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THE ADIABATIC STABILITY OF STARS CONTAINING MAGNETIC FIELDS—I

Toroidal Fields

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SUMMARY

Conditions for the stability against adiabatic perturbations of a star containing a purely toroidal magnetic field are established. These conditions are necessary and sufficient for stability, if the change in gravitational field produced by the perturbation is neglected. They remain necessary for stability if the change in gravitational field is included. The criteria are given in closed algebraic form so that they can be applied readily, if an equilibrium model of a star with a toroidal magnetic field is proposed.

Because the criteria are complicated, it is difficult to draw general conclusions from them. It is however possible to show that instability must occur close to the axis of symmetry of the star, if there is a non-zero electric current density on the axis. This proves that a large class of configurations are unstable. The occurrence of this instability depends only on the shape of the field and not on its strength, whereas in general the field strength will be important in determining instability.

It is not easy to predict the ultimate effect of such instabilities. They have very short growth times compared to characteristic times of stellar evolution and, even if the instability is initially contained as an oscillation of finite amplitude, it may lead to a considerably enhanced decay of the magnetic field. It is unlikely that stars exist with a purely toroidal field, although it may be stronger than the poloidal component. However, models have previously been obtained of stars with only a toroidal field and their oscillation properties have been discussed. Because these discussions by other authors have involved integrals over the whole star, the localized instabilities found in this paper have not been discovered.

1. INTRODUCTION

This paper is a sequel to three papers by Gough & Tayler (1966) and Moss & Tayler (1969, 1970); these papers which were concerned with the possibility that a sufficiently strong magnetic field might suppress thermal convection in a compressible fluid, will be referred to as Papers I, II and III in what follows. In Paper I, the influence of a magnetic field on the onset of convection in a plane parallel medium was studied. It was shown that a strong field could suppress convection provided that the field was not entirely horizontal. In Papers II and III, the influence of a poloidal magnetic field on convection in spheres and spherical shells was investigated. Such a poloidal magnetic field cannot have a vertical component at all points in a sphere and it was found that only a very large magnetic field could influence convection in a complete sphere or spherical shell. In Paper II it was suggested that, as a toroidal field is horizontal everywhere in an axisymmetric star, it is unlikely that it can have a significant effect on convection and, in addition, a strong toroidal field might introduce new hydromagnetic instabilities. This suggestion is investigated in the present paper.

In Papers I–III, because the aim was to discover whether any magnetic field could suppress convection, care was taken not to consider any field which might introduce new instabilities. In this and a succeeding paper, which will be concerned with poloidal magnetic fields, one of our purposes is to discover whether magnetic fields in radiative zones of stars are intrinsically stable or unstable. In fact, it will be shown that an arbitrary field configuration inside a star is likely to suffer from hydromagnetic instabilities and that these instabilities can have growth times which are very short on an astronomical time scale.

It is well known from the work of those attempting to produce controlled thermonuclear reactions in the laboratory that most configurations in which a plasma is confined by a magnetic field are unstable. Although the gravitational field inside a star makes stellar problems very different from laboratory problems, it is possible to show quite simply that some instabilities exist in which the gravitational field plays essentially no role. Furthermore instabilities occur even when the parameter

$$\beta \equiv P/(B^2/2) \tag{1.1}$$

which measures the relative importance of gas pressure and magnetic pressure is arbitrarily large. The growth time of the instability is of order the time taken by a hydromagnetic wave to cross the unstable region and this time is very long compared to the stellar dynamical time if β is large. However, even for a relatively weak magnetic field inside a star, the growth time is likely to be measured in years and to be very short compared to both the thermal and nuclear time scales of the star. For that reason, it makes sense to consider adiabatic disturbances in which all dissipative processes such as radiative conductivity and viscosity and electrical resistivity are neglected.

That there can exist instabilities which depend more on the topology of the magnetic field than on its strength has already been recognized independently by Mestel and Wright (Mestel, private communication; Wright 1970). Because the instabilities depend on very local properties of the magnetic field, they are not easily discovered by methods, such as the virial tensor techniques, which involve integrations over the whole star. In terms of any normal modes the instabilities represent very high overtones and they could only be discovered by taking the virial tensor method to a very high order. It is instead much easier to recognize that locally the magnetic field has a structure which is identical to one which exhibits a well known hydromagnetic instability.

In fact this general discussion is more appropriate to the poloidal fields studied in the succeeding paper than to the toroidal fields studied here as we shall show that a necessary and sufficient condition for the stability of purely toroidal fields can be obtained in closed form, provided only that the change in gravitational field produced by the perturbation is neglected. One adiabatic instability of toroidal fields has been studied in the past. This is the interchange instability in which two tubes of toroidal flux and the matter contained in them are interchanged adiabatically; the system is unstable to such a perturbation if energy is released by the interchange. In this paper it is shown that the interchange perturbations are the worst axisymmetric perturbations; this is not surprising as it is perhaps obvious that any arbitrary axisymmetric perturbation is equivalent to a succession of interchanges. What is more important is that our direct demonstration, that the interchange is the worst axisymmetric perturbation, can be exactly paralleled by a discussion of non-axisymmetric disturbances and this leads to the discovery of a necessary and sufficient condition for the stability of the system against non-axisymmetric perturbations. In the case of the configurations with a poloidal field discussed in Paper II, sufficient conditions for stability could be obtained but these conditions were not also necessary. It proves possible to see why the present problem is different and the discussion given of this point may be useful in other contexts.

The remainder of this paper is arranged as follows. The stability criteria are obtained in Section 2, although some of the mathematical details are given in two appendices. The application of the criteria to models of stars containing toroidal fields is discussed in Section 3 and some of the inadequacies of the present treatment are considered in Section 4.

