TO ADD: euler vector to matrix and back, inverse of coord system transform, quaternions, transformation composition, 3d transformation types, solve bayes specific example, feature detection, RANSAC

Probability and Gaussian Identities

$$x \sim N(\mu_x, \Sigma_x) \;\; y = Ax + b \Longrightarrow \mu_y = A\mu_x + b \;\; \Sigma_y = A\Sigma_x A^T$$

$$\sum_i \|A_i x \pm B_i\|^2_{\Sigma_i} = \left\| \begin{pmatrix} | \\ \Sigma_i^{-\frac{1}{2}} A_i \\ | \end{pmatrix} x \pm \begin{pmatrix} | \\ \Sigma_i^{-\frac{1}{2}} B_i \\ | \end{pmatrix} \right\|^2$$

$$I = \sum_i A_i^T \Sigma_i^{-1} A_i, \quad \mu = \mp I^{-1} \cdot \sum_i A_i^T \Sigma_i^{-1} B_i$$

$$x \sim N(x; \mu, \Sigma) = N^{-1}(x; \eta, \Lambda), \;\; \Lambda \doteq \Sigma^{-1}, \;\; \eta \doteq \Lambda\mu$$

$$N^{-1}(x; \eta, \Lambda) = \frac{\exp\left(-\frac{1}{2}\eta^T \Lambda^{-1} \eta\right)}{\sqrt{\det(2\pi\Lambda^{-1})}} \exp\left(-\frac{1}{2} x^T \Lambda x + \eta^T x\right)$$

$$p(x, y) = \mathcal{N}\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} \right) = \mathcal{N}^{-1}\left(\begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}, \begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix} \right)$$

	Marginalization	Conditioning
	$p(x) = \int p(x,y)\, dy$ $\doteq \mathcal{N}(\mu, \Sigma) \doteq \mathcal{N}^{-1}(\eta, I)$	$p(x \mid y) = \frac{p(x,y)}{p(y)}$ $\doteq \mathcal{N}(\mu', \Sigma') \doteq \mathcal{N}^{-1}(\eta', I')$
Covariance form	$\mu = \mu_x$ $\Sigma = \Sigma_{xx}$	$\mu' = \mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y)$ $\Sigma' = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}$
Information form	$\eta = \eta_x - I_{xy} I_{yy}^{-1} \eta_y$ $I = I_{xx} - I_{xy} I_{yy}^{-1} I_{yx}$	$\eta' = \eta_x - I_{xy} y$ $I' = I_{xx}$

Matrix identities

$$[t]_\times \doteq \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix}, \;\; J = \begin{pmatrix} \cdots & \frac{\partial f_1}{\partial x_i} & \cdots \\ & \cdots & \\ \cdots & \frac{\partial f_n}{\partial x_i} & \cdots \end{pmatrix} = \begin{pmatrix} | \\ (\nabla f_i)^T \\ | \end{pmatrix}$$

$$\|a\|^2_\Sigma = \|\Sigma^{-1/2} a\|^2, \quad \Sigma^{-1} = \Sigma^{-T/2}\Sigma^{-1/2}, \;\; A^\dagger = \left(A^T A\right)^{-1} A^T$$

$$f(x) = f(x_0) + J(x - x_0) + o(\|\Delta x\|^2)$$

SVD: $A = UDV^*$, columns of U, V - ort. eigenvectors of MM^* and M^*M

$$\text{QR: } \mathcal{A}\Delta\Theta = \breve{b}, \;\; Q^T \mathcal{A} = \begin{bmatrix} R \\ 0 \end{bmatrix}, \;\; Q^T \breve{b} = \begin{bmatrix} d \\ e \end{bmatrix}$$

Pose and Geometry

$$v^b = R_a^b v_a + t^b_{b \to a} \;\; \begin{pmatrix} v^b \\ 1 \end{pmatrix} = \begin{pmatrix} R_a^b & t^b_{b\to a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^a \\ 1 \end{pmatrix}$$

$$T_2 T_1 = \begin{pmatrix} R_2 R_1 & R_2 t_1 + t_2 \\ 0 & 1 \end{pmatrix}, \;\; T^{-1} = \begin{pmatrix} R^T & -R^T t \\ 0 & 1 \end{pmatrix}$$

$$R_a^b = \begin{pmatrix} x_a^b & y_a^b & z_a^b \end{pmatrix} = \begin{pmatrix} x_a \cdot x_b & y_a \cdot x_b & z_a \cdot x_b \\ x_a \cdot y_b & y_a \cdot y_b & z_a \cdot y_b \\ x_a \cdot z_b & y_a \cdot z_b & z_a \cdot z_b \end{pmatrix}$$

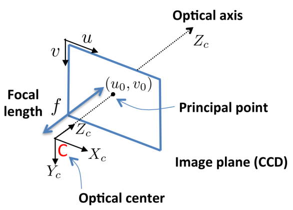
$$R_x(\phi) R_y(\theta) R_z(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{pmatrix} \begin{pmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{pmatrix} \begin{pmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

