**VAN – Homwork #3**

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1)

a.

$$p(x) = N(\hat{x}_0, \Sigma_0)$$

$$z_1 = h_1(x) + v_1, \qquad v \sim N(0, \Sigma_{v1})$$

$$z_2 = h_2(x) + v_2, \qquad v \sim N(0, \Sigma_{v2})$$

$$p(x|z_1, z_2) = \frac{p(z_2|x)p(x|z_1)}{p(z_2|z_1)} = \frac{p(z_2|x)p(z_1|x)p(x)}{p(z_2|z_1)p(z_1)} \propto p(z_2|x)p(z_1|x)p(x)$$

$$x = \tilde{x} + \Delta x$$

$$J(\tilde{x} + \Delta x) = \left\| \Sigma_{v2}^{-\frac{1}{2}} (H \Delta x + h_2(\tilde{x}) - z_2) \right\|^2 + \left\| \Sigma_{v1}^{-\frac{1}{2}} (H \Delta x + h_1(\tilde{x}) - z_1) \right\|^2$$

$$+ \left\| \Sigma_0^{-\frac{1}{2}} (\Delta x + \tilde{x} - \hat{x}_0) \right\|^2 = \left\| \overbrace{\begin{pmatrix} \Sigma_{v2}^{-\frac{1}{2}} H \\ \Sigma_{v1}^{-\frac{1}{2}} H \\ \Sigma_0^{-\frac{1}{2}} \end{pmatrix}}^{A} \Delta x - \overbrace{\begin{pmatrix} \Sigma_{v2}^{-\frac{1}{2}} (z_2 - h_2(\tilde{x})) \\ \Sigma_{v1}^{-\frac{1}{2}} (z_1 - h_1(\tilde{x})) \\ \Sigma_0^{-\frac{1}{2}} (\hat{x}_0 - \tilde{x}) \end{pmatrix}}^{b} \right\|^2$$

$$= \|A \Delta x - b\|^2 = \left\| (A^T A)^{\frac{1}{2}} \left( \Delta x - (A^T A)^{-\frac{1}{2}} b \right) \right\|^2$$

$$= \left\| \left( \Delta x - (A^T A)^{-\frac{1}{2}} b \right) \right\|^2_{(A^T A)^{-1}}$$

$$\Delta x^* = arg \min_{\Delta x} J(\tilde{x} + \Delta x) = arg \min_{\Delta x} \|A \Delta x - b\|^2$$

Update until convergence:

$$\tilde{x} \leftarrow \tilde{x} + \Delta x^*$$

After convergence:

$$\Sigma = (A^T A)^{-1}$$

The a posteriori information matrix:

$$I = \Sigma^{-1} = A^T A = H^T \Sigma_{v2}^{-1} H + H^T \Sigma_{v1}^{-1} H + \Sigma_0^{-1}$$

b.

$$p(x|z_1, z_2) = N(\mu, \Sigma)$$

$$p(l) = N(\hat{l}_0, \Sigma_{l0})$$

The initial re-projection error is:

$$z - \pi(\mu, \hat{l}_0)$$

c.

$$p(x, l|z_1, z_2, z) = \frac{p(x, l)p(z_1, z_2, z|x, l)}{p(z_1, z_2, z)} = \frac{\overbrace{p(x)p(l)}^{x,l\ independent}\ \overbrace{p(z_1|x, l)p(z_2|x, l)p(z|x, l)}^{z1,z2,z\ independent}}{p(z_1, z_2, z)}$$
$$= \frac{p(x)p(l)p(z_1|x)p(z_2|x)p(z|x, l)}{\underbrace{\iint p(z_1, z_2, z|x, l)p(x, l)dxdl}_{marginalization}}$$
$$= \frac{p(x)p(l)p(z_1|x)p(z_2|x)p(z|x, l)}{\iint p(x)p(l)p(z_1|x)p(z_2|x)p(z|x, l)dxdl}$$

d.

To find I' of p(x, l | z1, z2, z) , we need to find

$(\hat{x}, \hat{l})_{MAP} = arg \max_{x,l} \mathrm{p(x, l \mid z1, z2, z)}$ .

