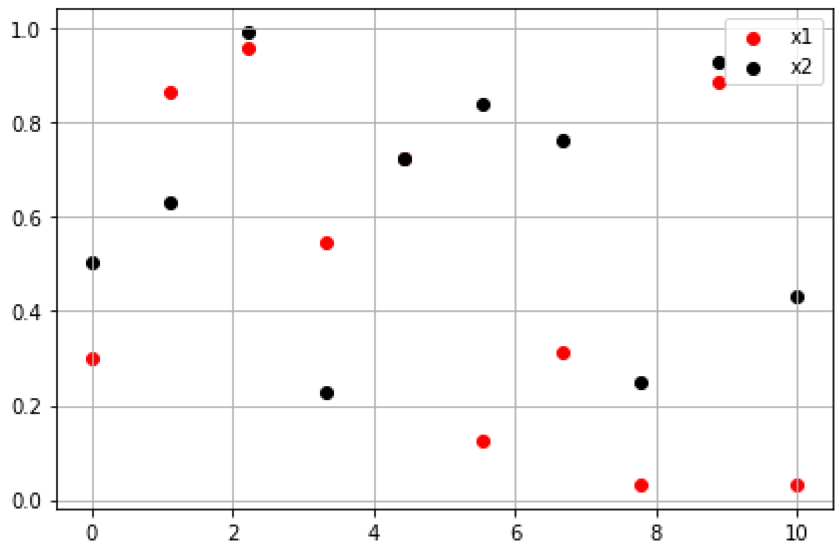
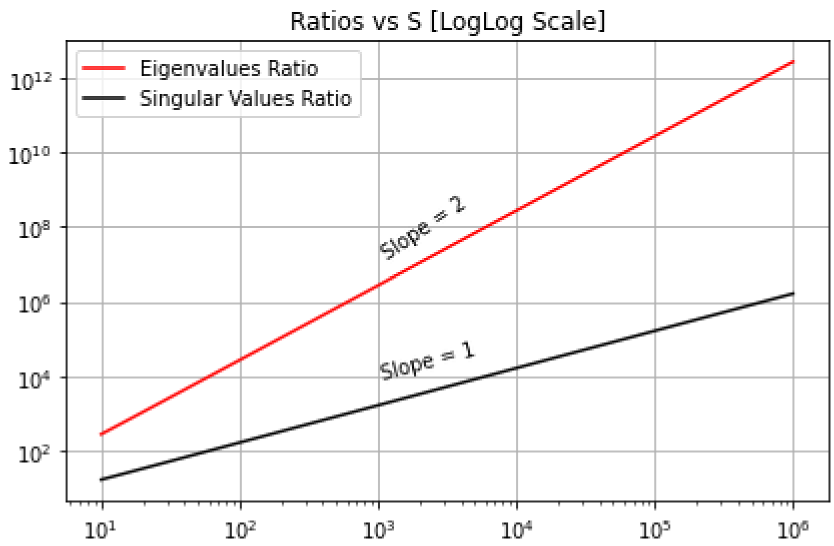
Mathematical Method for DS and PS

