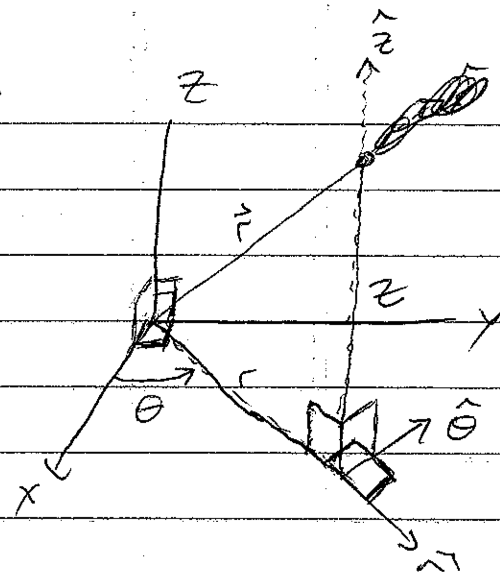
Cylindrical Coords



$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$
$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$
$$\hat{z} = \hat{z}$$
$$\frac{\partial \hat{r}}{\partial\theta} = \hat{\theta} \qquad \frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}$$
$$\vec{r} = r\hat{r} + z\hat{z}$$

$$\vec{\nabla} = \hat{r}\,\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \hat{z}\,\partial_z$$
$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\partial_r(rA_r) + \frac{1}{r}\partial_\theta A_\theta + \partial_z A_z$$
$$\vec{\nabla}\times\vec{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \partial_r & \partial_\theta & \partial_z \\ A_r & rA_\theta & A_z \end{vmatrix}$$
$$\vec{A}\cdot\vec{B} = A_r B_r + A_\theta B_\theta + A_z B_z \qquad \vec{A}\times\vec{B} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ A_r & A_\theta & A_z \\ B_r & B_\theta & B_z \end{vmatrix}$$

$$\nabla^2 f = \partial_r^2 f + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_\theta^2 f + \partial_z^2 f$$

$$\nabla^2\vec{A} = \hat{r}\left(\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\partial_\theta A_\theta\right) + \hat{\theta}\left(\nabla^2 A_\theta - A_\theta/r^2 + 2/r^2\,\partial_\theta A_r\right) + \hat{z}\,\nabla^2 A_z$$

$$\vec{v}\cdot\vec{\nabla} f = v_r\partial_r f + \frac{v_\theta}{r}\partial_\theta f + v_z\partial_z f$$

$$\vec\nabla\vec U = \left(\hat r\partial_r + \frac{\hat\theta}{r}\partial_\theta + \hat z\partial_z\right)\left(U_r\hat r + U_\theta\hat\theta + U_z\hat z\right)$$

$$= \hat r\partial_r(U_r\hat r) + \hat r\partial_r(U_\theta\hat\theta) + \hat r\partial_r(U_z\hat z)$$
$$+ \frac{\hat\theta}{r}\partial_\theta(U_r\hat r) + \frac{\hat\theta}{r}\partial_\theta(U_\theta\hat\theta) + \frac{\hat\theta}{r}\partial_\theta(U_z\hat z)$$
$$+ \hat z\partial_z(U_r\hat r) + \hat z\partial_z(U_\theta\hat\theta) + \hat z\partial_z(U_z\hat z)$$

$$= \hat r\hat r\,\partial_r U_r + \hat r\hat\theta\,\partial_r U_\theta + \hat r\hat z\,\partial_r U_z$$
$$+ \frac{\hat\theta\hat r}{r}\partial_\theta U_r + \frac{U_r}{r}\hat\theta(\hat\theta) + \frac{\hat\theta\hat\theta}{r}\partial_\theta U_\theta + \frac{U_\theta}{r}\hat\theta(-\hat r) + \frac{\hat\theta\hat z}{r}\partial_\theta U_z$$
$$+ \hat z\hat r\,\partial_z U_r + \hat z\hat\theta\,\partial_z U_\theta + \hat z\hat z\,\partial_z U_z$$

$$\vec\nabla\vec U = \hat r\hat r(\partial_r U_r) + \hat r\hat\theta(\partial_r U_\theta) + \hat r\hat z(\partial_r U_z)$$
$$+ \hat\theta\hat r\left(\frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r}\right) + \hat\theta\hat\theta\left(\frac{\partial_\theta U_\theta}{r} + \frac{U_r}{r}\right) + \frac{\hat\theta\hat z}{r}(\partial_\theta U_z)$$
$$+ \hat z\hat r(\partial_z U_r) + \hat z\hat\theta(\partial_z U_\theta) + \hat z\hat z(\partial_z U_z)$$

$$\Rightarrow \vec A\cdot\vec\nabla\vec U = (A_r\hat r + A_\theta\hat\theta + A_z\hat z)\cdot\vec\nabla\vec U$$

$$= A_r\hat r\cdot\begin{bmatrix} rr & r\theta & rz \\ & \vdots & \\ & & \end{bmatrix} + A_\theta\hat\theta\cdot\begin{bmatrix} rr & & \\ & \theta\theta & \\ & & zz \end{bmatrix} + A_z\hat z\cdot\begin{bmatrix} rr & & \\ \theta r & \cdots & \\ zr & & \end{bmatrix}$$

$$= A_r(\partial_r U_r)\hat r + A_r(\partial_r U_\theta)\hat\theta + A_r(\partial_r U_z)\hat z$$
$$+ A_\theta\left(\frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r}\right)\hat r + A_\theta\left(\frac{\partial_\theta U_\theta}{r} + \frac{U_r}{r}\right)\hat\theta + \frac{A_\theta}{r}(\partial_\theta U_z)\hat z$$
$$+ A_z(\partial_z U_r)\hat r + A_z(\partial_z U_\theta)\hat\theta + A_z(\partial_z U_z)\hat z$$

