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ON THE INSTABILITY OF TOROIDAL MAGNETIC FIELDS AND DIFFERENTIAL ROTATION IN STARS

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WITH AN APPENDIX ON THE AXISYMMETRIC DIFFUSIVE INSTABILITY OF TOROIDAL MAGNETIC FIELDS IN A ROTATING GAS

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CONTENTS

An electrically conducting spherical body of gas rotates in the presence of an azimuthal (toroidal) magnetic field B and its own gravitational field. Instabilities of the system due to either differential rotation or meridional gradients of B are examined by means of a local analysis. Account is taken of viscous, ohmic and thermal diffusion, the diffusivities being denoted by ν, η and κ respectively. Attention is mainly focused on the 'rapidly rotating' case in which the magnetic energy of the system is only a small fraction (ϵ) of the rotational energy.

A discussion is given of some overlooked aspects of Goldreich–Schubert instability, which is usually said to occur if the angular momentum (per unit mass) decreases with distance r from the rotation axis or varies with distance z parallel to that axis. It is then

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shown that a toroidal magnetic field is not only less capable of suppressing the instability than has hitherto been supposed (when $\nu \ll \eta$) but actually acts as a catalyst for *another quite different* differential rotation instability if η is sufficiently small. This one is *non-axisymmetric* and substantially precedes that of Goldreich & Schubert by developing rapidly and with large azimuthal wavenumber if the angular *velocity* decreases more than a very small amount $(O(\epsilon))$ with r. When the gas is strongly thermally stratified this instability still occurs if η is sufficiently small compared with κ.

When the rotation is uniform, instability may still occur owing to the (r, z) distribution of the toroidal magnetic field itself. Its nature depends crucially on whether the region of interest is inside or outside a certain 'critical radius', the latter case being typically the more important astrophysically. Other geometrical effects of this kind complicate the issue, and though summarized at the end of the paper are difficult to report concisely here. The following results apply to a considerably simpler *plane layer* model previously investigated by Gilman (1970) and Roberts & Stewartson (1977).

When the temperature gradient is almost adiabatic (as in a stellar convection zone) and rotation is absent, instability occurs (on the Alfvénic time scale) by Parker's mechanism of *magnetic buoyancy* if B decreases with height. Rapid uniform rotation, such that $\epsilon \ll 1$, stabilizes some field distributions, but those which decrease with height *faster than the density* ρ remain unstable (albeit with growth rates reduced by a factor of order $\epsilon^{\frac{1}{2}}$) provided η is sufficiently small. When the gas is strongly thermally stratified (as in a stellar radiative interior) these results still apply if the thermal diffusivity κ is large enough to annul the effects of buoyancy, and this is the case if $D_* \equiv \kappa V^2/\eta N^2 H^2$ is large. Here V denotes the Alfvén speed, H the scale height and N the (conventional) buoyancy frequency. In the rapidly rotating case the stability of the system behaves in a curious way as D_* is steadily decreased from an infinite value. The first significant effect of decreasing κ, or equivalently of *increasing the stratification* (!), is a *destabilizing* one, and only when D_* drops below about unity does the stratification exert a significant stabilizing influence.

The magnetic buoyancy instabilities above are all non-axisymmetric, but the possibility of axisymmetric instability, despite strong uniform rotation and stable stratification, is examined in an appendix. A somewhat novel instability, involving the simultaneous operation of two conceptually quite different doubly diffusive mechanisms, arises if ν/η is sufficiently small and κ/η is sufficiently large.

1. Introduction

We examine in this paper the stability of an electrically conducting gas rotating with angular velocity $\Omega(r, z)$ about the z-axis of a cylindrical polar coordinate system (r, θ, z) in the presence of an azimuthal (i.e. toroidal) magnetic field $B(r, z)$. The fluid is subject to a spherically symmetric gravitational body force (per unit mass) $\boldsymbol{g}^* = (-g^* \sin\Theta, 0, -g^* \cos\Theta)$, where Θ is the polar angle. It is convenient to use instead 'apparent' gravity $\boldsymbol{g} \equiv (-g_r^* + \Omega^2 r, 0, -g_z^*)$ in the stability analysis, which is a purely *local* one, formally valid only in the immediate neighbourhood of a particular station (r, z) (which may however be chosen arbitrarily) and then only for perturbations of wavelength λ in the $r-z$ plane such that $\lambda^2 \ll r^2 + z^2$. Both axisymmetric and non-axisymmetric disturbances are considered, and the azimuthal wavelength will usually be quite large, perhaps $O(r)$.

Before setting our problem in the context of previous work it is helpful to introduce some of the ingredients. Thus four dynamical speeds naturally arise, namely the rotation speed Ωr, the Alfvén speed V (proportional to the magnetic field), the (isothermal) sound speed a, and a typical speed of free fall under gravity $(gr)^{\frac{1}{2}}$, but hybrid combinations of these also turn out to be important. Three different irreversible processes are incorporated into the analysis, namely diffusion of vorticity, magnetic flux and heat, and respective measures of these are the kinematic viscosity ν, the magnetic diffusivity η (proportional to the electrical resistivity) and the thermal diffusivity κ, all of which have the same dimensions (length)^2 (time)$^{-1}$. These coefficients are taken to be so small that the influence of diffusion on the *mean* quantities, such as Ω and B, is negligible on the time scales of interest, while the perturbations, by virtue of a sufficiently short meridional wavelength λ, may feel diffusive effects, and these may be either stabilizing or destabilizing.

We shall throughout this paper confine attention to situations which are convectively stable (or, at least, neutral) by assuming no decrease of the specific entropy E in the vertical direction. In the stably stratified situation a fluid element displaced adiabatically would, in the absence of all the other effects we are attempting to consider, oscillate under buoyancy forces at the 'Brunt-Vaisala' frequency $N = (N_r^2 + N_z^2)^{\frac{1}{2}}$, where

$$N_r^2 \equiv \frac{g_r}{\gamma}\frac{\partial E}{\partial r}, \quad N_z^2 \equiv \frac{g_z}{\gamma}\frac{\partial E}{\partial z}, \tag{1.1}$$

and γ denotes the ratio of specific heats. The two possible sources of instability in our system are differential rotation $\Omega(r, z)$ and meridional gradients of the azimuthal magnetic field $B(r, z)$.

The motivation for this study is an astrophysical one, and we keep in mind throughout (while leaving detailed calculations for separate publication elsewhere) applications to the radiative interior of the Sun. The parameter values there are such that the régime

$$\left.\begin{array}{ll}\text{(i)} & V^2/r^2 \ll N^2; \quad \Omega^2 \ll N^2,\\ \text{(ii)} & a^2 \sim gr,\\ \text{(iii)} & \nu \ll \eta \ll \kappa,\end{array}\right\} \tag{1.2}$$

is an appropriate one to consider. We hope also, however, that our study may have some bearing on the dynamics of the upper layers of the Sun, despite the fact that they are in a state of turbulent convection. In that case (1.2) is not appropriate; a better (but still over-simplified) view is to take $N = 0$ (since a nearly adiabatic temperature gradient may be assumed to obtain) and recognize that the scale height $H \equiv a^2/g$ will be considerably smaller than r, in contrast to (ii) above. Depending, presumably, on the strength of the convection it may be appropriate to replace the diffusivities in (iii) by semi-empirical 'eddy' values, and these would probably not be widely disparate.

Instabilities of differential rotation have been reviewed in an astrophysical context by several authors, notably Spiegel & Zahn (1970), Fricke & Kippenhahn (1972) and Zahn (1974, 1975) (but see also Strittmatter 1969; Spiegel 1972*a*; Mestel 1975; Roxburgh 1975). Perhaps the best known result is that obtained by Goldreich & Schubert (1967) and Fricke (1968) in the wake of Dicke's (1964) suggestion that the deep interior of the Sun might be rotating roughly 20 times faster than the surface. We take the opportunity of emphasizing here a modified version of their result, namely that when $\Omega^2 \ll N^2$ but $\nu \ll \kappa$ axisymmetric instability occurs despite the heavy stratification if either

$$-\frac{1}{r^3}\frac{\partial}{\partial r}(\Omega r^2)^2 > \frac{\nu}{\kappa}\gamma N_r^2 \tag{1.3}$$

or

$$\left(\frac{r}{\Omega}\frac{\partial \Omega}{\partial z}\right)^2 > \frac{\nu}{\kappa}\frac{\gamma N^2}{\Omega^2}\sin 2\Theta \frac{\partial}{\partial \Theta}(\Omega \sin^2 \Theta). \tag{1.4}$$

These criteria are not wholly new, as noted in §4 where they are derived, but they do appear to have been totally neglected in most of the literature on the subject wherein the present author, at least, invariably finds quoted only the $\nu/\kappa \to 0$ limit of the instability criteria:

$$\partial(\Omega r^2)/\partial r < 0 \quad \text{or} \quad \partial\Omega/\partial z \neq 0. \tag{1.5a, b}$$

Clearly the extent to which (1.5) provides an adequate description of the criteria (1.3) and (1.4) depends on the parameter

$$\alpha \equiv \frac{\nu}{\kappa}\frac{N^2}{\Omega^2} \tag{1.6}$$

which, due to the two competing elements ($\nu \ll \kappa$, $\Omega^2 \ll N^2$), may or may not be small.

The possibility of suppressing Goldreich–Schubert instability by azimuthal magnetic fields has been explored by Fricke (1969), who concluded that very large fields, with $V^2 \gtrsim \Omega^2 r^2$, would be necessary. We re-examine this possibility in §4 and find that it is rendered even more remote by further doubly diffusive effects due to the disparity between ν and η.

The Goldreich–Schubert instability occurs when there is a radially outward decrease of *angular momentum* (per unit mass) Ωr^2, essentially by the classic centrifugal instability mechanism discovered by Lord Rayleigh (1916). It does so despite the stable stratification, the effectiveness of which is much reduced by the rapid heat exchange between a displaced fluid parcel and its surroundings, this being permitted by the size of the thermal diffusivity κ (Yih 1961). In §6 we demonstrate how in the presence of an azimuthal magnetic field, the (r, z) structure of which is largely irrelevant in the present context, instability can occur in a *non-axisymmetric* manner if there is more than a very weak radially outward decrease of angular *velocity* Ω.

Consider first the unstratified case $\nabla E = 0$. In the theory it is assumed that the magnetic energy is small compared with the rotational energy, i.e. $\epsilon \equiv V^2/\Omega^2 r^2 \ll 1$, and the differential rotation $r\Omega^{-1}\partial\Omega/\partial r$ is also assumed to be small compared with unity. A negative radial angular *velocity* gradient of only order ϵ is sufficient to give rise to instability. The most rapidly amplifying mode has an azimuthal wavenumber $m \sim (-\epsilon^{-1}\Omega^{-1}r\,\partial\Omega/\partial r)^{\frac{1}{2}}$ and a growth rate, which is independent of the magnetic field strength, of order $-r\,\partial\Omega/\partial r$. Thus if Ω decreases significantly with radius these instabilities develop, in an astrophysical context, very fast indeed and with large *azimuthal* wavenumber (in contrast to the Goldreich–Schubert instabilities, the fastest of which have large meridional wavenumber). This magnetically aided instability of differential rotation occurs only if the magnetic field and the electrical conductivity of the fluid are not too weak, and the above growth rate (estimated on the basis of a perfectly conducting theory) will clearly exceed the rate $(O(\eta m^2/r^2))$ at which the disturbance would otherwise decay due to ohmic dissipation only if

$$V^2/2\Omega\eta \gtrsim 1. \tag{1.7}$$

If the gas is strongly thermally stratified, in the sense that $\Omega^2 \ll N^2$, fully three-dimensional and almost adiabatic instabilities of the above kind are suppressed by buoyancy forces. There are two ways, however, in which the instability may still occur. The first is *via* almost horizontal motions, which experience practically no buoyancy force, and this happens if Ω decreases with distance from the rotation axis along a (almost spherical) surface of constant (specific) entropy. The second involves (spherically) radial motions as well as horizontal ones and works by a doubly diffusive reduction of the effectiveness of buoyancy forces if the thermal diffusivity κ is sufficiently larger than the *magnetic* diffusivity η. Thus instability of this latter kind occurs if, in addition to the previous requirements $-\Omega^{-1}r\partial\Omega/\partial r \gtrsim \epsilon$ and (1.7):

$$-r\frac{\partial\Omega^2}{\partial r} > \frac{\eta}{\kappa}\gamma N_r^2 \tag{1.8}$$

(cf. (1.3)).

It may be helpful at this point to contrast these magnetically aided differential rotation instabilities with a non-magnetic instability recently discussed by Zahn (1974, 1975) and Jones (1977), which is also essentially non-axisymmetric and may also occur for only small deviations from a state of uniform rotation. The basic mechanism is that of parallel flow or shear instability, and a necessary condition for this when $\Omega^2 \ll N^2$ and $\nu = \kappa = 0$ is the following natural cylindrical-geometry counterpart to the well known Richardson criterion (see, for example, Turner 1973, ch. 4):

$$\tfrac{1}{4}(r\mathrm{d}\Omega/\mathrm{d}r)^2 > \gamma N_r^2 \tag{1.9}$$

(Sung 1974; cf. Lalas 1975). Zahn (1974, 1975) applies the arguments of Townsend (1958) to derive a condition under which there is sufficient energy in the shear flow for turbulence, once generated, to be maintained, and this may roughly be expressed as follows:

$$\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2 \gtrsim \frac{\nu}{\kappa}Re_\mathrm{c}\, N_r^2. \tag{1.10}$$

Here Re_c denotes the critical Reynolds number, of order 10^3, obtained by viewing the differential rotation locally as a plane parallel flow and ignoring stratification. On the other hand Jones (1977) derives a condition under which the flow is unstable to infinitesimal disturbances, and this may roughly be expressed, for the sake of comparison, as follows:

$$\left(r\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2 \gtrsim \frac{\nu}{\kappa}\left(\frac{UL}{\nu}\right)\left(\frac{L}{\lambda}\right)N_r^2. \tag{1.11}$$

Here U is a typical flow speed relative to a frame rotating with the mean angular velocity, L is the length scale characteristic of the shear, and λ is the horizontal wavelength. Thus the *actual* Reynolds number $Re \equiv UL/\nu$, which may be enormous in a star, appears in (1.11) in contrast to the critical value in (1.10). Jones (1977) gives a full discussion of the complicated possibilities implied by these differing criteria. Our reason for presenting them is to emphasize, by comparison with the criteria listed above, that though they also refer to instability due to deviations from a state of uniform angular *velocity* (rather than angular momentum), it makes no essential difference with these instabilities whether Ω increases or decreases with radius; the mechanism by which they work can be adequately studied in a corresponding plane-parallel flow. By contrast, the magnetically aided differential rotation instabilities such as (1.8), though requiring nowhere near as strong a gradient of angular velocity as is needed for Goldreich–Schubert instability, *do need* $\partial\Omega/\partial r < 0$, and have no counterpart in plane-parallel flow; the curvature effects are crucial.

The other instability mechanism which we examine in this paper arises from certain (r, z) distributions of the azimuthal magnetic field itself, and the conditions giving instability are quite different depending on whether the local region under consideration lies inside or outside a *critical radius* given (somewhat implicitly) by

$$r_\mathrm{c} \sim 2a^2/g. \tag{1.12}$$

In the former case a fairly modest increase of magnetic field with distance from the rotation axis causes instability, even when the magnetic energy is small compared to the rotational energy ($V^2 \ll \Omega^2 r^2$). The mechanism works equally well in an incompressible fluid such as the Earth's liquid core, and has been advocated as a possible explanation of the geomagnetic westward drift (Acheson 1972). Outside the critical radius, which is perhaps the case of main astrophysical interest, compressibility effects play an essential rôle and instability is promoted by a *decrease* of the magnetic field with height. The mechanism is that of 'magnetic buoyancy', an idea first introduced as a possible explanation of sunspot formation by Parker (1955).