HW 1 - Question 5

```
import numpy as np
import matplotlib.pyplot as plt

# Drawing 2 random signals x1, x2 with dimention p

p = 10

x1 = np.random.rand(1,p)
x2 = np.random.rand(1,p)

x = np.linspace(0,p,p)

plt.scatter(x, x1.transpose(), c='r', label='x1')
plt.scatter(x, x2.transpose(), c='k', label='x2')

plt.grid()
plt.legend()
plt.tight_layout()
plt.show()
```



```
# Generate n observations of y

Y = []
n = 1000 # observations
mu = 0
sig = 1

a = np.random.normal(mu, sig, size = (n,1))
b = np.random.normal(mu, sig, size = (n,1))

pts = n//p
S = np.array(np.logspace(1,6,num=pts))
eigen_ratio = np.zeros(pts)
sing_ratio = np.zeros(pts)

for i,s in enumerate(S):

  Y = s * a.dot(x1) + b.dot(x2)

  # Covariance, Eigenvalues, Singular Values
  covar = np.cov(Y, rowvar = False)
  eigen_vals = np.sort(np.linalg.eigvals(covar))[::-1]
  sing_vals = np.linalg.svd(Y, compute_uv=False)

  # Ratio No.1
  eigen_ratio[i] = eigen_vals.real[0] / eigen_vals.real[1]

  # Ratio No.2
  sing_ratio[i] = sing_vals[0] / sing_vals[1]

plt.loglog(S, eigen_ratio, 'r', label='Eigenvalues Ratio')
plt.loglog(S, sing_ratio, 'k', label='Singular Values Ratio')

slope1, intercept1 = np.polyfit(np.log(S), np.log(eigen_ratio), 1)
slope2, intercept2 = np.polyfit(np.log(S), np.log(sing_ratio), 1)

plt.text(10e2, 10e7, "Slope = %.0f" %slope1, size=10, rotation=17*slope1,
         ha="left", va="center")

plt.text(10e2, 2.5*10e3, "Slope = %.0f" %slope2, size=10, rotation=15*slope2,
         ha="left", va="center")

plt.title('Ratios vs S [LogLog Scale]')
plt.grid()
plt.legend()
plt.tight_layout()
plt.show()
```



Conclusions

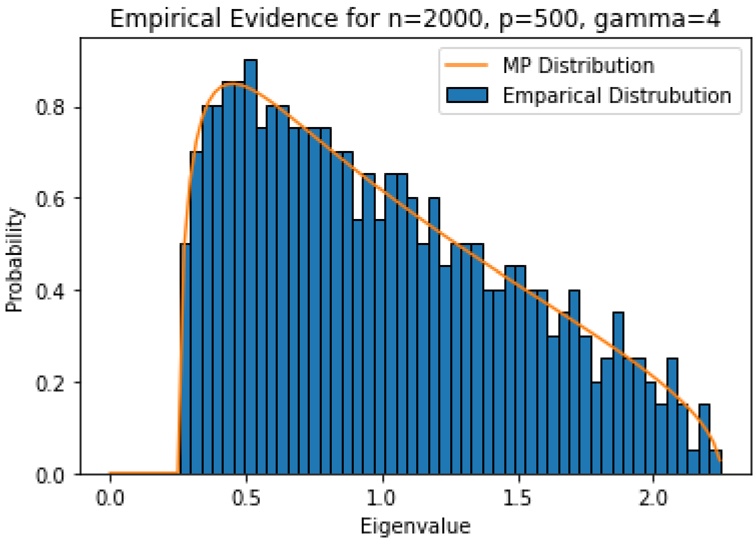
1. We can see that the slope of the ratio of the eigenvalues is 2 times greater than the ratio of the singular values. This result makes sense because a singular value is also known as the positive square root of the eigenvalue. Since our graph is on a logarithmic scale, this square root is represented as a multiple of 2.
2. The eigenvalues are very large and could be affected by numerical errors, so that it will be better to use the SVD.

   Note that the slopes of the graphs are rounded but very close to the values represented.

HW 1 - Question 6 - a

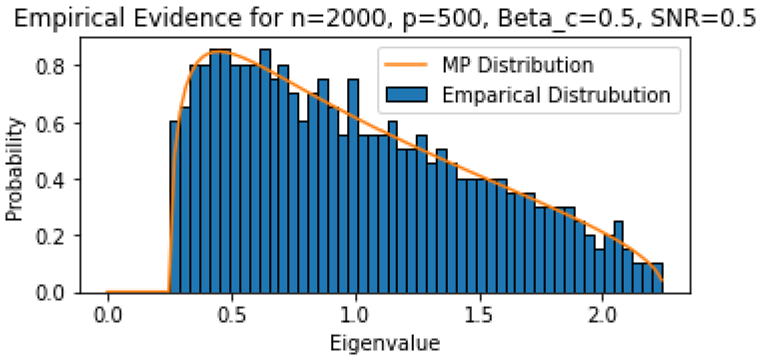
```
import numpy as np

def marchenko_pastur_mu(x, gamma, sigma2=1):
    x = np.atleast_1d(x).astype(float)
    gamma_p = sigma2 * (1 + np.sqrt(gamma)) ** 2
    gamma_m = sigma2 * (1 - np.sqrt(gamma)) ** 2
    mu = np.zeros_like(x)
    is_nonzero = (gamma_m < x) & (x < gamma_p)
    x_valid = x[is_nonzero]
    factor = 1 / (2 * np.pi * sigma2 * gamma)
    mu[is_nonzero] = factor / x_valid
    mu[is_nonzero] *= np.sqrt((gamma_p - x_valid) * (x_valid - gamma_m))
    if gamma > 1:
        mu[x == 0] = 1 - 1 / gamma
    return mu
```