2. DERIVATION OF STABILITY CRITERIA

As in the previous papers, the energy principle of Bernstein *et al.* (1958) is used. They showed that the stability of an ideally conducting system depends on the sign of the change of potential energy of the system, δW, produced by an arbitrary perturbation $\xi(\mathbf{x}, t)$. If δW is positive for all perturbations, the system is stable; otherwise it is unstable. δW takes the form

$$\delta W = \tfrac{1}{2}\int d\tau[\mathbf{Q}^2 - \mathbf{j}\cdot\mathbf{Q}\times\boldsymbol{\xi} + \gamma P(\operatorname{div}\boldsymbol{\xi})^2 + \boldsymbol{\xi}\cdot\operatorname{grad} P \operatorname{div}\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\operatorname{grad}\Phi \operatorname{div}\rho\boldsymbol{\xi}], \tag{2.1}$$

where $Q = \operatorname{curl}(\boldsymbol{\xi}\times\mathbf{B})$, P, ρ, $\mathbf{B}$ and $\mathbf{j}$ are the equilibrium pressure, density, magnetic field and current density, γ is the ratio of specific heats of the fluid and Φ is the gravitational potential defined so that the gravitational force on unit mass is $\mathbf{g} = +\operatorname{grad}\Phi$. In expression (2.1) the electromagnetic units are rationalized Gaussian units with the velocity of light put equal to unity, the integral is over the volume of the system and the variation in the gravitational field produced by the perturbation has been neglected. In Paper II it was shown that the additional term in δW depending on the change of gravitational field is necessarily negative so that by neglecting the term the stability of the system is being overestimated.

We assume that we have an axisymmetric configuration containing only a toroidal magnetic field. Thus in cylindrical polar coordinates (ϖ ϕ z) we can write

$$\mathbf{B} = (0, B(\varpi, z), 0), \tag{2.2}$$

$$\mathbf{j} = (-\partial B/\partial z, 0, (1/\varpi)\,\partial(\varpi B)/\partial\varpi), \tag{2.3}$$

$$P = P(\varpi, z),\ \rho = \rho(\varpi, z) \tag{2.4}$$

and

$$\mathbf{g} = (g_\varpi(\varpi, z), 0, g_z(\varpi, z)). \tag{2.5}$$

Using expressions (2.2)–(2.5), we obtain

$$\left.\begin{aligned} Q_\varpi &= \frac{B}{\varpi}\frac{\partial \xi_\varpi}{\partial \phi}, \\ Q_\phi &= -\frac{\partial}{\partial z}(\xi_z B) - \frac{\partial}{\partial \varpi}(\xi_\varpi B), \\ Q_z &= \frac{B}{\varpi}\frac{\partial \xi_z}{\partial \phi}. \end{aligned}\right\} \tag{2.6}$$

and

Then

$$\begin{aligned} \delta W = \frac{1}{2}\int \varpi\, d\varpi\, d\phi\, dz \Bigg[& \frac{B^2}{\varpi^2}\left(\frac{\partial \xi_\varpi}{\partial \phi}\right)^2 + \frac{B^2}{\varpi^2}\left(\frac{\partial \xi_z}{\partial \phi}\right)^2 + \left\{\frac{\partial}{\partial z}(\xi_z B) + \frac{\partial}{\partial \varpi}(\xi_\varpi B)\right\}^2 \\ & - \left\{\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi B)\,\xi_\varpi + \frac{\partial B}{\partial z}\,\xi_z\right\}\left\{\frac{\partial}{\partial z}(\xi_z B) + \frac{\partial}{\partial \varpi}(\xi_\varpi B)\right\} - \frac{B}{\varpi^2}\frac{\partial}{\partial \varpi}(\varpi B)\,\xi_\phi\frac{\partial \xi_\varpi}{\partial \phi} \\ & - \frac{B}{\varpi}\frac{\partial B}{\partial z}\,\xi_\phi\frac{\partial \xi_z}{\partial \phi} + \gamma P(\operatorname{div}\boldsymbol{\xi})^2 + \left\{\xi_\varpi\left(\frac{\partial P}{\partial \varpi} + \rho g_\varpi\right) + \xi_z\left(\frac{\partial P}{\partial z} + \rho g_z\right)\right\}\operatorname{div}\boldsymbol{\xi} \\ & + (\xi_\varpi g_\varpi + \xi_z g_z)\left(\xi_\varpi\frac{\partial \rho}{\partial \varpi} + \xi_z\frac{\partial \rho}{\partial z}\right)\Bigg]. \end{aligned} \tag{2.7}$$

Since no coefficients in expression (2.7) depend on the variable ϕ, it is possible to Fourier analyse the components of $\boldsymbol{\xi}$ in terms proportional to $\exp(im\phi)$ and to treat each value of m separately. The cases $m = 0$ and $m \neq 0$ must now be treated independently.

If $m = 0$, all ϕ derivatives of $\boldsymbol{\xi}$ vanish and

$$D \equiv \operatorname{div}\boldsymbol{\xi} = (1/\varpi)\,\partial(\varpi\xi_\varpi)/\partial\varpi + \partial\xi_z/\partial z. \tag{2.8}$$

Using these properties, it can be seen that (2.7) does not depend on ξ_ϕ and δW can be rewritten in the form

$$\begin{aligned} \delta W \propto \int \varpi\, d\varpi\, dz \Bigg[& \left\{BD + \xi_z\frac{\partial B}{\partial z} + \xi_\varpi\frac{\partial B}{\partial \varpi} - \frac{\xi_\varpi B}{\varpi}\right\}^2 \\ & - \left\{\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi B)\,\xi_\varpi + \frac{\partial B}{\partial z}\,\xi_z\right\}\left\{BD + \xi_z\frac{\partial B}{\partial z} + \xi_\varpi\frac{\partial B}{\partial \varpi} - \frac{\xi_\varpi B}{\varpi}\right\} \\ & + \gamma P D^2 + \left\{\xi_\varpi\left(\frac{\partial P}{\partial \varpi} + \rho g_\varpi\right) + \xi_z\left(\frac{\partial P}{\partial z} + \rho g_z\right)\right\} D \\ & + (\xi_\varpi g_\varpi + \xi_z g_z)\left(\xi_\varpi\frac{\partial \rho}{\partial \varpi} + \xi_z\frac{\partial \rho}{\partial z}\right)\Bigg]. \end{aligned} \tag{2.9}$$