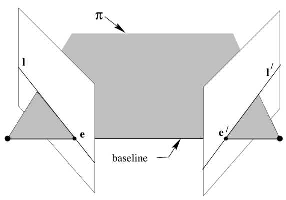
convention: $x^G \doteq \left(R_C^G, t^G_{G\to C}\right)$ re-projection error: $e = z_{observed} - \pi(x_{true}, l_{true})$

$$\lambda \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} = K \underbrace{\left(R_G^C \;\middle|\; t^C_{C\to G} \right)}_{M} \begin{pmatrix} x^G \\ y^G \\ z^G \\ 1 \end{pmatrix} = M \begin{pmatrix} x^G \\ y^G \\ z^G \\ 1 \end{pmatrix}, \;\; K = \begin{pmatrix} \alpha_x & 0 & u_0 \\ 0 & \alpha_y & v_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$u = \frac{m_{11}x + m_{12}y + m_{13}z + m_{14}}{m_{31}x + m_{32}y + m_{33}z + m_{34}} \quad v = \frac{m_{21}x + m_{22}y + m_{23}z + m_{24}}{m_{31}x + m_{32}y + m_{33}z + m_{34}}$$

$a_{x,y} = f k_{x,y}$, u_0, v_0 - PP in pixels





Epipolar Geometry

$$q_1 \triangleq K_1^{-1} x_1 \;\; q_2 \triangleq K_2^{-1} x_2 \quad \text{Epipolar constraint: } q_2^T [t^2_{2\to1}]_\times R_1^2 q_1 = 0$$

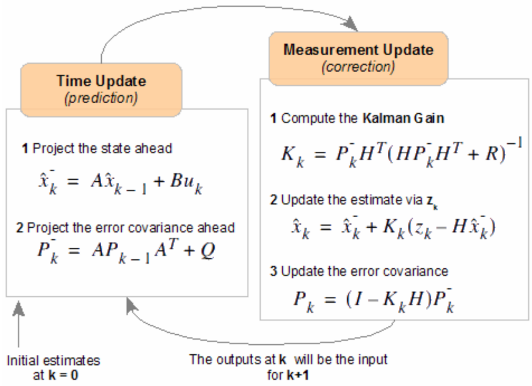
Essential M: $E = [t]_\times R \in \mathbb{R}^{3\times3}$ Fundamental M: $E = K_2^T F K_1$

$$\text{F estimation: } \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}^T F \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = 0, \;\; \left(x'x, x'y, x', y'x, y'y, y', x, y, 1\right) \boldsymbol{f} = 0$$

- F is a rank 2 homogeneous matrix with 7 degrees of freedom.
- **Point correspondence**: If $\mathbf{x}$ and $\mathbf{x}'$ are corresponding image points, then $\mathbf{x}'^{\mathsf{T}} \mathtt{F} \mathbf{x} = 0$.
- **Epipolar lines**:
  - ◇ $\mathbf{l}' = \mathtt{F}\mathbf{x}$ is the epipolar line corresponding to $\mathbf{x}$.
  - ◇ $\mathbf{l} = \mathtt{F}^{\mathsf{T}}\mathbf{x}'$ is the epipolar line corresponding to $\mathbf{x}'$.
- **Epipoles**:
  - ◇ $\mathtt{F}\mathbf{e} = \mathbf{0}$.
  - ◇ $\mathtt{F}^{\mathsf{T}}\mathbf{e}' = \mathbf{0}$.
- **Computation from camera matrices** $\mathtt{P}, \mathtt{P}'$:
  - ◇ General cameras, $\mathtt{F} = [\mathbf{e}']_\times \mathtt{P}'\mathtt{P}^+$, where $\mathtt{P}^+$ is the pseudo-inverse of $\mathtt{P}$, and $\mathbf{e}' = \mathtt{P}'\mathbf{C}$, with $\mathtt{P}\mathbf{C} = \mathbf{0}$.
  - ◇ Canonical cameras, $\mathtt{P} = [\mathtt{I} \mid \mathbf{0}]$, $\mathtt{P}' = [\mathtt{M} \mid \mathbf{m}]$, $\mathtt{F} = [\mathbf{e}']_\times \mathtt{M} = \mathtt{M}^{-\mathsf{T}}[\mathbf{e}]_\times$, where $\mathbf{e}' = \mathbf{m}$ and $\mathbf{e} = \mathtt{M}^{-1}\mathbf{m}$.
  - ◇ Cameras not at infinity $\mathtt{P} = \mathtt{K}[\mathtt{I} \mid \mathbf{0}]$, $\mathtt{P}' = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}]$, $\mathtt{F} = \mathtt{K}'^{-\mathsf{T}}[\mathbf{t}]_\times \mathtt{R}\mathtt{K}^{-1} = [\mathtt{K}'\mathbf{t}]_\times \mathtt{K}'\mathtt{R}\mathtt{K}^{-1} = \mathtt{K}'^{-\mathsf{T}}\mathtt{R}\mathtt{K}^{\mathsf{T}}[\mathtt{K}\mathtt{R}^{\mathsf{T}}\mathbf{t}]_\times$.

