Major Assignment: Autoregression Distriubances for MLE and LS

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Introduction

This report presents the results and analysis of MLE and LS estimates of a simple autoregressive model. C++ was used to write and optimize both the logliklihood and the LS functions. I first compute estimates for a simple version of the model, and then later on perform Monte Carlo simulations to observe more complicated versions of the model, and to see if the estimates converge overtime. I also hard coded all the optimization functions to not use any packages. I implemented a simple version of functions to generate gradients, to invert matrices, generate gradient descents and to generate Hessian matrices.

Model

This is an AR(2) model which the errors of the model are influence by both the $t-1$ period and the $t-2$ period. ϕ is an important variable of interest as it is the coeffìcent of the errors for each period. There will be cases where the model will be reduced to AR(1) by setting $\phi_2 = 0$.

$$
\begin{aligned}
&y_t = \alpha + \beta t + u_t \\
&u_t = \phi_1 u_{t-1} + \phi_2 u_{t-2} + \varepsilon_t \quad \varepsilon_t \sim \text{NIID}(0, \sigma_\varepsilon^2) \quad \text{for} \quad t = 1, \dots, T
\end{aligned}
$$

Maximum Likelihood and Non-Linear Least Squares

	ML	Standard Errors	t-statistics
α	-0.373115	0.749832	-0.497598
β	1.48195	0.102728	14.426
ϕ	-0.28469	0.573275	-0.496604
σ	2.22299	0.408475	5.44216

Table 1: MLE Estimation Results

The results of the initial estimation with $\phi_2 = 0$ is shown above. We see that β is statistically significant, and with a coeffîcent of 1.48195 we can confirm there is a clear relationship between y and t which makes sense in this context as we are neglecting the $t-2$ part of the error. This is also further confirmed by the ϕ value being insignificant, thus not impacting the AR(1) model much. However, σ also being significant means that there are parts of the estimation that is not captured by the model.

	NLS
α	-0.41569
β	1.47656
ϕ	-0.306232
σ	2.66469

Table 2: NLS AR(1) Estimation Results

We can confirm that MLE is an accurate estimator for NLS as the parameters values are relatively similar. It is important to note that in both cases the logliklihood function and the sum of squared residuals functions are both optimized using the same optimization function present in "func.cpp". The function *gradient_descent* primarly uses the finite difference method to calculate gradients. I ackwnoledge that this is not the most robust manner to optimize, but since I wrote the optimization functions from scratch I chose this method.

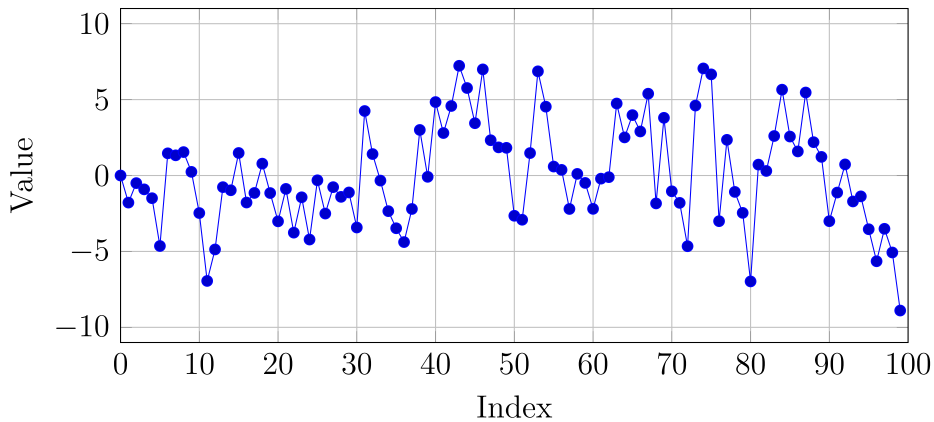
	OLS	NLS	MLE
α	-0.363624	-0.146065	-0.608293
β	1.47902	1.44913	1.50779
ϕ_1	-0.328802	-0.327517	-0.249232
ϕ_2	-0.0609359	-0.0586215	-0.033152
σ^2	3.26954	3.33666	0.452973