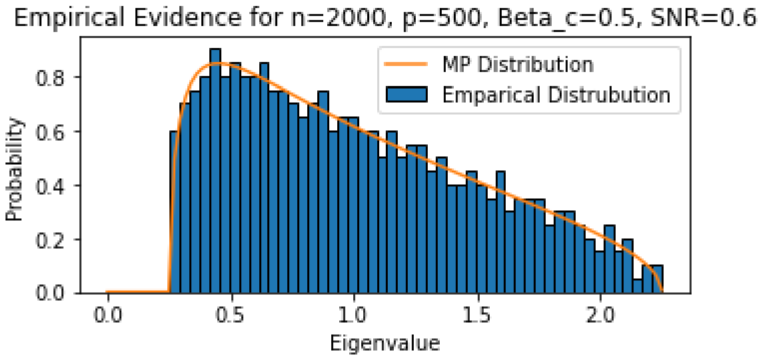
```
import numpy as np

p = 500   # feature length
n = 2000  # examples (observations)

mu = np.asarray([0] * p)
I = np.identity(p)

X = np.random.multivariate_normal(mu, I, size=n)

covar = np.cov(X, rowvar=False)
eigen = np.sort(np.linalg.eigvals(covar)).real
x_axis = np.linspace(0, np.max(eigen), 100)

plt.hist(eigen, 50, density=True, edgecolor='k', label='Emparical Distrubution')
plt.plot(x_axis, marchenko_pastur_mu(x_axis, gamma=p/n), label='MP Distribution')

plt.grid(False)
plt.legend()
plt.title('Empirical Evidence for n={}, p={}, gamma={}'.format(n,p,n//p))
plt.xlabel('Eigenvalue')
plt.ylabel('Probability')


plt.show()
```

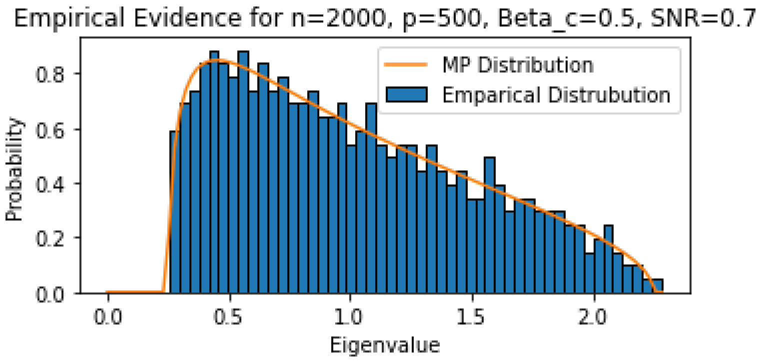

Empirical Evidence for n=2000, p=500, gamma=4
MP Distribution
Emparical Distrubution
Probability
0.0
0.2
0.4
0.6
0.8
0.0
0.5
1.0
1.5
2.0
Eigenvalue

