086761 - Homework 3

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1.1

$$\mathbb{P}(x \mid z_1, z_2) \propto \mathbb{P}(z_1, z_2 \mid x)\mathbb{P}(x) = \mathbb{P}(z_1 \mid x)\mathbb{P}(z_2 \mid x)\mathbb{P}(x) \tag{1.1}$$

$$\propto e^{-\frac{1}{2}\left(\|z_1-h_1(x)\|^2_{\Sigma_{v_1}}+\|z_2-h_2(x)\|^2_{\Sigma_{v_2}}+\|x-\hat{x}_0\|^2_{\Sigma_0}\right)} \tag{1.2}$$

Linearize the exponent (dropping constant factor) around $\bar{x}$, denote H_1, H_2 - the Jacobians of h_1, h_2 respectively at $\bar{x}$:

$$\|z_1 - h_1(x)\|^2_{\Sigma_{v_1}} + \|z_2 - h_2(x)\|^2_{\Sigma_{v_2}} + \|x - \hat{x}_0\|^2_{\Sigma_0} \approx \tag{1.3}$$

$$\|H_1 x + h_1(\bar{x}) - H_1\bar{x} - z_1\|^2_{\Sigma_{v_1}} + \|H_2 x + h_2(\bar{x}) - H_2\bar{x} - z_2\|^2_{\Sigma_{v_2}} + \|x - \hat{x}_0\|^2_{\Sigma_0} \tag{1.4}$$

The information matrix is the sum of quadratic coefficients (similar to the previous homework):

$$I = H_1^T \Sigma_{v_1}^{-1} H_1 + H_2^T \Sigma_{v_2}^{-1} H_2 + \Sigma_0^{-1} \tag{1.5}$$

$$\mu = -I^{-1}\left(H_1^T \Sigma_{v_1}^{-1}(h_1(\bar{x}) - H_1\bar{x} - z_1) + H_2^T \Sigma_{v_2}^{-1}(h_2(\bar{x}) - H_2\bar{x} - z_2) - \Sigma_0^{-1}\hat{x}_0\right) \tag{1.6}$$

similar to the previous homework, this can be refined by updating linearization point $\bar{x} \leftarrow \mu$ and recalculating Jacobians, until convergence.

1.2

Re-projection error for measurement z given belief over state variables x and l is defined as

$$e = z - \pi\left(\hat{x}, \hat{l}\right), \tag{1.7}$$

where

$$\hat{x}, \hat{l} = \arg\max_{x,l} \mathbb{P}(x, l \mid z_1, z_2) \tag{1.8}$$

Joint state belief (prior to measurement z):

$$\mathbb{P}(x, l \mid z_1, z_2) = \mathbb{P}(x \mid l, z_2, z_2)\mathbb{P}(l \mid z_2, z_2) = \mathbb{P}(x \mid z_1, z_2)\mathbb{P}(l), \tag{1.9}$$

whence

$$\hat{x} = \mu, \;\; \hat{l} = \hat{l}_0 \tag{1.10}$$

$$e = z - \pi(\mu, \hat{l}_0) \tag{1.11}$$

1.3

$$\mathbb{P}(x,l \mid z_1,z_2,z) = \frac{\mathbb{P}(z \mid x,l,z_1,z_2)\mathbb{P}(x,l \mid z_1,z_2)}{\mathbb{P}(z \mid z_1,z_2)} = \tag{1.12}$$

$$= \frac{\mathbb{P}(z \mid x,l)\mathbb{P}(x \mid z_1,z_2)\mathbb{P}(l)}{\int_{x,l}\mathbb{P}(z \mid x,l)\mathbb{P}(x \mid z_1,z_2)\mathbb{P}(l)dxdl} \propto \mathbb{P}(z \mid x,l)\mathbb{P}(x \mid z_1,z_2)\mathbb{P}(l) \tag{1.13}$$

1.4

From previous clause,

$$-\log(\mathbb{P}(x,l \mid z,z_1,z_2)) \propto \|z-\pi(x,l)\|^2_{\Sigma_v} + \|x-\mu\|^2_{\Sigma} + \|l-\hat{l}_0\|^2_{\Sigma_{l0}} + C \tag{1.14}$$

With C constant (normalizing factor). Linearizing π around $\mu,\hat{l}_0$, and denoting its Jacobian at that point as $H_\pi = \begin{pmatrix} H_\pi^{(x)} & H_\pi^{(l)} \end{pmatrix}$ we can rewrite

$$\approx \|\pi(\mu,\hat{l}_0) + H_\pi^{(x)}(x-\mu) + H_\pi^{(l)}(l-\hat{l}_0) - z\|^2_{\Sigma_v} + \|x-\mu\|^2_{\Sigma} + \|l-\hat{l}_0\|^2_{\Sigma_{l0}} + C \tag{1.15}$$

extracting the quadratic term coefficients (recall that variable is the joint state i.e. $(x^T\,l^T)^T$), we get that