Using the equations of equilibrium,

$$\left.\begin{aligned}\frac{\partial P}{\partial \varpi} &= \rho g_{\varpi} - \frac{B}{\varpi}\frac{\partial}{\partial \varpi}(\varpi B),\\ \frac{\partial P}{\partial z} &= \rho g_z - B\frac{\partial B}{\partial z},\end{aligned}\right\} \tag{2.10}$$

expression (2.9) can be rearranged in the form

$$\begin{aligned}\delta W \propto \int \varpi\, d\varpi\, dz \Bigg[&(B^2+\gamma P)\left\{D+\left[\rho g_z\xi_z+\left(\rho g_{\varpi}-\frac{2B^2}{\varpi}\right)\xi_{\varpi}\right]\Big/(B^2+\gamma P)\right\}^2\\ &-\left\{\rho g_z\xi_z+\left(\rho g_{\varpi}-\frac{2B^2}{\varpi}\right)\xi_{\varpi}\right\}^2\Big/(B^2+\gamma P)-\frac{2B\xi_{\varpi}}{\varpi}\left(\xi_z\frac{\partial B}{\partial z}+\xi_{\varpi}\frac{\partial B}{\partial \varpi}-\frac{\xi_{\varpi}B}{\varpi}\right)\\ &+(\xi_{\varpi}g_{\varpi}+\xi_z g_z)\left(\xi_{\varpi}\frac{\partial \rho}{\partial \varpi}+\xi_z\frac{\partial \rho}{\partial z}\right)\Bigg].\end{aligned} \tag{2.11}$$

In expression (2.11), the first term is certainly positive and the remaining terms in the integrand are a quadratic form in ξ_{ϖ} and ξ_z. It is clearly sufficient for stability that this quadratic form be positive at all points in the region of integration. The system is definitely stable if at all points

$$g_z\frac{\partial \rho}{\partial z}-\rho^2 g_z{}^2/(B^2+\gamma P)>0, \tag{2.12}$$

$$g_{\varpi}\frac{\partial \rho}{\partial \varpi}-\left(\rho g_{\varpi}-\frac{2B^2}{\varpi}\right)^2\Big/(B^2+\gamma P)-\frac{2B}{\varpi}\frac{\partial B}{\partial \varpi}+\frac{2B^2}{\varpi^2}>0 \tag{2.13}$$

and

$$\begin{aligned}&\left\{g_{\varpi}\frac{\partial \rho}{\partial z}+g_z\frac{\partial \rho}{\partial \varpi}-2\rho g_z\left(\rho g_{\varpi}-\frac{2B^2}{\varpi}\right)\Big/(B^2+\gamma P)-\frac{2B}{\varpi}\frac{\partial B}{\partial z}\right\}^2\\ &\quad<4\left\{g_z\frac{\partial \rho}{\partial z}-\rho^2 g_z{}^2/(B^2+\gamma P)\right\}\left\{g_{\varpi}\frac{\partial \rho}{\partial \varpi}-\left(\rho g_{\varpi}-\frac{2B^2}{\varpi}\right)^2\Big/(B^2+\gamma P)\right.\\ &\qquad\left.-\frac{2B}{\varpi}\frac{\partial B}{\partial \varpi}+\frac{2B^2}{\varpi^2}\right\}.\end{aligned} \tag{2.14}$$

It can in fact be shown that these conditions are necessary for stability but a proof of this will be deferred until the perturbations with $m \neq 0$ have been considered.

We return to expression (2.7) and substitute for the components of $\boldsymbol{\xi}$ the expressions

$$\xi_{\varpi} = \mathscr{R}[X\exp(im\phi)] = \tfrac{1}{2}[X\exp(im\phi)+X^*\exp(-im\phi)],$$

$$\xi_{\phi} = \mathscr{R}\left[\frac{iY}{m}\exp(im\phi)\right] = \frac{1}{2}\left[\frac{iY}{m}\exp(im\phi)-\frac{iY^*}{m}\exp(-im\phi)\right]$$

and

$$\xi_z = \mathscr{R}\Big[Z\exp(im\phi)\Big] = \frac{1}{2}\Big[Z\exp(im\phi)+Z^*\exp(-im\phi)\Big], \tag{2.15}$$

where the asterisk denotes complex conjugate. Then

$$\delta W \propto \int \varpi\, d\varpi\, dz \left[\frac{m^2B^2}{\varpi^2}(XX^*+ZZ^*)+\left\{\frac{\partial}{\partial z}(ZB)+\frac{\partial}{\partial \varpi}(XB)\right\}\right.$$
$$\times\left\{\frac{\partial}{\partial z}(Z^*B)+\frac{\partial}{\partial \varpi}(X^*B)\right\}-\left\{Z\frac{\partial B}{\partial z}+\frac{X}{\varpi}\frac{\partial}{\partial \varpi}(\varpi B)\right\}$$
$$\times\left\{\frac{\partial}{\partial z}(Z^*B)+\frac{\partial}{\partial \varpi}(X^*B)\right\}-\frac{B}{\varpi^2}\frac{\partial}{\partial \varpi}(\varpi B)YX^*-\frac{B}{\varpi}\frac{\partial B}{\partial z}YZ^*$$
$$+\gamma P\left\{\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi X)-\frac{Y}{\varpi}+\frac{\partial Z}{\partial z}\right\}\left\{\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi X^*)-\frac{Y^*}{\varpi}+\frac{\partial Z^*}{\partial z}\right\}$$
$$+\left\{X\left(\frac{\partial P}{\partial \varpi}+\rho g_\varpi\right)+Z\left(\frac{\partial P}{\partial z}+\rho g_z\right)\right\}\left\{\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi X^*)-\frac{Y^*}{\varpi}+\frac{\partial Z^*}{\partial z}\right\}$$
$$\left.+\{Xg_\varpi+Zg_z\}\left\{X^*\frac{\partial \rho}{\partial \varpi}+Z^*\frac{\partial \rho}{\partial z}\right\}+\text{complex conjugate}\right]. \qquad (2.16)$$