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$$\vec A\cdot\vec\nabla\vec U = \left(A_r\partial_r U_r + \frac{A_\theta}{r}\partial_\theta U_r + A_z\partial_z U_r - \frac{A_\theta U_\theta}{r}\right)\hat r$$
$$+ \left(A_r\partial_r U_\theta + \frac{A_\theta}{r}\partial_\theta U_\theta + A_z\partial_z U_\theta + \frac{A_\theta U_r}{r}\right)\hat\theta$$
$$+ \left(A_r\partial_r U_z + \frac{A_\theta}{r}\partial_\theta U_z + A_z\partial_z U_z\right)\hat z$$

r r z

$\vec A = \vec U$ gives $\vec U\cdot\vec\nabla\vec U$

$$\vec{\nabla}\vec{v} = \partial_i v_j \Rightarrow \partial_r v_\theta = \partial_r v_\theta \hat{r}\hat{\theta}$$

$$\vec{\nabla}\vec{v} = \hat{r}\hat{r}(\partial_r v_r) + \hat{r}\hat{\theta}(\partial_r v_\theta) + \hat{r}\hat{z}(\partial_r v_z)$$
$$+ \hat{\theta}\hat{r}\left(\frac{\partial_\theta v_r}{r} - \frac{v_\theta}{r}\right) + \hat{\theta}\hat{\theta}\left(\frac{\partial_\theta v_\theta}{r} + \frac{v_r}{r}\right) + \hat{\theta}\hat{z}\left(\frac{\partial_\theta v_z}{r}\right)$$
$$+ \hat{z}\hat{r}(\partial_z v_r) + \hat{z}\hat{\theta}(\partial_z v_\theta) + \hat{z}\hat{z}(\partial_z v_z)$$

$$(\vec{\nabla}\vec{v})^T = \hat{r}\hat{r}(\partial_r v_r) + \hat{r}\hat{\theta}\left(\frac{\partial_\theta v_r}{r} - \frac{v_\theta}{r}\right) + \hat{r}\hat{z}(\partial_z v_r)$$
$$+ \hat{\theta}\hat{r}(\partial_r v_\theta) + \hat{\theta}\hat{\theta}\left(\frac{\partial_\theta v_\theta}{r} + \frac{v_r}{r}\right) + \hat{\theta}\hat{z}(\partial_z v_\theta)$$
$$+ \hat{z}\hat{\theta}(\partial_r v_z) + \hat{z}\hat{\theta}\left(\frac{\partial_\theta v_z}{r}\right) + \hat{z}\hat{z}(\partial_z v_r)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) = \vec{\nabla}\left[\frac{1}{r}\partial_r(r v_r) + \frac{1}{r}\partial_\theta v_\theta + \partial_z v_z\right]$$
$$= \left(\hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \hat{z}\partial_z\right)\left(\partial_r v_r + \frac{v_r}{r} + \frac{1}{r}\partial_\theta v_\theta + \partial_z v_z\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) = \hat{r}\left[\partial_r^2 v_r - \frac{v_r}{r^2} + \frac{1}{r}\partial_r v_r - \frac{1}{r^2}\partial_\theta v_\theta + \frac{1}{r}\partial_r\partial_\theta v_\theta + \partial_r\partial_z v_z\right]$$
$$+ \hat{\theta}\left[\frac{1}{r}\partial_\theta\partial_r v_r + \frac{1}{r^2}\partial_\theta v_r + \frac{1}{r^2}\partial_\theta^2 v_\theta + \frac{1}{r}\partial_\theta\partial_z v_z\right]$$
$$+ \hat{z}\left[\partial_z\partial_r v_r + \frac{1}{r}\partial_z v_r + \frac{1}{r}\partial_z\partial_\theta v_\theta + \partial_z^2 v_z\right]$$

$$\nabla^2 \vec{A} = \nabla^2 \left(A_r \hat{r} + A_\theta \hat{\theta} + A_z \hat{z} \right)$$

$$= \nabla^2 (A_r \hat{r}) + \nabla^2 (A_\theta \hat{\theta}) + \nabla^2 (A_z \hat{z})$$

$$= \hat{r} \nabla^2 A_r + 2 \vec{\nabla} A_r \cdot \vec{\nabla} \hat{r} + A_r \nabla^2 \hat{r}$$

$$+ \hat{\theta} \nabla^2 A_\theta + 2 \nabla A_\theta \cdot \nabla \hat{\theta} + A_\theta \nabla^2 \hat{\theta}$$

$$+ \hat{z} \nabla^2 A_z + 2 \nabla A_z \cdot \nabla \hat{z} + A_z \nabla^2 \hat{z}$$

$$\cancel{= \hat{r} \nabla^2 A_r + 2 \left[\partial_r A_r \hat{r} + \tfrac{1}{r} \hat{\theta} \partial_\theta A_r + \partial_z A_r \hat{z} \right] \cdot \left[\partial_r \hat{r} \hat{r} + \tfrac{1}{r} \partial_\theta \right.}$$