$p(x|z_1, z_2) = N(\mu, \Sigma)$, p(x) and p(l) are given,

x and l are independent, therefore p(x, l) = p(x)p(l) .

$$p(x, l|z_1, z_2, z) = \frac{p(z_1, z_2, \mathrm{z}\ |x, l)p(x, l)}{p(z_1, z_2, \mathrm{z})} = \frac{p\left(z_{1,}z_2|x, l\right)p(\mathrm{z}|x, l)p(x)p(l)}{p(z_1, z_2)p(\mathrm{z})}$$
$$= \frac{p\left(z_{1,}z_2|x\right)p(\mathrm{z}|x, l)p(x)p(l)}{p(z_1, z_2)p(\mathrm{z})} = \frac{p\left(x|z_{1,}z_2\right)p\left(z_{1,}z_2\right)p(\mathrm{z}|x, l)p(x)p(l)}{p(x)p(z_1, z_2)p(\mathrm{z})}$$
$$= \frac{p\left(x|z_{1,}z_2\right)p(\mathrm{z}|x, l)p(l)}{p(\mathrm{z})} = \frac{p\left(x|z_{1,}z_2\right)p(\mathrm{z}|x, l)p(l)}{\int p(\mathrm{z|x, l})p(\mathrm{x, l})\mathrm{dxdl}}$$
$$\propto \eta_1 p\left(x|z_{1,}z_2\right)p(\mathrm{z}|x, l)p(l)$$
$$= \eta_2 exp^{(-\frac{1}{2}(\|(x-\mu)\|^2{}_{\Sigma} + \|(z-\pi(x,l))\|^2{}_{\Sigma_v} + \left\|(l-\hat{l}_0)\right\|^2{}_{\Sigma_{l_0}}))}$$

Therefore,

$$(\hat{x}, \hat{l})_{MAP} = arg \max_{x,l} \mathrm{p(x, l \mid z1, z2, z)}$$
$$= arg \min_{x,l} \mathrm{J(x, l)}$$
$$= arg \min_{x,l}(\|(x-\mu)\|^2{}_{\Sigma} + \|(z-\pi(x, l))\|^2{}_{\Sigma_v} + \left\|(l-\hat{l}_0)\right\|^2{}_{\Sigma_{l_0}})$$

Linearization:

$\hat{x}_{MAP} = x = \bar{x} + \Delta x$

$\hat{l}_{MAP} = l = \bar{l} + \Delta\mathrm{l}$

Therefore,

$\pi\left(\hat{x}_{MAP}, \hat{l}_{MAP}\right) = \pi\left(\bar{x} + \Delta x, \bar{l} + \Delta\mathrm{l}\right) = \pi\left(\bar{x}, \bar{l}\right) + \pi_x\Delta\mathrm{x} + \pi_l\Delta\mathrm{l}$

$\pi_x = \left[\frac{\partial\pi}{\partial x}\right]_{\bar{x},\bar{l}}, \pi_l = \left[\frac{\partial\pi}{\partial l}\right]_{\bar{x},\bar{l}}$