$$= \begin{pmatrix} x^T & l^T \end{pmatrix}\left(H_\pi^T\Sigma_v^{-1}H_\pi + \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & \Sigma_{l0}^{-1} \end{pmatrix}\right)\begin{pmatrix} x \\ l \end{pmatrix} + \ldots, \tag{1.16}$$

and so

$$I' = H_\pi^T\Sigma_v^{-1}H_\pi + \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & \Sigma_{l0}^{-1} \end{pmatrix} = \begin{pmatrix} H_\pi^{(x)T}\Sigma_v^{-1}H_\pi^{(x)} + \Sigma^{-1} & H_\pi^{(x)T}\Sigma_v^{-1}H_\pi^{(l)} \\ H_\pi^{(l)T}\Sigma_v^{-1}H_\pi^{(x)} & H_\pi^{(l)T}\Sigma_v^{-1}H_\pi^{(l)} + \Sigma_{l0}^{-1} \end{pmatrix} \tag{1.17}$$

The latter, similar to before, can be refined by iteritavely relinearizing around the minimum point of Eq. (1.15) (calculated e.g. using the linear term, or finding the derivative zero, or formulating as a linear min-squares solution).

To state explicitly, a min squares formulation of Eq. (1.15) is

$$\arg\min_{x,l}\left\| \underbrace{\begin{pmatrix} \Sigma_v^{-\frac{1}{2}}H_\pi^{(x)} & \Sigma_v^{-\frac{1}{2}}H_\pi^{(l)} \\ \Sigma^{-\frac{1}{2}} & 0 \\ 0 & \Sigma_{l0}^{-\frac{1}{2}} \end{pmatrix}}_{A}\begin{pmatrix} x \\ l \end{pmatrix} - \begin{pmatrix} \Sigma_v^{-\frac{1}{2}}\left(-\pi(\mu,\hat{l}_0) + H_\pi^{(x)}\mu + H_\pi^{(l)}\hat{l}_0 + z\right) \\ \Sigma^{-\frac{1}{2}}\mu \\ \Sigma_{l0}^{-\frac{1}{2}}\hat{l}_0 \end{pmatrix} \right\|^2, \tag{1.18}$$

whence we obtain the same result for the information matrix ($I' = A^TA$).

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2.1

Denote camera calibration matrices as K_1, K_2 respectively.

$$t \doteq t^{C_2}_{C_1 \to C_2} = R^{C_2}_G \cdot (t^G_{C_1 \to G} - t^G_{C_2 \to G}) = R_2^T \cdot (t_1 - t_2) \tag{2.1}$$

$$R \doteq R^{C_2}_{C_1} = R^{C_2}_G \cdot R^G_{C_1} = R_2^T \cdot R_1 \tag{2.2}$$

The epipolar constraint:

$$\begin{pmatrix} z_2^T & 1 \end{pmatrix} K_2^{-T} [t]_\times R K_1^{-1} \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0 \tag{2.3}$$

2.2

$$\mathbb{P}(x_1, x_2 \mid z_2, z_2) \propto \mathbb{P}(z_1, z_2 \mid x_1, x_2)\mathbb{P}(x_1)\mathbb{P}(x_2) \tag{2.4}$$

Treating the correspondence between z_1, z_2 as the measurement, we can model likelihood term as

$$\mathbb{P}(z_1, z_2 \mid x_1, x_2) = \mathbb{P}(h(x_1, x_2, z_1, z_2)) \tag{2.5}$$

where we assume that

$$\arg\max_{x_1, x_2} \mathbb{P}(z_1, z_2 \mid x_1, x_2) = \arg\max_{x_1, x_2} \mathbb{P}(h(x_1, x_2, z_1, z_2)) \tag{2.6}$$

and

$$h(x_1, x_2, z_1, z_2) \sim N(0, \Sigma_{ep}) \tag{2.7}$$

with the motivation being that given matching points z_1, z_2, epipolar constraint should satisfied, up to error, so h should be close to 0.

In summary, the MAP estimation can be written (in negative log-likelihood form, as before) as

$$\arg\max_{x_1, x_2} \mathbb{P}(x_1, x_2 \mid z_1, z_2) = \tag{2.8}$$

$$= \arg\min_{x_1, x_2} \|h(x_1, x_2, z_1, z_2)\|^2_{\Sigma_{ep}} + \|x_1 - \mu_{01}\|^2_{\Sigma_{01}} + \|x_2 - \mu_{02}\|^2_{\Sigma_{02}} \tag{2.9}$$

3

We start from the definition of fundamental matrix from the lecture:

$$F \doteq K_2^{-T}[t]_\times R K_1^{-1}, \tag{3.1}$$

with $K_{1,2}$, t and R defined as in question 2a. Denote $u = K_1 R^{-1} t$, where R^{-1} exists since R is a rotation matrix. Then

$$F\,u = K_2^{-T}[t]_\times R K_1^{-1} K_1 R^{-1} t = K_2^{-T}\,(t \times t) = 0, \tag{3.2}$$

whence F is singular.