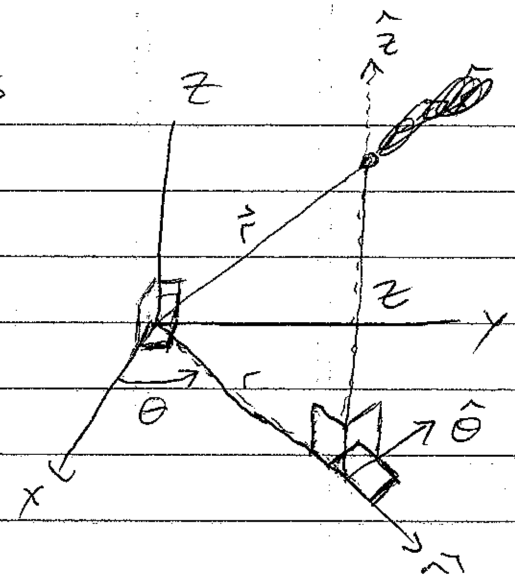
Cylindrical Coords



$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$\hat{z} = \hat{z}$$

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} \qquad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$$

$$\vec{r} = r\hat{r} + z\hat{z}$$

$$\vec{\nabla} = \hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \hat{z}\partial_z$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\partial_r(rA_r) + \frac{1}{r}\partial_\theta A_\theta + \partial_z A_z$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \partial_r & \partial_\theta & \partial_z \\ A_r & rA_\theta & A_z \end{vmatrix}$$

$$\vec{A}\cdot\vec{B} = A_r B_r + A_\theta B_\theta + A_z B_z \qquad \vec{A}\times\vec{B} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ A_r & A_\theta & A_z \\ B_r & B_\theta & B_z \end{vmatrix}$$

$$\nabla^2 f = \partial_r^2 f + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_\theta^2 f + \partial_z^2 f$$

$$\nabla^2\vec{A} = \hat{r}\left(\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\partial_\theta A_\theta\right) + \hat{\theta}\left(\nabla^2 A_\theta - A_\theta/r^2 + \frac{2}{r^2}\partial_\theta A_r\right) + \hat{z}\,\nabla^2 A_z$$

$$\vec{U}\cdot\vec{\nabla} f = U_r\partial_r f + \frac{U_\theta}{r}\partial_\theta f + U_z\partial_z f$$

$$\vec\nabla \vec U = \left(\hat r\partial_r + \frac{\hat\theta}{r}\partial_\theta + \hat z\partial_z\right)\left(U_r\hat r + U_\theta\hat\theta + U_z\hat z\right)$$

$$= \hat r\partial_r(U_r\hat r) + \hat r\partial_r(U_\theta\hat\theta) + \hat r\partial_r(U_z\hat z)$$
$$+ \frac{\hat\theta}{r}\partial_\theta(U_r\hat r) + \frac{\hat\theta}{r}\partial_\theta(U_\theta\hat\theta) + \frac{\hat\theta}{r}\partial_\theta(U_z\hat z)$$
$$+ \hat z\partial_z(U_r\hat r) + \hat z\partial_z(U_\theta\hat\theta) + \hat z\partial_z(U_z\hat z)$$

$$= \hat r\hat r\,\partial_r U_r + \hat r\hat\theta\,\partial_r U_\theta + \hat r\hat z\,\partial_r U_z$$
$$+ \frac{\hat\theta\hat r}{r}\partial_\theta U_r + \frac{U_r}{r}\hat\theta(\hat\theta) + \frac{\hat\theta\hat\theta}{r}\partial_\theta U_\theta + \frac{U_\theta}{r}\hat\theta(-\hat r) + \frac{\hat\theta\hat z}{r}\partial_\theta U_z$$
$$+ \hat z\hat r\,\partial_z U_r + \hat z\hat\theta\,\partial_z U_\theta + \hat z\hat z\,\partial_z U_z$$

$$\vec\nabla\vec U = \hat r\hat r(\partial_r U_r) + \hat r\hat\theta(\partial_r U_\theta) + \hat r\hat z(\partial_r U_z)$$
$$+ \hat\theta\hat r\left(\frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r}\right) + \hat\theta\hat\theta\left(\frac{\partial_\theta U_\theta}{r} + \frac{U_r}{r}\right) + \frac{\hat\theta\hat z}{r}(\partial_\theta U_z)$$
$$+ \hat z\hat r(\partial_z U_r) + \hat z\hat\theta(\partial_z U_\theta) + \hat z\hat z(\partial_z U_z)$$

$$\Rightarrow \vec A\cdot\vec\nabla\vec U = \left(A_r\hat r + A_\theta\hat\theta + A_z\hat z\right)\cdot\vec\nabla\vec U$$

$$= A_r\hat r\cdot\begin{bmatrix} rr & r\theta & rz \\ & \vdots & \\ & & \end{bmatrix} + A_\theta\hat\theta\cdot\begin{bmatrix} rr & & \\ & \theta\theta & \\ & & zz \end{bmatrix} + A_z\hat z\cdot\begin{bmatrix} rr & & \\ \theta r & \cdots & \\ zr & & \end{bmatrix}$$

$$= A_r(\partial_r U_r)\hat r + A_r(\partial_r U_\theta)\hat\theta + A_r(\partial_r U_z)\hat z$$
$$+ A_\theta\left(\frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r}\right)\hat r + A_\theta\left(\frac{\partial_\theta U_\theta}{r} + \frac{U_r}{r}\right)\hat\theta + \frac{A_\theta}{r}(\partial_\theta U_z)\hat z$$
$$+ A_z(\partial_z U_r)\hat r + A_z(\partial_z U_\theta)\hat\theta + A_z(\partial_z U_z)\hat z$$

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$$\vec A\cdot\vec\nabla\vec U = \left(A_r\partial_r U_r + \frac{A_\theta}{r}\partial_\theta U_r + A_z\partial_z U_r - \frac{A_\theta U_\theta}{r}\right)\hat r$$
$$+ \left(A_r\partial_r U_\theta + \frac{A_\theta}{r}\partial_\theta U_\theta + A_z\partial_z U_\theta + \frac{A_\theta U_r}{r}\right)\hat\theta$$
$$+ \left(A_r\partial_r U_z + \frac{A_\theta}{r}\partial_\theta U_z + A_z\partial_z U_z\right)\hat z$$

$\vec A = \vec U$ gives $\vec U\cdot\vec\nabla\vec U$

$$\vec{\nabla}\vec{v} = \partial_i v_j \Rightarrow \partial_r v_\theta = \partial_r v_\theta \hat{r}\hat{\theta}$$

$$\vec{\nabla}\vec{v} = \hat{r}\hat{r}\left(\partial_r v_r\right) + \hat{r}\hat{\theta}\left(\partial_r v_\theta\right) + \hat{r}\hat{z}\left(\partial_r v_z\right)$$

$$+\hat{\theta}\hat{r}\left(\frac{\partial_\theta v_r}{r} - \frac{v_\theta}{r}\right) + \hat{\theta}\hat{\theta}\left(\frac{\partial_\theta v_\theta}{r} + \frac{v_r}{r}\right) + \hat{\theta}\hat{z}\left(\frac{\partial_\theta v_z}{r}\right)$$

$$+\hat{z}\hat{r}\left(\partial_z v_r\right) + \hat{z}\hat{\theta}\left(\partial_z v_\theta\right) + \hat{z}\hat{z}\left(\partial_z v_z\right)$$

$$\left(\vec{\nabla}\vec{v}\right)^T = \hat{r}\hat{r}\left(\partial_r v_r\right) + \hat{r}\hat{\theta}\left(\frac{\partial_\theta v_r}{r} - \frac{v_\theta}{r}\right) + \hat{r}\hat{z}\left(\partial_z v_r\right)$$

$$+\hat{\theta}\hat{r}\left(\partial_r v_\theta\right) + \hat{\theta}\hat{\theta}\left(\frac{\partial_\theta v_\theta}{r} + \frac{v_r}{r}\right) + \hat{\theta}\hat{z}\left(\partial_z v_\theta\right)$$

$$+\hat{z}\hat{\theta}\left(\partial_r v_z\right) + \hat{z}\hat{\theta}\left(\frac{\partial_\theta v_z}{r}\right) + \hat{z}\hat{z}\left(\partial_z v_r\right)$$

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right) = \vec{\nabla}\left[\frac{1}{r}\partial_r(r v_r) + \frac{1}{r}\partial_\theta v_\theta + \partial_z v_z\right]$$

$$= \left(\hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \hat{z}\partial_z\right)\left(\partial_r v_r + \frac{v_r}{r} + \frac{1}{r}\partial_\theta v_\theta + \partial_z v_z\right)$$