$$= \hat{r} \nabla^2 A_r + 2 \left[\hat{r} \partial_r A_r + \frac{\hat{\theta}}{r} \partial_\theta A_r + \hat{z} \partial_z A_r \right] \cdot \left[\hat{r} \cancel{\partial_r \hat{r}} + \frac{\hat{\theta}}{r} \partial_\theta \hat{r} + \hat{z} \cancel{\partial_z \hat{r}} \right]$$

$$+ A_r \nabla^2 \hat{r} + \hat{\theta} \nabla^2 A_\theta + A_\theta \nabla^2 \hat{\theta} + \hat{z} \nabla^2 A_z + A_z \cancel{\nabla^2 \hat{z}}$$

$$+ 2 \left[\hat{r} \partial_r A_\theta + \frac{\hat{\theta}}{r} \partial_\theta A_\theta + \hat{z} \partial_z A_\theta \right] \cdot \left[\cancel{\hat{r} \partial_r \hat{\theta}} + \frac{\hat{\theta}}{r} \partial_\theta \hat{\theta} + \cancel{\hat{z} \partial_z \hat{\theta}} \right]$$

$$+ 2 \left[\hat{r} \partial_r A_z + \frac{\hat{\theta}}{r} \partial_\theta A_z + \hat{z} \partial_z A_z \right] \cdot \underbrace{\left[\hat{r} \partial_r \hat{z} + \frac{\hat{\theta}}{r} \partial_\theta \hat{z} + \hat{z} \partial_z \hat{z} \right]}_{0}$$

$$= \hat{r} \nabla^2 A_r + \hat{\theta} \nabla^2 A_\theta + \hat{z} \nabla^2 A_z + A_r \nabla^2 \hat{r} + A_\theta \nabla^2 \hat{\theta} + A_z \cancelto{0}{\nabla^2 \hat{z}}$$

$$+ 2 \left[\hat{r} \partial_r A_r + \frac{\hat{\theta}}{r} \partial_\theta A_r + \hat{z} \partial_z A_r \right] \cdot \left[\frac{\hat{\theta} \hat{\theta}}{r} \right]$$

$$+ 2 \left[\hat{r} \partial_r A_\theta + \frac{\hat{\theta}}{r} \partial_\theta A_\theta + \hat{z} \partial_z A_\theta \right] \cdot \left[- \frac{\hat{\theta} \hat{r}}{r} \right]$$

$$= \hat{r} \nabla^2 A_r + \hat{\theta} \nabla^2 A_\theta + \hat{z} \nabla^2 A_z + A_r \nabla^2 \hat{r} + A_\theta \nabla^2 \hat{\theta}$$

$$+ \frac{2}{r^2} \partial_\theta A_r \hat{\theta} - \frac{2}{r^2} \partial_\theta A_\theta \hat{r}$$

$$\nabla^2 \hat{r} = \frac{1}{r^2} \partial_\theta \partial_\theta \hat{r} = \frac{-\hat{r}}{r^2}$$

$$\nabla^2 \hat{\theta} = \frac{1}{r^2} \partial_\theta \partial_\theta \hat{\theta} = \frac{-\hat{\theta}}{r^2}$$

$$\Rightarrow \nabla^2 \vec{A} = \hat{r} \left(\nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2} \partial_\theta A_\theta \right)$$

$$+ \hat{\theta} \left(\nabla^2 A_\theta - \frac{A_\theta}{r^2} + \frac{2}{r^2} \partial_\theta A_r \right)$$

$$+ \hat{z} \left(\nabla^2 A_z \right)$$

$$\overleftrightarrow{\Pi} = \mu\left(\nabla\vec{u} + (\nabla\vec{u})^T - \frac{2}{3}\overleftrightarrow{I}(\nabla\cdot\vec{u})\right)$$

$$\Pi_{ij} = \mu\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{u}\right)$$

$$\vec{\nabla}\cdot\overleftrightarrow{\Pi} = \partial_i \Pi_{ij} = \mu\left[\partial_i(\partial_j u_i) + \partial_i\partial_i u_j - \frac{2}{3}\delta_{ij}\partial_i(\vec{\nabla}\cdot\vec{u})\right]$$

$$= \mu\left[\partial_j\partial_i u_i + \nabla^2 u_j - \frac{2}{3}\partial_j(\nabla\cdot\vec{u})\right]$$

$$= \mu\left[\partial_j(\nabla\cdot u) + \nabla^2 u_j - \frac{2}{3}\partial_j(\nabla\cdot u)\right]$$

$$\vec{\nabla}\cdot\overleftrightarrow{\Pi} = \mu\left[\nabla^2\vec{u} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{u})\right]$$

$$\frac{\vec{\nabla}\cdot\overleftrightarrow{\Pi}}{\mu} = \hat{r}\left(\nabla^2 u_r - \frac{u_r}{r^2} + \frac{2}{r^2}\partial_\theta u_\theta\right) + \hat{\theta}\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\partial_\theta u_r\right)$$

$$+ \hat{z}\,\nabla^2 u_z + \frac{1}{3}\Big[\hat{r}\Big\{\partial_r^2 u_r - \frac{u_r}{r^2} + \frac{1}{r}\partial_r u_r - \frac{1}{r^2}\partial_\theta u_\theta$$

$$+ \frac{1}{r}\partial_r\partial_\theta u_\theta + \partial_r\partial_z u_z\Big\} + \hat{\theta}\Big\{\frac{1}{r}\partial_\theta\partial_r u_r + \frac{1}{r^2}\partial_\theta u_r$$