The essence of magnetic buoyancy is best seen by considering an isolated horizontal tube of magnetic flux in a compressible gas. Dynamic equilibrium requires that the sum of the gas pressure p and 'magnetic pressure' $B^2/2\mu$ be the same both inside and just outside the tube, so $p_{\text{in}} < p_{\text{out}}$. If the temperature inside the tube is the same as that outside, the gas law implies $\rho_{\text{in}} < \rho_{\text{out}}$, so the tube feels lighter than its surroundings and tends to rise. The characteristic time scale on which the tube does so, while distorting (Parker 1955), is $O(H/V)$. In a number of subsequent papers Parker has pursued these ideas under a variety of different physical assumptions† and has recently (Parker 1975, 1976, 1977) argued that magnetic buoyancy brings up an azimuthal magnetic field of the order of 100G (1 gauss, G $= 10^{-4}$T) from most of the solar convection zone *too fast* for it to be regenerated from the poloidal (meridional) field by dynamo action. He concludes that if the solar dynamo is to function at all it must be in the very lowest levels of the convection zone.

Gilman (1970) pointed out that magnetic buoyancy might also be expected in stably stratified stellar *interiors*, since the rapid radiative heat transfer would facilitate the instability mechanism for modes of sufficiently short meridional wavelength by keeping the temperatures within individual flux tubes at the ambient value. The idea is an essentially doubly diffusive one ($\eta \ll \kappa$) akin to the examples presented above, but Gilman's stability analysis in fact contains neither the ratio η/κ nor the static stability N^2; the limit $\eta/\kappa \to 0$ is effectively taken by the device of dropping the thermodynamic energy equation and taking the temperature perturbation at every point to be zero, corresponding to a displaced fluid parcel instantaneously acquiring the temperature of its new surroundings. Such a parcel experiences no *conventional* buoyancy forces so N^2 does not appear. The analysis was for a plane inviscid layer of vanishingly small electrical resistivity ($\eta \to 0$) under gravity g and in the presence of a horizontal magnetic field $B(z)$. Gilman's main result was that the layer is unstable to disturbances of non-zero horizontal wavenumber k (in the direction of B) satisfying

$$-(g/a^2)\,\mathrm{d}\,(\ln B)/\mathrm{d}z > k^2, \tag{1.13}$$

so that the essential requirement is simply a *decrease of magnetic field with height*, the most easily excited mode having $k \to 0$, i.e. indefinitely long horizontal wavelength. The case $k = 0$, the counterpart of which in spherical geometry would be the axisymmetric case, requires separate consideration, and Gilman & Cadez (1970) showed that instability of this two-dimensional kind is only possible under the more demanding condition that the magnetic field decreases with height *faster than the density*, i.e.

$$-\mathrm{d}(B\rho^{-1})/\mathrm{d}z > 0. \tag{1.14}$$

That non-axisymmetric modes are the most readily excited by magnetic buoyancy was understood in terms of the dynamics of individual flux tubes by Parker (1955), and the otherwise

† Parker (1971) is notable, in particular, for the discussion in §IV (c) of the magnetically buoyant instability of a horizontal layer in the case when *pressure* perturbations are negligible, corresponding to a sound speed a very small compared with the Alfvén speed V. This is quite the opposite limit to that treated in the present paper.

curious distinction between $k \to 0$ and $k = 0$ revealed by (1.13) and (1.14) can also be understood from Parker's arguments. Of course in practice the azimuthal wavenumber m corresponding to k will be quantized so this feature does not arise. Cadez (1974) has investigated curvature effects of this kind on magnetic buoyancy under a variety of conditions, for both isothermal and adiabatic perturbations.

The magnetic buoyancy investigations in this paper, as in those of Roberts & Stewartson (1977) and Acheson & Gibbons (1978), were triggered by Gilman's extension of the above analysis (locally) to a uniformly *rotating* spherical body of gas. He assumed that the most unstable modes would be of short (formally zero) wavelength in the y-direction (i.e. northward), so that only the component of rotation perpendicular to gravity ($\Omega \sin \Theta$) would be significant. He examined in detail the particular case of an isothermal atmosphere of *constant Alfvén speed*, i.e. $B \propto \rho^{\frac{1}{2}}$. In the absence of rotation, we note, such an atmosphere does not satisfy (1.14) but is nevertheless unstable to *non*-axisymmetric disturbances by (1.13). Gilman found that as the rotation speed Ω was increased the growth rates were reduced until when

$$V^2/\Omega^2 H^2 < 8 \sin^2 \Theta, \tag{1.15}$$

magnetic buoyancy instability was totally suppressed. We have pointed out above that for many astrophysical situations of interest, $V^2 \ll \Omega^2 H^2$, so it is important to know whether this suppression of magnetic buoyancy by rotation occurs also in rather more general models.

Acheson & Gibbons (1978) retained Gilman's assumption (equivalent to $\kappa = \infty$) of instantaneous temperature adjustment (in fact taking an initially isothermal state so that $T =$ constant everywhere for all time), but included curvature effects, some investigation of finite ohmic diffusion ($\eta \neq 0$), and considered arbitrary azimuthal magnetic field distributions $B(r)$. They focused attention on the rapidly rotating case ($V^2 \ll \Omega^2 r^2$) and their main result, when stated in the context of a plane layer model, was that systems with magnetic field gradients slightly stronger than in Gilman's $B \propto \rho^{\frac{1}{2}}$ example were non-axisymmetrically unstable. The essential criterion for instability was in fact (1.14), namely that for *axisymmetric* disturbances in the *non-rotating* case!

Roberts & Stewartson (1977), on the other hand, retained Gilman's particular equilibrium state (isothermal plane layer, $B \propto \rho^{\frac{1}{2}}$) but restored finite thermal diffusion ($\kappa \neq \infty$) and non-zero ohmic diffusion ($\eta \neq 0$). They concentrated on the case (1.15) in which rotation is sufficiently large to suppress instability according to the $\kappa^{-1} = \eta = 0$ theory, and showed that the finite diffusive processes make instability possible if the *product* $\kappa\eta$ is sufficiently small, other parameters of the system being held constant.

Part of the motivation for the present paper was the obvious need to see how the theories of Acheson & Gibbons and Roberts & Stewartson are related. The answer is provided by figure 1 on page 493, which shows that each is correct in an appropriate region of parameter space and also delineates those respective regions. The other motive, however, came from observing that if κ and η are viewed as being held constant the Roberts & Stewartson results imply (see §7) that instability sets in when N^2 (see (1.1)) *exceeds* some critical *positive* value, i.e. when the fluid is sufficiently *strongly* stratified in a (so-called) 'statically stable' manner. This remarkable effect, which Roberts had noticed somewhat earlier in a related incompressible system (see Roberts 1978; Roberts & Loper 1978), clearly called for careful examination. While I am unable in this paper to offer much by way of physical explanation, figure 1 at least shows that the effect occurs only in a limited region of parameter space and that for large enough values of N^2 the system is stabilized by stratification, as one would intuitively expect.

The above remarks indicate the main background against which our study of magnetic buoyancy is set, but additional possibilities and complexities are implied, in my view, by the studies of Schubert (1968), Tayler (1973) and Cadez (1974). The picture that emerges after the analysis of §§2–8 is not, regrettably, a simple one, but nevertheless an attempt is made to summarize as clearly as possible the main results on magnetic buoyancy instability in §9.

2. Basic equations and approximations

The motion of a perfect gas of density ρ, pressure p and temperature T in the presence of a magnetic field $\boldsymbol{B}$ is described by the following equations:

$$\rho(\partial \boldsymbol{u}/\partial t+\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = -\nabla(p+\tfrac{1}{2}\mu^{-1}\boldsymbol{B}^2)+\mu^{-1}\boldsymbol{B}\cdot\nabla\boldsymbol{B}+\rho\boldsymbol{g}^*+\boldsymbol{F}, \tag{2.1}$$

$$\partial\rho/\partial t+\nabla\cdot(\rho\boldsymbol{u}) = 0, \tag{2.2}$$

$$\partial\boldsymbol{B}/\partial t = \nabla\wedge(\boldsymbol{u}\wedge\boldsymbol{B})-\nabla\wedge(\eta\nabla\wedge\boldsymbol{B}), \tag{2.3}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{2.4}$$

$$p = \rho\mathscr{R}T, \tag{2.5}$$

$$\rho c_{\mathrm{V}}\, T(\partial/\partial t+\boldsymbol{u}\cdot\nabla)\ln(p\rho^{-\gamma}) = F_{\mathrm{D}}+\eta\mu^{-1}(\nabla\wedge\boldsymbol{B})^2+\nabla\cdot(\rho c_{\mathrm{V}}\kappa\nabla T) \tag{2.6}$$

(see for example, Roberts 1967, ch. 1). Here $\boldsymbol{u}$ denotes fluid velocity, t time, μ magnetic permeability, $\boldsymbol{g}^*$ gravitational acceleration, $\boldsymbol{F}$ viscous force (per unit volume), $\mathscr{R}$ gas constant, c_{V} specific heat at constant volume, γ the ratio of specific heats and F_{D} is the rate of heating (per unit volume) due to viscous dissipation. The magnetic diffusivity η is defined as $(\sigma\mu)^{-1}$, where σ is the electrical conductivity, and has the same dimensions as the thermal diffusivity κ and the kinematic viscosity ν.

Equations (2.1) and (2.2) express conservation of momentum and mass, respectively. The Lorentz force (per unit volume) $\mu^{-1}(\nabla\wedge\boldsymbol{B})\wedge\boldsymbol{B}$ has been separated into two parts, one of which is the gradient of the 'magnetic pressure' $\tfrac{1}{2}\mu^{-1}\boldsymbol{B}^2$. The gravitational acceleration $\boldsymbol{g}^*$ is prescribed, thus excluding self-gravitational effects. In the equation of state (2.5) we shall treat $\mathscr{R}$ as a constant, thus restricting attention to a gas of uniform chemical composition. The electromagnetic induction equation (2.3) describes the way in which the magnetic field $\boldsymbol{B}$ changes as a result of the fluid motion and ohmic diffusion of magnetic flux. Finally, the energy equation (2.6) shows how the entropy of a moving fluid parcel changes as a result of irreversible processes, namely viscous and ohmic dissipation and the diffusion of heat.

All three diffusivities ν, η and κ are in general functions of the state of the gas, and for a discussion of this matter is an astrophysical context we refer the reader to Goldreich & Schubert (1967) and Schubert (1968). While the diffusivities will therefore depend on position in the basic equilibrium configuration, this will not complicate the stability analysis which follows, for two reasons. The first is that we assume all diffusive processes to be so small (in a way to be quantified shortly) that they have no significant effect on the basic equilibrium within the time scales of interest. The second is that the stability analysis will be an essentially local one, in which short-wavelength disturbances are considered. Diffusion of one kind or another may be effective on these short length scales, but to describe such effects it then suffices to use the local value of the diffusivity appropriate to the region under consideration. We shall now make these ideas more precise.

We shall study the stability of a gas rotating differentially with angular velocity $\Omega(r,z)$ in the presence of an azimuthal magnetic field $B(r,z)$. Here cylindrical polar coordinates are being used, and p, ρ and T are also functions of r and z. If all three diffusive processes are neglected, the basic equilibrium configuration is constrained simply by (2.5) together with the r and z components of (2.1), namely

$$\partial(p+\tfrac{1}{2}\mu^{-1}B^2)/\partial r = -B^2/\mu r+\rho(-g_r^*+\Omega^2 r), \tag{2.7}$$

$$\partial(p+\tfrac{1}{2}\mu^{-1}B^2)/\partial z = -\rho g_z^*. \tag{2.8}$$

Here we adopt the convention $\boldsymbol{g}^* = (-g_r^*, 0, -g_z^*)$ for the components of $\boldsymbol{g}^*$, thinking ahead to the case of a spherical mass of gas, for which both g_r^* and g_z^* as defined will (conveniently) be positive in the northern hemisphere (while g_z^* will be negative in the southern hemisphere). We introduce also 'apparent' gravity

$$\boldsymbol{g} = (-g_r, 0, -g_z) = (-g_r^*+\Omega^2 r, 0, -g_z^*), \tag{2.9}$$

although it is very important to note that, unlike $\boldsymbol{g}^*$, this will not be derivable from a potential unless $\partial\Omega/\partial z = 0$.

An important constraint on the basic state emerges if we cross-differentiate (2.7) and (2.8) to obtain

$$\frac{\partial}{\partial z}\left(\frac{B^2}{\mu r}-\rho\Omega^2 r\right) = \frac{\partial\rho}{\partial r}g_z^*-\frac{\partial\rho}{\partial z}g_r^*. \tag{2.10}$$

This is a magnetic equivalent of the meteorological 'thermal wind' equation (see, for example, Hide & Mason 1975) and may alternatively be written

$$\frac{\partial}{\partial z}\left(\frac{B^2}{\mu r}\right)-\rho r\frac{\partial\Omega^2}{\partial z} = \frac{\partial\rho}{\partial r}g_z-\frac{\partial\rho}{\partial z}g_r. \tag{2.11}$$

Unless the 'apparent' geopotentials and surfaces of constant density coincide there must be a variation with z of either Ω or B (or both).

At this point is it convenient to define the local Alfvén speed

$$V \equiv B/(\mu\rho)^{\frac{1}{2}}, \tag{2.12}$$

the local isothermal sound speed,

$$a \equiv (\mathscr{R}T)^{\frac{1}{2}}, \tag{2.13}$$

and the following nomenclature

$$R \equiv \ln(\Omega^2 r^4), \tag{2.14}$$

$$E \equiv \ln(p\rho^{-\gamma}), \tag{2.15}$$

$$F \equiv \ln(B/\rho r), \tag{2.16}$$

$$Q \equiv \ln(Br). \tag{2.17}$$

The first three of these represent useful measures of, respectively (and per unit mass), angular momentum, entropy and magnetic flux. It is easily shown that the basic balance equations (2.7) and (2.8) may be written in the form

$$\frac{\partial}{\partial r}\ln p+\frac{V^2}{a^2}\frac{\partial Q}{\partial r} = -\frac{g_r}{a^2}, \tag{2.18}$$

$$\frac{\partial}{\partial z}\ln p+\frac{V^2}{a^2}\frac{\partial Q}{\partial z} = -\frac{g_z}{a^2}, \tag{2.19}$$

and these are quite useful, in particular for demonstrating the following alternative way of writing the 'magnetic thermal wind equation':

$$\frac{V^2}{a^2}\left[\left(g_r-\frac{2\gamma a^2}{r}\right)\frac{\partial Q}{\partial z}-g_z\frac{\partial Q}{\partial r}\right]+\gamma\Omega^2 r\frac{\partial R}{\partial z}=g_z\frac{\partial E}{\partial r}-g_r\frac{\partial E}{\partial z}. \tag{2.20}$$

This relates the magnetic field and angular velocity gradients to the inclination between the 'apparent' geopotentials and surfaces of constant entropy (per unit mass). Note how even if B and Ω are independent of z, the magnetic field causes a non-zero inclination between the two, unless $B \propto r^{-1}$.

There are a number of natural dynamical time scales associated with the equilibrium configuration above, namely

$$\Omega^{-1}, r/V, r/a, \quad \text{and} \quad (r/g)^{\frac{1}{2}}, \tag{2.21}$$

where typical values of the various quantities are intended here. The first entry in (2.21) represents the rotation period, and the other three represent the time taken in crossing the system by, respectively, an Alfvén wave, a sound wave and a particle falling freely under gravity. In order to derive the most general dispersion relation appropriate to our present needs all these times are regarded as of comparable order, 'unity', in the stability analysis which follows in § 3, and disturbances oscillate or grow on this one time scale. Further, the scale heights of the various *basic state* quantities, such as density, angular velocity, etc., are taken to be comparable with the radius r in the neighbourhood of which our local analysis is to be valid. On the other hand the perturbations which we consider have very short wavelength (order $\lambda \ll r$, say) in the r and z directions, but $O(1)$ wavelength in the azimuthal direction. Thus differentiation of a basic state variable with respect to r or z leads to a term of much the same size as before, while differentiation of a *perturbation* variable with respect to r or z leads to a term bigger than before by a factor of order r/λ. In order, again, to keep the development as general as possible (given the constraints of the local analysis), all three diffusivities ν, η and κ are regarded as being of comparable magnitude and such that the diffusion time scales *based on the short meridional wavelength*, λ^2/ν, λ^2/η and λ^2/κ are $O(1)$. The diffusion times based on the equilibrium state scale, r^2/ν, r^2/η and r^2/κ are, therefore, very large in these same terms.