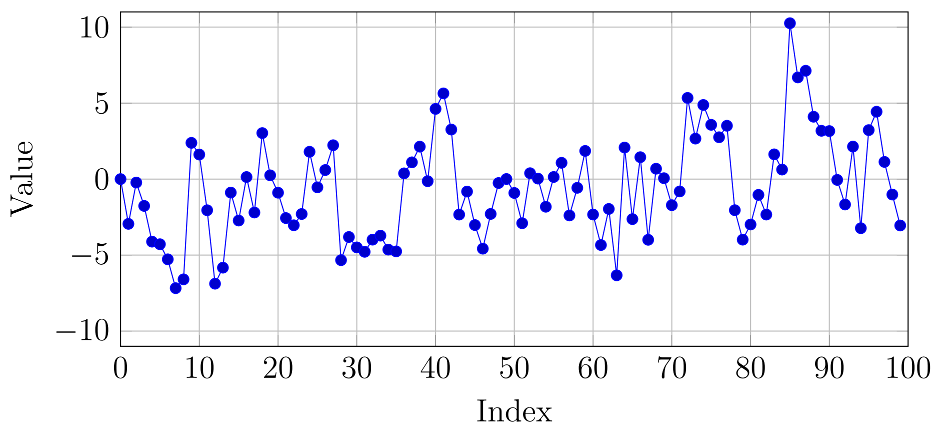
Table 3: Estimation Results for OLS, NLS, and MLE (AR(2))

We further see this patter continued when we move to AR(2). There is not significant difference among the results of the parameters. We also consistently see that ϕ_2 has a relatively small parameter values meaning that the $t-2$ errors cause less distrubances. However, we do notice that the σ^2 value is smaller for MLE which means that MLE captures the non linear model more consistently than OLS. Finally, we cannot use both t and $t-1$ as regressors as it would introduce multiconlinearity as $t-1$ is perfectally correlated with t.