$$+ \frac{1}{r^2}\partial_\theta^2 u_\theta + \frac{1}{r}\partial_\theta\partial_z u_z\Big\} + \hat{z}\Big\{\partial_z\partial_r u_r + \frac{1}{r}\partial_z u_r$$

$$+ \frac{1}{r}\partial_z\partial_\theta u_\theta + \partial_z^2 u_z\Big\}\Big]$$

$$\hat{r} = \nabla^2 u_r - \frac{u_r}{r^2} + \frac{2}{r^2}\partial_\theta u_\theta + \frac{1}{3}\partial_r^2 u_r - \frac{u_r}{3r^2} + \frac{1}{3r}\partial_r u_r$$

$$- \frac{1}{3r^2}\partial_\theta u_\theta + \frac{1}{3r}\partial_r\partial_\theta u_\theta + \frac{1}{3}\partial_r\partial_z u_z$$

$$\hat{\theta} = \nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\partial_\theta u_r + \frac{1}{3r}\partial_\theta\partial_r u_r + \frac{1}{3r^2}\partial_\theta u_r$$

$$+ \frac{1}{3r^2}\partial_\theta^2 u_\theta + \frac{2}{3r}\partial_\theta\partial_z u_z$$

$$\hat{z} = \nabla^2 u_z + \frac{1}{3}\partial_z\partial_r u_r + \frac{1}{3r}\partial_z u_r + \frac{1}{3r}\partial_z\partial_\theta u_\theta + \frac{1}{3}\partial_z^2 u_z$$

$$\cancel{\frac{\vec{\nabla}\,\overleftrightarrow{\Pi}}{\mu} = \Big[\overbrace{\partial_r^2 v_r + \frac{1}{r}\partial_r v_r + \frac{1}{r^2}\partial_\theta^2 v_r + \partial_z^2 v_r}^{\nabla^2 v_r} - \frac{v_\theta}{r^2} + \frac{2}{r^2}\partial_\theta v_r}$$

$$\cancel{+ \frac{1}{3r}\partial_\theta\partial_r v_r + \frac{1}{3r^2}\partial_\theta v_r + \frac{1}{3r^2}\partial_\theta^2 v_\theta + \frac{3}{r}\partial_\theta\partial_z v_z\Big]\hat{r}}$$

$$\cancel{+\Big[}$$

$$\begin{aligned}
\frac{\vec{\nabla}\cdot\overleftrightarrow{\Pi}}{\mu} = \hat{r}\Big[&\overbrace{\cancel{\partial_r^2 v_r} + \cancel{\frac{1}{r}\partial_r v_r} + \cancel{\frac{1}{r^2}\partial_\theta^2 v_r} + \cancel{\partial_z^2 v_r}}^{\nabla^2 v_r} - \cancel{\frac{3}{3r}v_r} + \cancel{\frac{2\cdot 3}{3r^2}\partial_\theta v_\theta} \\
&+ \cancel{\frac{1}{3}\partial_r^2 v_r} - \cancel{\frac{1}{3r^2}v_r} + \cancel{\frac{1}{3r}\partial_r v_r} - \cancel{\frac{1}{3r^2}\partial_\theta v_\theta} + \frac{1}{3r}\partial_r\partial_\theta v_\theta \\
&+ \cancel{\frac{1}{3}\partial_r\partial_z v_z}\Big] + \\
\hat{\theta}\Big[&\cancel{\partial_r^2 v_\theta} + \cancel{\frac{1}{r}\partial_r v_\theta} + \cancel{\frac{1}{r^2}\partial_\theta^2 v_\theta} + \cancel{\partial_z^2 v_\theta} - \cancel{\frac{1}{r^2}v_\theta} + \cancel{\frac{2}{r^2}\partial_\theta v_r} \\
&+ \frac{1}{3r}\partial_\theta\partial_r v_r + \cancel{\frac{1}{3r^2}\partial_\theta v_r} + \cancel{\frac{1}{3r^2}\partial_\theta^2 v_\theta} + \frac{1}{3r}\partial_\theta\partial_z v_z\Big] + \\
\hat{z}\Big[&\cancel{\partial_r^2 v_z} + \cancel{\frac{1}{r}\partial_r v_z} + \cancel{\frac{1}{r^2}\partial_\theta^2 v_z} + \partial_z^2 v_z + \frac{1}{3}\partial_z\partial_r v_r + \frac{1}{3r}\partial_z v_r \\
&+ \frac{1}{3r}\partial_z\partial_\theta v_\theta + \cancel{\frac{1}{3}\partial_z^2 v_z}\Big]
\end{aligned}$$