$$
\begin{aligned}
&J(\bar{x}+\Delta x, \bar{l}+\Delta \mathrm{l}) \\
&\qquad = \|\Delta x+(\bar{x}-\mu)\|^2{}_{\Sigma}+\left\|\left(z-\pi(\bar{x}, \bar{l})-\pi_x \Delta \mathrm{x}-\pi_l \Delta \mathrm{l}\right)\right\|^2{}_{\Sigma_v} \\
&\qquad + \left\|\Delta \mathrm{l}+\left(\bar{l}-\hat{l}_0\right)\right\|^2{}_{\Sigma_{l_0}} \\
&\qquad = \left\|\Sigma^{-\frac{1}{2}}(\Delta x+(\bar{x}-\mu))\right\|^2+\left\|\Sigma_v{}^{-\frac{1}{2}}\left(z-\pi(\bar{x}, \bar{l})-\pi_x \Delta \mathrm{x}-\pi_l \Delta \mathrm{l}\right)\right\|^2 \\
&\qquad + \left\|\Sigma_{l_0}{}^{-\frac{1}{2}}(\Delta \mathrm{l}+(\bar{l}-\hat{l}_0))\right\|^2 \\
&\qquad = \left\|\begin{matrix} \Sigma^{-\frac{1}{2}}\Delta x+\Sigma^{-\frac{1}{2}}(\bar{x}-\mu) \\ \Sigma_v{}^{-\frac{1}{2}}(\pi_x\Delta \mathrm{x}+\pi_l\Delta \mathrm{l})-\Sigma_v{}^{-\frac{1}{2}}\left(z-\pi(\bar{x}, \bar{l})\right) \\ \Sigma_{l_0}{}^{-\frac{1}{2}}\Delta l+\Sigma_{l_0}{}^{-\frac{1}{2}}(\bar{l}-\hat{l}_0) \end{matrix}\right\|^2 \\
&\qquad = \left\|\overbrace{\begin{bmatrix} \Sigma^{-\frac{1}{2}} & 0 \\ \Sigma_v{}^{-\frac{1}{2}}\pi_x & \Sigma_v{}^{-\frac{1}{2}}\pi_l \\ 0 & \Sigma_{l_0}{}^{-\frac{1}{2}} \end{bmatrix}}^{A}\begin{pmatrix}\Delta x\\ \Delta l\end{pmatrix}-\overbrace{\begin{pmatrix} \Sigma^{-\frac{1}{2}}(\mu-\bar{x}) \\ \Sigma_v{}^{-\frac{1}{2}}\left(z-\pi(\bar{x}, \bar{l})\right) \\ \Sigma_{l_0}{}^{-\frac{1}{2}}(\hat{l}_0-\bar{l}) \end{pmatrix}}^{b}\right\|^2 \\
&\qquad = \left\|A\begin{pmatrix}\Delta x\\ \Delta l\end{pmatrix}-b\right\|^2=\left\|(A^TA)^{\frac{1}{2}}\left(\begin{pmatrix}\Delta x\\ \Delta l\end{pmatrix}-(A^TA)^{-\frac{1}{2}}b\right)\right\|^2 \\
&\qquad = \left\|\left(\begin{pmatrix}\Delta x\\ \Delta l\end{pmatrix}-(A^TA)^{-\frac{1}{2}}b\right)\right\|^2_{(A^TA)^{-1}}
\end{aligned}
$$

We need:

$$A\begin{pmatrix}\Delta x\\ \Delta l\end{pmatrix}-b=0$$

$$\begin{pmatrix}\Delta x\\ \Delta l\end{pmatrix}=(A^TA)^{-1}A^Tb$$

Update until convergence:

$$\begin{pmatrix}\bar{x}\\ \bar{l}\end{pmatrix}\leftarrow\begin{pmatrix}\bar{x}\\ \bar{l}\end{pmatrix}+\begin{pmatrix}\Delta x\\ \Delta l\end{pmatrix}$$

After convergence:

$$\Sigma=(A^TA)^{-1}$$

$$I'=\Sigma^{-1}=A^TA=\begin{bmatrix}\Sigma^{-1}+\pi_x{}^T\Sigma_v^{-1}\pi_x & \pi_x{}^T\Sigma_v^{-1}\pi_l \\ \pi_l{}^T\Sigma_v^{-1}\pi_x & \pi_l{}^T\Sigma_v^{-1}\pi_l+\Sigma_{l_0}^{-1}\end{bmatrix}$$

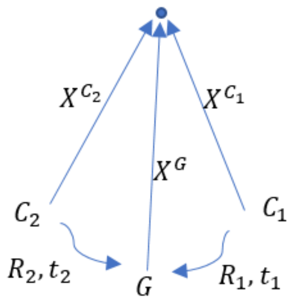
2)

a.

$$x_1=(R_1,t_1)\ ,\ \ x_2=(R_2,t_2)$$

Where:

$$R_i = R_{C_i}^G \ , \ t_i = t_{C_i \to G}^G \ , \ i = 1,2$$




Transforming a 3D point X between the two camera is done by:

$$X^{C_2} = R_{C_1}^{C_2} X^{C_1} + t_{C_2 \to C_1}^{C_2}$$

We multiply the left hand side by $\left[t_{C_2 \to C_1}^{C_2}\right]_\times$:

$$\left[t_{C_2 \to C_1}^{C_2}\right]_\times X^{C_2} = \left[t_{C_2 \to C_1}^{C_2}\right]_\times R_{C_1}^{C_2} X^{C_1} + \overbrace{\left[t_{C_2 \to C_1}^{C_2}\right]_\times t_{C_2 \to C_1}^{C_2}}^{a \times a = 0}$$

