Draft version February 16, 2015
Preprint typeset using LaTeX style emulateapj v. 12/16/11

EQUATION SET FOR TAYLER-SPRUIT DYNAMO

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Draft version February 16, 2015

ABSTRACT

We write down the equations to be used to study the Tayler-Spruit Dynamo problem.
Subject headings: MHD, Hydrodynamics, Computational Fluid Dynamics, Instability

1. GOVERNING EQUATIONS

We model the Tayler-Spruit dynamo in a fully compressible, resistive MHD framework with an Ideal gas equation of state:

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{u} \tag{1}$$

$$P = (\gamma-1)C_v\rho T = \rho^\gamma e^{S/C_v} \tag{2}$$

$$\rho T\frac{DS}{Dt} = \kappa\vec{\nabla}^2 T + \overleftrightarrow{\Pi} : \vec{\nabla}\vec{u} \tag{3}$$

$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{u}\times\vec{B}\right) + \eta\vec{\nabla}^2\vec{B} \tag{4}$$

$$\rho\frac{D\vec{u}}{Dt} = -\vec{\nabla}\left(P + \frac{B^2}{8\pi} + \phi_{eff}\right) + 2\vec{u}\times\vec{\Omega} + \frac{\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}}{4\pi} + \vec{\nabla}\cdot\overleftrightarrow{\Pi} \tag{5}$$

Where we have used the following definitions:

$$\overleftrightarrow{\Pi} = \mu\left(\vec{\nabla}\vec{u} + \left(\vec{\nabla}\vec{u}\right)^T - \frac{2}{3}\overleftrightarrow{I}\,\vec{\nabla}\cdot\vec{u}\right) \tag{6}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla} \tag{7}$$

$$\gamma = \frac{C_p}{C_v} \tag{8}$$

$$\phi_{eff} = \phi + \frac{1}{2}\lambda^2\Omega^2 \tag{9}$$

and λ is the distance from the rotation axis. We have also assumed that the thermal, viscous and magnetic diffusivities (κ, μ and η) are all constant.

2. THERMODYNAMICS

We start with the Ideal gas equation of state and solve it for the entropy:

$$\frac{S}{C_v} = \log P - \gamma\log\rho \Rightarrow \frac{S}{\gamma C_v} = \frac{1}{\gamma}\log\left((\gamma-1)C_v\rho T\right) - \log\rho \tag{10}$$

$$\frac{S}{C_p} = \frac{1}{\gamma}\log\left((\gamma-1)C_v\right) + \left(\frac{1}{\gamma}-1\right)\log\rho + \frac{1}{\gamma}\log T \tag{11}$$

Assuming both C_v and γ are constant,

$$\frac{1}{C_v}\frac{DS}{Dt} = (1-\gamma)\frac{D\log\rho}{Dt} + \frac{D\log T}{Dt} = (\gamma-1)\vec{\nabla}\cdot\vec{u} + \frac{D\log T}{Dt} \tag{12}$$

Where we have used the continuity equation for the last equality.

3. LINEARIZED EQUATIONS

To linearize the equations, we decompose the fields into background and fluctuating parts: $x(\vec{r},t) = x_0(\vec{r}) + x_1(\vec{r},t)$. We further assume that the fluctuating parts are small compared to the background: $|x_1| \ll |x_0|$. This allows us to drop any terms that are quadratic in the fluctuating quantities, i.e. terms that would appear as $x_1 y_1$. We also impose that the background quantites satisfy the equations. This allows us to subtract the background state after plugging in the decomposed fields, leaving equations for the evolution of the fluctuations.

First we linearize the equation of state using the assumption that x_1/x_0 is a small quantity and the approximation $\log(1+\epsilon) \approx \epsilon$.

$$P = (\gamma - 1) C_v \rho T \tag{13}$$

$$P_1 + P_0 = (\gamma - 1) C_v \left(\rho_0 + \rho_1\right)\left(T_0 + T_1\right) \tag{14}$$

$$P_0 \left(1 + \frac{P_1}{P_0}\right) = (\gamma - 1) C_v \rho_0 T_0 \left(1 + \frac{\rho_1}{\rho_0}\right)\left(1 + \frac{T_1}{T_0}\right) \tag{15}$$

$$\left(1 + \frac{P_1}{P_0}\right) = \left(1 + \frac{\rho_1}{\rho_0}\right)\left(1 + \frac{T_1}{T_0}\right) \tag{16}$$

$$\log\left(1 + \frac{P_1}{P_0}\right) = \log\left(1 + \frac{\rho_1}{\rho_0}\right) + \log\left(1 + \frac{T_1}{T_0}\right) \tag{17}$$

$$\frac{P_1}{P_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0} \tag{18}$$

The final set of linearized equations is:

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \vec{\nabla} \cdot \vec{u}_1 - \vec{u}_1 \cdot \vec{\nabla} \rho_0 \tag{19}$$

$$\frac{\partial \vec{B}_1}{\partial t} = \vec{\nabla} \times \left(\vec{u}_1 \times \vec{B}_0\right) + \eta \vec{\nabla}^2 \vec{B}_1 \tag{20}$$

$$\rho_0 \frac{\partial T_1}{\partial t} = -\rho_0 \vec{u}_1 \cdot \vec{\nabla} T_0 - \rho_0 T_0 (\gamma - 1) \vec{\nabla} \cdot \vec{u}_1 + \frac{\kappa}{C_v} \vec{\nabla}^2 T_1 \tag{21}$$

$$\rho_0 \frac{\partial \vec{u}_1}{\partial t} = -\vec{\nabla}\left(P_1 + \frac{B_0 B_1}{4\pi} + \phi_{eff}\right) + 2\vec{u}_1 \times \vec{\Omega} + \frac{\left(\vec{\nabla} \times \vec{B}_0\right) \times \vec{B}_1}{4\pi} + \frac{\left(\vec{\nabla} \times \vec{B}_1\right) \times \vec{B}_0}{4\pi} + \mu \vec{\nabla}^2 \vec{u}_1 + \frac{\mu}{3} \vec{\nabla}\left(\vec{\nabla} \cdot \vec{u}_1\right) \tag{22}$$

$$\frac{P_1}{P_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0} \tag{23}$$