$$\boxed{\begin{aligned}
\frac{\vec{\nabla}\cdot\overleftrightarrow{\Pi}}{\mu} = \Big[&\frac{4}{3}\partial_r^2 v_r + \frac{4}{3r}\partial_r v_r + \frac{5}{3r^2}\partial_\theta v_\theta - \frac{4}{3r^2}v_r + \frac{1}{r^2}\partial_\theta^2 v_r + \partial_z^2 v_r \\
&+ \frac{1}{3}\partial_r\partial_z v_z + \frac{1}{3r}\partial_r\partial_\theta v_\theta\Big]\hat{r} + \\
\Big[&\partial_r^2 v_r + \frac{1}{r}\partial_r v_\theta + \frac{4}{3}\partial_\theta^2 v_\theta + \partial_z^2 v_\theta - \frac{1}{r^2}v_\theta \\
&+ \frac{7}{3r^2}\partial_\theta v_r + \frac{1}{3r}\partial_\theta\partial_r v_r + \frac{1}{3r}\partial_\theta\partial_z v_z\Big]\hat{\theta} + \\
\Big[&\partial_r^2 v_z + \frac{1}{r}\partial_r v_z + \frac{1}{r^2}\partial_\theta^2 v_z + \frac{4}{3}\partial_z^2 v_z + \frac{1}{3}\partial_z\partial_r v_r \\
&+ \frac{1}{3r}\partial_z v_r + \frac{1}{3r}\partial_z\partial_\theta v_\theta\Big]\hat{z}
\end{aligned}}$$

$$(\nabla U)_{ij} = \partial_i U_j$$

$$\overleftrightarrow{\Pi} : \vec{\nabla}\vec{U} \quad \text{where } \overleftrightarrow{A} : \overleftrightarrow{B} = A_{ij}B_{ij}$$

$$\overleftrightarrow{\Pi}_{ij}(\nabla U)_{ij} = \Pi_{11}\nabla U_{11} + \Pi_{12}\nabla U_{12} + \Pi_{13}\nabla U_{13} + \Pi_{21}\nabla U_{21} + \Pi_{22}\nabla U_{22} + \Pi_{23}\nabla U_{23} + \Pi_{31}\nabla U_{31} + \Pi_{32}\nabla U_{32} + \Pi_{33}\nabla U_{33}$$

$$\Pi_{ij} = \mu\Big(\underbrace{\partial_i U_j}_{\nabla U} + \underbrace{\partial_j U_i}_{(\nabla U)^T} - \tfrac{2}{3}\delta_{ij}\partial_k U_k\Big)$$

$$\left(-\tfrac{2}{3}\delta_{ij}\partial_k U_k\right)\partial_i U_j = -\tfrac{2}{3}\partial_k U_k \partial_i U_j \delta_{ij} = -\tfrac{2}{3}(\vec{\nabla}\cdot\vec{U})\,\partial_j U_j = -\frac{2}{3}(\vec{\nabla}\cdot\vec{U})^2$$

$$+\frac{2}{3}(\vec{\nabla}\cdot\vec{U})^2 + \frac{\overleftrightarrow{\Pi}:\vec{\nabla}\vec{U}}{\mu} = \partial_1 U_1\partial_1 U_1 + \partial_1 U_1^T\partial_1 U_1 + \partial_1 U_2\partial_1 U_2 + \partial_2 U_1^T\partial_1 U_2$$
$$+ \partial_1 U_3\partial_1 U_3 + \partial_3 U_1^T\partial_1 U_3 +$$
$$\partial_2 U_1\partial_2 U_1 + \partial_1 U_2^T\partial_2 U_1 + \partial_2 U_2\partial_2 U_2 + \partial_2 U_2^T\partial_2 U_2$$
$$+ \partial_2 U_3\partial_2 U_3 + \partial_3 U_2^T\partial_2 U_3 +$$
$$\partial_3 U_1\partial_3 U_1 + \partial_1 U_3^T\partial_3 U_1 + \partial_3 U_2\partial_3 U_2 + \partial_2 U_3^T\partial_3 U_2$$
$$+ \partial_3 U_3\partial_3 U_3 + \partial_3 U_3^T\partial_3 U_3$$

T superscript denotes it is from the $\nabla U^T : \nabla U$ term, not the ∇U^T matrix

i.e. $\partial_3 U_2^T = \partial_3 U_2$ (based on the indices of $\partial_i U_j + \partial_j U_i$)

so the transpose was taken when the indices were reversed so it still refers to elements of $\vec{\nabla}\vec{U}$ matrix different matrix $\Leftarrow$ same indices

$$\nabla U_{ij}\left[\nabla U_{ij} + \left(\nabla U^T\right)_{ij}\right] = \pi_{ij}\nabla U_{ij}\ \frac{2}{3}(\vec{\nabla}\cdot\vec{U})^2$$

$$\cancel{\nabla U_{ij}} \qquad = \left[\partial_i U_j + \partial_j U_i\right]\partial_i U_j$$

same matrix, different indices

$$\frac{\frac{2}{3}(\vec{\nabla}\cdot\vec{U})^2 + \overleftrightarrow{\pi}:\vec{\nabla}\vec{U}}{\mu} = 2(\partial_r U_r)^2 + (\partial_r U_\theta)^2 + \left(\frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r}\right)(\partial_r U_\theta)$$

$$+ (\partial_r U_z)^2 + (\partial_z U_r)(\partial_r U_z) + \left(\frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r}\right)^2$$

$$+ (\partial_r U_\theta)\left(\frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r}\right) + 2\left(\frac{\partial_\theta U_\theta}{r} + \frac{U_r}{r}\right)^2$$

$$+ \left(\frac{\partial_\theta U_z}{r}\right)^2 + (\partial_z U_\theta)\left(\frac{\partial_\theta U_z}{r}\right) + (\partial_z U_r)^2$$

$$+ (\partial_r U_z)(\partial_z U_r) + (\partial_z U_\theta)^2 + \left(\frac{\partial_\theta U_z}{r}\right)(\partial_z U_\theta)$$