Clearly, as m only enters in the first positive definite term and, for given X, Y, Z, this term has its least value when $m = 1$, the most unstable modes correspond to $m = 1$. If we henceforth consider only $m = 1$ and split X, Y, Z into real and imaginary parts by writing $X = X_R + iX_I$ etc., it can be shown that the expression for δW consists of two identical integrals, one involving X_R, Y_R, Z_R and the other X_I, Y_I, Z_I. We need then only consider the minimum value of one expression of the form

$$\delta W \propto \int \varpi\, d\varpi\, dz \left[\frac{B^2}{\varpi^2}(X^2+Z^2)+\left\{\frac{\partial}{\partial z}(ZB)+\frac{\partial}{\partial \varpi}(XB)\right\}^2\right.$$
$$-\left\{\frac{\partial B}{\partial z}Z+\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi B)X\right\}\left\{\frac{\partial}{\partial z}(ZB)+\frac{\partial}{\partial \varpi}(XB)\right\}-\frac{B}{\varpi^2}\frac{\partial}{\partial \varpi}(\varpi B)YX$$
$$-\frac{B}{\varpi}\frac{\partial B}{\partial z}YZ+\gamma P\left\{\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi X)-\frac{Y}{\varpi}+\frac{\partial Z}{\partial z}\right\}^2+\{Xg_\varpi+Zg_z\}\left\{X\frac{\partial \rho}{\partial \varpi}+Z\frac{\partial \rho}{\partial z}\right\}$$
$$\left.+\left\{X\left(\frac{\partial P}{\partial \varpi}+\rho g_\varpi\right)+Z\left(\frac{\partial P}{\partial z}+\rho g_z\right)\right\}\left\{\frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi X)-\frac{Y}{\varpi}+\frac{\partial Z}{\partial z}\right\}\right]. \qquad (2.17)$$

As Y only appears algebraically in expression (2.17), it is possible to perform an explicit minimization. Using the equations of equilibrium (2.10), the expression obtained for Y is

$$Y = \frac{\partial}{\partial \varpi}(\varpi X)+\varpi\frac{\partial Z}{\partial z}+\frac{\varpi\rho}{\gamma P}(g_\varpi X+g_z Z). \qquad (2.18)$$

If this value for Y is substituted into expression (2.17) and some rearrangement is made, there results

$$\delta W \propto \int \varpi\, d\varpi\, dz \left[B^2\left\{\varpi\frac{\partial}{\partial \varpi}\left(\frac{X}{\varpi}\right)+\frac{\partial Z}{\partial z}\right\}^2-\frac{B^2X^2}{\varpi^2}-\frac{2B}{\varpi}\frac{\partial B}{\partial \varpi}X^2\right.$$
$$-\frac{2B}{\varpi}\frac{\partial B}{\partial z}XZ+\frac{B^2}{\varpi^2}Z^2+(g_\varpi X+g_z Z)\left(\frac{\partial \rho}{\partial \varpi}X+\frac{\partial \rho}{\partial z}Z\right)$$
$$\left.-\frac{1}{\gamma P}(\rho g_\varpi X+\rho g_z Z)^2\right]. \qquad (2.19)$$

Expression (2.19) is now similar in form to expression (2.11); the first term in the integrand is positive and the remainder of the integrand is a quadratic form in X and Z. It is then sufficient for stability that this quadratic form be positive everywhere. This leads to the criteria

$$g_\varpi \frac{\partial \rho}{\partial \varpi} - \frac{\rho^2 g_\varpi{}^2}{\gamma P} - \frac{B^2}{\varpi^2} - \frac{2B}{\varpi}\frac{\partial B}{\partial \varpi} > 0, \tag{2.20}$$

$$g_z \frac{\partial \rho}{\partial z} - \frac{\rho^2 g_z{}^2}{\gamma P} + \frac{B^2}{\varpi^2} > 0 \tag{2.21}$$

and

$$\left\{g_\varpi \frac{\partial \rho}{\partial z} + g_z \frac{\partial \rho}{\partial \varpi} - \frac{2\rho^2 g_\varpi g_z}{\gamma P} - \frac{2B}{\varpi}\frac{\partial B}{\partial z}\right\}^2$$

$$< 4\left\{g_\varpi \frac{\partial \rho}{\partial \varpi} - \frac{\rho^2 g_\varpi{}^2}{\gamma P} - \frac{B^2}{\varpi^2} - \frac{2B}{\varpi}\frac{\partial B}{\partial \varpi}\right\}\left\{g_z \frac{\partial \rho}{\partial z} - \frac{\rho^2 g_z{}^2}{\gamma P} + \frac{B^2}{\varpi^2}\right\}. \tag{2.22}$$

The system is certainly stable against all perturbations if all of (2.12)–(2.14) and (2.20)–(2.22) are satisfied everywhere. In fact all of these conditions are also necessary for stability and a proof of this property will now be given.

To prove the conditions necessary for stability, we must demonstrate that, if any of them is violated in the neighbourhood of any point, there exists a perturbation of the system which leads to a negative δW. We can give a mathematical argument to show that this is true which is valid for both $m = 0$ and $m = 1$ perturbations but we can in addition give a well known physical argument for the $m = 0$ case. We discuss this first.

A particular type of $m = 0$ perturbation is the interchange perturbation whereby a tube of flux centred at the point (ϖ_1, z_1) is exchanged with a tube of flux centred at (ϖ_2, z_2). It is possible to calculate the change of energy produced by such an interchange in terms of the cross-sections of the two tubes of flux and the values of all of the physical variables at the two points. The change in energy can then be minimized with respect to the relative cross-sections of the two tubes of flux and an expression for the change in energy results in the form

$$\delta W = A\delta\varpi^2 + B\delta\varpi\delta z + C\delta z^2, \tag{2.23}$$

where $\delta\varpi = \varpi_2 - \varpi_1$ and $\delta z = z_2 - z_1$ and A, B, C are constants. The system is then unstable to an interchange perturbation if the conditions

$$A > 0, \quad C > 0, \quad B^2 < 4AC \tag{2.24}$$

are not all satisfied.