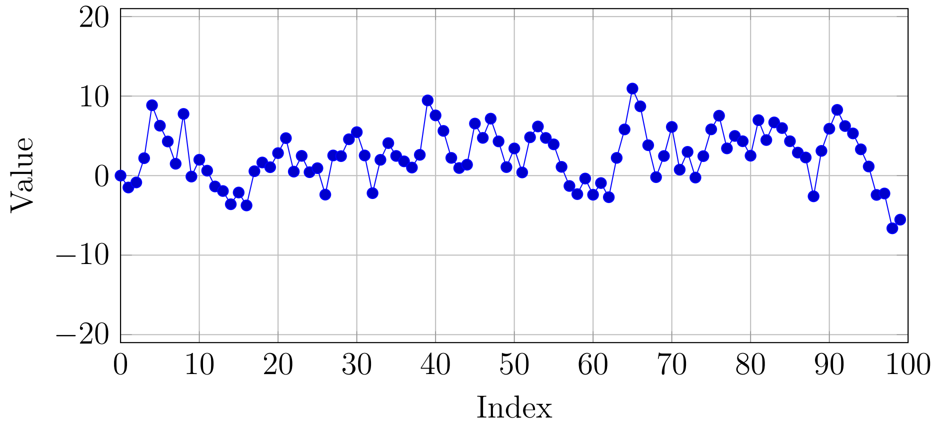
Simulation

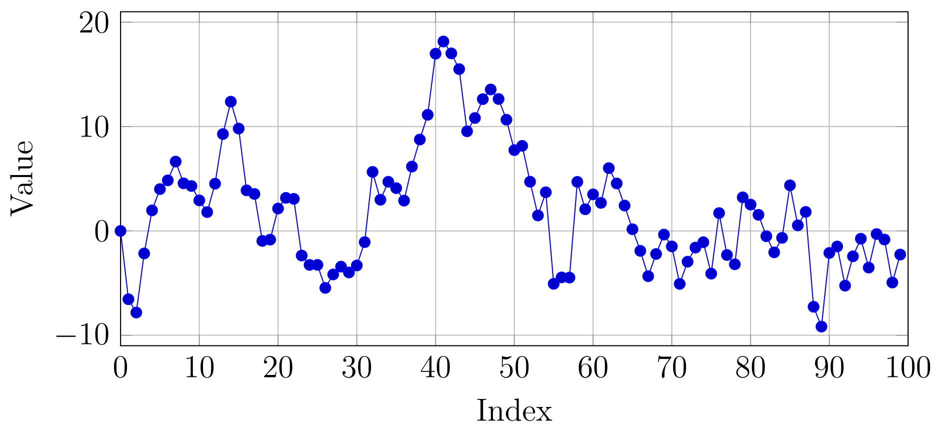
$\phi = 0.5$



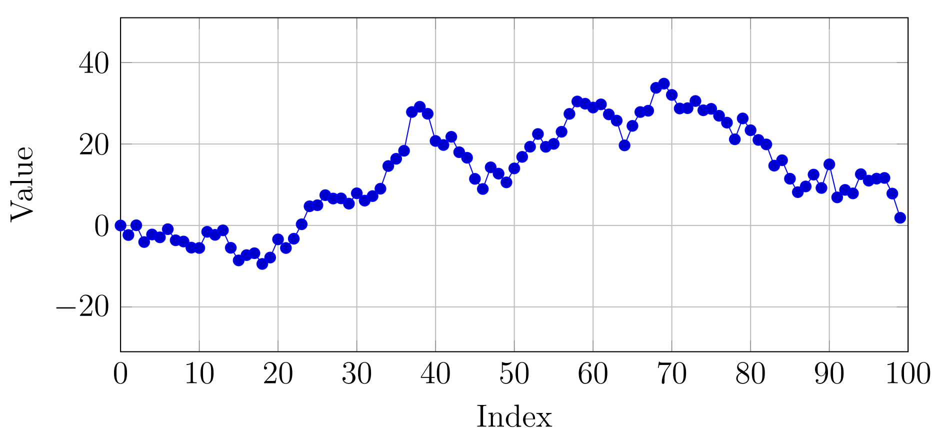


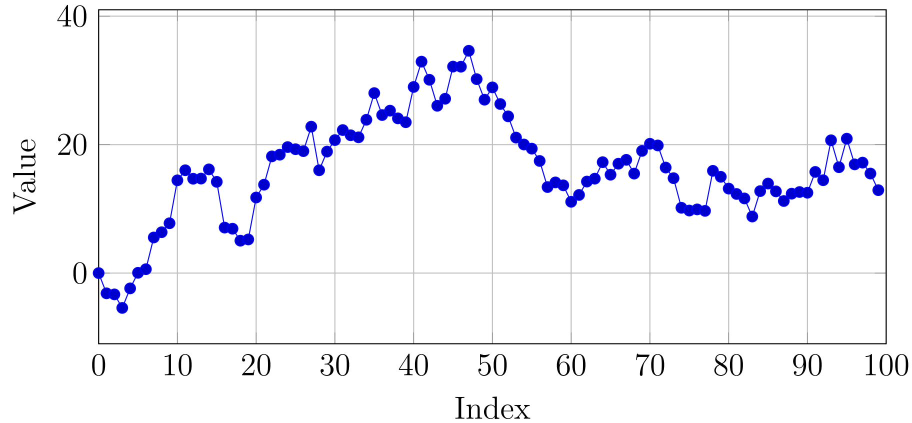
$\phi = 0.9$



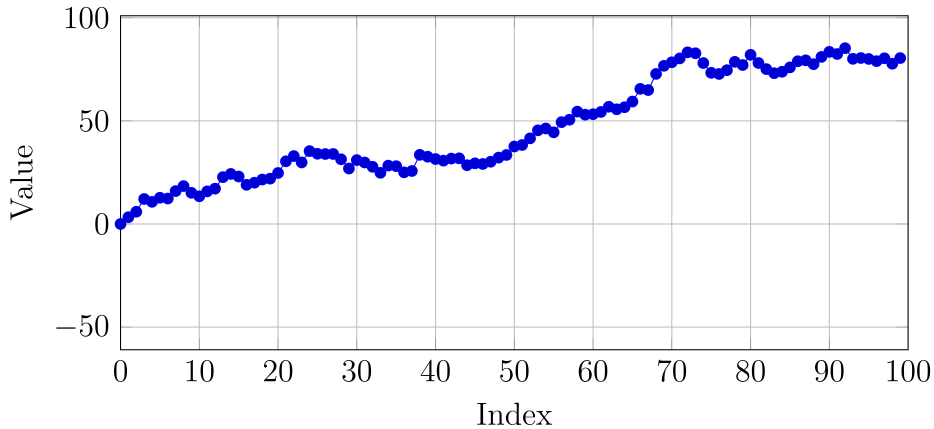


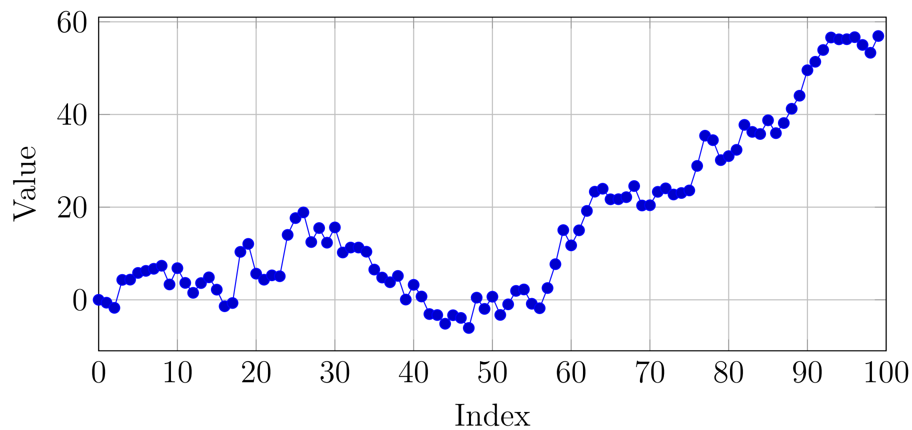
$\phi = 0.99$





$\phi = 1$





Here, we clearly see how the more we increase the value of *phi* the persistence of the shocks continue. For example in 0.5 we see the model act more jagged in the sense that positive shocks are followed by negative shocks. However in 1.0 we the error propegate more and more as the number of iterations increases.

ML Estimates	DGP 1	DGP 3	DGP 5	DGP 7
α	1.00444	-3.26174	-1.56724	-1.10345
β	-0.0155939	-0.0261957	0.127633	-0.466311
ϕ	0.527131	0.790891	0.833133	0.907961
OLS	0.532125	0.797716	0.840822	0.902399
OLS σ	9.00884	8.97009	8.865	10.2996

Table 4: ML Estimates for Different DGPs