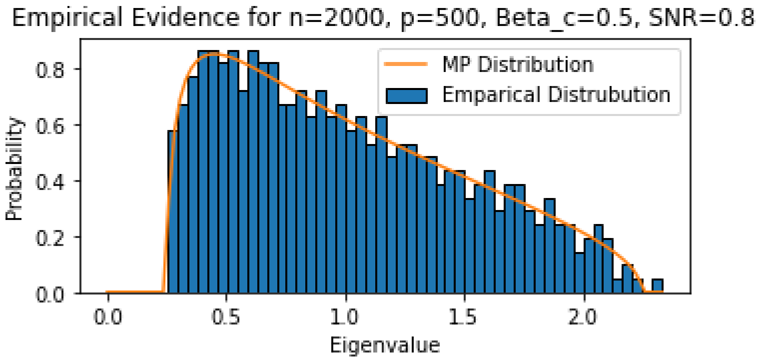
HW 1 - Question 6 - b

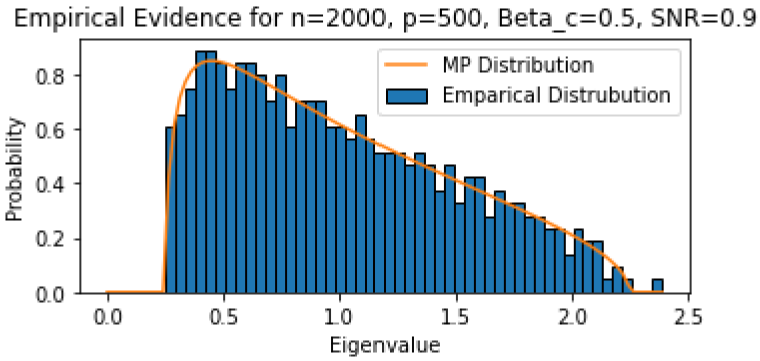
```
n, p = 2000, 500

beta_c = np.sqrt(p/n)

Beta = np.linspace(beta_c,0.9,5)

fig, ax = plt.subplots(len(Beta), figsize=(5.5, 13))

g = np.random.normal(0, 1, [1, n])
g = np.tile(g, (p, 1))
g0 = np.random.normal(0, 1, [p, n])
u = np.random.normal(0, 1, [p, 1])
u = u / np.linalg.norm(u)
u_n = np.tile(u, (1, n))

for i,beta in enumerate(Beta):

  X_spike = np.sqrt(beta) * g * u + g0
  X_spike = X_spike.T

  covar_spike = np.cov(X_spike, rowvar=False)
  eigen_spike = np.sort(np.linalg.eigvals(covar_spike))

  x_axis = np.linspace(0, np.max(eigen_spike), 100)

  ax[i].hist(eigen_spike, 50, density=True, edgecolor='k', label='Emparical Distrub
  ax[i].plot(x_axis, marchenko_pastur_mu(x_axis, gamma=p/n), label='MP Distribution

  ax[i].grid(False)
  ax[i].legend()
  ax[i].set_title('Empirical Evidence for n={}, p={}, Beta_c={}, SNR={}'.format(n,p
  ax[i].set_xlabel('Eigenvalue')
  ax[i].set_ylabel('Probability')

plt.tight_layout()
plt.show()
```


Empirical Evidence for n=2000, p=500, Beta_c=0.5, SNR=0.5
MP Distribution
Emparical Distrubution
Probability
Eigenvalue
0.0
0.2
0.4
0.6
0.8
0.5
1.0
1.5
2.0


Empirical Evidence for n=2000, p=500, Beta_c=0.5, SNR=0.6
MP Distribution
Emparical Distrubution
Probability
Eigenvalue
0.0
0.2
0.4
0.6
0.8
0.5
1.0
1.5
2.0


Empirical Evidence for n=2000, p=500, Beta_c=0.5, SNR=0.7
MP Distribution
Emparical Distrubution
Probability
Eigenvalue
0.0
0.2
0.4
0.6
0.8
0.5
1.0
1.5
2.0


Empirical Evidence for n=2000, p=500, Beta_c=0.5, SNR=0.8
MP Distribution
Emparical Distrubution
Probability
Eigenvalue
0.0
0.2
0.4
0.6
0.8
0.5
1.0
1.5
2.0


Empirical Evidence for n=2000, p=500, Beta_c=0.5, SNR=0.9
MP Distribution
Emparical Distrubution
Probability
Eigenvalue
0.0
0.2
0.4
0.6
0.8
0.5
1.0
1.5
2.0
2.5

Conclusions

The minimal SNR (β) value which we can see the signal is ~0.7-0.8 which is greater than the theorical value (βc) that we expecterd to see in the theorical case and this is probably because in our case n is not infinity.