Having outlined the formal basis for the derivation of the local dispersion relation in § 3, we make three points about the way in which the analysis of this highly idealized model should be viewed in the context of applications to a real star such as the Sun.

First, we have to note that in the presence of weak diffusive effects most distributions of angular velocity, magnetic field and so on, satisfying (2.5), (2.7) and (2.8), will not quite satisfy the full equations and will slowly evolve. Processes contributing to such evolution in a real star will include, for example, ohmic decay of the magnetic field, Eddington–Sweet circulation currents (see Mestel 1975; Mestel & Moss 1977), and 'spin-down' (see Spiegel 1972*b*; Benton & Clark 1974) which would be enhanced by the effects of a weak poloidal magnetic field (see Loper 1976). Any instabilities we may discuss must be viewed against such a background of other effects, and clearly only those with sufficiently rapid growth rates will be physically significant.

Secondly, when we discuss instabilities in the absence of stratification, i.e. with $\nabla E = 0$, it is with a stellar convection zone in mind. While (2.18) and (2.19) may still hold individually to a high degree of approximation, the magnetic thermal wind relation (2.20) obtained by cross-differentiation is easily upset by the convective transport processes, and we would not wish to restrict attention to distributions of $\Omega(r, z)$ and $B(r, z)$ satisfying the $\nabla E = 0$ version of that equation. Thus in the non-magnetic case, for example, while the Taylor–Proudman-like constraint of rotation is strongly effective when the convection is weak, convection at more super-critical Rayleigh numbers drives a differential rotation which is distinctly z-dependent (see Gilman 1976; Durney 1976).

Finally it must be emphasized that the approximation of short meridional wavelength, $\lambda \ll r$, is made primarily to facilitate analytical progress. It is true that the doubly diffusive instabilities are often such that modes of very short wavelength possess the fastest growth rates, and the validity of the $\lambda \ll r$ approximation can in such cases be justified *a posteriori*. Instabilities which do not rely on diffusive processes (but are, indeed, often somewhat hindered by them) may well not possess this property, however, and we do not then suggest for a moment that the way in which they will actually occur is necessarily with $\lambda \ll r$. We *do* note that in such mainly diffusionless cases the eigenvalue ω in the stability equation (3.20) is dependent on the orientation of the meridional wavenumber but *virtually independent of its magnitude*. There is nothing, therefore, in such cases to suggest that the results of the local analysis should be inadequate as a qualitative guide to the response of the system to large-scale disturbances. Indeed, in the special case of an incompressible fluid with no diffusive processes it is known that a local analysis, when properly interpreted, provides results in full qualitative agreement with the global theory (Acheson 1972).

3. The local dispersion relation

We first make a few points about the way in which dissipation terms will enter the linearized versions of (2.1), (2.3) and (2.6). The frictional term in the momentum equation is conveniently written in tensor notation

$$F_i = \frac{\partial}{\partial x_j}\left[\mu_S\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\mu_B\,\delta_{ij}\,\nabla\cdot\boldsymbol{u}\right], \tag{3.1}$$

where μ_S and μ_B denote the coefficients of shear and bulk viscosity. In view of the short wavelengths of the disturbances in the r and z directions, however, together with the various other scaling assumptions outlined in § 2, it follows from the conservation of mass equation (2.2) that the perturbation motions are *to a first approximation* non-divergent, i.e.

$$\partial u_r/\partial r + \partial u_z/\partial z = 0 \tag{3.2}$$

to first order in λ/r. Thus the second contribution to (3.1) is smaller than the first by a factor of order λ/r. Further, since μ_S will vary (principally due to the variations of T) over a length scale of order r, while u_i will vary on a scale of order λ, a similarly small error is committed by taking μ_S outside the differential operator in (3.1). We are then left with the viscous term in the conventional form of incompressible flow theory, $\boldsymbol{F} = \mu_S\,\nabla^2\boldsymbol{u}$, and on neglecting certain contributions arising from the cylindrical polar coordinate system, again with error of order λ/r, we find that for present purposes

$$\boldsymbol{F} = \mu_S(\partial^2/\partial r^2 + \partial^2/\partial z^2)\,\boldsymbol{u}. \tag{3.3}$$

Similar remarks apply to the ohmic diffusion term in (2.3), which is adequately replaced by

$$\eta(\partial^2/\partial r^2 + \partial^2/\partial z^2)\,\boldsymbol{b} \tag{3.4}$$

where $\boldsymbol{b}$ is the magnetic field perturbation.

In the energy equation (2.6) the viscous heating term F_D is adequately replaced, in view of (3.2), by that appropriate to an incompressible fluid (see, for example, Batchelor 1967, p. 153), and when the right hand side of the equation is linearized and simplified according to the short-wavelength approximation, it becomes

$$2\mu_S r\left(\frac{\partial\Omega}{\partial z}\frac{\partial u_\theta}{\partial z}+\frac{\partial\Omega}{\partial r}\frac{\partial u_\theta}{\partial r}\right)+2\eta\mu^{-1}\left[\frac{\partial B}{\partial z}\frac{\partial b_\theta}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(Br)\frac{\partial b_\theta}{\partial r}\right]+\rho c_V\kappa\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right)T_1, \tag{3.5}$$

where T_1 denotes the temperature perturbation. It is noteworthy that with the scaling assumptions discussed above, the first two terms are smaller than the third by a factor of order λ/r, and will therefore be neglected. The entropy of a fluid parcel will, in our analysis, change only by thermal diffusion, and though essentially a consequence of the short-wavelength approximation this will be particularly appropriate when, as in a typical radiative stellar interior, κ is the largest of the three diffusivities.

We now consider, then, small perturbations (u_r, u_θ, u_z), (b_r, b_θ, b_z), p_1, ρ_1 and T_1 to the basic variables of the system, and it is convenient to use the notation

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}, \quad \frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\theta}, \tag{3.6}$$

the latter representing rate-of-change as measured by an observer rotating with the local mean angular velocity. Thus from (2.5) and (2.6) we have

$$p_1/p = \rho_1/\rho+T_1/T \tag{3.7}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_1}{p}-\frac{\gamma\rho_1}{\rho}\right)+\left(u_r\frac{\partial}{\partial r}+u_z\frac{\partial}{\partial z}\right)\ln(p\rho^{-\gamma}) = \kappa\nabla^2\left(\frac{T_1}{T}\right), \tag{3.8}$$

while the r and z components of (2.3) give

$$(\mathrm{d}/\mathrm{d}t-\eta\nabla^2)(b_r, b_z) = (B/r)\,\partial(u_r, u_z)/\partial\theta. \tag{3.9), (3.10}$$

To leading order in λ/r equation (2.2) and the θ-component of (2.3) give the same result, namely

$$\partial u_r/\partial r+\partial u_z/\partial z = 0, \tag{3.11}$$

and independent information is obtained by eliminating the combination $\partial u_r/\partial r+\partial u_z/\partial z$ between them, whence

$$\frac{\mathrm{d}b_\theta}{\mathrm{d}t}-\eta\nabla^2 b_\theta-\frac{B}{\rho}\frac{\mathrm{d}\rho_1}{\mathrm{d}t} = \frac{B}{r}\frac{\partial u_\theta}{\partial\theta}+r\left(b_r\frac{\partial}{\partial r}+b_z\frac{\partial}{\partial z}\right)\Omega-\rho r\left(u_r\frac{\partial}{\partial r}+u_z\frac{\partial}{\partial z}\right)\left(\frac{B}{\rho r}\right). \tag{3.12}$$

Similarly, to leading order, the r and z components of (2.1) both give the result that the *total* pressure perturbation is zero:

$$p_1+\mu^{-1}Bb_\theta = 0, \tag{3.13}$$

and independent information is obtained by eliminating $p_1+\mu^{-1}Bb_\theta$ by cross-differentiation, whence

$$\rho\left(\frac{\mathrm{d}}{\mathrm{d}t}-\nu\nabla^2\right)\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)-2\rho\Omega\frac{\partial u_\theta}{\partial z} = \mu^{-1}\left[\frac{B}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial b_r}{\partial z}-\frac{\partial b_z}{\partial r}\right)-\frac{2B}{r}\frac{\partial b_\theta}{\partial z}\right]-g_r\frac{\partial\rho_1}{\partial z}+g_z\frac{\partial\rho_1}{\partial r}. \tag{3.14}$$

In view of (3.13) and the scaling in the azimuthal direction, the θ-component of (2.1) becomes

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}-\nu\nabla^2\right)u_\theta+\frac{1}{r}\left(u_r\frac{\partial}{\partial r}+u_z\frac{\partial}{\partial z}\right)(\Omega r^2) = (\mu\rho)^{-1}\left[\frac{B}{r}\frac{\partial b_\theta}{\partial\theta}+\frac{1}{r}\left(b_r\frac{\partial}{\partial r}+b_z\frac{\partial}{\partial z}\right)(Br)\right]. \tag{3.15}$$

Equations (3.7)–(3.15) are nine independent linear equations for the nine independent variables (u_r, u_θ, u_z), (b_r, b_θ, b_z), p_1, ρ_1, and T_1. The final step is to replace all the coefficients by their local values at a particular position (r, z) and seek solutions in which all perturbation quantities are a (local) constant multiple of

$$\exp \mathrm{i}(lr + m\theta + nz - \sigma t) \tag{3.16}$$

where l and n are $O(\lambda^{-1})$ and m is an $O(1)$ integer. The Doppler-shifted frequency, $w \equiv \sigma - m\Omega$, naturally emerges as the important one, and it is convenient to define

$$s^2 \equiv l^2 + n^2, \tag{3.17}$$

$$G \equiv g_r - (l/n)\, g_z, \tag{3.18}$$

$$\frac{\partial}{\partial h} \equiv \frac{\partial}{\partial r} - \frac{l}{n}\frac{\partial}{\partial z}, \tag{3.19}$$

the last of these being proportional to the spatial rate of change along 'crests', i.e. lines of constant phase, in the meridional plane. The dispersion relation that must be satisfied if equations (3.7)–(3.15) are to admit solutions of the form (3.16) is found, after a great deal of algebra that is omitted here, to be

$$\begin{aligned} V^2\left[\frac{2\Omega m}{r} + (\omega + \mathrm{i}\nu s^2)\left\{\frac{2}{r} - \frac{G}{a^2}\left(\frac{\omega + \mathrm{i}\kappa s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\right)\right\}\right]\left[\frac{m}{\omega + \mathrm{i}\eta s^2}\frac{\partial\Omega}{\partial h} + \frac{\partial F}{\partial h} - \frac{\omega}{\omega\gamma + \mathrm{i}\kappa s^2}\frac{\partial E}{\partial h}\right] \\ + \left[\frac{s^2}{n^2}\left(\omega + \mathrm{i}\nu s^2 - \frac{m^2V^2/r^2}{\omega + \mathrm{i}\eta s^2}\right) - \frac{G}{\omega\gamma + \mathrm{i}\kappa s^2}\frac{\partial E}{\partial h}\right]\left[(\omega + \mathrm{i}\nu s^2)\,(\omega + \mathrm{i}\eta s^2) - \frac{m^2V^2}{r^2}\right. \\ \left. + \frac{V^2}{a^2}\left(\frac{\omega + \mathrm{i}\kappa s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\right)\omega(\omega + \mathrm{i}\nu s^2)\right] - \left[\frac{\partial}{\partial h}(\Omega r^2) + \frac{mV^2}{\omega + \mathrm{i}\eta s^2}\frac{\partial Q}{\partial h}\right] \\ \times\left[\frac{2\Omega}{r}\left\{\omega + \mathrm{i}\eta s^2 + \frac{V^2}{a^2}\left(\frac{\omega + \mathrm{i}\kappa s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\right)\omega\right\} + \frac{mV^2}{r^2}\left\{\frac{2}{r} - \frac{G}{a^2}\left(\frac{\omega + \mathrm{i}\kappa s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\right)\right\}\right] = 0. \end{aligned} \tag{3.20}$$

In view of the remarks made toward the end of §2 we note that the magnetic thermal wind relation (2.20) has *not* been used in the derivation of this equation. Before we explore its properties in various limiting cases it is worth noting what we obtain if the magnetic field is zero, i.e. $V = 0$. The dispersion relation then becomes

$$\frac{\Omega^2 r\, \partial R/\partial h}{\omega + \mathrm{i}\nu s^2} + \frac{G\, \partial E/\partial h}{\omega\gamma + \mathrm{i}\kappa s^2} - (\omega + \mathrm{i}\nu s^2)\frac{s^2}{n^2} = 0. \tag{3.21}$$

A most important feature of (3.21) is the absence of the azimuthal wavenumber m, so that the equation is indistinguishable from that for purely axisymmetric disturbances. Thus *any non-axisymmetric instability which we extract from* (3.20) *will owe its existence, in part at least, to the* (*originally*) *azimuthal magnetic field.* Our scaling assumptions have filtered out all non-magnetic non-axisymmetric instabilities such as the shear ones described by (1.9)–(1.11), for example. Sound waves have also been filtered out by the scaling assumptions, which result in motions that are to a first approximation divergence-free.

4. Axisymmetric instabilities

When $m = 0$ equation (3.20) simplifies to

$$\begin{aligned} \left[\frac{s^2}{n^2}(\omega + \mathrm{i}\nu s^2) - \frac{\Omega^2 r\, \partial R/\partial h}{\omega + \mathrm{i}\nu s^2}\right]\left[\omega + \mathrm{i}\eta s^2 + \frac{V^2}{a^2}\left(\frac{\omega + \mathrm{i}\kappa s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\right)\omega\right] \\ - \left[G(\omega + \mathrm{i}\eta s^2) + \frac{2V^2}{r}\omega\right]\frac{1}{\omega\gamma + \mathrm{i}\kappa s^2}\frac{\partial E}{\partial h} + \left[\frac{2}{r} - \frac{G}{a^2}\left(\frac{\omega + \mathrm{i}\kappa s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\right)\right]V^2\frac{\partial F}{\partial h} = 0. \end{aligned} \tag{4.1}$$

We shall first examine axisymmetric instability on the assumption that diffusive effects are absent, i.e. $\nu = \eta = \kappa = 0$.