Standard Errors	DGP 1	DGP 3	DGP 5	DGP 7
SE(α)	1.24659	3.25732	3.29341	15.3566
SE(β)	0.0212527	0.0520038	0.0545495	0.194296
SE(ϕ)	0.232713	0.395076	0.363669	0.922397

Table 5: Standard Errors for Different DGPs

These results are supported estimates as well. We see the standard error considerably increased for $\phi = 1$ as the is more error caused by ϕ increasing.

Replications

α_{ML}	Mean	StdDev	Mean Bias	RMSE	5% Quantile	Median	95% Quantile
$\phi = 0.5$	0.01	0.82	0.01	0.67	-1.27	0.0034	1.29
$\phi = 0.9$	0.16	3.20	0.16	10.25	-5.10	-0.0065	5.68
$\phi = 0.99$	0.19	4.16	0.19	17.29	-6.89	0.0038	8.80
$\phi = 1$	-0.10	2.76	-0.10	7.62	-4.40	-0.0009	0.27

Table 6: Alpha Estimates

β_{ML}	Mean	StdDev	Mean Bias	RMSE	5% Quantile	Median	95% Quantile
$\phi = 0.5$	0.00	0.01	0.00	0.00	-0.01	-0.005	0.01
$\phi = 0.9$	-0.00	0.03	-0.00	0.00	-0.05	-0.0046	0.04
$\phi = 0.99$	-0.00	0.11	-0.00	0.01	-0.17	-0.01	0.18
$\phi = 1$	0.01	0.21	0.01	0.05	-0.31	0.01	0.37