If the interchange calculation is performed it is found that conditions (2.24) are identical with the three conditions (2.12), (2.13) and (2.14). A proof of this result is given in Appendix 1. This implies two things. It means that the interchange perturbations are the worst $m = 0$ perturbations and it means that conditions (2.12)–(2.14) are necessary for the stability of the system. This immediately suggests that the criteria (2.20)–(2.22) are also necessary for stability. The reason for this is that expression (2.19) is mathematically very similar to expression (2.11). To demonstrate that conditions (2.20)–(2.22) are necessary for stability we must show that, if any one of the conditions is violated it is possible to find a pair of functions X and Z which make expression (2.19) negative. In Appendix 2 such functions are constructed.

In the discussion of the stability of a system containing a poloidal magnetic field, Moss & Tayler (1969) obtained sufficient conditions for stability by a method similar to that used in the present paper. In that case the sufficient conditions were far from being necessary for stability. It is clearly of interest to ask what properties the integrand in δW must possess for sufficient conditions also to be necessary. It can in fact be shown, by an extension of the argument given in Appendix 2, that, if an expression for δW can be obtained containing only two components of $\boldsymbol{\xi}$ and if the integrand consists of a positive definite term containing derivatives of $\boldsymbol{\xi}$ and a quadratic form in the components of $\boldsymbol{\xi}$, the sufficient conditions for stability obtained by demanding that the quadratic form be positive definite are also necessary, when the derivatives of $\boldsymbol{\xi}$ enter only in the combination $\partial X/\partial \varpi + \partial Z/\partial z$, or the equivalent in a different coordinate system. This was not true in the poloidal field problem studied by Moss and Tayler but it is true in both expressions (2.11) and (2.19).

It should be stressed that all of the criteria obtained in this Section are necessary for stability even when the change in gravitational field produced by the perturbation is included. This means that any instabilities which they predict will certainly occur but if the present criteria suggest that a system is stable, it may in fact be unstable.

3. DISCUSSION OF STABILITY CRITERIA

In the previous section necessary and sufficient conditions for the stability of a non-rotating star containing a purely toroidal magnetic field have been obtained. It is improbable that there exist any stars with precisely this property but it is possible that there may exist stars in which the toroidal component of the magnetic field is stronger than the poloidal component and in which rotation is slow. In that case conclusions drawn from the criteria obtained in this paper should be valid to a first approximation. In addition, as will be discussed below, several authors have previously discussed both the equilibrium and the oscillations of stars containing purely toroidal fields. As none of these discussions has obtained all of our criteria, we have been able to discover instabilities which they have not found.

Ultimately it is hoped that the results of this paper and the following one on purely poloidal fields will be followed by a discussion of more general fields. Purely toroidal fields have one peculiar property. If it is supposed that a star is surrounded by a vacuum, it is easy to show that the toroidal field must be entirely confined to the star as there is no vacuum toroidal field that can fill space outside the star. As the medium outside a star is not a vacuum this result is only true to a first approximation but it suggests that toroidal fields are lacking in observational interest unless they are so strong that they distort the surface of the star without themselves being visible.

Having indicated that the results may not be immediately applicable to any real stars, we now turn to a discussion of the stability criteria. It is not easy to discuss the stability criteria in a general case. The reason for this is that the structure of a star is inevitably altered by the introduction of a toroidal magnetic field, which cannot be force-free everywhere. The alteration in the structure of the star will be of order B^2 and this is also the order of the terms which alter the stability criteria. Thus even when, as in the case of criterion (2.12)

$$g_z \frac{\partial \rho}{\partial z} - \frac{\rho^2 g_z^2}{\gamma P + B^2} > 0 \tag{2.12}$$

the magnetic field formally has a stabilizing influence, the change in the structure of the star and in the expressions for ρ, g_z and P is also of order B^2 and this must be considered before a definite decision can be taken.

It is possible to obtain one general result. Near the axis of the star the field configuration resembles that of a cylindrical gas discharge. If it were not for the presence of gravity such a system would be unstable to perturbations of both the $m = 0$ and $m = 1$ type. (See, for example, Tayler 1957). In the presence of the force of gravitation it is possible to find $m = 1$ perturbations of the unstable type which primarily involve motions along surfaces of constant gravity but this is not true for $m = 0$ perturbations which inevitably involve motions along the axis; the two types of perturbation are illustrated in Fig. 1. It thus appears likely that the system will be unstable to $m = 1$ perturbations close to the axis of symmetry.

This result is readily verified by a study of criterion (2.20) for $m = 1$ modes,

$$g_\varpi \frac{\partial \rho}{\partial \varpi} - \frac{\rho^2 g_\varpi^2}{\gamma P} - \frac{B^2}{\varpi^2} - \frac{2B}{\varpi}\frac{\partial B}{\partial \varpi} > 0. \tag{2.20}$$

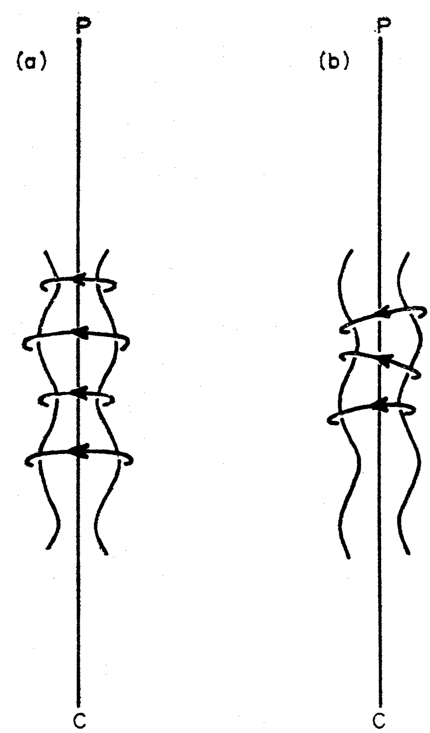

FIG. 1. *Possible instabilities near the axis of a star. CP is the axis (C, centre; P, pole). (a) Shows an $m = 0$ disturbance which requires motion along the axis. (b) Shows an $m = 1$ disturbance which involves motions mainly perpendicular to the axis. Magnetic field lines are marked by arrows.*

