frac{p(z|x,a)\,p(x|a)}{p(z|a)}$$

Markov assumption:

$$p(z_n|x) = p(z_n|x, z_1, \ldots, z_{n-1})$$

$$p(x|z_1,\ldots,z_n) = \frac{p(z_n|x)\,p(x|z_1,\ldots,z_{n-1})}{p(z_n|z_1,\ldots,z_{n-1})} = \eta p(z_n|x)\,p(x|z_1,\ldots,z_{n-1}) = \eta_{1:n}\prod_{i=1}^{n} p(z_i|x)\,p(x)$$

Measurement Likelihood, Prior

$$p(x|z) = \frac{p(z|x)\,p(x)}{p(z)} = \eta p(z|x)\,p(x) \propto p(z|x)\,p(x)$$

Prior on sensors

-motion model: $x_{k+1} = f(x_k, u_k) + w_k \quad p(x_{k+1}|x_k, u_k)$

-observation model: $z_k = h(x_k) + v_k \quad p(z_k|x_k)$

-a posteriori: $p(x_k|u_{0:k-1}, z_{1:k})$

-objective: $x^\star_k = \arg\max_{x_k} p(x_k|u_{0:k-1}, z_{1:k})$

-Markov: $p(x_k|x_{0:k-1}, z_{1:k-1}, u_{0:k-1}) = p(x_k|x_{k-1}, u_{k-1})$

$$p(z_k|x_{0:k}, u_{0:k-1}, z_{1:k-1}) = p(z_k|x_k)$$

Recursive Bayesian Update

-objective: given a posteriori, calculate current state

-prediction: $p(x_k|u_{0:k-1}, z_{1:k-1}) =$

$$= \int \underbrace{p(x_{k-1}|u_{0:k-2}, z_{1:k-1})}_{\text{previous belief}}\,p(x_k|x_{k-1}, u_{k-1})\,dx_{k-1}$$

-update:

$$p(x_k|u_{0:k-1}, z_{1:k}) = \eta p(z_k|x_k)\,p(x_k|u_{0:k-1}, z_{1:k-1})$$

$$= \eta p(z_k|x_k)\int p(x_k|x_{k-1}, u_{k-1})\,\underbrace{p(x_{k-1}|u_{0:k-2}, z_{0:k-1})}_{\text{previous belief}}\,dx_{k-1}$$

-without marginalization (smoothing):

$$p(x_{0:k}|u_{0:k-1}, z_{1:k}) = \eta p(x_0)\prod_{i=1}^{k} p(x_i|x_{i-1}, u_{i-1})\,p(z_i|x_i)$$