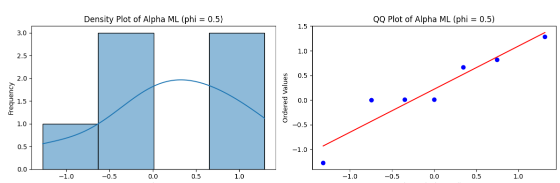
Table 7: Beta Estimates

ϕ_{ML}	Mean	StdDev	Mean Bias	RMSE	5% Quantile	Median	95% Quantile
$\phi = 0.5$	0.48	0.06	0.48	0.23	0.38	0.48	0.58
$\phi = 0.9$	0.87	0.04	0.87	0.75	0.80	0.87	0.92
$\phi = 0.99$	0.95	0.03	0.95	0.91	0.90	0.96	0.99
$\phi = 1$	0.96	0.02	0.96	0.92	0.91	0.97	0.99

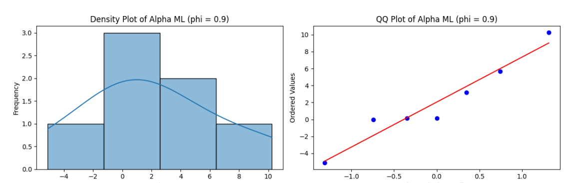
Table 8: Phi (ML) Estimates

ϕ_{OLS}	Mean	StdDev	Mean Bias	RMSE	5% Quantile	Median	95% Quantile
$\phi = 0.5$	0.05	0.16	0.05	1.14	0.80	1.06	1.32
$\phi = 0.9$	0.27	0.32	0.27	7.47	2.19	2.70	3.27
$\phi = 0.99$	0.40	0.72	0.40	16.76	2.94	3.99	5.23
$\phi = 1$	0.42	0.75	0.42	18.52	3.13	4.18	5.54

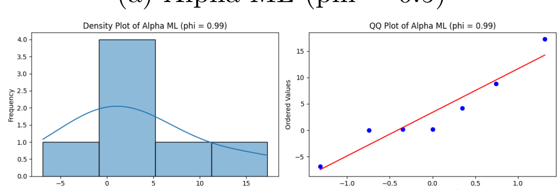
Table 9: Phi (OLS) Estimates



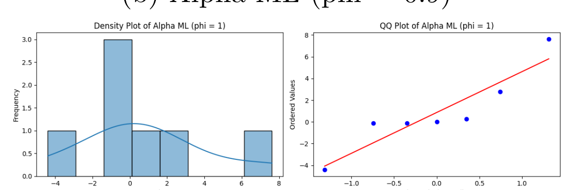
(a) Alpha ML (phi = 0.5)



(b) Alpha ML (phi = 0.9)

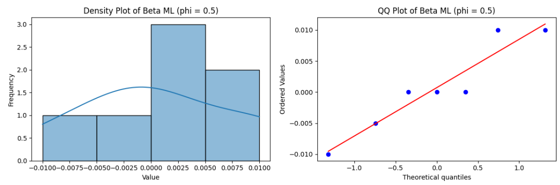


(c) Alpha ML (phi = 0.99)

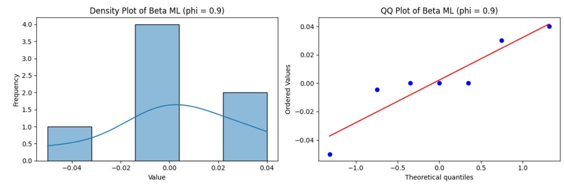


(d) Alpha ML (phi = 1)