Diffusionless theory

In place of (4.1) we then have the much simpler stability equation

$$\gamma \frac{s^2}{n^2}\left(1+\frac{V^2}{\gamma a^2}\right)\omega^2 = \gamma \Omega^2 r\left(1+\frac{V^2}{\gamma a^2}\right)\frac{\partial R}{\partial h}+\left(G+\frac{2V^2}{r}\right)\frac{\partial E}{\partial h}+\left(G-\frac{2\gamma a^2}{r}\right)\frac{V^2}{a^2}\frac{\partial F}{\partial h}. \tag{4.2}$$

In view of (3.18) and (3.19) the right hand side is an expression quadratic in l/n, and for stability we must clearly have $\omega^2 > 0$ for *all* l/n. The requirement that ax^2+bx+c be positive for all x is easily shown to be equivalent to the conditions $a > 0$, $c > 0$ and $b^2 < 4ac$, and application of this result to the right hand side of (4.2) leads to the conclusion that the system is dynamically *un*stable to axisymmetric disturbances if *any* of the following three inequalities is satisfied:

$$\left(1+\frac{V^2}{\gamma a^2}\right)\Omega^2 r\frac{\partial R}{\partial r}+\left(g_r+\frac{2V^2}{r}\right)\frac{1}{\gamma}\frac{\partial E}{\partial r}+\left(\frac{g_r}{\gamma}-\frac{2a^2}{r}\right)\frac{V^2}{a^2}\frac{\partial F}{\partial r} < 0, \tag{4.3}$$

$$g_z\left(\frac{\partial E}{\partial z}+\frac{V^2}{a^2}\frac{\partial F}{\partial z}\right) < 0, \tag{4.4}$$

$$g_z\left[\frac{\partial R}{\partial r}\frac{\partial E}{\partial z}-\frac{\partial R}{\partial z}\frac{\partial E}{\partial r}+\frac{2V^2}{\Omega^2 r^2}\left(\frac{\partial E}{\partial r}\frac{\partial F}{\partial z}-\frac{\partial E}{\partial z}\frac{\partial F}{\partial r}\right)+\frac{V^2}{a^2}\left(\frac{\partial R}{\partial r}\frac{\partial F}{\partial z}-\frac{\partial R}{\partial z}\frac{\partial F}{\partial r}\right)\right] < 0. \tag{4.5}$$

To obtain the final one, (4.5), a certain amount of algebraic manipulation is involved. It is convenient, having written down the expression '$b^2 < 4ac$', to subtract from both sides that quantity which will change the left hand side from b^2 to d^2, where d is identical with b except that g_z is replaced, where it occurs, by $-g_z$. One can then show by using the basic balance equations that $d = 0$, whence (4.5), and it is helpful to establish from (2.11)–(2.20) the supplementary equations

$$\frac{\partial E}{\partial r}+\frac{V^2}{a^2}\frac{\partial F}{\partial r} = -\frac{1}{a^2}\left(g_r+\frac{2V^2}{r}\right)-\gamma\left(1+\frac{V^2}{\gamma a^2}\right)\frac{\partial}{\partial r}\ln\rho, \tag{4.6}$$

$$\frac{\partial E}{\partial z}+\frac{V^2}{a^2}\frac{\partial F}{\partial z} = -\frac{g_z}{a^2}-\gamma\left(1+\frac{V^2}{\gamma a^2}\right)\frac{\partial}{\partial z}\ln\rho, \tag{4.7}$$

for this purpose.

We shall throughout this paper suppose that the entropy (per unit mass) is a non-decreasing function of height (so that $g_r\,\partial E/\partial r$ and $g_z\,\partial E/\partial z$ are non-negative), thus excluding thermal convection. The instabilities we consider are due to differential rotation and/or magnetic field gradients. Equation (4.3) clearly shows how axisymmetric instability of the former kind may arise if the angular momentum (per unit mass) decreases with distance from the rotation axis. Magnetic instability, on the other hand, may be of two quite different kinds, depending on whether the region under discussion is located inside or outside a 'critical radius'

$$r_c = 2\gamma a^2/g_r. \tag{4.8}$$

In the former case an increase of magnetic flux (per unit mass) with radius r (i.e. $\partial F/\partial r > 0$) makes the third term of (4.3) negative and promotes instability, while outside the critical radius it is a *decrease* of magnetic flux with r that promotes instability, essentially by the mechanism of 'magnetic buoyancy'.

Typical values of the basic parameters in the (convectively stable) radiative zone of a star (such as the Sun) appear to be such that

$$V^2 \ll \Omega^2 r^2 \ll a^2 \lesssim gr, \tag{4.9}$$

so that we may pay little regard to (4.4) as a source of instability and (4.3) may be simplified a little:

$$\Omega^2 r \frac{\partial R}{\partial r} + \frac{g_r}{\gamma}\frac{\partial E}{\partial r} + \left(\frac{g_r}{\gamma} - \frac{2a^2}{r}\right)\frac{V^2}{a^2}\frac{\partial F}{\partial r} < 0. \tag{4.10}$$

If a significant radial decrease of angular momentum is present, therefore, an azimuthal magnetic field can do little to stop the instability in the parameter régime (4.9). The buoyancy restoring forces associated with the vertical entropy gradient, on the other hand, apparently suppress centrifugal instability most effectively, essentially since $\Omega^2 r \ll g$.

We must not ignore (4.5), however, which in the parameter régime (4.9), reduces approximately to its non-magnetic form, giving instability if

$$g_z\left(\frac{\partial R}{\partial r}\frac{\partial E}{\partial z} - \frac{\partial R}{\partial z}\frac{\partial E}{\partial r}\right) < 0, \tag{4.11}$$

i.e. if the specific angular momentum decreases outward along a surface of constant specific entropy. The reason is that the stratification has no stabilizing effect whatsoever on fluid motions which take place entirely along isentropic surfaces (cf. Goldreich & Schubert 1967; Roxburgh 1975); for these motions, in view of (3.2), l/n is such that $\partial E/\partial h = 0$, and when $V = 0$ one may in fact derive (4.11) directly from (4.2) by asking in what circumstances the right hand side is negative for such modes. There is, however, a further subtlety. We recall that a non-cylindrical rotation law $\partial\Omega/\partial z \neq 0$ will, by (2.20), be accompanied by an inclination between the apparent geopotentials and the surfaces of constant specific entropy. It may be possible, therefore, for fluid motions to even make the term $G\,\partial E/\partial h$ *negative* (implying a concomitant *release* of potential energy) by taking place on planes sandwiched (angle-wise) between the (nearly coincident) isentropes and apparent geopotentials. Even in the non-magnetic case the theory of such '*baroclinic*' instabilities is quite complicated; the simple criterion (4.11) was due originally to Høiland (see Wasiutynski 1946, ch. 2; Eliassen & Kleinschmidt 1957, pp. 64–72), but at large Richardson numbers such as in the solar radiative interior one must certainly expect non-axisymmetric instabilities of this kind to occur much more readily (Eady 1949; Hide & Mason 1975). Thus, as with thermal convection, I postpone a discussion of the magnetic baroclinic instabilities evidently contained in (3.20) to a companion paper, while referring the reader to Gilman (1967) and Braginsky & Roberts (1975).

Finally, consider diffusionless axisymmetric instability due to a magnetic field gradient. Taking Ω = constant as the prototype of a centrifugally *stable* angular velocity distribution we see by inspection of (4.10) that in the 'weak magnetic field' parameter régime (4.9), the rotation and stratification act together to make such instability quite impossible.

Viscous and magnetic effects on Goldreich–Schubert instability

We now restore all three diffusive processes, and on setting $w = iq$ in (4.1) we obtain a quartic in q with real coefficients. A necessary condition for the stability of the system is therefore that the constant term be positive, that is

$$\Omega^2 r \frac{\partial R}{\partial h} + \frac{\nu}{\kappa} G \frac{\partial E}{\partial h} + \frac{\nu}{\eta}\left(G - \frac{2a^2}{r}\right)\frac{V^2}{a^2}\frac{\partial F}{\partial h} + \nu^2 \frac{s^6}{n^2} > 0, \tag{4.12}$$

for all disturbance orientations l/n. The ratio of the first term to the last is of order $\Omega^2\lambda^4/\nu^2$, where λ is a characteristic wavelength, and this quantity is truly enormous in stars unless λ is exceptionally small (e.g. 1 m). We shall therefore neglect the final term of (4.12) in what follows, and when written out in full, using (3.18) and (3.19), the left hand side of (4.12) is an expression quadratic in l/n. By the same procedure as in the previous subsection we thus find that the system is diffusively unstable to axisymmetric disturbances if any of the following criteria are satisfied:

$$\Omega^2 r\frac{\partial R}{\partial r}+\frac{\nu}{\kappa}g_r\frac{\partial E}{\partial r}+\frac{\nu}{\eta}\frac{V^2}{a^2}\left(g_r-\frac{2a^2}{r}\right)\frac{\partial F}{\partial r}<0, \tag{4.13}$$

$$g_z\left(\frac{\partial E}{\partial z}+\frac{\kappa}{\eta}\frac{V^2}{a^2}\frac{\partial F}{\partial z}\right)<0, \tag{4.14}$$

$$4g_z\left[\frac{\nu}{\kappa}\Omega^2 r\left(\frac{\partial R}{\partial r}\frac{\partial E}{\partial z}-\frac{\partial R}{\partial z}\frac{\partial E}{\partial r}\right)+\frac{2V^2}{r}\frac{\nu}{\kappa}\frac{\nu}{\eta}\left(\frac{\partial E}{\partial r}\frac{\partial F}{\partial z}-\frac{\partial E}{\partial z}\frac{\partial F}{\partial r}\right)+\frac{\nu}{\eta}\frac{V^2}{a^2}\Omega^2 r\left(\frac{\partial R}{\partial r}\frac{\partial F}{\partial z}-\frac{\partial R}{\partial z}\frac{\partial F}{\partial r}\right)\right]$$
$$<\left[\Omega^2 r\frac{\partial R}{\partial z}+\frac{\nu}{\kappa}g_r\frac{\partial E}{\partial z}-\frac{\nu}{\kappa}g_z\frac{\partial E}{\partial r}+\frac{\nu}{\eta}\frac{V^2}{a^2}\left(g_r-\frac{2a^2}{r}\right)\frac{\partial F}{\partial z}-\frac{\nu}{\eta}\frac{V^2}{a^2}g_z\frac{\partial F}{\partial r}\right]^2. \tag{4.15}$$

It is of interest to first consider the non-magnetic case $V=0$, in which case (4.14) will certainly not be satisfied, since the system is supposed statically stable, while (4.13) and (4.15) reduce to

$$-\Omega^2 r\frac{\partial R}{\partial r}>\frac{\nu}{\kappa}g_r\frac{\partial E}{\partial r} \tag{4.16}$$

and

$$\left(1-\gamma\frac{\nu}{\kappa}\right)^2\left(\frac{\partial R}{\partial z}\right)^2>4\frac{\nu}{\kappa}\frac{g_z}{\Omega^2 r}\left(\frac{\partial R}{\partial r}\frac{\partial E}{\partial z}-\frac{\partial R}{\partial z}\frac{\partial E}{\partial r}\right). \tag{4.17}$$

The thermal wind equation (2.20) has been used here to eliminate the entropy gradient on the right hand side of (4.15). For comparison we note that the corresponding adiabatic criteria (4.3) and (4.5) are, when $V=0$,

$$-\Omega^2 r\frac{\partial R}{\partial r}>\frac{g_r}{\gamma}\frac{\partial E}{\partial r}, \tag{4.18}$$

$$0>g_z\left(\frac{\partial R}{\partial r}\frac{\partial E}{\partial z}-\frac{\partial R}{\partial z}\frac{\partial E}{\partial r}\right). \tag{4.19}$$

In the limit $\nu/\kappa\to 0$, (4.16) and (4.17) reduce to the well known criteria due to Goldreich & Schubert (1967) and Fricke (1968), namely that the system is secularly unstable if

$$\text{either}\quad -\partial R/\partial r>0\quad\text{or}\quad \partial R/\partial z\neq 0\quad\text{(or both)}. \tag{4.20}$$

When ν/κ is small *but non-zero* the stratification exerts a stabilizing influence, according to (4.16), albeit a very weak one compared with its influence on adiabatic perturbations (cf. (4.18)). This is because when $\nu\ll\kappa$ there exist motions with either an appropriately long time scale or an appropriately short length scale that heat exchange between a displaced fluid parcel and its surroundings effectively annuls the stabilizing buoyancy forces, while viscous diffusion is too weak to significantly reduce the effectiveness of the *de*stabilizing angular momentum gradient. This particular 'doubly diffusive' instability mechanism was first noted in a related incompressible flow problem (with $\partial\Omega/\partial z=0$) by Yih (1961), and is in turn a simple analogue of the classic instability which can take place when (light) hot salty water overlies (heavy) cold fresh water, by virtue of the difference in diffusivities of salt and heat (see, for example, Stern 1960; Turner 1973).

Turning now to (4.17) we see that a decrease of angular momentum (per unit mass) with distance from the rotation axis along a surface of constant specific entropy automatically leads to instability; the mechanism here has nothing to do with double-diffusive effects but simply involves fluid motions confined to isentropic surfaces, and was discussed in the previous subsection.

Finally, given an angular momentum increasing with r *and* along isentropic surfaces we still have to ensure, according to (4.17), that $\partial\Omega/\partial z$ is not too large if the system is to be stable. In the stellar situations which we envisage $g \gg \Omega^2 r$ and the entropy gradient is very nearly aligned with gravity, so a good approximation to (4.17) is, with $\nu \ll \kappa$:

$$\left(\frac{r}{\Omega}\frac{\partial\Omega}{\partial z}\right)^2 > \frac{2g_z}{\Omega^2 r}\frac{\nu}{\kappa}r\,|\nabla E|\sin\Theta\frac{\partial}{\partial\Theta}\ln(\Omega\sin^2\Theta), \tag{4.21}$$

where Θ denotes the polar angle.

If we consider the entropy scale height to be comparable with the local radius, which is true for example in the solar interior, the extent to which the Goldreich–Schubert criteria (4.20) are good approximations to (4.16) and (4.17) clearly depends on the dimensionless parameter

$$\alpha \equiv \frac{\nu}{\kappa}\frac{g}{\Omega^2 r}, \tag{4.22}$$

agreement between the two sets being best when α is small. In particular, according to (4.21) instability due to variation of Ω with z will only arise if $r\Omega^{-1}\partial\Omega/\partial z$ exceeds some definite value of order $\alpha^{\frac{1}{2}}$. The way in which this comes about may be simply seen by inspection of the pertinent quadratic expression in (4.12):

$$\Omega^2 r\left(\frac{\partial R}{\partial r}-\frac{l}{n}\frac{\partial R}{\partial z}\right)+\frac{\nu}{\kappa}\left(g_r-\frac{l}{n}g_z\right)\left(\frac{\partial E}{\partial r}-\frac{l}{n}\frac{\partial E}{\partial z}\right). \tag{4.23}$$

If $|\partial R/\partial z| \ll \partial R/\partial r \sim 1$ instability will only occur for disturbance orientations having a large value of l/n, i.e.

$$|(l/n)\,\partial R/\partial z| \gtrsim 1; \tag{4.24}$$

otherwise the stabilizing effects of the radially outward increase in angular momentum prevail. A good approximation to the buoyancy term in (4.23) is then the contribution *quadratic* in l/n, whence l/n must not be too large or the stabilizing effects of stratification prevail. For instability we need, in fact,

$$\Omega^2 r\left|\frac{\partial R}{\partial z}\right| \gtrsim \frac{\nu}{\kappa}g\left|\frac{l}{n}\right|, \tag{4.25}$$

and combining (4.24) and (4.25) we obtain $(\partial R/\partial z)^2 \gtrsim \alpha$.

Goldreich & Schubert (1967) derive their criteria by taking the limit $\nu/\kappa \to 0$ in (4.23), thus ignoring the second term. Fricke (1968) effectively derived (4.21) (see his equation (3.33)), but it appears to have been totally disregarded in the subsequent literature (e.g. Fricke 1969; Fricke & Schubert 1970; Spiegel & Zahn 1970; Fricke & Kippenhahn 1972; Zahn 1974; Roxburgh 1975) where one invariably finds the criterion $\partial\Omega/\partial z \neq 0$. Our point is simply that it is dangerous to interpret this criterion too strictly, the key quantity α being the product of a small number (ν/κ) and a large one $(g/\Omega^2 r)$. Whether or not α is small will depend crucially on the particular system under consideration. Taking $\nu/\kappa \sim 10^{-6}$ and $g/\Omega^2 r \sim 10^4$ for a moderately rotating solar interior (see, for example, Spiegel 1972*b*), α *is* small, $O(10^{-2})$, but even here a 10% change in Ω as one crosses the zone in the z-direction might well be stable, on the present theory, at least.