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right) = \hat{r}\left[\partial_r^2 v_r - \frac{v_r}{r^2} + \frac{1}{r}\partial_r v_r - \frac{1}{r^2}\partial_\theta v_\theta + \frac{1}{r}\partial_r\partial_\theta v_\theta + \partial_r\partial_z v_z\right]$$
$$+ \hat{\theta}\left[\frac{1}{r}\partial_\theta\partial_r v_r + \frac{1}{r^2}\partial_\theta v_r + \frac{1}{r^2}\partial_\theta^2 v_\theta + \frac{1}{r}\partial_\theta\partial_z v_z\right]$$
$$+ \hat{z}\left[\partial_z\partial_r v_r + \frac{1}{r}\partial_z v_r + \frac{1}{r}\partial_z\partial_\theta v_\theta + \partial_z^2 v_z\right]$$

$$\nabla^2 \vec{A} = \nabla^2 (A_r \hat{r} + A_\theta \hat{\theta} + A_z \hat{z})$$
$$= \nabla^2 (A_r \hat{r}) + \nabla^2 (A_\theta \hat{\theta}) + \nabla^2 (A_z \hat{z})$$
$$= \hat{r} \nabla^2 A_r + 2 \vec{\nabla} A_r \cdot \vec{\nabla} \hat{r} + A_r \nabla^2 \hat{r}$$
$$+ \hat{\theta} \nabla^2 A_\theta + 2 \nabla A_\theta \cdot \nabla \hat{\theta} + A_\theta \nabla^2 \hat{\theta}$$
$$+ \hat{z} \nabla^2 A_z + 2 \nabla A_z \cdot \nabla \hat{z} + A_z \nabla^2 \hat{z}$$

$$\cancel{= \hat{r} \nabla^2 A_r + 2 [\partial_r A_r \hat{r} + \tfrac{1}{r} \hat{\theta} \partial_\theta A_r + \partial_z A_r \hat{z}] \cdot [\partial_r \hat{r} \hat{r} + \tfrac{1}{r} \partial_\theta}$$

$$= \hat{r} \nabla^2 A_r + 2 \left[\hat{r} \partial_r A_r + \frac{\hat{\theta}}{r} \partial_\theta A_r + \hat{z} \partial_z A_r \right] \cdot \left[\hat{r} \cancel{\partial_r \hat{r}} + \frac{\hat{\theta}}{r} \partial_\theta \hat{r} + \hat{z} \cancel{\partial_z \hat{r}} \right]$$
$$+ A_r \nabla^2 \hat{r} + \hat{\theta} \nabla^2 A_\theta + A_\theta \nabla^2 \hat{\theta} + \hat{z} \nabla^2 A_z + A_z \nabla^2 \hat{z}$$
$$+ 2 \left[\hat{r} \partial_r A_\theta + \frac{\hat{\theta}}{r} \partial_\theta A_\theta + \hat{z} \partial_z A_\theta \right] \cdot \left[\cancel{\hat{r} \partial_r \hat{\theta}} + \frac{\hat{\theta}}{r} \partial_\theta \hat{\theta} + \cancel{\hat{z} \partial_z \hat{\theta}}^{\,0} \right]$$
$$+ 2 \left[\hat{r} \partial_r A_z + \frac{\hat{\theta}}{r} \partial_\theta A_z + \hat{z} \partial_z A_z \right] \cdot \underbrace{\left[\cancel{\hat{r} \partial_r \hat{z}} + \frac{\hat{\theta}}{r} \cancel{\partial_\theta \hat{z}} + \hat{z} \cancel{\partial_z \hat{z}} \right]}_{0}$$

$$= \hat{r} \nabla^2 A_r + \hat{\theta} \nabla^2 A_\theta + \hat{z} \nabla^2 A_z + A_r \nabla^2 \hat{r} + A_\theta \nabla^2 \hat{\theta} + A_z \cancel{\nabla^2 \hat{z}}^{\,0}$$
$$+ 2 \left[\hat{r} \partial_r A_r + \frac{\hat{\theta}}{r} \partial_\theta A_r + \hat{z} \partial_z A_r \right] \cdot \left[\frac{\hat{\theta} \hat{\theta}}{r} \right]$$
$$+ 2 \left[\hat{r} \partial_r A_\theta + \frac{\hat{\theta}}{r} \partial_\theta A_\theta + \hat{z} \partial_z A_\theta \right] \cdot \left[-\frac{\hat{\theta} \hat{r}}{r} \right]$$

$$= \hat{r} \nabla^2 A_r + \hat{\theta} \nabla^2 A_\theta + \hat{z} \nabla^2 A_z + A_r \nabla^2 \hat{r} + A_\theta \nabla^2 \hat{\theta}$$
$$+ \frac{2}{r^2} \partial_\theta A_r \hat{\theta} - \frac{2}{r^2} \partial_\theta A_\theta \hat{r}$$

$$\nabla^2 \hat{r} = \frac{1}{r^2} \partial_\theta \partial_\theta \hat{r} = -\frac{\hat{r}}{r^2}$$

$$\nabla^2 \hat{\theta} = \frac{1}{r^2} \partial_\theta \partial_\theta \hat{\theta} = -\frac{\hat{\theta}}{r^2}$$

$$\Rightarrow \nabla^2 \vec{A} = \hat{r} \left(\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2} \partial_\theta A_\theta \right) ✓$$
$$+ \hat{\theta} \left(\nabla^2 A_\theta - \frac{A_\theta}{r^2} + \frac{2}{r^2} \partial_\theta A_r \right)$$
$$+ \hat{z} \left(\nabla^2 A_z \right)$$

$$\overleftrightarrow{\Pi} = \mu\left(\nabla\vec{u} + (\nabla\vec{u})^T - \frac{2}{3}\overleftrightarrow{I}(\nabla\cdot\vec{u})\right)$$

$$\Pi_{ij} = \mu\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{u}\right)$$

$$\vec{\nabla}\cdot\overleftrightarrow{\Pi} = \partial_i\Pi_{ij} = \mu\left[\partial_i(\partial_j u_i) + \partial_i\partial_i u_j - \frac{2}{3}\delta_{ij}\partial_i(\vec{\nabla}\cdot\vec{u})\right]$$

$$= \mu\left[\partial_j\partial_i u_i + \nabla^2 u_j - \frac{2}{3}\partial_j(\nabla\cdot\vec{u})\right]$$

$$= \mu\left[\partial_j(\nabla\cdot u) + \nabla^2 u_j - \frac{2}{3}\partial_j(\nabla\cdot u)\right]$$

$$\vec{\nabla}\cdot\overleftrightarrow{\Pi} = \mu\left[\nabla^2\vec{u} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{u})\right]$$

$$\frac{\vec{\nabla}\cdot\overleftrightarrow{\Pi}}{\mu} = \hat{r}\left(\nabla^2 u_r - \frac{u_r}{r^2} + \frac{2}{r^2}\partial_\theta u_\theta\right) + \hat{\theta}\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\partial_\theta u_r\right)$$
$$+ \hat{z}\,\nabla^2 u_z + \frac{1}{3}\Big[\hat{r}\Big\{\partial_r^2 u_r - \frac{u_r}{r^2} + \frac{1}{r}\partial_r u_r - \frac{1}{r^2}\partial_\theta u_\theta$$
$$+ \frac{1}{r}\partial_r\partial_\theta u_\theta + \partial_r\partial_z u_z\Big\} + \hat{\theta}\Big\{\frac{1}{r}\partial_\theta\partial_r u_r + \frac{1}{r^2}\partial_\theta u_r$$
$$+ \frac{1}{r^2}\partial_\theta^2 u_\theta + \frac{1}{r}\partial_\theta\partial_z u_z\Big\} + \hat{z}\Big\{\partial_z\partial_r u_r + \frac{1}{r}\partial_z u_r$$
$$+ \frac{1}{r}\partial_z\partial_\theta u_\theta + \partial_z^2 u_z\Big\}\Big]$$

$$\hat{r} = \nabla^2 u_r - \frac{u_r}{r^2} + \frac{2}{r^2}\partial_\theta u_\theta + \frac{1}{3}\partial_r^2 u_r - \frac{u_r}{3r^2} + \frac{1}{3r}\partial_r u_r$$
$$- \frac{1}{3r^2}\partial_\theta u_\theta + \frac{1}{3r}\partial_r\partial_\theta u_\theta + \frac{1}{3}\partial_r\partial_z u_z$$

$$\hat{\theta} = \nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\partial_\theta u_r + \frac{1}{3r}\partial_\theta\partial_r u_r + \frac{1}{3r^2}\partial_\theta u_r$$
$$+ \frac{1}{3r^2}\partial_\theta^2 u_\theta + \frac{1}{3r}\partial_\theta\partial_z u_z$$

$$\hat{z} = \nabla^2 u_z + \frac{1}{3}\partial_z\partial_r u_r + \frac{1}{3r}\partial_z u_r + \frac{1}{3r}\partial_z\partial_\theta u_\theta + \frac{1}{3}\partial_z^2 u_z$$

$$\cancel{\frac{\vec{\nabla}\cdot\overleftrightarrow{\Pi}}{\mu} = \Big[\underbrace{\partial_r^2 u_r + \frac{1}{r}\partial_r u_r + \frac{1}{r^2}\partial_\theta^2 u_r + \partial_z^2 u_r}_{\nabla^2 u_r} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\partial_\theta u_r + \frac{1}{3r}\partial_\theta\partial_r u_r + \frac{1}{3r^2}\partial_\theta u_r + \frac{1}{3r^2}\partial_\theta^2 u_\theta + \frac{1}{3r}\partial_\theta\partial_z u_z\Big]\hat{r} + \Big[}$$