Near the axis of the star, B can be expanded

$$B = B_1 \frac{\varpi}{\varpi_0} + B_2 \left(\frac{\varpi}{\varpi_0}\right)^2 + \dots \tag{3.1}$$

where B_1, B_2 and ϖ_0 are constants. Whatever the effect of the magnetic field near to the axis, for purely geometrical reasons it is clear that

$$\left.\begin{aligned} g_\varpi &\propto \varpi \\ \partial\rho/\partial\varpi &\propto \varpi \end{aligned}\right\} \tag{3.2}$$

for small ϖ. By inspection it can now be seen that if B_1 is non-zero the magnetic term is dominant in criterion (2.20) and that this term is $-3B_1{}^2/\varpi_0{}^2$, which is negative. Thus, if there is a non-zero current density near the axis of the star, a purely toroidal field is necessarily unstable. This instability *may* be stabilized by a strong enough poloidal field or by rapid rotation; a detailed discussion of these possibilities is outside the scope of this paper but they will be discussed briefly shortly. It may be noted that in criterion (2.13) for $m = 0$ modes,

$$g_\varpi \frac{\partial\rho}{\partial\varpi} - \frac{(\rho g_\varpi - 2B^2/\varpi)^2}{B^2 + \gamma P} - \frac{2B}{\varpi}\frac{\partial B}{\partial\varpi} + \frac{2B^2}{\varpi^2} > 0, \tag{2.13}$$

the contribution in $B_1{}^2$ from the last two terms in the criterion vanishes identically so that as predicted there is no obvious $m = 0$ instability.

Roxburgh (1963) has obtained a model of a star containing a toroidal magnetic field of the form

$$B = K\rho\varpi, \tag{3.3}$$

where K is a constant. Although the star rotates and contains a poloidal magnetic field, the toroidal field is much stronger than the poloidal field and the magnetic energy is much greater than the rotational energy. The toroidal field (3.3) is not altered by meridional circulation and, in the absence of instability, the configuration should be very long lived. Sinha (1968) has also constructed models of polytropes with a field of the form (3.3); these are non-rotating and contain no poloidal field. If we apply criterion (2.20) to this field it can be seen that instability must occur close enough to the axis of the star.

The oscillation properties of stars and polytropes of the above type have been studied by Roxburgh & Durney (1967), Anand (1969) and Sood & Trehan (1972). All of these treatments have involved variational methods or the use of virial tensor techniques. In each case the influence of a relatively weak magnetic field on the lowest modes of oscillation has been found and the oscillation frequencies have been given as a small correction to the frequencies of the non-magnetic star. These variational and virial tensor methods are ill suited to finding the very localized instabilities which we have just been discussing and I do not believe that they have been described before.

If we restrict our present attention to stars containing a purely toroidal field, we must conclude that, if a star containing a toroidal field is to be stable to short time-scale instabilities, the toroidal field must go to zero with ϖ at least as rapidly as ϖ^2. If, indeed,

$$B = B_2(\varpi/\varpi_0)^2 \tag{3.4}$$

and the magnetic field is weak enough that its effect on the structure of the star is slight, it is easy to see that criteria (2.13) and (2.20) can be written near to the axis:

$$g|\nabla\rho| - (\rho^2 g^2/\gamma P) - (2B_2{}^2 z^2/\varpi_0{}^4) > 0 \tag{3.5}$$

and

$$g|\nabla\rho| - (\rho^2 g^2/\gamma P) - (5B_2{}^2 z^2/\varpi_0{}^4) > 0, \tag{3.6}$$

where all of the quantities are, of course, functions of z and g is taken to be positive. Both of these criteria show that a magnetic field of the form (3.4) tends to lead to instability if the star is already on the verge of becoming unstable to convection and it can be seen that once again the $m = 1$ perturbations are the most severe.

It is difficult to obtain any more definite conclusions from the stability criteria without studying a definite field configuration and applying the criterion throughout the star. No models other than the ones proposed above, which are unstable, are available. The necessary and sufficient conditions for stability given in this paper can however be applied if any further model is proposed. Although we have been primarily concerned with the possibility that a toroidal field will lead to instability in a region which is stable to convection, the criteria do of course describe the influence of a toroidal field on the onset of convection.

4. CONCLUSIONS

In the previous sections we have obtained necessary and sufficient conditions for the stability of a star containing a toroidal magnetic field and have shown that a general class of field configurations is unstable. In our discussion two important questions have been left unanswered. The first is concerned with the ultimate effect of the instabilities discovered. The second is concerned with whether in a real star the effects of rotation and weak poloidal magnetic fields can ever be neglected.

It is not easy to estimate what will be the ultimate result of the instabilities. This would be much easier if the $m = 0$ disturbances were usually the most unstable. The effect of a succession of interchanges of flux tubes would be to redistribute the toroidal magnetic flux through the star until (hopefully) all tendency to instability was removed. The non-linear growth of these interchange instabilities is easily envisaged but the same is not true of the $m = 1$ perturbations which, from our limited investigation, seem likely to be the worst instabilities in the linear regime. It seems likely that, at any rate if the magnetic field is not very strong, the $m = 1$ instabilities will only occur close to the axis of the star, unless the star is on the brink of convective instability. The instabilities are therefore likely to be limited at finite amplitude and they will presumably gradually be damped. If they are damped by radiative conductivity or radiative viscosity, this will have no effect on the structure of the magnetic field, so that the instability will once again be excited. Although damping due to electrical resistivity seems much less probable than damping due to radiative conductivity, the period of the finite amplitude oscillations of the magnetic field is likely to be so small compared to either the thermal or nuclear time scale of the star that, even if very slight field dissipation occurs in each cycle of the oscillation, the ultimate effect of the $m = 1$ instabilities may be a substantial destruction of magnetic flux. This possibility, which has been mentioned by Mestel (1971), will be discussed further in the following paper on poloidal fields where the field configuration is *a priori* more probable.

It is also difficult to decide whether slow rotation or a weak poloidal field will seriously influence the results of this paper. There is a temptation to say that, if the energy of the poloidal field and the rotational energy are small compared to the energy of the toroidal field, the results will be essentially unaltered. However, it is not clear that this is valid since we have just shown that a toroidal field can induce instabilities when *its* energy is very small compared to the thermal energy and the gravitational energy.