We multiply the left hand side by ${X^{C_2}}^T$:

$$\overbrace{{X^{C_2}}^T \left[t_{C_2 \to C_1}^{C_2}\right]_\times X^{C_2}}^{a^T (b \times a) = 0} = {X^{C_2}}^T \left[t_{C_2 \to C_1}^{C_2}\right]_\times R_{C_1}^{C_2} X^{C_1}$$

The epipolar constraint:

$${X^{C_2}}^T \left[t_{C_2 \to C_1}^{C_2}\right]_\times R_{C_1}^{C_2} X^{C_1} = 0$$

Where:

$$R_{C_1}^{C_2} = R_G^{C_2} R_{C_1}^G = R_2^T R_1$$

$$t_{C_2 \to C_1}^{C_2} = R_{C_2}^G \left(t_{C_2 \to G}^G - t_{C_1 \to G}^G\right) = R_2^T (t_2 - t_1)$$

$$\tilde{\omega}_i \begin{pmatrix} z_i \\ 1 \end{pmatrix} = \tilde{\omega}_i \begin{pmatrix} u_i \\ v_i \\ 1 \end{pmatrix} = \begin{pmatrix} \tilde{u}_i \\ \tilde{v}_i \\ \tilde{\omega}_i \end{pmatrix} = K_i \left[R_{C_i}^G \quad t_{C_i \to G}^{C_i}\right] \begin{pmatrix} X_G \\ Y_G \\ Z_G \\ 1 \end{pmatrix} = K_i X^{C_i}$$

$$X^{C_1} = \tilde{\omega}_1 K_1^{-1} \begin{pmatrix} z_1 \\ 1 \end{pmatrix} \ , \ X^{C_2} = \tilde{\omega}_2 K_2^{-1} \begin{pmatrix} z_2 \\ 1 \end{pmatrix}$$

So, the epipolar constraint can now be written as:

$$\tilde{\omega}_2 \left(K_2^{-1}\binom{z_2}{1}\right)^T [R_2^T(t_2-t_1)]_\times R_2^T R_1 \tilde{\omega}_1 K_1^{-1}\binom{z_1}{1} = 0 \backslash : (\tilde{\omega}_1\tilde{\omega}_2)$$

$$\left(K_2^{-1}\binom{z_2}{1}\right)^T [R_2^T(t_2-t_1)]_\times R_2^T R_1 K_1^{-1}\binom{z_1}{1} = 0$$

b.

We have a prior on each camera:

$$p(x_1) = N(\mu_{01}, \Sigma_{01}) \ , \ \ p(x_2) = N(\mu_{02}, \Sigma_{02})$$

In addition:

$$p(h(x_1, x_2, z_1, z_2)|x_1, x_2) = p(h|x_1, x_2) = N(0, \Sigma_{ep})$$

$$p(x_1, x_2|z_1, z_2) = p(x_1, x_2|h) = \frac{p(x_1, x_2, h)}{p(h)} = \frac{p(h|x_1, x_2)p(x_1, x_2)}{p(h)}$$
$$= \frac{p(h|x_1, x_2)p(x_1)p(x_2)}{p(h)} \propto p(h|x_1, x_2)p(x_1)p(x_2)$$

$$\mathrm{x}_{\mathrm{map}} = \arg max\, p(x_1, x_2|z_1, z_2)$$
$$= \arg min\, (\|h(x_1, x_2, z_1, z_2)\|^2_{\Sigma_{ep}} + \|x_1 - \mu_{01}\|^2_{\Sigma_{01}} + \|x_2 - \mu_{02}\|^2_{\Sigma_{02}})$$

3)

the fundamental matrix is defined as:

$$F \triangleq K_2^{-\mathrm{T}}[t]_\times R K_1^{-1}$$

$[t]_\times$ is:

$$[t]_\times = \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix} \ , \ \ t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

To show singularity we need to show that $\det(\mathrm{F}) = 0$

$$det(F) = det(K_2^{-\mathrm{T}})det([t]_\times)det(R)det(K_1^{-1})$$

Notice that

$$det([t]_\times) = 0 \cdot (0 + t_1^2) - (-t_3)(0 \cdot t_3 - t_1 t_2) + t_2(t_1 t_3 + 0 \cdot t_2) =$$

$$= -t_1 t_2 t_3 + t_1 t_2 t_3 = 0$$

Therefore:

$$det(F) = 0$$

$$F - singular$$