$$\frac{\vec{\nabla}\cdot\overleftrightarrow{\Pi}}{\mu} = \hat{r}\Big[\underbrace{\partial_r^2 u_r + \frac{1}{r}\partial_r u_r + \frac{1}{r^2}\partial_\theta^2 u_r + \partial_z^2 u_r}_{\nabla^2 u_r} - \frac{3}{3r^2}u_r + \frac{2\cdot 3}{3r^2}\partial_\theta u_\theta$$
$$+ \frac{1}{3}\partial_r^2 u_r - \frac{1}{3r^2}u_r + \frac{1}{3r}\partial_r u_r - \frac{1}{3r^2}\partial_\theta u_\theta + \frac{1}{3r}\partial_r\partial_\theta u_\theta$$
$$+ \frac{1}{3}\partial_r\partial_z u_z\Big] +$$
$$\hat{\theta}\Big[\partial_r^2 u_\theta + \frac{1}{r}\partial_r u_\theta + \frac{1}{r^2}\partial_\theta^2 u_\theta + \partial_z^2 u_\theta - \frac{1}{r^2}u_\theta + \frac{2}{r^2}\partial_\theta u_r$$
$$+ \frac{1}{3r}\partial_\theta\partial_r u_r + \frac{1}{3r^2}\partial_\theta u_r + \frac{1}{3r^2}\partial_\theta^2 u_\theta + \frac{1}{3r}\partial_\theta\partial_z u_z\Big] +$$
$$\hat{z}\Big[\partial_r^2 u_z + \frac{1}{r}\partial_r u_z + \frac{1}{r^2}\partial_\theta^2 u_z + \partial_z^2 u_z + \frac{1}{3}\partial_z\partial_r u_r + \frac{1}{3r}\partial_z u_r$$
$$+ \frac{1}{3r}\partial_z\partial_\theta u_\theta + \frac{1}{3}\partial_z^2 u_z\Big]$$

$$\boxed{\begin{aligned}\frac{\vec{\nabla}\cdot\overleftrightarrow{\Pi}}{\mu} = {} & \Big[\frac{4}{3}\partial_r^2 u_r + \frac{4}{3r}\partial_r u_r + \frac{5}{3r^2}\partial_\theta u_\theta - \frac{4}{3r^2}u_r + \frac{1}{r^2}\partial_\theta^2 u_r + \partial_z^2 u_r \\ & + \frac{1}{3}\partial_r\partial_z u_z + \frac{1}{3r}\partial_r\partial_\theta u_\theta\Big]\hat{r} + \\ & \Big[\partial_r^2 u_r + \frac{1}{r}\partial_r u_\theta + \frac{4}{3}\partial_\theta^2 u_\theta + \partial_z^2 u_\theta - \frac{1}{r^2}u_\theta \\ & + \frac{7}{3r^2}\partial_\theta u_r + \frac{1}{3r}\partial_\theta\partial_r u_r + \frac{1}{3r}\partial_\theta\partial_z u_z\Big]\hat{\theta} + \\ & \Big[\partial_r^2 u_z + \frac{1}{r}\partial_r u_z + \frac{1}{r^2}\partial_\theta^2 u_z + \frac{4}{3}\partial_z^2 u_z + \frac{1}{3}\partial_z\partial_r u_r \\ & + \frac{1}{3r}\partial_z u_r + \frac{1}{3r}\partial_z\partial_\theta u_\theta\Big]\hat{z}\end{aligned}}$$

above assumes $\mu = \text{const}$

$$\vec{\nabla}\cdot\overleftrightarrow{\Pi} = \vec{\nabla}\cdot\mu\overleftrightarrow{S} = \vec{\nabla}\mu\cdot\overleftrightarrow{S} + \mu\vec{\nabla}\cdot\overleftrightarrow{S} \qquad (\mu \neq \text{const})$$

$$(\nabla U)_{ij} = \partial_i U_j$$

$\overleftrightarrow{\Pi} : \vec{\nabla} \vec{U}$ where $\hat{A} : \hat{B} = A_{ij} B_{ij}$

$$\overleftrightarrow{\Pi}_{ij} (\nabla U)_{ij} = \Pi_{11} \nabla U_{11} + \Pi_{12} \nabla U_{12} + \Pi_{13} \nabla U_{13} +$$
$$\Pi_{21} \nabla U_{21} + \Pi_{22} \nabla U_{22} + \Pi_{23} \nabla U_{23} +$$
$$\Pi_{31} \nabla U_{31} + \Pi_{32} \nabla U_{32} + \Pi_{33} \nabla U_{33}$$

$$\Pi_{ij} = \mu \left(\underbrace{\partial_i U_j}_{\nabla U} + \underbrace{\partial_j U_i}_{(\nabla U)^T} - \frac{2}{3} \delta_{ij} \partial_k U_k \right)$$

$$\left(-\frac{2}{3} \delta_{ij} \partial_k U_k \right) \partial_i U_j = -\frac{2}{3} \partial_k U_k \partial_i U_j \delta_{ij}$$
$$= -\frac{2}{3} (\vec{\nabla} \cdot \vec{U}) \partial_j U_j$$
$$= -\frac{2}{3} (\vec{\nabla} \cdot \vec{U})^2$$

$$+\frac{2}{3}(\vec{\nabla} \cdot \vec{U})^2 + \frac{\overleftrightarrow{\Pi} : \vec{\nabla} \vec{U}}{\mu} = \partial_1 U_1 \partial_1 U_1 + \partial_1 U_1^T \partial_1 U_1 + \partial_1 U_2 \partial_1 U_2 + \partial_2 U_1^T \partial_1 U_2$$
$$+ \partial_1 U_3 \partial_1 U_3 + \partial_3 U_1^T \partial_1 U_3 +$$
$$\partial_2 U_1 \partial_2 U_1 + \partial_1 U_2^T \partial_2 U_1 + \partial_2 U_2 \partial_2 U_2 + \partial_2 U_2^T \partial_2 U_2$$
$$+ \partial_2 U_3 \partial_2 U_3 + \partial_3 U_2^T \partial_2 U_3 +$$
$$\partial_3 U_1 \partial_3 U_1 + \partial_1 U_3^T \partial_3 U_1 + \partial_3 U_2 \partial_3 U_2 + \partial_2 U_3^T \partial_3 U_2$$
$$+ \partial_3 U_3 \partial_3 U_3 + \partial_3 U_3^T \partial_3 U_3$$

T superscript denotes it is from the $\nabla U^T : \nabla U$ term, not the ∇U^T matrix

i.e. $\partial_3 U_2^T = \partial_3 U_2$ (based on the indices of $\partial_i U_j + \partial_j U_i$)

so the transpose was taken when the indices were reversed so it still refers to elements of $\vec{\nabla} \vec{U}$ matrix different matrix $\Leftarrow$ same indices

$$\nabla u_{ij}\left[\nabla u_{ij} + \left(\nabla u^{T}\right)_{ij}\right] = \Pi_{ij}\nabla u_{ij}\ \frac{2}{3}(\vec{\nabla}\cdot\vec{u})^2$$

$$\cancel{\nabla u_{ij}} \qquad = \left[\partial_i u_j + \partial_j u_i\right]\partial_i u_j$$

same matrix, different indices

$$\frac{2}{3}(\vec{\nabla}\cdot\vec{u})^2 + \frac{\overleftrightarrow{\Pi}:\vec{\nabla}\vec{u}}{\mu} = 2(\partial_r u_r)^2 + (\partial_r u_\theta)^2 + \left(\frac{\partial_\theta u_r}{r} - \frac{u_\theta}{r}\right)(\partial_r u_\theta)$$

$$+ (\partial_r u_z)^2 + (\partial_z u_r)(\partial_r u_z) + \left(\frac{\partial_\theta u_r}{r} - \frac{u_\theta}{r}\right)^2$$

$$+ (\partial_r u_\theta)\left(\frac{\partial_\theta u_r}{r} - \frac{u_\theta}{r}\right) + 2\left(\frac{\partial_\theta u_\theta}{r} + \frac{u_r}{r}\right)^2$$

$$+ \left(\frac{\partial_\theta u_z}{r}\right)^2 + (\partial_z u_\theta)\left(\frac{\partial_\theta u_z}{r}\right) + (\partial_z u_r)^2$$

$$+ (\partial_r u_z)(\partial_z u_r) + (\partial_z u_\theta)^2 + \left(\frac{\partial_\theta u_z}{r}\right)(\partial_z u_\theta)$$

$$+2(\partial_z u_z)^2$$

$$\frac{\overleftrightarrow{\Pi}:\vec{\nabla}\vec{u}}{\mu} = -\frac{2}{3}(\vec{\nabla}\cdot\vec{u})^2 + 2(\partial_r u_r)^2 + (\partial_r u_\theta)^2 + 2(\partial_r u_\theta)\left(\frac{\partial_\theta u_r}{r} - \frac{u_\theta}{r}\right)$$

$$+ (\partial_r u_z)^2 + 2(\partial_z u_r)(\partial_r u_z) + \left(\frac{\partial_\theta u_r}{r} - \frac{u_\theta}{r}\right)^2$$

$$+ 2\left(\frac{\partial_\theta u_\theta}{r} + \frac{u_\theta}{r}\right)^2 + \left(\frac{\partial_\theta u_z}{r}\right)^2 + 2(\partial_z u_\theta)\left(\frac{\partial_\theta u_z}{r}\right)$$