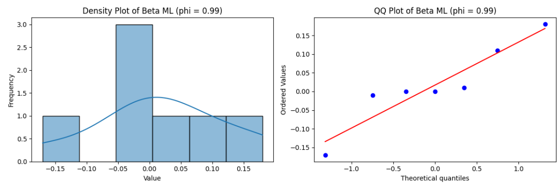
Figure 1: Alpha ML Estimates



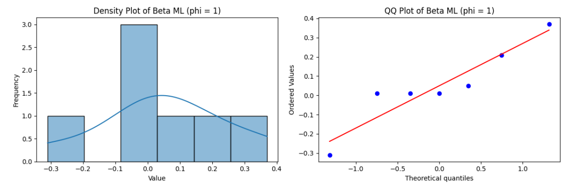
(a) Beta ML (phi = 0.5)



(b) Beta ML (phi = 0.9)

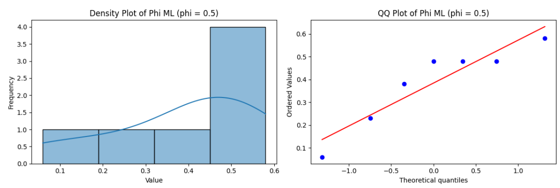


(c) Beta ML (phi = 0.99)

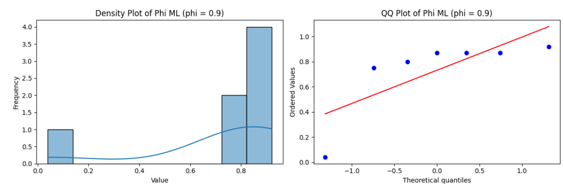


(d) Beta ML (phi = 1)

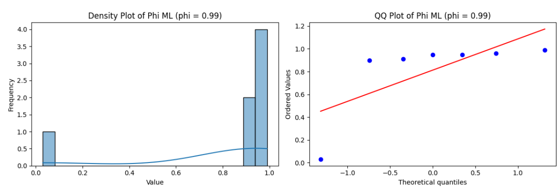
Figure 2: Beta ML Estimates



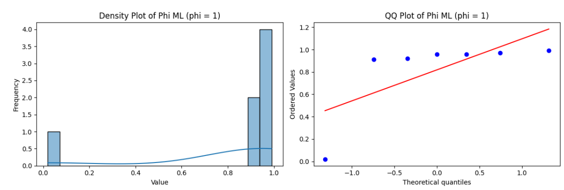
(a) Phi ML (phi = 0.5)



(b) Phi ML (phi = 0.9)

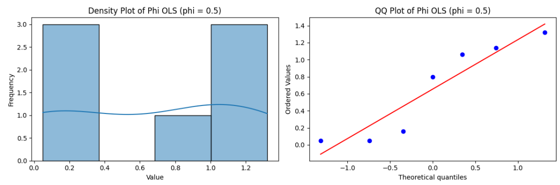


(c) Phi ML (phi = 0.99)

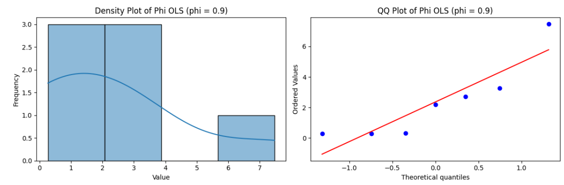


(d) Phi ML (phi = 1)

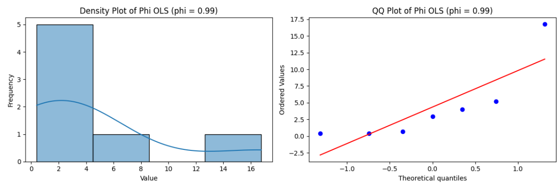
Figure 3: Phi ML Estimates



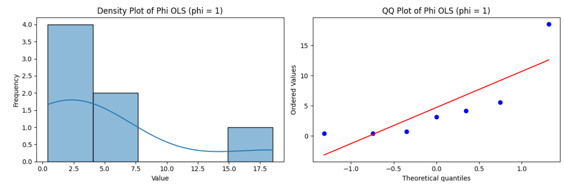
(a) Phi OLS (phi = 0.5)



(b) Phi OLS (phi = 0.9)



(c) Phi OLS (phi = 0.99)



(d) Phi OLS (phi = 1)

Figure 4: Phi OLS Estimates