In one case the effect of rotation can be discussed. Interchange perturbations can be considered provided that there is no poloidal magnetic field. We will mention only the analogue of criterion (2.13) for a uniformly rotating star. This can be shown to be

$$\frac{\partial \rho}{\partial \varpi}(g_\varpi+\Omega^2\varpi)-\{\rho(g_\varpi+\Omega^2\varpi)-2B^2/\varpi\}^2/(B^2+\gamma P)-\frac{2B}{\varpi}\frac{\partial B}{\partial \varpi}+\frac{2B^2}{\varpi^2}+4\rho\Omega^2>0, \tag{4.1}$$

where Ω is the angular velocity of the star. If this criterion is applied near the axis of the star, the leading term in ϖ is the term $4\rho\Omega^2$ which is definitely positive, showing that rotation tends to stabilize near axis interchanges. It is not so clear that rotation stabilizes $m = 1$ perturbations, because it is not possible to write down an $m = 1$ analogue of (4.1). It is in some sense an accident that the terms from B in (4.1) which are independent of ϖ cancel identically and we know that they do not cancel for $m = 1$ perturbations in the absence of rotation. It therefore seems likely that rotation and magnetic fields contribute in the same order to the $m = 1$ criteria and which effect is more important may depend on the ratio of rotational energy to magnetic energy.

It is known from the study of cylindrical plasmas that an axial field (or in this case an poloidal field) exerts a stabilizing influence, particularly on short wavelength instabilities. However, unless the poloidal field is comparable in strength with the maximum value of the toroidal field in the unstable region, the stabilization will not be very great and, even if the poloidal field is strong, some instabilities are likely to remain.

At the beginning of this paper it was suggested that, as a toroidal field is horizontal, it might not have much effect on convection and that a *sufficiently strong* field might introduce new hydromagnetic instabilities. Probably the main result of the paper is the demonstration that an *arbitrarily weak* toroidal field can introduce new instabilities, provided its shape is correct, and a similar result for poloidal fields will be found in the succeeding paper.

After this paper was ready for submission my attention was drawn to a recent paper by Vandakurov (1972) in which he discusses the stability of stars with toroidal magnetic fields. Although he does not obtain the precise criteria given in this paper, he also draws attention to the possibility of $m = 1$ modes being unstable and suggests that such instabilities might be particularly important in superdense stars and white dwarfs.

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APPENDIX 1

DERIVATION OF INTERCHANGE CRITERION

We consider the interchange of the material and magnetic flux contained in two axisymmetric flux tubes of volumes V and $V+\delta V$ respectively centred at the points ϖ, z and $\varpi+\delta\varpi$, $z+\delta z$. We calculate the energy content of these flux tubes before and after the interchange and obtain the condition for the change of energy consequent on the interchange to be positive. The interchange is assumed to be adiabatic. Thus in the interchange the mass, magnetic flux and P/ρ^γ are conserved. Thus the values of the physical quantities in the two flux tubes before and after interchange may be written:

Tube at ϖ, z.

Before	$P,$	$\rho,$	$B,$
After	$(P+\delta P)\left(\dfrac{V+\delta V}{V}\right)^\gamma,$	$\dfrac{(\rho+\delta\rho)(V+\delta V)}{V},$	$\dfrac{(B+\delta B)(V+\delta V)\,\varpi}{V(\varpi+\delta\varpi)}.$

Tube at $\varpi+\delta\varpi$, $z+\delta z$.

Before	$P+\delta P$	$\rho+\delta\rho,$	$B+\delta B,$
After	$P\left(\dfrac{V}{V+\delta V}\right)^\gamma,$	$\dfrac{\rho V}{V+\delta V},$	$\dfrac{BV(\varpi+\delta\varpi)}{(V+\delta V)\,\varpi}.$

In these expressions

$$\delta P = \frac{\partial P}{\partial \varpi}\,\delta\varpi + \frac{\partial P}{\partial z}\,\delta z$$

and similarly for $\delta\rho$ and δB.

Three types of energy must be considered; internal energy, magnetic energy and gravitational energy. Without loss of generality the zero of gravitational energy can be taken at ϖ, z. Then we have

$$(\text{Energy})_{\text{before}} = \frac{PV}{\gamma-1} + \frac{B^2V}{2} + \frac{(P+\delta P)(V+\delta V)}{\gamma-1} + \frac{(B+\delta B)^2(V+\delta V)}{2} - (\rho+\delta\rho)(V+\delta V)(g_z\delta z + g_\varpi\delta\varpi).$$

$$(\text{Energy})_{\text{after}} = \frac{(P+\delta P)\,V}{\gamma-1}\left(\frac{V+\delta V}{V}\right)^{\gamma} + \frac{(B+\delta B)^2(V+\delta V)^2\varpi^2}{V^2(\varpi+\delta\varpi)^2}\,\frac{V}{2}$$
$$+\frac{P(V+\delta V)}{\gamma-1}\left(\frac{V}{V+\delta V}\right)^{\gamma} + \frac{B^2V^2(\varpi+\delta\varpi)^2}{(V+\delta V)^2\varpi^2}\,\frac{(V+\delta V)}{2}$$
$$-\rho V(g_z\delta z+g_\varpi\delta\varpi).$$