$$+ (\partial_z u_r)^2 + (\partial_z u_\theta)^2 + 2(\partial_z u_z)^2$$

$$\left(\vec{\nabla}\cdot\vec{v}\right)^2 = \left(\partial_r A_r + \frac{A_r}{r} + \frac{1}{r}\partial_\theta A_\theta + \partial_z A_z\right)^2$$

$$= (\partial_r A_r)^2 + \left(\frac{A_r}{r}\right)^2 + \left(\frac{\partial_\theta A_\theta}{r}\right)^2 + \cancel{2}(\partial_z A_z)^2$$

$$+ \frac{2A_r}{r}\partial_r A_r + 2\frac{\partial_r A_r \partial_\theta A_\theta}{r} + 2\partial_r A_r \partial_z A_z$$

$$+ \frac{2A_r}{r^2}\partial_\theta A_\theta + \frac{2A_r}{r}\partial_z A_z + \frac{2\partial_\theta A_\theta \partial_z A_z}{r}$$

$$\Rightarrow \frac{\overleftrightarrow{\pi}:\vec{\nabla}\vec{v}}{\mu} = \cancel{-\frac{2}{3}(\partial_r v_r)^2} \cancel{-\frac{2}{3}\left(\frac{v_r}{r}\right)^2} - \frac{2}{3}\left(\frac{\partial_\theta v_\theta}{r}\right)^2 - \frac{2}{3}(\partial_z v_z)^2$$

$$-\frac{4}{3}\frac{v_r}{r}\partial_r v_r - \frac{4}{3}\frac{\partial_r v_r \partial_\theta v_\theta}{r} - \frac{4}{3}\partial_r v_r \partial_z v_z$$

$$\cancel{-\frac{4}{3}\frac{v_r}{r^2}\partial_\theta v_\theta} \cancel{-\frac{4}{3}\frac{v_r}{r}\partial_z v_z} - \frac{4}{3}\frac{\partial_\theta v_\theta \partial_z v_z}{r}$$

$$+ \cancel{2(\partial_r v_r)^2} + (\partial_r v_\theta)^2 + 2(\partial_r v_\theta)\left(\frac{\partial_\theta v_r}{r} - \frac{v_\theta}{r}\right)$$

$(\partial_\theta v_r)^2$ $v_\theta \partial_\theta v_r$ v_θ^2

$$+ (\partial_r v_z)^2 + 2\partial_z v_r \partial_r v_z + \left(\frac{\partial_\theta v_r}{r} - \frac{v_\theta}{r}\right)^2$$

$$+ 2\left(\frac{\partial_\theta v_\theta}{r} + \frac{v_\theta}{r}\right)^2 + \left(\frac{\partial_\theta v_z}{r}\right)^2 + 2\partial_z v_\theta \frac{\partial_\theta v_z}{r}$$

$$+ (\partial_z v_r)^2 + (\partial_z v_\theta)^2 + 2(\partial_z v_z)^2$$

$$= \cancel{\frac{4}{3}(\partial_r v_r)^2 - \frac{2}{3}\frac{v_r^2}{r^2}}$$

$$= \frac{4}{3}(\partial_r v_r)^2 - \cancel{\frac{2}{3}\frac{v_r^2}{r^2}} - \cancel{\frac{2}{3}\frac{(\partial_\theta v_\theta)^2}{r^2}} - \frac{2}{3}(\partial_z v_z)^2 - \frac{4}{3}\frac{v_r \partial_r v_r}{r}$$

$$\cancel{-\frac{4}{3}\frac{\partial_r v_r \partial_\theta v_\theta}{r}} - \frac{4}{3}\partial_r v_r \partial_z v_z - \cancel{\frac{4}{3}\frac{v_r}{r^2}\partial_\theta v_\theta} - \cancel{\frac{4}{3}\frac{v_r}{r}\partial_z v_z}$$

$$\cancel{-\frac{4}{3}\frac{\partial_\theta v_\theta \partial_z v_z}{r}} + \cancel{(\partial_r v_\theta)^2} + \cancel{\frac{2\partial_r v_\theta \partial_\theta v_r}{r}} - \cancel{\frac{2 v_\theta \partial_r v_\theta}{r}}$$

$$+ (\partial_r v_z)^2 + \cancel{2\partial_z v_r \partial_r v_z} + \cancel{\frac{(\partial_\theta v_r)^2}{r^2}} - \cancel{\frac{2 v_\theta \partial_\theta v_r}{r^2}} + \cancel{\frac{v_\theta^2}{r^2}}$$

$$+ \cancel{2\frac{(\partial_\theta v_\theta)^2}{r^2}} + \cancel{4\frac{v_\theta \partial_\theta v_\theta}{r^2}} + \cancel{2\frac{v_\theta^2}{r^2}} + \frac{(\partial_\theta v_z)^2}{r^2} + \frac{2\partial_z v_\theta \partial_\theta v_z}{r}$$

$$+ (\partial_z v_r)^2 + (\partial_z v_\theta)^2 + 2(\partial_z v_z)^2$$

$$\frac{\overleftrightarrow{\Pi}:\vec{\nabla}\vec{U}}{\mu} = \frac{4}{3}(\partial_r U_r)^2 - \frac{2}{3}\frac{U_r^2}{r^2} + \frac{4}{3}\frac{(\partial_\theta U_\theta)^2}{r^2} + \frac{4}{3}(\partial_z U_z)^2$$

$$- \frac{4}{3}\frac{U_r \partial_r U_r}{r} - \frac{4}{3}\frac{\partial_r U_r \partial_\theta U_\theta}{r} - \frac{4}{3}\partial_r U_r \partial_z U_z$$

$$- \frac{4}{3}\frac{U_r}{r^2}\partial_\theta U_\theta - \frac{4}{3}\frac{U_r}{r}\partial_z U_z - \frac{4}{3}\frac{\partial_\theta U_\theta \partial_z U_z}{r}$$

$$+ (\partial_r U_\theta)^2 + 2\frac{\partial_r U_\theta \partial_\theta U_r}{r} - 2\frac{U_\theta \partial_r U_\theta}{r} + (\partial_r U_z)^2$$

$$+ 2\partial_z U_r \partial_r U_z + \frac{(\partial_\theta U_r)^2}{r^2} - 2\frac{U_\theta \partial_\theta U_r}{r^2} + 3\frac{U_\theta^2}{r^2}$$

$$+ 4\frac{U_\theta \partial_\theta U_\theta}{r^2} + \frac{(\partial_\theta U_z)^2}{r^2} + \frac{2\partial_z U_\theta \partial_\theta U_z}{r} + (\partial_z U_r)^2 + (\partial_z U_\theta)^2$$

Fluid Equations

MHD: μ, k, η need not be const:

1) $P = (\gamma-1) C_v \rho T = \rho^{\gamma} e^{S/C_v} = \rho k T / \bar{m}$

2) $\rho T \frac{DS}{Dt} = -\vec{\nabla}\cdot\vec{Q} + \overleftrightarrow{\pi} : \vec{\nabla}\vec{U} + \frac{\eta}{4\pi}(\vec{\nabla}\times\vec{B})^2 \qquad \vec{Q} = -k\vec{\nabla}T$

3) $\partial_t \rho + \vec{U}\cdot\vec{\nabla}\rho + \rho\vec{\nabla}\cdot\vec{U} = 0$

4) $\rho \frac{D\vec{U}}{Dt} = -\vec{\nabla}P + \rho\vec{g} + \vec{\nabla}\cdot\overleftrightarrow{\pi} + \frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi}$

5) $\partial_t \vec{B} = \vec{\nabla}\times\vec{U}\times\vec{B} - \vec{\nabla}\times(\eta\vec{\nabla}\times\vec{B})$

Hydro:

1) $P = (\gamma-1) C_v \rho T$

2) $\rho T \frac{DS}{Dt} = -\vec{\nabla}\cdot\vec{Q} + \overleftrightarrow{\pi} : \vec{\nabla}\vec{U}$

3) $\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{U}$

4) $\rho \frac{D\vec{U}}{Dt} = -\vec{\nabla}P + \rho\vec{g} + \vec{\nabla}\cdot\overleftrightarrow{\pi}$

Pressure Eqn (derived on pg 14)

$$\frac{DP}{Dt} = -\gamma P\vec{\nabla}\cdot\vec{U} - (\gamma-1)\vec{\nabla}\cdot\vec{Q} + (\gamma-1)\overleftrightarrow{\pi} : \vec{\nabla}\vec{U} + \frac{(\gamma-1)}{4\pi}\eta(\vec{\nabla}\times\vec{B})^2$$

Derivation of $T\alpha_T \frac{ds}{c_P} = \frac{1}{\Gamma_1}\frac{dP}{P} - \frac{d\rho}{\rho}$

(1) $\left.\frac{\partial a}{\partial b}\right|_c \left.\frac{\partial b}{\partial a}\right|_c = 1$ (2) $\left.\frac{\partial a}{\partial b}\right|_c \left.\frac{\partial b}{\partial c}\right|_a \left.\frac{\partial c}{\partial a}\right|_b = -1$