$$+ 2(\partial_z U_z)^2$$

$$\frac{\overleftrightarrow{\pi}:\vec{\nabla}\vec{U}}{\mu} = -\frac{2}{3}(\vec{\nabla}\cdot\vec{U})^2 + 2(\partial_r U_r)^2 + (\partial_r U_\theta)^2 + 2(\partial_r U_\theta)\left(\frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r}\right)$$

$$+ (\partial_r U_z)^2 + 2(\partial_z U_r)(\partial_r U_z) + \left(\frac{\partial_\theta U_r}{r} - \frac{U_\theta}{r}\right)^2$$

$$+ 2\left(\frac{\partial_\theta U_\theta}{r} + \frac{U_\theta}{r}\right)^2 + \left(\frac{\partial_\theta U_z}{r}\right)^2 + 2(\partial_z U_\theta)\left(\frac{\partial_\theta U_z}{r}\right)$$

$$+ (\partial_z U_r)^2 + (\partial_z U_\theta)^2 + 2(\partial_z U_z)^2$$

$$(\vec{\nabla}\cdot\vec{v})^2 = \left(\partial_r A_r + \frac{A_r}{r} + \frac{1}{r}\partial_\theta A_\theta + \partial_z A_z\right)^2$$

$$= (\partial_r A_r)^2 + \left(\frac{A_r}{r}\right)^2 + \left(\frac{\partial_\theta A_\theta}{r}\right)^2 + (\partial_z A_z)^2$$

$$+ \frac{2A_r}{r}\partial_r A_r + 2\frac{\partial_r A_r \partial_\theta A_\theta}{r} + 2\partial_r A_r \partial_z A_z$$

$$+ 2\frac{A_r}{r^2}\partial_\theta A_\theta + 2\frac{A_r}{r}\partial_z A_z + 2\frac{\partial_\theta A_\theta \partial_z A_z}{r}$$

$$\Rightarrow \frac{\overleftrightarrow{\pi}:\vec{\nabla}\vec{v}}{\mu} = -\frac{2}{3}(\partial_r v_r)^2 - \frac{2}{3}\left(\frac{v_r}{r}\right)^2 - \frac{2}{3}\left(\frac{\partial_\theta v_\theta}{r}\right)^2 - \frac{2}{3}(\partial_z v_z)^2$$

$$-\frac{4}{3}\frac{v_r}{r}\partial_r v_r - \frac{4}{3}\frac{\partial_r v_r \partial_\theta v_\theta}{r} - \frac{4}{3}\partial_r v_r \partial_z v_z$$

$$-\frac{4}{3}\frac{v_r}{r^2}\partial_\theta v_\theta - \frac{4}{3}\frac{v_r}{r}\partial_z v_z - \frac{4}{3}\frac{\partial_\theta v_\theta \partial_z v_z}{r}$$

$$+ 2(\partial_r v_r)^2 + (\partial_r v_\theta)^2 + 2(\partial_r v_\theta)\left(\frac{\partial_\theta v_r}{r} - \frac{v_\theta}{r}\right)$$

$(\partial_\theta v_r)^2$ $v_\theta \partial_\theta v_r$ v_θ^2

$$+ (\partial_r v_z)^2 + 2\partial_z v_r \partial_r v_z + \left(\frac{\partial_\theta v_r}{r} - \frac{v_\theta}{r}\right)^2$$

$$+ 2\left(\frac{\partial_\theta v_\theta}{r} + \frac{v_\theta}{r}\right)^2 + \left(\frac{\partial_\theta v_z}{r}\right)^2 + 2\partial_z v_\theta \frac{\partial_\theta v_z}{r}$$

$$+ (\partial_z v_r)^2 + (\partial_z v_\theta)^2 + 2(\partial_z v_z)^2$$

$$= \cancel{\frac{4}{3}(\partial_r v_r)^2 - \frac{2}{3}\frac{v_r^2}{r^2}}$$

$$= \frac{4}{3}(\partial_r v_r)^2 - \frac{2}{3}\frac{v_r^2}{r^2} - \frac{2}{3}\frac{(\partial_\theta v_\theta)^2}{r^2} - \frac{2}{3}(\partial_z v_z)^2 - \frac{4}{3}\frac{v_r \partial_r v_r}{r}$$

$$-\frac{4}{3}\frac{\partial_r v_r \partial_\theta v_\theta}{r} - \frac{4}{3}\partial_r v_r \partial_z v_z - \frac{4}{3}\frac{v_r}{r^2}\partial_\theta v_\theta - \frac{4}{3}\frac{v_r}{r}\partial_z v_z$$

$$-\frac{4}{3}\frac{\partial_\theta v_\theta \partial_z v_z}{r} + (\partial_r v_\theta)^2 + \frac{2\partial_r v_\theta \partial_\theta v_r}{r} - \frac{2 v_\theta \partial_r v_\theta}{r}$$

$$+ (\partial_r v_z)^2 + 2\partial_z v_r \partial_r v_z + \frac{(\partial_\theta v_r)^2}{r^2} - \frac{2 v_\theta \partial_\theta v_r}{r^2} + \frac{v_\theta^2}{r^2}$$

$$+ \frac{2(\partial_\theta v_\theta)^2}{r^2} + \frac{4 v_\theta \partial_\theta v_\theta}{r^2} + \frac{2 v_\theta^2}{r^2} + \frac{(\partial_\theta v_z)^2}{r^2} + \frac{2\partial_z v_\theta \partial_\theta v_z}{r}$$