The question of whether an appropriate toroidal magnetic field can suppress the Goldreich–Schubert instability has been considered by Fricke (1969), who assumes that stratification effects have been wholly annihilated by fast thermal diffusion (formally $\kappa = \infty$). He takes $\nu = \eta = 0$ and thus concludes, roughly speaking, that magnetic fields increasing outwards and satisfying

$$V \gtrsim (\Omega^2 r/g)^{\frac{1}{2}}a \tag{4.26}$$

are needed to stop the instability. It is clear from (4.12), however, that when $\nu \ll \eta$, this is a considerable underestimate of the magnetic field required, for the ohmic diffusion diminishes its effectiveness in the same way that κ annihilated that of the stratification. Thus, following Fricke by letting $\kappa \to \infty$ in (4.13), we find instability if

$$\Omega^2 r \frac{\partial R}{\partial r} + \frac{\nu}{\eta} \frac{V^2}{a^2} \left(g_r - \frac{2a^2}{r} \right) \frac{\partial F}{\partial r} < 0, \tag{4.27}$$

and this means that magnetic fields stronger than (4.26) by a factor $O(\eta/\nu)^{\frac{1}{2}}$, and increasing outward (for $r > r_c$), are needed to stop the instability.

Multiply-diffusive magnetic instabilities

Outside the critical radius r_c magnetic fields *decreasing* with r *promote* instability, of course, by magnetic buoyancy. We have seen that on a diffusionless theory, strong stratification and rapid rotation prevent such an instability, but here again multiply-diffusive effects, this time of a somewhat novel kind, can come into play and render it possible provided the diffusivities are sufficiently disparate. This part of the paper is presented in the appendix under joint authorship with Dr M. P. Gibbons.

5. Non-axisymmetric instability; the low-frequency approximation

Experience of related non-axisymmetric problems in the diffusionless ($\nu = \eta = \kappa = 0$) and incompressible limit (Acheson 1972) leads us to expect little in terms of concrete and practically useful results except in two limiting cases. The first is that of zero rotation, $\Omega = 0$, which we investigate in §8. The second, fortunately, coincides with the parameter régime apparently of most astrophysical interest:

$$V^2 \ll \Omega^2 r^2 \ll a^2 \lesssim gr, \tag{5.1}$$

and is a 'rapidly rotating' one from a *magnetohydrodynamic* point of view. Provided the *differential* rotation is weak in the sense that

$$r(\partial/\partial r, \partial/\partial z) \ln \Omega \ll 1 \tag{5.2}$$

we anticipate that instability may occur with frequency and growth rate small enough that

$$|\omega|^2 \ll m^2 V^2/r^2. \tag{5.3}$$

If we approximate (3.20) accordingly, still allowing for as wide a variety of diffusive effects as possible by regarding $|\omega|/s^2 \sim \nu \sim \eta \sim \kappa$ in the process, we obtain

$$\frac{2\Omega m V^2}{r}\left[\frac{m}{\omega + \mathrm{i}\eta s^2}\frac{\partial \Omega}{\partial h} + \frac{\partial F}{\partial h} - \frac{\omega}{\omega\gamma + \mathrm{i}\kappa s^2}\frac{\partial E}{\partial h}\right] + \frac{m^2 V^2}{r^2}\left[\frac{s^2}{n^2 r^2(\omega + \mathrm{i}\eta s^2)} \frac{m^2 V^2}{} + \frac{G}{\omega\gamma + \mathrm{i}\kappa s^2}\frac{\partial E}{\partial h}\right]$$
$$- \left(2\Omega r + \frac{m V^2}{\omega + \mathrm{i}\eta s^2}\frac{\partial Q}{\partial h}\right)\left[\frac{2\Omega}{r}(\omega + \mathrm{i}\eta s^2) + \frac{m V^2}{r^2}\left\{\frac{2}{r} - \frac{G}{a^2}\left(\frac{\omega + \mathrm{i}\kappa s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\right)\right\}\right] = 0. \tag{5.4}$$

Notably, the viscosity ν does not appear in (5.4); this is not surprising in view of studies of similar low-frequency instabilities in incompressible fluids, where ν has to exceed η by a factor of order $\Omega^2 r^2/V^2$ to have much effect (Acheson & Hide 1973, p. 178). It turns out to be quite instructive to multiply (5.4) by $\omega + \mathrm{i}\eta s^2$, arrange it in the form of a quadratic equation for $\Omega(\omega + \mathrm{i}\eta s^2)$ and 'solve' it, despite the fact that the result is an *implicit* expression for ω:

$$\omega = \left[\frac{G}{\gamma a^2} + \frac{G}{a^2}\left(\frac{\omega + \mathrm{i}\kappa s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\right) - \frac{4}{r} + \frac{\mathrm{i}\kappa s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\frac{1}{\gamma}\frac{\partial E}{\partial h}\right]\frac{mV^2}{4\Omega r}$$
$$- \mathrm{i}\eta s^2 \pm \frac{mV}{2\Omega r}\left[\left\{\frac{G}{a^2}\left(\frac{\omega + \mathrm{i}\kappa s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\right) - \frac{2}{r}\right\} V^2 \left\{\frac{\partial F}{\partial h} - \frac{\omega}{\omega\gamma + \mathrm{i}\kappa s^2}\frac{\partial E}{\partial h}\right\}\right.$$
$$\left. + \Omega^2 r \frac{\partial}{\partial h}\ln \Omega^2 + \frac{m^2V^2}{r^2}\frac{s^2}{n^2} - \frac{V^2\kappa^2 s^4}{(\omega\gamma + \mathrm{i}\kappa s^2)^2}\left(\frac{1}{a}\frac{\partial a}{\partial h}\right)^2 + \left(\frac{\omega + \mathrm{i}\eta s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\right) G \frac{\partial E}{\partial h}\right]^{\frac{1}{2}}. \tag{5.5}$$

We have used here the following identity, which may readily be established from the basic balance equations (2.18) and (2.19):

$$\left(1 + \frac{V^2}{\gamma a^2}\right)\frac{\partial Q}{\partial h} = \frac{\partial F}{\partial h} - \frac{1}{\gamma}\frac{\partial E}{\partial h} + \frac{2}{r} - \frac{G}{\gamma a^2}, \tag{5.6}$$

although the $V^2/\gamma a^2$ part is consistently neglected, in view of the approximations above stemming from (5.1). Indeed, all (5.6) is actually then saying is that the basic balance is almost hydrostatic in the parameter régime (5.1):

$$\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right)\ln p \approx -\left(\frac{g_r}{a^2}, \frac{g_z}{a^2}\right). \tag{5.7}$$

This has allowed us, at one convenient point in deriving (5.5) from (5.4), to express the entropy gradient in terms of the temperature, or isothermal sound speed gradient:

$$\frac{\partial E}{\partial h} \approx (\gamma - 1)\frac{G}{a^2} + \gamma\frac{\partial}{\partial h}(\ln a^2). \tag{5.8}$$

We emphasize that the more delicate 'magnetic thermal wind' balance (2.20) has not been used anywhere in the non-axisymmetric analysis so far.

By substituting (5.5) into (5.3) we find a useful set of conditions under which our pursuit of a 'low-frequency' instability has been a self-consistent procedure. They are, in addition to (5.1) and (5.2):

$$\frac{V^2}{\Omega^2 H^2} \ll 1, \quad \frac{\eta^2 s^4}{\Omega^2} \ll \frac{m^2 V^2}{\Omega^2 r^2} \ll 1, \tag{5.9}$$

and
$$\left|\frac{\omega + \mathrm{i}\eta s^2}{\omega\gamma + \mathrm{i}\kappa s^2}\right| G \frac{\partial E}{\partial h} \ll \Omega^2. \tag{5.10}$$

The first two of these will turn out to be comparatively innocuous, but when the fluid is strongly stratified ($N^2 \gg \Omega^2$), the condition (5.10) may evidently be satisfied only if the motions are sufficiently horizontal or sufficiently effective doubly diffusive mechanisms are at work.

Before going on to discuss adiabatic instabilities of the above kind in § 6, and fast-heat-exchanging instabilities in § 7, we emphasize their essentially *non-axisymmetric* and *magnetic* nature. If either $m = 0$ or $V = 0$, (5.3) cannot possibly be satisfied and (5.5) has no validity. Another important point to bear in mind is that there could be 'high-frequency' non-axisymmetric instabilities for which (5.3) would be false, and which would therefore go undetected by the present approximate method. The results in the appendix (especially (A 11) and (A 20)) on diffusive *axisymmetric* instabilities suggest that this will only be the case if the viscosity (which plays no significant part in the low-frequency instabilities (5.4), as noted above) is very small, i.e. $\nu/\eta \lesssim V^2/\Omega^2 r^2$. This inequality is unlikely to be satisfied in the astrophysical applications of the theory which we have in mind, but further work will be needed before high-frequency instabilities can be ignored with confidence.

6. Non-axisymmetric instability; adiabatic modes

For modes only weakly affected by diffusion, with $|\omega| \gg \eta s^2$ and $|\omega| \gg \kappa s^2$, (5.10) can be satisfied in the parameter régime (5.1) only if either (*a*) the temperature gradient is almost adiabatic or (*b*) the motions are almost entirely horizontal, in which case $G \approx 0$ because of the fact that the motions are almost non-divergent, i.e. $lu_r + nu_z \approx 0$ (see (3.11)).

Adiabatic temperature gradient

We shall consider first the case when $\eta = \kappa = 0$, and (5.5) then reduces to

$$\omega = \left(\frac{G}{\gamma a^2} - \frac{2}{r}\right)\frac{mV^2}{2\Omega r} \pm \frac{mV}{2\Omega r}\left[\left(\frac{G}{\gamma a^2} - \frac{2}{r}\right) V^2 \frac{\partial F}{\partial h} + \Omega^2 r \frac{\partial}{\partial h}(\ln \Omega^2) + \frac{m^2 V^2}{r^2}\frac{s^2}{n^2}\right]^{\frac{1}{2}} \tag{6.1}$$

The expression in square brackets is quadratic in l/n, and by a now familiar procedure we conclude that the system is unstable if either of the following criteria is satisfied:

$$-\Omega^2 r \frac{\partial}{\partial r} \ln \Omega^2 - \left(\frac{g_r}{\gamma a^2} - \frac{2}{r}\right) V^2 \frac{\partial F}{\partial r} > \frac{m^2 V^2}{r^2}, \tag{6.2}$$

$$-\frac{g_z}{\gamma}\frac{V^2}{a^2}\frac{\partial F}{\partial z} > \frac{m^2 V^2}{r^2}. \tag{6.3}$$

For the present we shall isolate differential rotation as the instability mechanism by taking the magnetic flux gradient as zero. According to the axisymmetric criterion (4.3) a decrease of angular momentum (per unit mass) with radius is then needed for instability. The criterion (6.2) for *non*-axisymmetric instability in the 'rapidly rotating' régime (5.1), on the other hand, reduces to

$$-r\partial(\ln \Omega^2)/\partial r > V^2/\Omega^2 r^2 \tag{6.4}$$

when one notes that $m = 1$ is the mode most easily generated. Evidently a tiny decrease in angular *velocity* with radius is sufficient for non-axisymmetric instability! Although the magnetic field appears to be playing a stabilizing rôle in (6.4) it is very important to note that it is only due to its presence that we have this instability; the growth rate, according to (6.1) evidently decreases with V and vanishes when V is zero.

One can easily estimate growth rates from (6.1); by some elementary differentiation we find that for a given super-critical angular velocity distribution the maximum growth rate is $\frac{1}{2}r|\nabla\Omega|$. This is attained by a mode having

$$l\frac{\partial\Omega}{\partial r} + n\frac{\partial\Omega}{\partial z} = 0, \quad m = \frac{\Omega r}{V}\left(-\frac{r}{\Omega}\frac{\partial\Omega}{\partial r}\right)^{\frac{1}{2}}, \tag{6.5}$$

i.e. with 'crests' aligned with $\nabla\Omega$. We refer to these as 'estimates' because no account has been taken in their evaluation of the fact that m ought really to be an integer. Their accuracy will be best when the angular velocity gradient is highly super-critical so that, according to (6.5), m is large (but, note, not so large that the second of (5.9) would be violated). Note that the instabilities take the form of propagating waves, and since in any application of the present analysis to a stellar convective envelope we anticipate a rather small scale height, $a^2/g \ll r$, the azimuthal propagation speed of the most rapidly growing mode can be written

$$\frac{\omega_R}{m} \approx \frac{V^2}{2\Omega r}\frac{1}{\gamma a^2}\left(g_r + g_z\frac{\partial\Omega/\partial z}{\partial\Omega/\partial r}\right). \tag{6.6}$$

Thus the amplifying wave propagates east or west *relative to the local rotation* according as the angular velocity decreases or increases with height. When the Ω-gradient is only slightly unstable by (6.2) this propagation speed significantly exceeds (by a factor $O(gr/a^2)$) the deviations of the fluid from uniform rotation.

With only the *caveat*, therefore, that our approximation procedure requires the angular velocity gradient to be weak in the sense of (5.2), very rapidly growing instabilities will quickly accompany the establishment of an angular *velocity* distribution which decreases more than a very small amount with radius.

We turn now to the magnetic flux gradient instabilities, and confine attention for simplicity to the case of uniform rotation. Comparing (6.2) with the corresponding axisymmetric result (4.3) we see that exactly the same remarks about the types of field distribution which promote instability (i.e. $\partial F/\partial r < 0$ outside the critical radius $r_c = 2\gamma a^2/g_r$, $\partial F/\partial r > 0$ inside) still apply; the essential difference is simply that the enormous stabilizing term due to rotation in (4.3), $4\Omega^2$ when the rotation is uniform, has completely disappeared and been replaced by a much smaller new term on the right hand side of (6.2). The reason is that the physical origin of the $4\Omega^2$ term was the conservation of angular momentum of axially symmetric fluid rings; in the present context a ring does not remain axially symmetric when the system is disturbed, the magnetic field lines threading it are distorted, and an azimuthal component of Lorentz force arises which breaks the angular-momentum-conservation constraint, thus facilitating instability. A comparatively modest amount of work is, however, required to twist the field lines against the resistance of their own tension, and this is what the right hand side of (6.2) represents (see Acheson & Hide 1973, pp. 184–188).

Turning now to (6.3) and comparing it with (4.4) we see that a decrease of F with z (or, in the southern hemisphere, with $-z$) can again cause instability, but axisymmetric modes are evidently more easily excited by this mechanism. The physical reason is clear: rotation exerts no stabilizing influence at all on the modes ($l/n \rightarrow \infty$; see (4.2)) to which (6.3), taken in isolation, refers, for they do not involve radial motions. There is nothing to be gained in this case, therefore, by twisting the field lines, for this only requires an additional expenditure of work.

As far as estimating growth rates and phase speeds is concerned the fully three-dimensional case is rather complicated, and we confine attention now to a cylindrically symmetric configuration and the equatorial plane, i.e. $g_z = 0$. Then (6.1) becomes

$$\omega = \frac{mV^2}{2\Omega r}\left[\frac{g_r}{\gamma a^2} - \frac{2}{r} \pm \left\{\left(\frac{g_r}{\gamma a^2} - \frac{2}{r}\right)\frac{dF}{dr} + \frac{m^2 s^2}{r^2 n^2}\right\}^{\frac{1}{2}}\right], \tag{6.7}$$

and it is easily shown that the maximum growth rate is

$$-\frac{V^2}{4\Omega}\left(\frac{g_r}{\gamma a^2} - \frac{2}{r}\right)\frac{dF}{dr}, \tag{6.8}$$

this being attained by a mode having $l/n = 0$ and

$$m^2 = -\tfrac{1}{2}r^2\left(\frac{g_r}{\gamma a^2} - \frac{2}{r}\right)\frac{dF}{dr}. \tag{6.9}$$

We finally note that the frequency of these waves is typically comparable with their growth rate, and the azimuthal direction of propagation is westward inside the critical radius but *eastward* outside the critical radius, as one typically would be in the convective envelope of a star, where the scale height is rather small.