Kalman Filter

$$x_{k+1} = A_k x_k + B_k u_k + \omega_k, \;\; z_k = H_k x_k + v_k$$



SLAM

$$J_{BA}(X, L) \doteq \sum_{i=1}^{N} \sum_{j \in \mathcal{M}_i} \|z_{i,j} - \pi(x_i, l_j)\|^2_\Sigma$$

Solve: Gauss-Newton: (1) Linearize (Taylor 1st)

$$J(\bar{X} + \Delta X, \bar{L} + \Delta L) \approx \sum_{i=1}^{N} \sum_{j \in \mathcal{M}_i} \left\|z_{i,j} - \pi(\bar{x}_i, \bar{l}_j) - \nabla_{x_i}\pi \cdot \Delta x_i - \nabla_{l_j}\pi \cdot \Delta l_j\right\|^2_\Sigma$$

$$J(\bar{\Theta} + \Delta\Theta) \approx \sum_{i=1}^{N} \sum_{j \in \mathcal{M}_i} \left\|z_{i,j} - \pi(\bar{x}_i, \bar{l}_j) - A_{i,j}\Delta\Theta\right\|^2_\Sigma = \sum_{i=1}^{N} \sum_{j \in \mathcal{M}_i} \|A_{i,j}\Delta\Theta - b_{i,j}\|^2_\Sigma \quad \Theta \doteq \{X, L\}$$

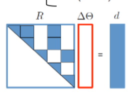
$$J(\bar{\Theta} + \Delta\Theta) \approx \sum_{i=1}^{N} \sum_{j \in \mathcal{M}_i} \Big\|\underbrace{\Sigma^{-1/2}A_{i,j}}_{\doteq \mathcal{A}_{i,j}}\Delta\Theta - \underbrace{\Sigma^{-1/2}b_{i,j}}_{\doteq \breve{b}_{i,j}}\Big\|^2 \rightarrow J(\bar{\Theta} + \Delta\Theta) \approx \left\|\mathcal{A}\Delta\Theta - \breve{b}\right\|^2 \quad \mathcal{A} \doteq \begin{pmatrix} \vdots \\ \mathcal{A}_{i,j} \\ \vdots \end{pmatrix}, \;\; \breve{b} = \begin{pmatrix} \vdots \\ \breve{b}_{i,j} \\ \vdots \end{pmatrix}$$

(2) Calculate optimal increment $\Delta\Theta$, update, repeat $\bar{\Theta} \leftarrow \bar{\Theta} + \Delta\Theta$

Calculating the increment: (a) for dense matrixes: $\Delta\Theta = (\mathcal{A}^T\mathcal{A})^{-1}\mathcal{A}^T\breve{b}$

(b) sparse matrixes – QR factorization

$$\left\|\mathcal{A}\Delta\Theta - \breve{b}\right\|_2^2 = \|R\Delta\Theta - d\|_2^2 + \|e\|_2^2 \qquad Q^T\mathcal{A} = \begin{bmatrix} R \\ 0 \end{bmatrix} \qquad Q^T\breve{b} \doteq \begin{bmatrix} d \\ e \end{bmatrix}$$



- obtain least squares solution via back-substitution

Smoothing and Mapping (SAM) $x^\star_{0:k}, L^\star_k = \underset{x_{0:k}, L_k}{\arg\max}\, p(x_{0:k}, L_k \mid u_{0:k-1}, z_{0:k})$

- full joint pdf :

$$p(x_{0:k}, L_k \mid u_{0:k-1}, z_{0:k}) = \eta p(x_0) \prod_i \left[p(x_i \mid x_{i-1}, u_{i-1}) \prod_{j \in \mathcal{M}_i} p(z_{i,j} \mid x_i, l_j) \right]$$

- MAP, sol: least sq. problem

$$x^\star_{0:k}, L^\star_k = \underset{x_{0:k}, L_k}{\arg\min} \left\{ \|x_0 - \hat{x}_0\|^2_{\Sigma_0} + \sum_i \left[\|x_i - f(x_{i-1}, u_{i-1})\|^2_{\Sigma_w} + \sum_{j \in \mathcal{M}_i} \|z_{i,j} - h(x_i, l_j)\|^2_{\Sigma_v} \right] \right\}$$

– Identify nodes (variables) in new factor graph that are **involved** in new factors

– Find all paths in the **previous Bayes net** that lead from the last eliminated node (**the root**) to each of the involved nodes

– Nodes on the found paths should be re-eliminated