$\Gamma_1 \equiv \left.\frac{\partial P}{\partial \rho}\right|_s \frac{\rho}{P}$ $\qquad ds = \left.\frac{\partial s}{\partial P}\right|_\rho dP + \left.\frac{\partial s}{\partial \rho}\right|_P d\rho$

$$-\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_P ds = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_P \left.\frac{\partial s}{\partial P}\right|_\rho dP - \frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_P \left.\frac{\partial s}{\partial \rho}\right|_P d\rho$$

$$= -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_P \left.\frac{\partial s}{\partial P}\right|_\rho dP - \frac{d\rho}{\rho} \quad \text{using (1)}$$

$$= \frac{1}{\rho}\left.\frac{\partial \rho}{\partial P}\right|_s dP - \frac{d\rho}{\rho} \quad \text{using (2)}$$

$$= \frac{P}{P}\frac{1}{\rho}\left.\frac{\partial \rho}{\partial P}\right|_s dP - \frac{d\rho}{\rho} = \boxed{\frac{1}{\Gamma_1}\frac{dP}{P} - \frac{d\rho}{\rho}}$$

$dh = T\,ds + \frac{1}{\rho}dP \Rightarrow T = \left.\frac{\partial h}{\partial s}\right|_P \quad \frac{1}{\rho} = \left.\frac{\partial h}{\partial P}\right|_s$

$$\frac{\partial}{\partial s}\left.\frac{\partial h}{\partial P}\right|_s = \frac{\partial^2 h}{\partial s \partial P} = \frac{\partial}{\partial s}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\left.\frac{\partial \rho}{\partial s}\right|_P \Rightarrow \rho\frac{\partial^2 h}{\partial s\partial P} = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_P$$

$$-\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_P = \rho\frac{\partial^2 h}{\partial s \partial P} = \rho\frac{\partial}{\partial P}\left(\frac{\partial h}{\partial s}\right) = \rho\left.\frac{\partial T}{\partial P}\right|_s$$

$$= -\rho\left.\frac{\partial s}{\partial P}\right|_T \left.\frac{\partial T}{\partial s}\right|_P \quad \text{using (2)}$$

$$= -\rho\left.\frac{\partial s}{\partial P}\right|_T \frac{T}{c_P} \quad \text{using } c_P \equiv T\left.\frac{\partial s}{\partial T}\right|_P$$

$$g = h - Ts \Rightarrow dg = dh - T\,ds - s\,dT = \frac{1}{\rho}\,dp - s\,dT$$

$$\left.\frac{\partial g}{\partial p}\right|_T = \frac{1}{\rho} \Rightarrow \frac{\partial^2 g}{\partial p\,\partial T} = -\frac{1}{\rho^2}\left.\frac{\partial \rho}{\partial T}\right|_p$$

$$\left.\frac{\partial g}{\partial T}\right|_p = -s \Rightarrow -\left.\frac{\partial s}{\partial p}\right|_T = \frac{\partial^2 g}{\partial p\,\partial T}$$

$$-\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_p = -\frac{\rho T}{c_p}\left.\frac{\partial s}{\partial p}\right|_T = \frac{\rho T}{c_p}\frac{\partial^2 g}{\partial p\,\partial T} = \frac{-\rho T}{c_p \rho^2}\left.\frac{\partial \rho}{\partial T}\right|_p$$

$$= -\frac{T}{c_p}\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right|_p \equiv \frac{\alpha_T T}{c_p}$$

$$\alpha_T \equiv -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right|_p$$

$$-\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_p ds = \frac{dp}{\Gamma_1 p} - \frac{d\rho}{\rho}$$

$$\boxed{\frac{\alpha_T T}{c_p}\,ds = \frac{dp}{\Gamma_1 p} - \frac{d\rho}{\rho}}$$

Ideal gas $\Rightarrow \alpha_T = 1/T,\ \Gamma_1 = \gamma$

$$\frac{ds}{c_p} = \frac{dp}{\gamma p} - \frac{d\rho}{\rho}$$

$$\frac{1}{c_p}\frac{Ds}{Dt} = \frac{1}{\gamma}\frac{D\log p}{Dt} - \frac{D\log \rho}{Dt}$$

Derivation of Pressure Eqn

Ideal gas $\Rightarrow$ $\alpha = 1/T$ $\Gamma_1 = \gamma$ $P = (\gamma-1)c_v \rho T$ $c_p = \gamma c_v$

$$\therefore \frac{dS}{c_p} = \frac{dP}{\gamma P} - \frac{d\rho}{\rho} \Rightarrow \frac{1}{c_p}\frac{DS}{Dt} = \frac{D\log P}{\gamma Dt} - \frac{D\log\rho}{Dt}$$

$$\frac{DS}{Dt} = \frac{1}{\rho T}\vec{\nabla}\cdot(k\vec{\nabla}T) + \frac{1}{\rho T}\overleftrightarrow{\Pi}:\vec{\nabla}\vec{U} + \frac{1}{\rho T}\frac{\eta}{4\pi}(\vec{\nabla}\times\vec{B})^2$$

$$\frac{1}{\gamma}\frac{D\log P}{Dt} - \frac{D\log\rho}{Dt} = \frac{1}{\rho T c_p}\vec{\nabla}\cdot(k\vec{\nabla}T) + \frac{1}{\rho T c_p}\overleftrightarrow{\Pi}:\vec{\nabla}\vec{U} + \frac{1}{\rho T c_p}\frac{\eta}{4\pi}(\vec{\nabla}\times\vec{B})^2$$

$$\frac{1}{P}\frac{DP}{Dt} = -\gamma\vec{\nabla}\cdot\vec{U} + \frac{\gamma}{\rho T c_p}\vec{\nabla}\cdot(k\vec{\nabla}T) + \frac{\gamma}{\rho T c_p}\overleftrightarrow{\Pi}:\vec{\nabla}\vec{U} + \frac{\gamma}{\rho T c_p}\frac{\eta}{4\pi}(\vec{\nabla}\times\vec{B})^2$$

$$\boxed{\frac{DP}{Dt} = -\gamma P\vec{\nabla}\cdot\vec{U} + (\gamma-1)\vec{\nabla}\cdot(k\vec{\nabla}T) + (\gamma-1)\overleftrightarrow{\Pi}:\vec{\nabla}\vec{U} + \frac{(\gamma-1)\eta}{4\pi}(\vec{\nabla}\times\vec{B})^2}$$

Ohmic Heating Term

$\vec{J} = \sigma \vec{E}$ $\qquad \vec{E} = -\frac{\vec{v}}{c} \times \vec{B} + \vec{J}/\sigma$

$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \partial_t \vec{B}$

$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \vec{\nabla} \times (\vec{v} \times \vec{B}) + \vec{\nabla} \times \left(\frac{1}{\sigma} \vec{J}\right) = -\frac{1}{c} \partial_t \vec{B}$

$\partial_t \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}) - c \vec{\nabla} \times \left[\frac{1}{\sigma} \frac{c}{4\pi} \vec{\nabla} \times \vec{B} \right]$ $\qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$

$\partial_t \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}) - \vec{\nabla} \times \left(\frac{c^2}{4\pi\sigma} \vec{\nabla} \times \vec{B} \right)$ $\qquad \eta \equiv \frac{c^2}{4\pi\sigma}$

$$\boxed{\partial_t \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}) - \vec{\nabla} \times (\eta \vec{\nabla} \times \vec{B})}$$

$\eta = \text{const}$

$$\partial_t \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B}$$

$$\text{Ohmic Heating} = \vec{J} \cdot \vec{E} = \vec{J} \cdot \vec{J}/\sigma = J^2 / \tfrac{c^2}{4\pi\eta}$$

$$= \frac{4\pi\eta}{c^2} J^2 = \frac{4\pi\eta}{c^2} \frac{c^2}{(4\pi)^2} (\vec{\nabla} \times \vec{B})^2$$

$$= \frac{\eta}{4\pi} (\vec{\nabla} \times \vec{B})^2$$

Dedalus Friendly (Hydro) Equations

1) $$\partial_t \rho + \vec{U}\cdot\vec{\nabla}\rho + \rho\vec{\nabla}\cdot\vec{U} = 0$$

$$\partial_t\rho_0 + \partial_t\rho_1 + (V\hat{\theta} + \vec{U})\cdot\nabla(\rho_0+\rho_1) + (\rho_0+\rho_1)\vec{\nabla}\cdot(V\hat{\theta}+\vec{U}) = 0$$

$$\cancel{\partial_t\rho_0} + \partial_t\rho_1 + \cancel{V\hat{\theta}\cdot\nabla\rho_0} + V\hat{\theta}\cdot\nabla\rho_1 + U\cdot\nabla\rho_0 + U\cdot\nabla\rho_1 + \rho_0\cancel{(\nabla\cdot V\hat{\theta})}$$

$$+ \rho_0(\nabla\cdot\vec{U}) + \rho_1\nabla\cdot V\hat{\theta} + \rho_1\nabla\cdot\vec{U} = 0$$

$$\partial_t\rho_1 + \frac{V}{r}\partial_\theta\rho_1 + \vec{U}\cdot\nabla\rho_0 + \rho_0\vec{\nabla}\cdot\vec{U} + \underbrace{\rho_1\cancel{\frac{1}{r}\partial_\theta V}}_{0} = -\vec{U}\cdot\vec{\nabla}\rho_1 - \rho_1(\vec{\nabla}\cdot\vec{U})$$

$$\partial_t\rho_1 + \frac{V\partial_\theta\rho_1}{r} + \vec{U}\cdot\vec{\nabla}\rho_0 + \frac{\rho_0}{r}\partial_r(rU_r) + \frac{\rho_0}{r}\partial_\theta U_\theta + \rho_0\partial_z U_z$$

$$= -\vec{U}\cdot\vec{\nabla}\rho_1 - \frac{\rho_1}{r}\partial_r(rU_r) - \frac{\rho_1}{r}\partial_\theta U_\theta - \rho_1\partial_z U_z$$

$$\boxed{\begin{aligned} r\partial_t\rho_1 + V\partial_\theta\rho_1 + r\vec{U}\cdot\vec{\nabla}\rho_0 + \rho_0\partial_r(rU_r) + \rho_0\partial_\theta U_\theta + r\rho_0\partial_z U_z \\ = -\rho_1\partial_r(rU_r) - \rho_1\partial_\theta U_\theta - r\rho_1\partial_z U_z \\ - rU_r\partial_r\rho_1 - U_\theta\partial_\theta\rho_1 - rU_z\partial_z\rho_1 \end{aligned}}$$