Strongly sub-adiabatic temperature gradient

In this case almost diffusionless modes can be unstable only with almost horizontal motions. We thus set $\partial E/\partial h \approx 0$ and $G \approx 0$ in (5.5) (since, by virtue of (5.1), ∇E and $\boldsymbol{g}$ must be almost parallel in the equilibrium state) and $l/n \approx \tan\Theta$. In this way we obtain

$$\omega = -\frac{mV^2}{\Omega r^2} - \mathrm{i}\eta s^2 \pm \frac{mV}{2\Omega r}\left[\tan\Theta\left\{2\Omega^2\frac{\partial}{\partial\Theta}\ln\Omega - \frac{2V^2}{r^2}\frac{\partial F}{\partial\Theta}\right\} + \frac{m^2V^2}{r^2}\sec^2\Theta\right]^{\frac{1}{2}} \tag{6 10}$$

where $s^2 = 4\pi^2/\lambda^2$, λ being the meridional wavelength. Thermal diffusion plays no part at all, and instability occurs if

$$2\tan\Theta\left[-\Omega^2\frac{\partial}{\partial\Theta}(\ln\Omega) + \frac{V^2}{r^2}\frac{\partial F}{\partial\Theta}\right] > \frac{m^2V^2}{r^2}\sec^2\Theta + \frac{4\Omega^2r^2\eta^2}{m^2V^2}\left(\frac{2\pi}{\lambda}\right)^4. \tag{6.11}$$

It therefore sets in first for λ as large as is possible compatible with whatever constraints are appropriate,† and we shall denote its maximum value by $2d$. If

$$\mathscr{C} \equiv (V^2/2\Omega\eta)\,(d^2/r^2) \tag{6.12}$$

is large, diffusive effects are unimportant in determining the marginal stability point, and instability first sets in when

$$2\tan\Theta\left[-\Omega^2\frac{\partial}{\partial\Theta}(\ln\Omega) + \frac{V^2}{r^2}\frac{\partial F}{\partial\Theta}\right] > \frac{V^2}{r^2}\sec^2\Theta \tag{6.13}$$

for the mode $m = 1$. If the ohmic diffusivity is high enough, or the field small enough, that $\mathscr{C} \lesssim 1$, however, we find by differentiation that instability first sets in when

$$2\tan\Theta\left[-\Omega^2\frac{\partial}{\partial\Theta}\ln\Omega + \frac{V^2}{r^2}\frac{\partial F}{\partial\Theta}\right] > \frac{4\Omega\eta\pi^2}{d^2}\sec\Theta \tag{6.14}$$

for the mode
$$m = (\pi r/d)\,(2\Omega\eta\cos\Theta/V^2)^{\frac{1}{2}}. \tag{6.15}$$

Clearly if $\mathscr{C} \ll 1$ much stronger gradients of angular velocity/magnetic flux than predicted by the diffusionless theory are necessary for instability, and when it does occur the azimuthal wavenumber m is much larger. We note in particular that when $\mathscr{C} \ll 1$ the magnetic instability is unlikely to occur, since from (6.14) it requires $\partial F/\partial\Theta \gg 1$.

From (6.13) we see that when $\mathscr{C} \gg 1$ instability due to differential rotation occurs if

$$-\partial(\ln\Omega)/\partial\Theta > (V^2/\Omega^2r^2)\,\mathrm{cosec}\,2\Theta, \tag{6.16}$$

i.e. if the angular velocity decreases slightly with r along a spherical surface. The maximum growth rate is roughly

$$-\tfrac{1}{2}\sin\Theta\,\partial\Omega/\partial\Theta \tag{6.17}$$

† We could somewhat artificially introduce spherical boundaries at r and $r+d$, where $d \ll r$, but this really only postpones the usual problem with all local analyses of this type of what *is* the largest allowable value of λ. It should certainly not exceed a scale height, for example, if the analysis is to be self-consistent.

and occurs for an azimuthal wavenumber

$$m = \frac{\Omega r}{V}\left(-\frac{\tan\Theta}{\Omega}\frac{\partial\Omega}{\partial\Theta}\right)^{\frac{1}{2}}\cos\Theta, \tag{6.18}$$

although the mode most easily generated is $m = 1$.

Instability due to a magnetic flux gradient occurs, provided $\mathscr{C} \gg 1$, if $B^2/\tan\Theta\sin^2\Theta$ increases with r along a spherical surface, and again the mode most easily generated is $m = 1$. The mechanism is not, of course, magnetic buoyancy, and the kinetic energy of the perturbations comes directly from the magnetic energy of the field itself. The maximum growth rate is roughly

$$\frac{V^2}{2\Omega r^2}\cos\Theta\tan\Theta\frac{\partial}{\partial\Theta}\ln\left(\frac{B}{\sin\Theta}\right) \tag{6.19}$$

and occurs for an azimuthal wavenumber

$$m = \left\{\frac{\tan\Theta}{\sec^2\Theta}\frac{\partial}{\partial\Theta}\ln\left(\frac{B}{\sin\Theta}\right)\right\}^{\frac{1}{2}}, \tag{6.20}$$

although this is again only a rough estimate, because m ought to be an integer.

7. Non-axisymmetric instability by fast heat diffusion

Even with a strong stratification such that $N^2 \gg \Omega^2$ it may be possible for three-dimensional (i.e. not purely horizontal) motions to satisfy the inequality (5.10), which is necessary for the validity of our low-frequency approximation procedure, provided that $|\omega| \ll \kappa s^2$. Fluid parcels then rapidly adjust their temperature to that of the surroundings, thus annulling much of their buoyancy. Equation (5.5) reduces to

$$\omega = \left(\frac{G}{a^2}-\frac{2}{r}+\frac{1}{a}\frac{\partial a}{\partial h}\right)\frac{mV^2}{2\Omega r}-\mathrm{i}\eta s^2 \pm \frac{mV}{2\Omega r}\left[\left(\frac{G}{a^2}-\frac{2}{r}\right)V^2\frac{\partial F}{\partial h}+\Omega^2 r\frac{\partial}{\partial h}(\ln\Omega^2)+\frac{m^2V^2}{r^2}\frac{s^2}{n^2}\right.$$
$$\left.+\frac{\eta}{\kappa}G\frac{\partial E}{\partial h}+V^2\left(\frac{1}{a}\frac{\partial a}{\partial h}\right)^2-\frac{\mathrm{i}\omega}{\kappa s^2}G\frac{\partial E}{\partial h}\right]^{\frac{1}{2}}, \tag{7.1}$$

and the important inequality (5.10) may be conveniently written

$$\frac{|\omega|}{\kappa s^2} \ll \frac{\Omega^2}{N^2}, \quad \frac{\eta}{\kappa} \ll \frac{\Omega^2}{N^2}. \tag{7.2), (7.3}$$

We seek the state of marginal stability, for which ω is real, and introduce X and Y by writing (7.1) in the form

$$\omega - Y + \mathrm{i}\eta s^2 = \pm\frac{mV}{2\Omega r}\left(X-\frac{\mathrm{i}\omega}{\kappa s^2}\gamma N_h^2\right)^{\frac{1}{2}}, \tag{7.4}$$

where we have defined

$$N_h^2 \equiv (G/\gamma)\,\partial E/\partial h. \tag{7.5}$$

Squaring both sides of (7.4) and taking real and imaginary parts we find

$$(\omega - Y)^2 - \eta^2 s^4 = (m^2V^2/4\Omega^2r^2)\,X \tag{7.6}$$

and

$$\omega\left(1+\frac{\gamma N_h^2 m^2V^2}{8\Omega^2r^2\kappa\eta s^4}\right) = Y. \tag{7.7}$$

Using (7.7) to eliminate ω from (7.6) and recalling the definitions of X and Y, we thus find instability if

$$-\Omega^2 r\frac{\partial}{\partial h}(\ln\Omega^2)-\left(\frac{G}{a^2}-\frac{2}{r}\right)V^2\frac{\partial F}{\partial h}>\frac{m^2V^2}{r^2}\frac{s^2}{n^2}+\frac{4\Omega^2r^2\eta^2s^4}{m^2V^2}+V^2\left(\frac{1}{a}\frac{\partial a}{\partial h}\right)^2$$
$$+\frac{\eta}{\kappa}\gamma N_h^2-\left(\frac{G}{a^2}-\frac{2}{r}+\frac{1}{a}\frac{\partial a}{\partial h}\right)^2V^2\Big/\left[1+\frac{8\Omega^2r^2\eta\kappa s^4}{m^2V^2\gamma N_h^2}\right]^2 \qquad (7.8)$$

Assuming for simplicity that $a^2 \sim gr$, we find on substituting (7.7) directly back into (7.2) that (7.2) is automatically satisfied when

$$\eta s^2/\Omega \ll 1, \qquad (7.9)$$

which we already has as a restriction from the second of (5.9). It is also instructive to record the order of magnitude of the parameter which decides whether or not the modes are fast diffusers of magnetic flux as well as of heat:

$$\frac{\eta s^2}{|\omega|}\sim\frac{mN^2}{\Omega\kappa s^2}+\frac{\Omega\eta r^2s^2}{mV^2}. \qquad (7.10)$$

In order to obtain some clear-cut results we confine attention to a cylindrically symmetric configuration and the equatorial plane ($g_z = 0$), and rewrite (7.8) in the following way:

$$\Delta\equiv-\left(\frac{\Omega^2r^2}{V^2}\right)r\frac{\mathrm{d}}{\mathrm{d}r}(\ln\Omega^2)-\left(\frac{g_r}{a^2}-\frac{2}{r}\right)r^2\frac{\mathrm{d}F}{\mathrm{d}r}>m^2\frac{s^2}{n^2}+\frac{(sd)^4}{m^2\mathscr{C}^2}+\left(\frac{r}{a}\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2$$
$$+\frac{1}{D}-r^2\left(\frac{g_r}{a^2}-\frac{2}{r}+\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2\Big/\left(1+\frac{2Ds^4d^4}{m^2\mathscr{C}^2}\right)^2, \qquad (7.11)$$

where the important dimensionless parameter D is defined as follows:

$$D\equiv\frac{\kappa}{\eta}\frac{V^2}{\gamma N_r^2r^2}, \qquad \mathscr{C}\equiv\frac{V^2}{2\Omega\eta}\left(\frac{d^2}{r^2}\right), \qquad (7.12)$$

and we have noted again the definition of $\mathscr{C}$, as by (6.12), for convenience.

The question we are asking is, given all the other local parameters of the system, what is the smallest *gradient* Δ, whether of magnetic field or angular velocity, that will give instability? Three general points are worth making: First, the minimum value of Δ is always achieved by taking l as small as possible, merely by inspection of (7.11). Second, the smallest possible value of l is π/d, by definition, so that in no circumstances can the combination sd be much smaller than unity (though it could be large). Finally, we note that the last two terms of (7.11) owe their presence entirely to finite thermal diffusivity and vanish when $D = \infty$. The first is positive and stabilizing while the second is negative and *de*stabilizing. It is natural, therefore, to expect quite complicated changes in the stability properties of the system as we change D.

Infinitely fast heat diffusion; $D = \infty$

The last two terms of (7.11) vanish and when $\mathscr{C} \gg 1$ the value of Δ is minimized by taking

$$n = (\mathscr{C}^2/2\pi^4)^{\frac{1}{6}}\,\pi/d, \quad m = 1. \qquad (7.13)$$

(The fact that m is quantized is crucial here; a minimization of Δ with respect to *both* m and n by differentiation attributes to m an impossibility low value, since $\mathscr{C} \gg 1$.) This gives the instability criterion

$$\Delta > 1+\left(\frac{r}{a}\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2+O(\mathscr{C}^{-\frac{2}{3}}), \qquad (7.14)$$

and by (7.7) the frequency at the onset of instability is

$$\omega = \left(\frac{g_r}{a^2} - \frac{2}{r} + \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}r}\right)\frac{V^2}{2\Omega r}. \tag{7.15}$$

The criterion (7.14) is a slight generalization of that obtained by Acheson & Gibbons (1978), who considered the case of uniform rotation and an isothermal atmosphere (in addition to assuming $\kappa = \infty$), and also attempted to derive some results without resorting to the purely local treatment of this paper. The essential result was that despite the rapid uniform rotation of the system magnetically buoyant instability may still occur for a quite modest decrease of magnetic field with radius r, although a decrease of F, i.e. $B/\rho r$, is necessary (cf. (7.14) and (7.11)). The instability has, from (7.1), a growth rate of order $V^2/\Omega r^2$ and takes the form of eastward-propagating waves with azimuthal angular phase speed of the same order.

The study by Acheson & Gibbons (1978) extends that of Gilman (1970) in a way which has already been discussed in the Introduction. The important point in the present context is that both theories assume infinitely fast thermal diffusion, which totally wipes out the *conventionally* buoyant effects that would otherwise occur and makes for great simplification because the thermodynamic energy equation (2.6) is effectively dropped. Further, ohmic diffusion effects are largely ignored (except very briefly in Acheson & Gibbons 1978). Gilman (1970) actually presents an estimate, based on an elementary time-scale argument, for the approximate validity of those assumptions:

$$\eta s^2 \ll |\omega| \ll \kappa s^2, \tag{7.16}$$

which, applied to the rapidly-rotating system we are currently considering, gives

$$\eta s^2 \ll V^2/\Omega r^2 \ll \kappa s^2. \tag{7.17}$$

While intuitively (7.17) is certainly necessary, it can hardly be sufficient since it contains no reference to the potential strength of the buoyancy forces which the large κ is meant to be reducing. From the following discussion it will become clear that the condition $D \gg \mathscr{C}^2$ is required for the approximate validity of the '$\kappa = \infty$' theories at rapid rotation speeds, and these theories therefore fail, given everything else constant and finite, if the stratification is too strong.

Before proceeding it is useful to take stock of the various approximations we have made. They are (i) the low-frequency approximation in passing from (3.20) to (5.4), and (ii) the fast-thermal-diffusion approximation in passing from (5.5) to (7.1). The restrictions implied by these approximations are (5.1), (5.2), (5.3), (5.9) and (7.3). (Recall that (5.10) is covered by (7.2) and (7.3), and (7.2) is covered by (7.9), which is a less strict requirement than (5.9).) We shall take $a^2 \sim gr$ and the entropy scale height of order r for simplicity, and also take $m \sim 1$ because we shall find below that this is the important case. We shall also find that $\omega \lesssim V^2/\Omega r^2$, so that (5.3) gives a less strict requirement than (5.1). Inequalities (5.1), (5.9) and (7.3) are then satisfied if

$$V^2/r^2 \ll \Omega^2 \ll N^2, \tag{7.18}$$

$$\frac{V}{r} \gg \eta s^2, \quad \frac{\eta}{\kappa} \ll \frac{\Omega^2}{N^2}. \tag{7.19a, b}$$

If we assume, for the present, that the departure from uniform rotation is at most $O(V^2/\Omega^2 r^2)$, then (5.2) is encompassed by (7.18). The inequalities (7.19) may usefully be cast in terms of the parameters $\mathscr{C}$ and D:

$$\mathscr{C} \gg (s^2 d^2)\, V/\Omega r, \quad D \gg V^2/\Omega^2 r^2, \tag{7.20a, b}$$

and (7.18) and (7.20a, b) are the key approximations we have made. The parameters $\mathscr{C}$ and D may be thought of as measures (of a kind) of ohmic and thermal diffusion, and may be chosen independently of each other and of the diffusionless parameters in (7.18). In what follows we shall take $\mathscr{C}$ moderately large, and consider what happens as D is continually decreased from an initial value much larger than $\mathscr{C}^2$. ($\mathscr{C}$ must not be too large, in fact, if the analysis is to be self-consistent, but we shall return to this point later.) We may note that having chosen $\mathscr{C} \gg 1$ and $D \gtrsim 1$, (7.20a, b) is in fact automatically satisfied by virtue of (7.18), because although s^2d^2 is occasionally large in what follows, it is never so large that (7.20a) is violated.