It is now possible to expand both of these expressions to second order in the quantities δV, $\delta\varpi$, δz and to evaluate $E_{\text{after}}-E_{\text{before}}$ to that order. If that is done there results:

$$E_{\text{after}}-E_{\text{before}} = (B^2+\gamma P)\frac{\delta V^2}{V}+\delta V\left[\frac{\partial P}{\partial\varpi}\delta\varpi+\frac{\partial P}{\partial z}\delta z+B\frac{\partial B}{\partial\varpi}\delta\varpi+B\frac{\partial B}{\partial z}\delta z\right.$$
$$\left.+\rho g_z\delta z+\rho g_\varpi\delta\varpi-\frac{3B^2}{\varpi^2}\delta\varpi\right]$$
$$+V\left[\left(\frac{\partial\rho}{\partial\varpi}\delta\varpi+\frac{\partial\rho}{\partial z}\delta z\right)(g_z\delta z+g_\varpi\delta\varpi)+\frac{2B^2\delta\varpi^2}{\varpi^2}\right.$$
$$\left.-\frac{2B\delta\varpi}{\varpi}\left(\frac{\partial B}{\partial\varpi}\delta\varpi+\frac{\partial B}{\partial z}\delta z\right)\right],$$

or, using the equations of equilibrium (2.10),

$$E_{\text{after}}-E_{\text{before}} = (B^2+\gamma P)\frac{\delta V^2}{V}+2\delta V\left[\left(\rho g_\varpi-\frac{2B^2}{\varpi}\right)\delta\varpi+\rho g_z\delta z\right]$$
$$+V\left[\left(\frac{\partial\rho}{\partial\varpi}\delta\varpi+\frac{\partial\rho}{\partial z}\delta z\right)(g_z\delta z+g_\varpi\delta\varpi)+\frac{2B^2\delta\varpi^2}{\varpi^2}-\frac{2B\delta\varpi}{\varpi}\right.$$
$$\left.\times\left(\frac{\partial B}{\partial\varpi}\delta\varpi+\frac{\partial B}{\partial z}\delta z\right)\right].$$

We can now minimize this expression with respect to $\delta V/V$ to obtain the worst interchange. This gives

$$(B^2+\gamma P)\frac{\delta V}{V} = \left(\frac{B^2}{\varpi}-\rho g_\varpi\right)\delta\varpi-\rho g_z\delta z.$$

With this choice of $\delta V/V$, the energy change becomes

$$E_{\text{after}}-E_{\text{before}} = V\left\{\left(\frac{\partial\rho}{\partial\varpi}\delta\varpi+\frac{\partial\rho}{\partial z}\delta z\right)(g_z\delta z+g_\varpi\delta\varpi)\right.$$
$$+\frac{2B^2}{\varpi^2}\delta\varpi^2-\frac{2B}{\varpi}\delta\varpi\left(\frac{\partial B}{\partial\varpi}\delta\varpi+\frac{\partial B}{\partial z}\delta z\right)$$
$$\left.-\left[\left(\frac{2B^2}{\varpi}-\rho g_\varpi\right)\delta\varpi-\rho g_z\delta z\right]^2\Big/(B^2+\gamma P)\right\}.$$

The condition that this expression be positive for all $\delta\varpi$, δz now leads directly to the three conditions (2.12)–(2.14) as required.

APPENDIX 2

In this Appendix a form of X and Z is found which makes $I<0$, where

$$I \equiv \int \varpi \, d\varpi \, dz \left[B^2 \left\{\varpi \frac{\partial}{\partial \varpi}\left(\frac{X}{\varpi}\right)+\frac{\partial Z}{\partial z}\right\}^2 - \frac{B^2 X^2}{\varpi^2} - \frac{2B}{\varpi}\frac{\partial B}{\partial \varpi} X^2 \right.$$

$$- \frac{2B}{\varpi}\frac{\partial B}{\partial z} XZ + \frac{B^2}{\varpi^2} Z^2 + (g_\varpi X + g_z Z)\left(\frac{\partial \rho}{\partial \varpi} X + \frac{\partial \rho}{\partial z} Z\right)$$

$$\left. - \frac{1}{\gamma P}(\rho g_\varpi X + \rho g_z Z)^2 \right], \tag{A2.1}$$

whenever one or more of inequalities (2.20)–(2.22) is violated inside the region of integration.

We first introduce new variables

$$\bar{X} = X/\varpi, \quad \bar{Z} = Z/\varpi, \tag{A2.2}$$

so that the positive definite term in expression (A2.1) becomes

$$B^2\varpi^2(\partial \bar{X}/\partial \varpi + \partial \bar{Z}/dz)^2.$$

We now suppose that the sufficient conditions for stability are violated near the point (ϖ_0, z_0). In evaluating integral (A2.1) in a small region of the (ϖ, z) plane we can think of (ϖ, z) as Cartesian coordinates. We can now introduce a new set of Cartesian coordinates (ϖ', z') with origin at (ϖ_0, z_0) which are inclined to (ϖ, z) in such a way that the quadratic form in (A2.1) is diagonalized at the origin. If $(\bar{X}, \bar{Z})$ becomes (X', Z') in the new coordinate system, $\partial \bar{X}/\partial \varpi + \partial \bar{Z}/\partial z$ becomes $\partial X'/\partial \varpi' + \partial Z'/\partial z'$. For a region in the neighbourhood of the origin, I becomes

$$I = \int d\varpi' \, dz' \left[AX'^2 + BX'Z' + CZ'^2 + D\left(\frac{\partial X'}{\partial \varpi'} + \frac{\partial Z'}{\partial z'}\right)^2\right], \tag{A2.3}$$

where the coefficient B must not be confused with the magnetic induction and where D is positive and B vanishes at the origin. If the sufficient conditions for stability are violated at the origin, one or both of A and C must be negative there. We suppose that the coordinate system has been chosen so that A is negative.

We now consider a perturbation

$$X' = X_0 \sin k\varpi' \cos lz', \tag{A2.4}$$

$$Z' = Z_0 \cos k\varpi' \sin lz'.$$

We choose

$$kX_0 + lZ_0 = 0, \tag{A2.5}$$

so that the positive term in expression (A2.3) vanishes identically. In addition we choose

$$|l| \gg |k| \gg 1/h, \quad |X_0| \gg |Z_0|, \tag{A2.6}$$

where h is a typical scale height of the physical quantities. The perturbation can be confined to the region

$$-\frac{\pi}{k} < \varpi' < \frac{\pi}{k}, \quad -\frac{\pi}{l} < z' < \frac{\pi}{l}; \tag{A2.7}$$

as the normal component of $\boldsymbol{\xi}$ vanishes on the boundary of the region, all necessary boundary conditions are satisfied if $\boldsymbol{\xi}$ vanishes identically outside the region. Because of the first set of inequalities (A2.6), A, B and C are slowly varying in the region and, because of the second inequality, the integral is dominated by the negative term $\int AX'^2\, d\varpi'\, dz'$. Thus I is negative and the system is unstable.