Analyzing the Spruit-Tayler Dynamo

As implemented in stellar structure codes such as MESA, the Spruit-Tayler (ST) dynamo gives a prescription for the solid angle averaged magnetic torque $\langle B_r B_\phi \rangle/(4\pi))$. This prescription is based on values for B_r/B_ϕ and B_ϕ itself which are derived from linear and nonlinear theory of the Tayler instability. This motivates us to see whether the instability operates as assumed.

There are other reasons to go deeper into the ST dynamo. The estimate of the magnetic torque seems to be based on the mean magnetic fields, but the averaged fluctuating magnetic stresses $\langle \delta B_r \delta B_\phi \rangle$ and Reynolds stresses $\langle \delta v_r \delta v_\phi \rangle$ will also play a role. The question of whether the ST mean field is itself stable hasn't been addressed. And finally, all the analysis is local, and the global structure of the magnetic field in the radiative zone is not discussed.

Review of the ST Torque

Here is a brief review of how the ST torque is calculated, with arguments taken from Spruit 2002 (S02). For the moment I assume inefficient thermal diffusion (case 0 of S02). The maximum radial unstable wavelength l_c satisfies

$$l_c \sim \frac{v_{A\phi}}{N}, \tag{1}$$

where N is the buoyancy frequency (this estimate can be derived by assuming incompressible displacements, $\xi_r/\xi_h \sim l_c/r$, and equating the work done against the stable stratfication, $\xi_r^2 N^2$, with the magnetic energy released, $v_{A\phi}^2(\xi_h^2/\xi_r^2)$. It is assumed that the lengthscale l_c survives into the nonlinear stage to set the mean field ratio $B_r/B_\phi \sim l_c/r \ll 1$.

The next step in calculating the torque is calculating the amplitude of the saturated field B_ϕ. Here, we consider the $\hat{\phi}$ component of the magnetic induction equation

$$\frac{\partial B_\phi}{\partial t} = r\mathbf{B} \cdot \nabla\Omega + \lambda \nabla^2 B_\phi, \tag{2}$$

where λ is the magnetic diffusivity. In a steady state, we have $B_r/B_\phi \sim \lambda/(l_c^2 r\Omega')$ (where we assume $l_c/r \ll 1$). The second major step in the ST analysis is to assume λ is an eddy quantity produced by the small scale Tayler instability fields themselves, such that

$$\frac{\lambda}{l_c^2} \sim \frac{v_{A\phi}^2}{r^2\Omega}. \tag{3}$$

The right hand side of eqn. (3) is the growth rate of the Tayler instability $v_{A\phi}^2/(r^2\Omega)$ in the rapidly rotating limit $v_{A\phi}/(r\Omega) \ll 1$. Combining eqns. (1), (2), and (3) yields the saturated toroidal ST field

$$B_\phi \sim \frac{r^2\Omega\Omega'}{N}(4\pi\rho)^{1/2}. \tag{4}$$

From eqns. (1) and (4), the characteristic radial scale is

$$l_c \sim \frac{r^2\Omega\Omega'}{N}, \tag{5}$$

and the poloidal ST field is B_r is

$$B_r \sim B_\phi \frac{l_c}{r} \sim \frac{(\Omega\Omega')^2 r^3}{N^3}(4\pi\rho)^{1/2}. \tag{6}$$

Note that the results above only make sense if N is large compared to other frequencies, such as Ω and the global Alfven travel time r/v_A. This suggests that the problem of transfering angular momentum from a radiative core to a convective envelope is not fully addressed by the S02 analysis, another reason to work on this problem.

Simulation of the Tayler Instability

Differentially rotating, magnetized, stratified fluids are subject to a great many instabilities. Spruit (1999) discusses a number of them in the stellar context, drawing on more complete (and more complicated) work by Acheson (1978). Before embarking on a simulation beginning from any particular equilibrium state, I think it would be useful to find the linear modes of that equilibrium (using GYRE?) to sort out which instabilities are present. Candidates include $m = 0$ modes, magnetorotational instability (MRI), and buoyancy (Parker type) instabilities. Another advantage of solving the linear problem is that the lineareigenfunctions can be used to calculate the stresses, and give insight into the nonlinear results.

In an ideal fluid, the Taylor instability has no threshold in magnetic fieldstrength. However, in a resistive medium, the growth rate should be less than the damping rate λ/l_c^2. Using eqn. (1) this leads to the instability condition

$$\frac{v_{A\phi}}{\Omega r} > \left(\frac{\lambda}{r^2\Omega}\right)^{1/4}\left(\frac{N}{\Omega}\right)^{1/2}, \tag{7}$$

which can also be written as a condition on λ. If λ is small, N^2 is large, and rotation is weak, this threshold field is much less than required for buoyancy instabilities, while Spruit (1999) argues that the MRI is inefficient because of the stable stratification. These are Spruit's argument for why the Tayler instability is the important one.

If someone has a favorite model to analyze, they should go ahead and suggest it, but one possibility is a cylindrical equilibrium, say an annulus with $\varpi_i < \varpi < \varpi_o$ and (ϕ, z) ignorable coordinates. If gravity is provided by a line mass at the origin then we can write

$$\mathbf{g} = -\hat{\varpi}\frac{V^2}{\varpi} \tag{8}$$

Let the ratio of centrifugal to gravitational acceleration be a constant;

$$\Omega = \Omega_0 \frac{\varpi_0}{\varpi}. \tag{9}$$

Assume the gas is isothermal; $P = \rho a^2$, and the plasma $\beta \equiv 8\pi P/B^2$ is a constant. Then equilibrium can be achieved for

$$\rho = \rho_0 \left(\frac{\varpi}{\varpi_0}\right)^{-p}, \tag{10}$$

as long as the following relationship holds

$$p\left(1 + \frac{1}{\beta}\right) - \frac{V^2}{a^2}\left(1 - \frac{V_0^2}{V^2} + \frac{2a^2}{\beta V^2}\right) = 0. \tag{11}$$

Without magnetic field or rotation, this reduces to $p = V^2/a^2$. If the magnetic is very strong $(\beta \to 0)$, $p \to 2$, a current free field. Rotation reduces the pressure (and density) gradient. I believe this equilibrum would be stable according to the Schwarzschild criterion (at least in the plane parallel limit) for $\gamma > 1$.