Now consider D to be finite. The final (*de*stabilising) term in (7.11) can *at best* decrease Δ (from its $D = \infty$ value of (7.14)) by an amount of order unity, and this amount is biggest when $Ds^4d^4/m^2\mathscr{C}^2$ is small. When $D \gg \mathscr{C}^2$ it follows that $Ds^4d^4/m^2\mathscr{C}^2$ is *large*, so that this effect is very weak, since $sd \gtrsim 1$ and $m \sim 1$. (We evidently have no cause to consider $m \gg 1$, for although it may help diminish Δ by the above mechanism it does so at the absurd cost of making the first term on the right hand side of (7.11), m^2s^2/n^2, large, thus greatly hindering instability.) The fourth (stabilizing) term (D^{-1}) is also small when $D \gg 1$, so while $D \gg \mathscr{C}^2 \gg 1$ the thermal diffusion has only a very small effect on the $D = \infty$ instability criterion.

When $D \sim \mathscr{C}^2 \gg 1$ it is clear by inspection of (7.11) that m and n may now be chosen so that the final term can reduce Δ from its $D = \infty$ value by an $O(1)$ amount. The value of m must still be $O(1)$, or the first term on the right hand side of (7.11) will be large, and nd must also be $O(1)$ or $2Ds^4d^4/m^2\mathscr{C}^2$ will be large and the last term in (7.11) will be small. Actual values of m and n will depend upon the precise values of $D/\mathscr{C}^2$ and the numerator of the last term. Let us now pass on and reduce D still further.

The parameter range $\mathscr{C}^2 \gg D \gg 1$

It is clear in the paragraphs above why we shall consider $m \sim 1$. When $\mathscr{C}^2 \gg D$ it therefore follows that $Ds^4d^4/m^2\mathscr{C}^2 \ll 1$ unless $n^4d^4 \gtrsim \mathscr{C}^2/D$. Let us suppose nd were large in this way, so as to make $Ds^4d^4/m^2\mathscr{C}^2 \gtrsim 1$. The opportunity of decreasing Δ by an $O(1)$ number, via the last term of (7.11), would not have been used to best advantage, but would anything be gained elsewhere? The second term on the right hand side of (7.11) would have been made a lot *larger* than it was, and an insignificant (order $D^{\frac{1}{2}}/\mathscr{C}$ at most) decrease would have been effected in the first term, so the answer is no. Thus $Ds^4d^4/m^2\mathscr{C}^2 \ll 1$, and we continue our minimization procedure by expanding the last term of (7.11) binomially. Having done this it is quite helpful to rewrite the inequality in a rather different form:

$$-\left(\frac{\Omega^2r^2}{V^2}\right)r\frac{\mathrm{d}}{\mathrm{d}r}(\ln\Omega^2) - \left(\frac{g_r}{a^2}-\frac{2}{r}\right)r^2\left(\frac{\mathrm{d}F}{\mathrm{d}r}-\frac{g_r}{a^2}+\frac{2}{r}-\frac{2}{a}\frac{\mathrm{d}a}{\mathrm{d}r}\right) > \frac{m^2s^2}{n^2}+\frac{s^4d^4}{m^2\mathscr{C}^2}+\frac{1}{D}+r^2\left(\frac{g_r}{a^2}-\frac{2}{r}+\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2\frac{4Ds^4d^4}{m^2\mathscr{C}^2}. \tag{7.21}$$

The second term on the right hand side is negligible compared with the fourth, since $D \gg 1$, and since $\mathscr{C}^2 \gg D$ minimization takes place in exactly the same way as in the $D = \infty$ case (7.13), but with different numerical results:

$$n = \mathscr{C}^{\frac{1}{3}}(8\pi^4D)^{-\frac{1}{6}}\left|\frac{rg_r}{a^2}-2+\frac{r}{a}\frac{\mathrm{d}a}{\mathrm{d}r}\right|^{\frac{1}{3}}\frac{\pi}{d}, \quad m = 1. \tag{7.22}$$

The right hand side is then unity to a first approximation, yielding an instability criterion which can be simplified even further on using $F = \ln(B/\rho r)$ and noting that in the present parameter régime (5.1) the basic balance is almost hydrostatic so that (5.7) obtains. The final result is

$$-\left(\frac{\Omega^2r^2}{V^2}\right)r\frac{\mathrm{d}}{\mathrm{d}r}(\ln\Omega^2) - \left(\frac{g_r}{a^2}-\frac{2}{r}\right)r^2\frac{\mathrm{d}}{\mathrm{d}r}\ln(Br) > 1, \tag{7.23}$$

and from (7.7) the frequency at marginal stability is

$$\omega = \frac{2D}{\mathscr{C}^2}(n^4d^4)\left(\frac{g_r}{a^2}-\frac{2}{r}+\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}r}\right)\frac{V^2}{2\Omega r}. \tag{7.24}$$

Since from (7.22), nd is of order $(\mathscr{C}^2/D)^{\frac{1}{6}}$ we have $\omega \sim (D/\mathscr{C}^2)^{\frac{1}{3}}\, V^2/\Omega r^2$, and the frequency is thus much smaller than that $(V^2/\Omega r^2)$ when $D \gg \mathscr{C}^2$ (see (7.15)). It follows that $\omega/\eta s^2$ is of order $(D/\mathscr{C}^{\frac{1}{2}})^{\frac{2}{3}}$.

In the present régime, $\mathscr{C}^2 \gg D \gg 1$, the finite thermal diffusivity effects a very significant decrease in the right hand side of (7.11), via the last term, from its $D = \infty$ value. Perhaps this can best be seen in a simple example; we shall take the rotation to be uniform, and suppose an isothermal basic state with scale height $H = a^2/g_r$ rather small compared with r, so that (7.23) essentially becomes

$$\mathrm{d}(Br)/\mathrm{d}r < 0 \tag{7.25}$$

and is to be compared with the corresponding result (7.14) for $D = \infty$:

$$\mathrm{d}[B(\rho r)^{-1}]/\mathrm{d}r < 0. \tag{7.26}$$

Since the scale height is rather small, ρ will decrease quite rapidly with r, and to satisfy (7.26) B then has to decrease with r considerably faster than it does to satisfy (7.25). The distinction between the two is even more clear when curvature effects vanish, as in the plane layer model shortly to be considered.

Clearly Δ does not continue to decrease as D does once D is sufficiently small for the third term in (7.21) to be comparable with the fourth, and in view of (7.22) this occurs when $D \sim \mathscr{C}^{\frac{1}{2}}$. Notably, this point at which Δ begins to *increase* as D decreases is also the point at which the oscillation frequency ω drops below the ohmic decay rate ηs^2 (cf. discussion of doubly diffusive phenomena in appendix).

The parameter range $D \ll 1$

As D drops below a value of order unity the stratification (measured by D^{-1}) exerts a strongly stabilizing influence, for the fourth term on the right hand side of (7.11) begins to substantially exceed unity and the fifth term is powerless to off-set this effect. When $D \ll 1$ we find from (7.21), which still holds, that the fourth term on the right hand side is negligible compared with the second, and the critical mode is thus (7.13), as in the $D = \infty$ case. The criterion for instability, however, is then essentially

$$-\left(\frac{\Omega^2r^2}{V^2}\right)r\frac{\mathrm{d}}{\mathrm{d}r}\ln\Omega^2-\left(\frac{g_r}{a^2}-\frac{2}{r}\right)r^2\frac{\mathrm{d}}{\mathrm{d}r}\ln B > \frac{\eta}{\kappa}\frac{\gamma N_r^2r^2}{V^2}, \tag{7.27}$$

so that without an extremely steep gradient of magnetic flux magnetic buoyancy will not give rise to instability. A decrease in *angular velocity* with radius might suffice:

$$-r\frac{\mathrm{d}}{\mathrm{d}r}\ln\Omega^2 > \frac{\eta}{\kappa}\frac{\gamma N_r^2}{\Omega^2}, \tag{7.28}$$

and this doubly diffusive instability criterion has a rather novel form, when compared with (1.3) (say), in that the magnetic diffusivity η appears where one might (naively) expect the viscosity ν. This serves as a reminder that although the magnetic field does not explicitly appear in (7.28) it is only by virtue of its presence that an angular velocity gradient satisfying (7.28) would give rise to instability (and recall, also, that we are assuming throughout this section that $\mathscr{C} \gg 1$).

It was remarked earlier that for the self-consistency of the above analysis $\mathscr{C}$ should not be too large, and the reason is that the minimization procedures have involved attributing significance to small terms on the right hand side of (7.11) and (7.21), and we must therefore make sure that such terms greatly exceed the small errors in these expressions due to the low-frequency and fast-thermal-diffusion approximations. We can give upper bounds to these errors, bearing in mind (i) we have found above that m is $O(1)$ at marginal stability, (ii) ω does not exceed $O(V^2/\Omega r^2)$ (see (7.7)), and (iii) we are interested in D greater than about unity (for if it is not, a good approximation to the right hand side of (7.11) is simply D^{-1}). The errors made by the various approximations in passing from (3.20) to (5.4) may be seen to be *at most* of order max $(V^2/\Omega^2r^2, \omega\eta s^2r^2/V^2)$ in these circumstances. The error made by the fast-thermal-diffusion approximation, i.e. passing from (5.5) to (7.1), is $O(\omega/\kappa s^2)$. We can now go back to (7.14) and (7.21), therefore, and check for the self-consistency of the minimization procedures; this turns out to be guaranteed provided that $\mathscr{C}^{\frac{2}{3}} \ll \Omega^2r^2/V^2$, which is only a very weak restriction on the size of $\mathscr{C}$.

The results established so far in this section are illustrated very schematically in figure 1 on page 493. The first significant effect of decreasing the thermal diffusivity from a very large value, or equivalently of *increasing the* ('bottom-heavy') *stratification* from a value such that $D_*^{-1} \ll \mathscr{C}_*^{-2}$, is to *destabilize* the system, in the sense that a less pronounced magnetic field or angular velocity gradient is needed for instability. Eventually, however, when the stratification is large enough (i.e. D small enough) it plays its more usual rôle as a stabilizing mechanism. Also on the figure are indicated the results established in the related studies of Gilman (1970), Roberts & Stewartson (1977) and Acheson & Gibbons (1978). The first two of these were derived on the basis of a plane layer model, so we now briefly discuss the corresponding results to those above for such a model.

A plane layer model

We confine attention to the uniformly rotating and isothermal case, with a small scale height $H = a^2/g_r \ll r$ and curvature effects neglected. This is effected by replacing the combination m/r by an equivalent plane wavenumber k in (7.11) but otherwise letting $r \to \infty$. To avoid confusion we continue to use $\mathrm{d}/\mathrm{d}r$, but this is to mean simply differentiation with respect to the vertical coordinate. We are considering, therefore, the locally plane layer limit of our cylindrically symmetric and equatorially located ($g_z = 0$) model, the stability of which is governed by (7.11). If we introduce the dimensionless wavenumbers $k' = kH$, $l' = ld$, $n' = nd$ and $s' = sd$, we can then write the planar equivalent of (7.11) in the following way, on dropping primes:

$$-H\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(\frac{B}{\rho}\right) > k^2\frac{s^2}{n^2}+\frac{s^4}{\mathscr{C}_*^2k^2}+\frac{1}{D_*}-\left(1+\frac{2D_*s^4}{\mathscr{C}_*^2k^2}\right)^{-2}. \tag{7.29}$$

Here $\mathscr{C}_*$ and D_* are identical with $\mathscr{C}$ and D as defined by (7.12), except that the scale height H replaces r where it occurs. We suppose $\mathscr{C}_* \gg 1$. The differences between the results of this model and those of the one considered above arise primarily because there is no lower limit on k here corresponding to that (of unity) on m in the previous case.

When $D_* = \infty$ the last two terms of (7.29) vanish, and instability first occurs for

$$n^2 = \tfrac{1}{2}\pi^2, \quad k^2 = \tfrac{\sqrt{3}}{2}\pi^2/\mathscr{C}_* \tag{7.30}$$

when

$$-H\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(\frac{B}{\rho}\right) > \frac{3\pi^2\sqrt{3}}{\mathscr{C}_*} \tag{7.31}$$

(Acheson & Gibbons 1978). The frequency at marginal stability is $kV^2/2\Omega H^2$. The mechanism of stabilization is conceptually quite different to that in (7.14); there the 'elasticity' of the field lines was called into play by their twisting and ohmic effects were negligible. Here the twisting of the field lines can be made arbitrarily weak, but the growth rate then eventually (by (7.1)) falls below the ohmic decay rate, so that ohmic effects provide the stabilizing influence in (7.31). Note from the second of (7.30) that k at marginal stability is small, since $\mathscr{C}_* \gg 1$.

When D_* is finite inspection of (7.29) reveals that stratification has little effect until D_* drops to about $\mathscr{C}_*^{\frac{2}{3}}$. Similar arguments to those used earlier then show that, via the fourth term in (7.29), it has a significant *destabilizing* effect. This is particularly evident when $\mathscr{C}_*^{\frac{2}{3}} \gg D_* \gg 1$, in which case we deduce from (7.29) that instability sets in when

$$-H\frac{\mathrm{d}}{\mathrm{d}r}\ln B > 6\sqrt{3}\,\pi^2\frac{D_*^{\frac{1}{2}}}{\mathscr{C}_*}+\frac{1}{D_*}, \tag{7.32}$$

the critical mode being $$n^2 = \tfrac{1}{2}\pi^2, \quad k^2 = \sqrt{3}\pi^2 D_*^{\frac{1}{2}}/\mathscr{C}_*. \tag{7.33}$$

Note that the azimuthal wavenumber has increased (cf. (7.30)). The frequency at marginal stability is

$$\omega = \frac{9\pi^3}{3^{\frac{1}{4}}}\left(\frac{D_*}{\mathscr{C}_*^{\frac{2}{3}}}\right)^{\frac{3}{4}}\frac{V^2}{4\Omega H^2}, \tag{7.34}$$

so $\omega/\eta s^2 \sim (D_*/\mathscr{C}_*^{\frac{2}{3}})^{\frac{3}{4}}$.

Bearing in mind the approximations involved, we see that (7.31) essentially says that B/ρ must decrease with height, while (7.32) simply says that B must decrease with height, and this shows how much the effects of stratification have destabilized the system.

As D_* decreases from $\mathscr{C}_*^{\frac{2}{3}}$ towards unity it is clear, however, from (7.32) that this curious behaviour does not continue. The right hand side of (7.32) has a minimum when $D_* = (\mathscr{C}_*/3\pi^2\sqrt{3})^{\frac{2}{3}}$, and a further decrease of D_* has a stabilizing effect as the stratification begins to play its more usual rôle. As in the cylindrical case, the point at which this happens is just that at which the oscillation frequency ω drops below the ohmic decay rate, although here that point is $D_* \sim \mathscr{C}_*^{\frac{2}{3}}$, while in the cylindrical case it is $D \sim \mathscr{C}^{\frac{1}{2}}$, the difference arising because m is quantized in that case. This does not *explain*, in any sense, the curious response of this system to stratification, but it does at least tie in with the author's expectations on the basis of other doubly diffusive problems (see the appendix), and can be seen to have an important effect on (5.5). The curious rôle of stratification undoubtedly arises from the fact that the very last term in the radical may be significantly *complex*, but when ω is small compared with *both* diffusion rates ηs^2 and κs^2 it simply plays a stabilizing rôle by providing an extra positive contribution of order D_*^{-1} compared with those already present.

Finally, let us return to the régime $D_* \sim \mathscr{C}_*^{\frac{2}{3}}$. Inspection of (7.29) reveals that the minimization of the right hand side is achieved, as (7.33) would suggest, by taking k and n of order unity. By such a choice the second and third terms are rendered negligible, while the destabilizing potential of the fourth term is well exploited. It could be exploited better still by taking k large, but the effect of this would be more than off-set by the concomitant increase in the first term.