$$+ (\partial_z v_r)^2 + (\partial_z v_\theta)^2 + 2(\partial_z v_z)^2$$

$$
\begin{aligned}
\frac{\overleftrightarrow{\Pi} : \vec{\nabla}\vec{U}}{\mu} = {} & \frac{4}{3}(\partial_r U_r)^2 - \frac{2}{3}\frac{U_r^2}{r^2} + \frac{4}{3}\frac{(\partial_\theta U_\theta)^2}{r^2} + \frac{4}{3}(\partial_z U_z)^2 \\
& - \frac{4}{3}\frac{U_r \partial_r U_r}{r} - \frac{4}{3}\frac{\partial_r U_r \partial_\theta U_\theta}{r} - \frac{4}{3}\partial_r U_r \partial_z U_z \\
& - \frac{4}{3}\frac{U_r}{r^2}\partial_\theta U_\theta - \frac{4}{3}\frac{U_r}{r}\partial_z U_z - \frac{4}{3}\frac{\partial_\theta U_\theta \partial_z U_z}{r} \\
& + (\partial_r U_\theta)^2 + 2\frac{\partial_r U_\theta \partial_\theta U_r}{r} - 2\frac{U_\theta \partial_r U_\theta}{r} + (\partial_r U_z)^2 \\
& + 2\partial_z U_r \partial_r U_z + \frac{(\partial_\theta U_r)^2}{r^2} - 2\frac{U_\theta \partial_\theta U_r}{r^2} + 3\frac{U_\theta^2}{r^2} \\
& + 4\frac{U_\theta \partial_\theta U_\theta}{r^2} + \frac{(\partial_\theta U_z)^2}{r^2} + 2\frac{\partial_z U_\theta \partial_\theta U_z}{r} + (\partial_z U_r)^2 + (\partial_z U_\theta)^2
\end{aligned}
$$

Fluid Equations

1) $P = (\gamma-1) C_v \rho T = \rho^\gamma e^{S/C_v} = \rho kT/\bar{m}$

2) $\rho T \frac{DS}{Dt} = -\nabla(k\nabla T) + \overleftrightarrow{\Pi} : \vec{\nabla}\vec{u} + \eta \left(\frac{c}{4\pi}\right)^2 |\vec{\nabla}\times\vec{B}|^2$

3) $\partial_t \rho + \vec{u}\cdot\nabla\rho + \rho\nabla\cdot\vec{u} = 0$

4) $\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla}\left[P + \frac{B^2}{8\pi} + \phi_{eff}\right] + 2\vec{u}\times\vec{\Omega} + \vec{\nabla}\cdot\overleftrightarrow{\Pi} + \frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi}$

5) $\partial_t \vec{B} = \vec{\nabla}\times(\vec{u}\times\vec{B}) + \eta\nabla^2\vec{B}$ (assumes $\eta = \text{const}$)

$\vec{B} = 0$:

1) $P = (\gamma-1) C_v \rho T$

2) $\rho T \frac{DS}{Dt} = -\nabla\cdot k\nabla T + \overleftrightarrow{\Pi} : \vec{\nabla}\vec{u}$

3) $\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{u}$

4) $\rho\frac{D\vec{u}}{Dt} = -\nabla[P + \phi_{eff}] + 2\vec{u}\times\vec{\Omega} + \vec{\nabla}\cdot\overleftrightarrow{\Pi}$

Derivation of $T\alpha_T \dfrac{ds}{c_p} = \dfrac{1}{\Gamma_1}\dfrac{dP}{P} - \dfrac{d\rho}{\rho}$

(1) $\left.\dfrac{\partial a}{\partial b}\right|_c \left.\dfrac{\partial b}{\partial a}\right|_c = 1$ (2) $\left.\dfrac{\partial a}{\partial b}\right|_c \left.\dfrac{\partial b}{\partial c}\right|_a \left.\dfrac{\partial c}{\partial a}\right|_b = -1$

$\Gamma_1 \equiv \left.\dfrac{\partial P}{\partial \rho}\right|_s \dfrac{\rho}{P}$ $ds = \left.\dfrac{\partial s}{\partial p}\right|_\rho dp + \left.\dfrac{\partial s}{\partial \rho}\right|_p d\rho$

$$-\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_p ds = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_p \left.\frac{\partial s}{\partial p}\right|_\rho dp - \frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_p \left.\frac{\partial s}{\partial \rho}\right|_p d\rho$$

$$= -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_p \left.\frac{\partial s}{\partial p}\right|_\rho dp - \frac{d\rho}{\rho} \quad \text{using (1)}$$

$$= \frac{1}{\rho}\left.\frac{\partial \rho}{\partial p}\right|_s dp - \frac{d\rho}{\rho} \quad \text{using (2)}$$

$$= \frac{P}{P}\frac{1}{\rho}\left.\frac{\partial \rho}{\partial p}\right|_s dp - \frac{d\rho}{\rho} = \boxed{\frac{1}{\Gamma_1}\frac{dp}{P} - \frac{d\rho}{\rho}}$$