Continuity

2) $$\rho\partial_t\vec{U} + \rho\vec{U}\cdot\vec{\nabla}\vec{U} = -\nabla P + \rho\vec{g} + \vec{\nabla}\cdot\overleftrightarrow{\pi}$$

$$\partial_t\vec{U} + \vec{U}\cdot\nabla\vec{U} = -\frac{1}{\rho}\nabla P + \vec{g} + \frac{1}{\rho}\nabla\cdot\overleftrightarrow{\pi}$$

$$\partial_t V\hat{\Theta} + \partial_t\vec{U}_1 + V\hat{\Theta}\cdot\nabla V\hat{\Theta} + V\hat{\Theta}\cdot\nabla\vec{U}_1 + \vec{U}_1\cdot\nabla V\hat{\Theta} + \vec{U}_1\cdot\nabla\vec{U}_1 =$$
$$-\frac{1}{\rho}\nabla P_0 - \frac{1}{\rho}\nabla P_1 + \frac{\rho_0\vec{g}}{\rho} + \frac{\rho_1}{\rho}\vec{g} + \frac{1}{\rho}\vec{\nabla}\cdot\overleftrightarrow{\pi}_0 + \frac{1}{\rho}\vec{\nabla}\cdot\overleftrightarrow{\pi}_1$$

HSE $\Rightarrow$
$$V\hat{\Theta}\cdot\nabla V\hat{\Theta} = -\frac{1}{\rho}\nabla P_0 + \frac{\rho_0\vec{g}}{\rho} + \frac{1}{\rho}\vec{\nabla}\cdot\overleftrightarrow{\pi}_0$$

$$\Rightarrow \partial_t\vec{U}_1 + V\hat{\Theta}\cdot\nabla\vec{U}_1 + \vec{U}_1\cdot\nabla V\hat{\Theta} + \vec{U}_1\cdot\nabla\vec{U}_1 =$$
$$-\frac{1}{\rho}\nabla P_1 + \frac{\rho_1}{\rho}\vec{g} + \frac{1}{\rho}\vec{\nabla}\cdot\overleftrightarrow{\pi}_1$$

HSE subtracted off

$$(\rho_0+\rho_1)\partial_t\vec{U}_1 + (\rho_0+\rho_1)\left[V\hat{\Theta}\cdot\nabla\vec{U}_1 + \vec{U}_1\cdot\nabla\hat{\Theta}V + \vec{U}_1\cdot\nabla\vec{U}_1\right] =$$
$$-\nabla P_1 + \rho_1\vec{g} + \nabla\cdot\overleftrightarrow{\pi}_1$$

$$\rho_0\partial_t\vec{U}_1 + \rho_0\left[V\hat{\Theta}\cdot\nabla\vec{U}_1 + \vec{U}_1\cdot\nabla V\hat{\Theta} + \vec{U}_1\cdot\nabla\vec{U}_1\right] + \rho_1\left[\partial_t\vec{U}_1 + V\hat{\Theta}\cdot\nabla\vec{U}_1 + \vec{U}_1\cdot\nabla V\hat{\Theta} + \vec{U}_1\cdot\nabla\vec{U}_1\right]$$
$$= -\nabla P_1 + \rho_1\vec{g} + \nabla\cdot\overleftrightarrow{\pi}_1$$

$$\rho_0\partial_t\vec{U}_1 + \rho_0\left[V\hat{\Theta}\cdot\nabla\vec{U}_1 + \vec{U}_1\cdot\nabla V\hat{\Theta} + \vec{U}_1\cdot\nabla\vec{U}_1\right] + \rho_1\left[-\frac{1}{\rho}\nabla P_1 + \frac{\rho_1}{\rho}\vec{g} + \frac{1}{\rho}\vec{\nabla}\cdot\overleftrightarrow{\pi}_1\right]$$
$$= -\nabla P_1 + \rho_1\vec{g} + \nabla\cdot\overleftrightarrow{\pi}_1$$

$$\rho_0\partial_t\vec{U}_1 + \nabla P_1 - \rho_1\vec{g} - \vec{\nabla}\cdot\overleftrightarrow{\pi}_1 = -\rho_0\left[V\hat{\Theta}\cdot\nabla\vec{U}_1 + \vec{U}_1\cdot\nabla V\hat{\Theta}\right] - \rho_0\vec{U}_1\cdot\nabla\vec{U}_1 - \rho_1[\quad]$$

$$\boxed{\begin{aligned}\rho_0\partial_t\vec{U}_1 + \nabla P_1 - \rho_1\vec{g} - \nabla\cdot\overleftrightarrow{\pi}_1 + \rho_0 V\hat{\Theta}\cdot\nabla\vec{U}_1 + \rho_0\vec{U}_1\cdot\nabla V\hat{\Theta} &= -\rho_0\vec{U}_1\cdot\nabla\vec{U}_1 \\ &- \rho_1\left[-\frac{1}{\rho}\nabla P_1 + \frac{\rho_1}{\rho}\vec{g} + \frac{1}{\rho}\vec{\nabla}\cdot\overleftrightarrow{\pi}_1\right]\end{aligned}}$$

$$\rho_0 \partial_t \vec{U}_1 + \hat{r}\partial_r P_1 + \frac{\hat{\theta}}{r}\partial_\theta P_1 + \hat{z}\partial_z P_1 - \rho_1 \vec{g} - \vec{\nabla}\cdot\overleftrightarrow{\pi}_1$$

$$+ \left(\rho_0 \frac{V}{r}\partial_\theta U_r - \rho_0 \frac{V U_\theta}{r}\right)\hat{r} + \left(\rho_0 \frac{V}{r}\partial_\theta U_\theta + \rho_0 \frac{V U_r}{r}\right)\hat{\theta}$$

$$+ \left(\rho_0 \frac{V}{r}\partial_\theta U_z\right)\hat{z} + \left(-\rho_0 \frac{U_\theta V}{r}\right)\hat{r} + \left(\rho_0 U_r \partial_r V\right)\hat{\theta}$$

$$= -\rho_0\left[U_r\partial_r U_r + \frac{U_\theta}{r}\partial_\theta U_r + U_z\partial_z U_r - \frac{U_\theta U_\theta}{r}\right]\hat{r}$$

$$-\rho_0\left[U_r\partial_r U_\theta + \frac{U_\theta}{r}\partial_\theta U_\theta + U_z\partial_z U_\theta + \frac{U_\theta U_r}{r}\right]\hat{\theta}$$

$$-\rho_0\left[U_r\partial_r U_z + \frac{U_\theta}{r}\partial_\theta U_z + U_z\partial_z U_z\right]\hat{z}$$

$$-\frac{\rho_1}{\rho}\left[\hat{r}\partial_r P_1 + \frac{\hat{\theta}}{r}\partial_\theta P_1 + \hat{z}\partial_z P_1 + \rho_1\vec{g} + \vec{\nabla}\cdot\overleftrightarrow{\pi}_1\right]$$

$$\rho_0\partial_t\vec{U}_1 + \hat{r}\left[\partial_r P_1 + \rho_0\frac{V}{r}\partial_\theta U_r - \rho_0\frac{V U_\theta}{r} - \rho_0\frac{V U_\theta}{r}\right] +$$

$$\hat{\theta}\left[\frac{1}{r}\partial_\theta P_1 + \rho_0\frac{V}{r}\partial_\theta U_\theta + \rho_0\frac{V U_r}{r} + \rho_0 U_r\partial_r V\right] +$$

$$\hat{z}\left[\partial_z P_1 + \rho_0\frac{V}{r}\partial_\theta U_z\right] - \rho_1\vec{g} - \vec{\nabla}\cdot\overleftrightarrow{\pi}_1 =$$

$$= \left[-\rho_0 U_r\partial_r U_r - \rho_0\frac{U_\theta}{r}\partial_\theta U_r - \rho_0 U_z\partial_z U_r + \rho_0\frac{U_\theta^2}{r}\right]\hat{r}$$

$$+ \left[-\rho_0 U_r\partial_r U_\theta - \rho_0\frac{U_\theta}{r}\partial_\theta U_\theta - \rho_0 U_z\partial_z U_\theta - \rho_0\frac{U_\theta U_r}{r}\right]\hat{\theta}$$

$$+ \left[-\rho_0 U_r\partial_r U_z - \rho_0\frac{U_\theta}{r}\partial_\theta U_z - \rho_0 U_z\partial_z U_z\right]\hat{z} - \frac{\rho_1}{\rho}\left[\hat{r}\partial_r P_1 + \frac{\hat{\theta}}{r}\partial_\theta P_1 + \hat{z}\partial_z P_1\right]$$

$$- \frac{\rho_1}{\rho}\left[\rho_1\vec{g} + \vec{\nabla}\cdot\overleftrightarrow{\pi}_1\right]$$

Momentum Eqn by Components

$$\rho_0 \partial_t U_r + \partial_r P_1 + \frac{\rho_0 V}{r}\partial_\theta U_r - \rho_0 \frac{V U_\theta}{r} - \rho_0 \frac{V U_\theta}{r} - \rho_1 \vec{g}\cdot\hat{r}$$

$$-(\vec{\nabla}\cdot\overleftrightarrow{\pi}_1)\cdot\hat{r} = -\rho_0 U_r \partial_r U_r - \rho_0 \frac{U_\theta}{r}\partial_\theta U_r - \rho_0 U_z \partial_z U_r$$

$$+\rho_0 \frac{U_\theta^2}{r} - \frac{\rho_1}{\rho_1}\partial_r P_1 - \frac{\rho_1^2}{\rho}\vec{g}\cdot\hat{r}$$

$$-\frac{\rho_1}{\rho}(\vec{\nabla}\cdot\overleftrightarrow{\pi}_1)\cdot\hat{r}$$

$$\rho_0 \partial_t U_\theta + \frac{1}{r}\partial_\theta P_1 + \frac{\rho_0 V}{r}\partial_\theta U_\theta + \rho_0 \frac{V U_r}{r} + \rho_0 U_r \partial_r V - \rho_1 \vec{g}\cdot\hat{\theta} - (\vec{\nabla}\cdot\overleftrightarrow{\pi}_1)\cdot\hat{\theta}$$

$$= -\rho_0 U_r \partial_r U_\theta - \rho_0 \frac{U_\theta}{r}\partial_\theta U_\theta - \rho_0 U_z \partial_z U_\theta - \rho_0 \frac{U_\theta U_r}{r}$$

$$-\frac{\rho_1}{\rho}\frac{\partial_\theta P_1}{r} - \frac{\rho_1^2}{\rho}\vec{g}\cdot\hat{\theta} - \frac{\rho_1}{\rho}(\vec{\nabla}\cdot\overleftrightarrow{\pi}_1)\cdot\hat{\theta}$$

$$\rho_0 \partial_t U_z + \partial_z P_1 + \frac{\rho_0 V}{r}\partial_\theta U_z - \rho_1 \vec{g}\cdot\hat{z} - (\nabla\cdot\overleftrightarrow{\pi}_1)\cdot\hat{z} =$$

$$-\rho_0 U_r \partial_r U_z - \rho_0 \frac{U_\theta}{r}\partial_\theta U_z - \rho_0 U_z \partial_z U_z - \frac{\rho_1}{\rho}\partial_z P_1$$

$$-\frac{\rho_1^2}{\rho}\vec{g}\cdot\hat{z} - \frac{\rho_1}{\rho}(\nabla\cdot\overleftrightarrow{\pi}_1)\cdot\hat{z}$$

c

3) $$\frac{DP}{Dt} = -\gamma P \vec{\nabla}\cdot\vec{U} - (\gamma-1)\vec{\nabla}\cdot\vec{Q} + (\gamma-1)\overleftrightarrow{\Pi}:\vec{\nabla}\vec{U}$$

$$\partial_t P_0 + \partial_t P_1 = -\gamma(P_0+P_1)\vec{\nabla}\cdot(\vec{U}_0+\vec{U}_1) - (\vec{U}_0+\vec{U}_1)\cdot\vec{\nabla}(P_0+P_1) - (\gamma-1)\vec{\nabla}\cdot\vec{Q}_0 - (\gamma-1)\vec{\nabla}\cdot\vec{Q}_1$$

$$+ (\gamma-1)(\overleftrightarrow{\Pi}_0+\overleftrightarrow{\Pi}_1):\vec{\nabla}(\vec{U}_0+\vec{U}_1)$$

$$\partial_t P_0 + \partial_t P_1 = \overbrace{-\vec{U}_0\cdot\nabla P_0 - \gamma P_0\vec{\nabla}\cdot\vec{U}_0 - (\gamma-1)\vec{\nabla}\cdot\vec{Q}_0 + (\gamma-1)\overleftrightarrow{\Pi}_0:\nabla\vec{U}_0}^{\text{zero order terms satisfy the Eqn}}$$

$$+ (\gamma-1)\overleftrightarrow{\Pi}_0:\nabla\vec{U}_1 + (\gamma-1)\overleftrightarrow{\Pi}_1:\nabla\vec{U}_0 + (\gamma-1)\overleftrightarrow{\Pi}_1:\vec{\nabla}\vec{U}_1$$

$$-\gamma P_0\vec{\nabla}\cdot\vec{U}_1 - \gamma P_1\vec{\nabla}\cdot\vec{U}_0 - \gamma P_1\vec{\nabla}\cdot\vec{U}_1 - \vec{U}_0\cdot\vec{\nabla}P_1 - \vec{U}_1\cdot\vec{\nabla}P_0$$

$$-\vec{U}_1\cdot\vec{\nabla}P_1 - (\gamma-1)\vec{\nabla}\cdot\vec{Q}_1$$

$$\vec{Q} = \vec{Q}_0 + \vec{Q}_1 = \underbrace{-K_0\nabla T_0}_{Q_0} \underbrace{- K_1\nabla T_0 - K_0\nabla T_1 - K_1\nabla T_1}_{Q_1}$$

if $K = K(r)$ then $K_1 = 0$ $\Rightarrow$ $\vec{Q} = \underbrace{-K\nabla T_0}_{Q_0} + \underbrace{-K\nabla T_1}_{Q_1}$

$$\partial_t P_1 + \vec{U}_0\cdot\nabla P_1 + \vec{U}_1\cdot\nabla P_0 + \gamma P_0\nabla\cdot\vec{U}_1 + \gamma P_1\nabla\cdot\vec{U}_0 + (\gamma-1)\vec{\nabla}\cdot\vec{Q}_1 =$$

$$-\gamma P_1\nabla\cdot\vec{U}_1 - \vec{U}_1\cdot\nabla P_1 + (\gamma-1)\overleftrightarrow{\Pi}_0:\nabla U_1 + (\gamma-1)\overleftrightarrow{\Pi}_1:\nabla\vec{U}_0$$

$$+ (\gamma-1)\overleftrightarrow{\Pi}_1:\nabla U_1$$

linear in $\vec{U}_1$

$$\boxed{\begin{aligned}&\partial_t P_1 + \vec{U}_0\cdot\nabla P_1 + U_1\cdot\nabla P_0 + \gamma P_0\nabla\cdot\vec{U}_1 + \gamma P_1\nabla\cdot\vec{U}_0 + (\gamma-1)\nabla\cdot\vec{Q}_1\\ &\quad - (\gamma-1)\overleftrightarrow{\Pi}_0:\nabla\vec{U}_1 - (\gamma-1)\overleftrightarrow{\Pi}_1:\nabla\vec{U}_0 = -\gamma P_1\nabla\cdot\vec{U}_1\\ &\quad - \vec{U}_1\cdot\nabla P_1 + (\gamma-1)\overleftrightarrow{\Pi}_1:\nabla\vec{U}_1\end{aligned}}$$

$$\vec{\nabla}\cdot V\hat{\theta} = 0$$

$$\partial_t P_1 + \frac{V}{r}\partial_\theta P_1 + \vec{U}_1\cdot\nabla P_0 + \frac{\gamma P_0}{r}\partial_r(rU_r) + \frac{\gamma P_0}{r}\partial_\theta U_\theta + \gamma P_0 \partial_z U_z$$

$$+ \gamma P_1 \cancel{\nabla\cdot\vec{U}_0}^{\,0} - (\gamma-1)\vec{\nabla}\cdot(K\vec{\nabla}T_1) - (\gamma-1)\overleftrightarrow{\Pi}_0 : \nabla\vec{U}_1 - (\gamma-1)\overleftrightarrow{\Pi}_1 : \nabla\vec{U}_0$$

$$= -\frac{\gamma P_1}{r}\partial_r(rU_r) - \frac{\gamma P_1}{r}\partial_\theta U_\theta - \gamma P_1 \partial_z U_z - U_r\partial_r P_1 - \frac{U_\theta}{r}\partial_\theta P_1$$

$$- U_z\partial_z P_1 + (\gamma-1)\overleftrightarrow{\Pi}_1 : \nabla\vec{U}_1$$

$$\overleftrightarrow{\Pi}_0 : \nabla\vec{U}_1 = \mu\left[\partial_r V\partial_r U_\theta - \frac{V}{r}\partial_r U_\theta + \frac{\partial_r V\partial_\theta U_r}{r} - \frac{U_\theta}{r}\partial_r V - \frac{V}{r^2}\partial_\theta U_r + \frac{VU_\theta}{r^2}\right]$$