$dh = T\,ds + \frac{1}{\rho}dp \Rightarrow T = \left.\dfrac{\partial h}{\partial s}\right|_p$ $\dfrac{1}{\rho} = \left.\dfrac{\partial h}{\partial p}\right|_s$

$$\frac{\partial}{\partial s}\left.\frac{\partial h}{\partial p}\right|_s = \frac{\partial^2 h}{\partial s \partial p} = \frac{\partial}{\partial s}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\left.\frac{\partial \rho}{\partial s}\right|_p \Rightarrow \rho\frac{\partial^2 h}{\partial s\partial p} = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_p$$

$$-\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_p = \rho\frac{\partial^2 h}{\partial s \partial p} = \rho\frac{\partial}{\partial p}\left(\frac{\partial h}{\partial s}\right) = \rho\left.\frac{\partial T}{\partial p}\right|_s$$

$$= -\rho\left.\frac{\partial s}{\partial p}\right|_T \left.\frac{\partial T}{\partial s}\right|_p \quad \text{using (2)}$$

$$= -\rho\left.\frac{\partial s}{\partial p}\right|_T \frac{T}{c_p} \quad \text{using } c_p \equiv T\left.\frac{\partial s}{\partial T}\right|_p$$

$$g = h - Ts \Rightarrow dg = dh - Tds - SdT = \frac{1}{\rho}dp - sdT$$

$$\left.\frac{\partial g}{\partial p}\right|_T = \frac{1}{\rho} \Rightarrow \frac{\partial^2 g}{\partial p \partial T} = -\frac{1}{\rho^2}\left.\frac{\partial \rho}{\partial T}\right|_p$$

$$\left.\frac{\partial g}{\partial T}\right|_p = -s \Rightarrow -\left.\frac{\partial s}{\partial p}\right|_T = \frac{\partial^2 g}{\partial p \partial T}$$

$$-\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_p = -\frac{\rho T}{c_p}\left.\frac{\partial s}{\partial p}\right|_T = \frac{\rho T}{c_p}\frac{\partial^2 g}{\partial p \partial T} = \frac{-\rho T}{c_p \rho^2}\left.\frac{\partial \rho}{\partial T}\right|_p$$

$$= -\frac{T}{c_p}\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right|_p \equiv \frac{\alpha_T T}{c_p}$$

$$\alpha_T \equiv -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right|_p$$

$$-\frac{1}{\rho}\left.\frac{\partial \rho}{\partial s}\right|_p ds = \frac{dp}{\Gamma_1 p} - \frac{d\rho}{\rho}$$

$$\boxed{\frac{\alpha_T T}{c_p} ds = \frac{dp}{\Gamma_1 p} - \frac{d\rho}{\rho}}$$

Ideal gas $\Rightarrow \alpha_T = 1/T, \; \Gamma_1 = \gamma$

$$\frac{ds}{c_p} = \frac{dp}{\gamma p} - \frac{d\rho}{\rho}$$

$$\frac{1}{c_p}\frac{Ds}{Dt} = \frac{1}{\gamma}\frac{D\log p}{Dt} - \frac{D\log \rho}{Dt}$$

Derivation of Pressure Eqn

Ideal gas $\Rightarrow$ $\alpha = 1/T$, $\Gamma_1 = \gamma$, $P = (\gamma-1) c_v \rho T$ $\quad c_p = \gamma c_v$

$$\therefore \frac{ds}{c_p} = \frac{dp}{\gamma P} - \frac{d\rho}{\rho} \Rightarrow \frac{1}{c_p}\frac{DS}{Dt} = \frac{1}{\gamma}\frac{D\log P}{Dt} - \frac{D\log\rho}{Dt}$$

$$\frac{DS}{Dt} = + \frac{1}{\rho T}\nabla\cdot(k\nabla T) + \frac{1}{\rho T}\overleftrightarrow{\Pi} : \nabla\vec{U} + \eta\left(\frac{c^2}{4\pi}\right)(\vec{\nabla}\times\vec{B})^2$$

$$\frac{1}{\gamma}\frac{D\log P}{Dt} - \frac{D\log\rho}{Dt} = + \frac{1}{\rho T c_p}\vec{\nabla}\cdot(k\vec{\nabla}T) + \frac{1}{\rho T c_p}\overleftrightarrow{\Pi} : \vec{\nabla}\vec{U} + \frac{\eta}{c_p}\left(\frac{c}{4\pi}\right)^2(\vec{\nabla}\times\vec{B})^2$$

$$\frac{1}{\gamma}\frac{D\log P}{Dt} = -\vec{\nabla}\cdot\vec{U} + \frac{1}{\rho T c_p}\vec{\nabla}\cdot(k\nabla T) + \frac{1}{\rho T c_p}\overleftrightarrow{\Pi} : \vec{\nabla}\vec{U} + \frac{\eta}{c_p}\left(\frac{c}{4\pi}\right)^2|\vec{\nabla}\times\vec{B}|^2$$

$$\frac{DP}{Dt} = -\gamma P\vec{\nabla}\cdot\vec{U} + \frac{P}{c_v\rho T}\nabla\cdot k\nabla T + \frac{P}{\rho T c_v}\overleftrightarrow{\Pi} : \vec{\nabla}\vec{U} + \frac{P\eta}{c_v}\left(\frac{c}{4\pi}\right)^2|\vec{\nabla}\times\vec{B}|^2$$

$$\boxed{\frac{DP}{Dt} = -\gamma P\vec{\nabla}\cdot\vec{U} + (\gamma-1)\vec{\nabla}\cdot(k\vec{\nabla}T) + (\gamma-1)\overleftrightarrow{\Pi} : \nabla\vec{U} + \frac{P\eta}{c_v}\left(\frac{c}{4\pi}\right)^2|\vec{\nabla}\times\vec{B}|^2}$$