$$\overleftrightarrow{\Pi}_1 : \nabla\vec{U}_0 = \mu\left[\partial_r V\partial_r U_\theta + \frac{\partial_r V\partial_\theta U_r}{r} - \frac{U_\theta}{r}\partial_r V - \frac{V}{r^2}\partial_\theta U_r + \frac{VU_\theta}{r^2} - \frac{V\partial_r U_\theta}{r}\right]$$

$$\Rightarrow \overleftrightarrow{\Pi}_0 : \nabla\vec{U}_1 = \overleftrightarrow{\Pi}_1 : \nabla\vec{U}_0$$

$$\boxed{\begin{aligned}
&\partial_t P_1 + \frac{V}{r}\partial_\theta P_1 + \vec{U}_1\cdot\nabla P_0 + \frac{\gamma P_0}{r}\partial_r(rU_r) + \frac{\gamma P_0}{r}\partial_\theta U_\theta + \gamma P_0\partial_z U_z \\
&\quad -(\gamma-1)\vec{\nabla}\cdot(K\vec{\nabla}T_1) - 2(\gamma-1)\mu\left[\partial_r V\partial_r U_\theta + \frac{\partial_r V\partial_\theta U_r}{r}\right. \\
&\quad \left. - \frac{U_\theta}{r}\partial_r V - \frac{V}{r^2}\partial_\theta U_r + \frac{VU_\theta}{r^2} - \frac{V\partial_r U_\theta}{r}\right] \\
&= -\frac{\gamma P_1}{r}\partial_r(rU_r) - \frac{\gamma P_1}{r}\partial_\theta U_\theta - \gamma P_1\partial_z U_z - U_r\partial_r P_1 \\
&\quad - \frac{U_\theta}{r}\partial_\theta P_1 - U_z\partial_z P_1 + (\gamma-1)\overleftrightarrow{\Pi}_1 : \nabla\vec{U}_1
\end{aligned}}$$

$$\overleftrightarrow{\Pi}_1 : \nabla \vec{U}_0 \qquad \vec{U}_0 = V(r)\,\hat{\theta}$$

$$\nabla \vec{U}_0 = (0)\hat{r}\hat{r} + \hat{r}\hat{\theta}\,\partial_r V + \hat{r}\hat{z}(0)$$

$$+ \hat{\theta}\hat{r}\left(-\frac{V}{r}\right) + \hat{\theta}\hat{\theta}(0) + \hat{\theta}\hat{z}(0)$$

$$+ \hat{z}\hat{r}(0) + \hat{z}\hat{\theta}(0) + \hat{z}\hat{z}(0)$$

$$\Rightarrow \nabla \vec{U}_0 = \hat{r}\hat{\theta}\,\partial_r V - \hat{\theta}\hat{r}\,\frac{V}{r}$$

$$\overleftrightarrow{\Pi}_1 : \nabla \vec{U}_0 = \Pi_{r\theta} \nabla U_{r\theta} + \Pi_{\theta r} \nabla U_{\theta r}$$

$$= \mu\left(\nabla U_{1,r\theta} + \nabla U_{1,\theta r} - \tfrac{2}{3}(0)\nabla\cdot\vec{U}\right)\left[\partial_r V\right]$$

$$+ \mu\left(\nabla U_{\theta r} + \nabla U_{r\theta} - \tfrac{2}{3}(0)\nabla\cdot\vec{U}\right)\left[-\frac{V}{r}\right]$$

$$= \mu\left(\partial_r U_\theta + \frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r}\right)\partial_r V$$

$$+ \mu\left(\frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r} + \partial_r U_\theta\right)\left(-\frac{V}{r}\right)$$

$$\boxed{\overleftrightarrow{\Pi}_1 : \nabla \vec{U}_0 = \mu\left[\partial_r U_\theta \partial_r V + \frac{\partial_\theta U_r}{r}\partial_r V - \frac{U_\theta}{r}\partial_r V - \frac{V}{r^2}\partial_\theta U_r + \frac{U_\theta V}{r^2} - \frac{V\partial_r U_\theta}{r}\right]}$$

$\overleftrightarrow{\Pi}_0 : \nabla\vec{U}_1 \qquad \vec{U}_0 = V(r)\hat{\theta}$

$$\overleftrightarrow{\Pi}_0 = \mu\left(\nabla\vec{U}_0 + \nabla\vec{U}_0^{\,T} - \tfrac{2}{3}\overleftrightarrow{I}\,\cancel{(\vec{\nabla}\cdot\vec{U}_0)}\right)$$

$$= \mu\underbrace{\left(\hat{r}\hat{\theta}\,\partial_r V - \hat{\theta}\hat{r}\frac{V}{r}\right)}_{\nabla U_0} + \mu\underbrace{\left(-\frac{V}{r}\hat{r}\hat{\theta} + \partial_r V\,\hat{\theta}\hat{r}\right)}_{\nabla U_0^{\,T}}$$

$$\overleftrightarrow{\Pi}_0 = \mu\left[\hat{r}\hat{\theta}\left(\partial_r V - \frac{V}{r}\right) + \hat{\theta}\hat{r}\left(\partial_r V - \frac{V}{r}\right)\right]$$

$$\overleftrightarrow{\Pi}_0 : \nabla\vec{U}_1 = \Pi_{r\theta}\nabla U_{r\theta} + \Pi_{\theta r}\nabla U_{\theta r}$$

$$= \mu\left(\partial_r V - \frac{V}{r}\right)\partial_r U_\theta + \mu\left(\partial_r V - \frac{V}{r}\right)\left(\frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r}\right)$$

$$= \mu\left[\partial_r V\,\partial_r U_\theta - \frac{V}{r}\partial_r U_\theta + \frac{\partial_r V\,\partial_\theta U_r}{r} - \frac{\partial_r V\,U_\theta}{r} + \frac{V}{r^2}U_\theta - \frac{V}{r}\frac{\partial_\theta U_r}{r}\right]$$

$$\boxed{\overleftrightarrow{\Pi}_0 : \nabla\vec{U}_1 = \mu\left[\partial_r V\,\partial_r U_\theta - \frac{V}{r}\partial_r U_\theta + \frac{\partial_r V\,\partial_\theta U_r}{r} - \frac{U_\theta}{r}\partial_r V - \frac{V}{r^2}\partial_\theta U_r + \frac{V U_\theta}{r^2}\right]}$$

4) $P = (\gamma - 1) C_v \rho T$

$$\nabla P_0 + \nabla P_1 = \nabla[(\gamma-1)C_v]\rho T + (\gamma-1)C_v\rho\nabla T + (\gamma-1)C_v T\nabla\rho$$

$$\nabla P_0 + \nabla P_1 = \rho T\nabla[(\gamma-1)C_v] + (\gamma-1)C_v\rho\nabla T_1 + (\gamma-1)C_v\rho\nabla T_0$$
$$+ (\gamma-1)C_v T\nabla\rho_0 + (\gamma-1)C_v T\nabla\rho_1$$

$$\cancel{\frac{\nabla P_0}{P}} + \frac{\nabla P_1}{P} = \cancel{\frac{\nabla T_0}{T}} + \frac{\nabla T_1}{T} + \cancel{\frac{\nabla\rho_0}{\rho}} + \frac{\nabla\rho_1}{\rho} + \cancel{\frac{\nabla[(\gamma-1)C_v]}{(\gamma-1)C_v}}$$

$$\frac{\nabla P_1}{P} = \frac{\nabla T_1}{T} + \frac{\nabla\rho_1}{\rho}$$

if γ, C_v are only func of r, then "γ_1" and "C_{v_1}" $= 0$

$$\boxed{\nabla T_1 = \frac{T_0+T_1}{P_0+P_1}\nabla P_1 - \frac{T_0+T_1}{\rho_0+\rho_1}\nabla\rho_1}$$

$$\frac{A}{a(b+c)} = \frac{B}{a} + \frac{C}{b+c}$$

know $B, C, a, b, c \Rightarrow B(b+c) + Ca = A$

$$\frac{A}{P_0(P_0+P_1)} = \frac{T_0}{P_0} + \frac{T_0+T_1}{P_0+P_1} \Rightarrow A = T_1P_0 - T_0P_1$$

$$\Rightarrow \frac{T_0+T_1}{P_0+P_1} = \frac{T_1P_0 - T_0P_1}{P_0(P_0+P_1)} + \frac{T_0}{P_0}$$

$$\frac{T_0+T_1}{\rho_0+\rho_1} = \frac{T_1\rho_0 - T_0\rho_1}{\rho_0(\rho_0+\rho_1)} + \frac{T_0}{\rho_0}$$

$$\Rightarrow \boxed{\nabla T_1 = \frac{T_0}{P_0}\nabla P_1 + \frac{T_0}{\rho_0}\nabla\rho_1 = \frac{T_1P_0 - T_0P_1}{P_0(P_0+P_1)}\nabla P_1 - \frac{T_1\rho_0 - T_0\rho_1}{\rho_0(\rho_0+\rho_1)}\nabla\rho_1}$$