In order to relate the above theory to that of Roberts & Stewartson (1977) we finally take, as did those authors, a basic state of *constant Alfvén speed*, $B \propto \rho^{\frac{1}{2}}$. In that case the left hand side of (7.29) is approximately $-\frac{1}{2}$, and instability evidently cannot occur in this *rapidly rotating* system if $D_* = \infty$, as Gilman (1970) found. When $\mathscr{C}_*^{\frac{2}{3}} \sim D_*$ the neglect, as discussed in the previous paragraph, of the second and third terms on the right hand side of (7.29) and a little re-arrangement leads to

$$\frac{\mathscr{C}_*^2}{2D_*} > \frac{s^4/k^2}{(\frac{1}{2}+k^2s^2/n^2)^{-\frac{1}{2}}-1} \tag{7.35}$$

as the condition for instability. Roberts & Stewartson's analysis is *not* restricted, as ours is from the outset, to low frequency solutions in the rapidly rotating $V^2 \ll \Omega^2H^2$ limit, but if we take this special limit in their equations (3.3) and (3.20) (in their notation the limit is $\lambda \gg 1$, $\omega_0 \sim \lambda^{-1}$) we obtain precisely (7.35). With our notation but their result (the last entry in their table 1) the critical value below which κ has to fall for, as they term it, a 'conductive' instability, is given by

$$\mathscr{C}_*^2/D_* \approx 344\pi^4 \quad \text{for} \quad k = 0.27, l = \pi, n = \pi/\sqrt{2}, \tag{7.36}$$

and this corresponds to travelling from A to B along a horizontal line† in figure 1 (on page 493).

Some insight into the instability when $\mathscr{C}_*^2 \sim D_*$ may be obtained by noting from (7.7) that since (reverting to dimensional wavenumbers) $k \sim H^{-1}$ and $l \sim n \sim d^{-1}, \omega \sim V^2/\Omega H^2$. In the plane layer, uniform rotation and isothermal limit (7.1) may be written

$$\omega = \frac{kV^2}{2\Omega H} - \mathrm{i}\eta s^2 \pm \frac{Vk}{2\Omega}\left[\frac{V^2}{H}\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(\frac{B}{\rho}\right) + V^2k^2\frac{s^2}{n^2} + \frac{\eta}{\kappa}\gamma N^2 - \frac{\mathrm{i}\omega}{\kappa s^2}\gamma N^2\right]^{\frac{1}{2}}, \tag{7.37}$$

and if B/ρ does not decrease with height the system is stable when $\kappa = \infty$. Further, when $\kappa \neq \infty$ instability can then arise solely through the final (imaginary) term in square brackets. Now in the present régime the third and fourth terms in square brackets are order $\mathscr{C}_*^{-2}$ and $\mathscr{C}_*^{-1}$ respectively compared to the first two, and expanding the square brackets binomially we thus obtain a small growth rate term, $O(\eta d^{-2})$, which given precisely the right circumstances indicated by (7.36) is large enough to off-set the term in (7.37) representing ohmic dissipation. Thus although instability may occur for smaller magnetic field gradients than are needed when $D_* \gg \mathscr{C}_*^2$, one should note that the growth rates are much smaller, by a factor of order $\mathscr{C}_*$, than those $(O(V^2/\Omega H^2))$ that result when B/ρ decreases with height so as to make the real part of the expression in square brackets in (7.37) negative.

8. Magnetic instabilities of a non-rotating gas

Our main interest throughout most of this paper has been the instability of rapidly rotating $(V^2 \ll \Omega^2r^2)$ systems. We now discuss non-axisymmetric instabilities in *non*-rotating systems, the corresponding axisymmetric results already being available in §4.

Diffusionless theory

When $\nu = \eta = \kappa = 0$ and $\Omega = 0$ we obtain from (3.20) an equation which is biquadratic in ω (provided $m \neq 0$). Letting $\omega = \mathrm{i}q$ it becomes a biquadratic in q with real coefficients, and it follows that a sufficient condition for instability is that the constant term is negative, i.e.

$$\left(\frac{G}{\gamma a^2}-\frac{2}{r}\right)V^2\frac{\partial Q}{\partial h}+\frac{m^2V^2s^2}{r^2\ n^2}+\frac{G}{\gamma}\frac{\partial E}{\partial h} < 0, \tag{8.1}$$

where we recall that

$$Q \equiv \ln(Br). \tag{8.2}$$

† One cannot 'travel' much to the right of B and eventually find stability again with the Roberts & Stewartson analysis, which involves an expansion in their small parameter δ for fixed (formally $O(1)$) values of their parameters S and λ. This implies $\delta \sim D_*^{-\frac{1}{2}}$ in our terminology, so that the expansion procedure breaks down once D_* ceases to be large.

The left hand side of (8.1) is quadratic in l/n, and we thus obtain by the standard procedure employed in this paper the result that the system is unstable if any of the following conditions are satisfied:

$$\left(\frac{g_r}{\gamma a^2}-\frac{2}{r}\right) V^2\frac{\partial Q}{\partial r}+\frac{g_r}{\gamma}\frac{\partial E}{\partial r}+\frac{m^2V^2}{r^2} < 0, \tag{8.3}$$

$$\frac{g_z}{\gamma a^2} V^2\frac{\partial Q}{\partial z}+\frac{g_z}{\gamma}\frac{\partial E}{\partial z}+\frac{m^2V^2}{r^2} < 0, \tag{8.4}$$

$$\begin{aligned}\frac{1}{4}\left[\left(\frac{g_r}{\gamma a^2}-\frac{2}{r}\right) V^2\frac{\partial Q}{\partial z}+\frac{g_z}{\gamma a^2} V^2\frac{\partial Q}{\partial r}+\frac{g_z}{\gamma}\frac{\partial E}{\partial r}+\frac{g_r}{\gamma}\frac{\partial E}{\partial z}\right]^2 \\ > \left[\frac{g_z}{\gamma a^2} V^2\frac{\partial Q}{\partial z}+\frac{g_z}{\gamma}\frac{\partial E}{\partial z}+\frac{m^2V^2}{r^2}\right]\left[\left(\frac{g_r}{\gamma a^2}-\frac{2}{r}\right) V^2\frac{\partial Q}{\partial r}+\frac{g_r}{\gamma}\frac{\partial E}{\partial r}+\frac{m^2V^2}{r^2}\right]\end{aligned} \tag{8.5}$$

Instability occurs most easily for the $m = 1$ mode, and these conditions have been obtained by Tayler (1973) (see his equations (2.20)–(2.22)). In addition, and more significantly, he showed that the satisfaction of at least one of these is *necessary* for instability of the system by applying the energy principle of Bernstein, Frieman, Kruskal & Kulsrud (1958). Comparison of (8.3)–(8.5) with the corresponding axisymmetric criteria (4.3)–(4.5) (setting $\Omega = 0$) is interesting, for it shows how Q has effectively replaced F as the key quantity as far as the instability is concerned. In particular if $a^2/g \ll r$ and $V^2 \lesssim a^2$, then (4.3) becomes, in the non-rotating case,

$$-\frac{g_r}{\gamma a^2}\frac{\partial}{\partial r}\ln\left(\frac{B}{\rho r}\right) > \frac{N_r^2}{V^2}, \tag{8.6}$$

while (8.3) becomes

$$-\frac{g_r}{\gamma a^2}\frac{\partial}{\partial r}\ln(Br) > \frac{m^2}{r^2}+\frac{N_r^2}{V^2}. \tag{8.7}$$

Because $a^2/g \ll r$, the over-riding difference between the two left hand sides above is the presence of ρ (which will have scale height of order $H \equiv a^2/g$) in the first but not in the second, so that instability (due to magnetic buoyancy) occurs most readily in a non-axisymmetric way. The stratification in the radiative zone of a typical star, however, is such that $N^2H^2 \gg V^2$ and in general exerts a very strongly stabilizing influence. Tayler (1973) showed that instability will nevertheless always occur *sufficiently close to the axis of symmetry of the star* if there is a non-zero electric current density on that axis. We now investigate an alternative possibility, namely the relaxation of buoyancy forces by doubly diffusive effects.

Doubly diffusive reduction of buoyancy effects

We now restore all three diffusive processes and set $\omega = 0$ as well as $\Omega = 0$ in (3.20), thus obtaining

$$-\left(\frac{G}{a^2}-\frac{2}{r}\right) V^2\frac{\partial Q}{\partial h} > \frac{s^2}{n^2}\frac{m^2V^2}{r^2}+\frac{\eta}{\kappa}G\frac{\partial E}{\partial h}+\nu\eta\frac{s^6}{n^2}+V^2\left(\frac{2}{r}-\frac{G}{a^2}\right)^2\left(1+\frac{m^2V^2}{r^2\nu\eta s^4}\right)^{-1} \tag{8.8}$$

as the inequality governing the onset of monotonic instability. By the same procedure as usual we find that in the absence of viscous effects the system is unstable if any of the following conditions are satisfied:

$$\left(\frac{g_r}{a^2}-\frac{2}{r}\right) V^2\frac{\partial Q}{\partial r}+\frac{\eta}{\kappa}g_r\frac{\partial E}{\partial r}+\frac{m^2V^2}{r^2} < 0, \tag{8.9}$$

$$g_z\frac{V^2}{a^2}\frac{\partial Q}{\partial z}+\frac{\eta}{\kappa}g_z\frac{\partial E}{\partial z}+\frac{m^2V^2}{r^2} < 0, \tag{8.10}$$

$$\frac{1}{4}\left[\left(\frac{g_r}{a^2}-\frac{2}{r}\right)V^2\frac{\partial Q}{\partial z}+\frac{g_z}{a^2}V^2\frac{\partial Q}{\partial r}+\frac{\eta}{\kappa}g_z\frac{\partial E}{\partial r}+\frac{\eta}{\kappa}g_r\frac{\partial E}{\partial z}\right]^2$$
$$>\left[\frac{g_z}{a^2}V^2\frac{\partial Q}{\partial z}+\frac{\eta}{\kappa}g_z\frac{\partial E}{\partial z}+\frac{m^2V^2}{r^2}\right]\left[\left(\frac{g_r}{a^2}-\frac{2}{r}\right)V^2\frac{\partial Q}{\partial r}+\frac{\eta}{\kappa}g_r\frac{\partial E}{\partial r}+\frac{m^2V^2}{r^2}\right]. \quad (8.11)$$

Comparison of these criteria with (8.3)–(8.5) immediately reveals the new effects; adiabatic sound speeds have simply been replaced by isothermal ones, and the effectiveness of any stratification has been reduced by a factor $\gamma\eta/\kappa$ due to doubly diffusive effects. The quantity Q remains the important one for instability, but when r is sufficiently large, in the sense $a^2/g \ll r$ and $B(\partial B/\partial r)^{-1} \ll r$, curvature effects are negligible and, on using the notation $k \equiv m/r$, we have the planar analogue of (8.9):

$$-\frac{g_r}{a^2}\frac{\partial}{\partial r}\ln B > k^2+\frac{\eta}{\kappa}\frac{\gamma N_r^2}{V^2}. \quad (8.12)$$

With $\kappa = \infty$ this reduces to the criterion for magnetic buoyancy instability obtained by Gilman (1970). A decrease of magnetic *field*, rather than (as required when $k = 0$) magnetic *flux* B/ρ with height is the essential requirement for instability, which takes place most readily for (indefinitely) small but non-zero k. The reason is simply that a little twisting of the lines of force helps magnetic buoyancy instability, since it permits the flow of gas down the rising portions of the distorted flux tubes to the sinking portions, which enhances the magnetic buoyancy effect (Parker 1955), while too much twisting (i.e. k too large) results in the restoring forces arising from the 'elasticity' of the field lines outweighing the magnetic buoyancy effects.

When κ is *finite*, however, apparent once again is the definite limit to the extent to which it can relax the conventional buoyancy forces, and only a very steep magnetic field gradient will give magnetic buoyancy instability unless

$$D_* \gtrsim 1, \quad (8.13)$$

a familiar criterion from §7. Roberts & Stewartson (1977) have also noted this result in their isothermal and constant-Alfvén-speed model, and the $k \to 0$ limit of (8.12) reduces to their equation (A 11) in that case. In their plane layer model viscous effects place a lower bound on the wavenumber k for which instability can occur. If the viscosity is sufficiently small, however, a situation analogous to that in the $D = \infty$ subsection of §7 (see also Acheson & Gibbons 1978) then arises, and the *geometry* imposes the lower limit (of unity) on m. Thus in a cylindrically symmetric configuration we find from (8.8) that when

$$\chi \equiv \frac{V^2d^4}{\nu\eta r^2}\left[1+\left(2-\frac{r}{H}\right)^2\right]^{-1} \quad (8.14)$$

is large instability sets in essentially when (8.9) is satisfied, the critical mode being

$$l = \frac{\pi}{d}, \quad m = 1, \quad n = \left(\frac{\chi}{2\pi^4}\right)^{\frac{1}{6}}\frac{\pi}{d}. \quad (8.15)$$

Almost horizontal motions

Even in the presence of a vertical stratification so strong that (8.13) is violated, so that neither adiabatic nor large-thermal-diffusion magnetic buoyancy instability can occur, there remains of course the possibility that a toroidal magnetic field distribution may be unstable to almost horizontal motions, for they encounter only very weak buoyancy forces.

We consider, in the first instance, adiabatic *axisymmetric* perturbations, and know by setting $\Omega = 0$ in (4.5) that instability will result if

$$g_z\left(\frac{\partial E}{\partial r}\frac{\partial F}{\partial z}-\frac{\partial E}{\partial z}\frac{\partial F}{\partial r}\right) < 0, \tag{8.16}$$

i.e. if $B/\rho r$ increases outward along a surface of constant specific entropy. This is evidently a magnetic analogue of the Høiland criterion for axisymmetric *baroclinic* instability, and we are immediately moved by subsequent developments in that theory to examine *non*-axisymmetric disturbances. The appropriate criterion is (8.5), and we shall re-write it in the following way:

$$\frac{1}{4}\left[\left(\frac{g_r}{\gamma a^2}-\frac{2}{r}\right)V^2\frac{\partial Q}{\partial z}-\frac{g_z}{\gamma a^2}V^2\frac{\partial Q}{\partial r}+\frac{g_r}{\gamma}\frac{\partial E}{\partial z}-\frac{g_z}{\gamma}\frac{\partial E}{\partial r}\right]^2$$
$$> \frac{2V^2}{r}\frac{g_z}{\gamma}\left(\frac{\partial E}{\partial r}\frac{\partial Q}{\partial z}-\frac{\partial E}{\partial z}\frac{\partial Q}{\partial r}\right)+\frac{m^4V^4}{r^4}+\frac{m^2V^2}{r^2}\left[\left(\frac{g_r}{\gamma a^2}-\frac{2}{r}\right)V^2\frac{\partial Q}{\partial r}+\frac{g_z}{\gamma a^2}V^2\frac{\partial Q}{\partial z}+\frac{g_r}{\gamma}\frac{\partial E}{\partial r}+\frac{g_z}{\gamma}\frac{\partial E}{\partial z}\right], \tag{8.17}$$

Although further simplification is afforded by the fact that the left hand side vanishes, as may be seen from (2.20), in order to obtain some physically intelligible results we confine attention to the case $V^2 \ll a^2 \sim gr$, in which event the magnetic field causes only a small asymmetry to the system and, by (2.11) and (2.20), surfaces of constant density and specific entropy are almost spherical. The terms involving the entropy gradient on the right hand side of (8.17) are then bigger by a factor $O(a^2/V^2)$ than those which do not, and we have as the criterion for instability to non-axisymmetric disturbances

$$\frac{2g_z}{r}\left(\frac{\partial E}{\partial r}\frac{\partial Q}{\partial z}-\frac{\partial E}{\partial z}\frac{\partial Q}{\partial r}\right)+\frac{m^2}{r^2}\left(g_r\frac{\partial E}{\partial r}+g_z\frac{\partial E}{\partial z}\right) < 0. \tag{8.18}$$

This in turn is more easily written in the form

$$\partial Q/\partial \Theta > m^2/\sin 2\Theta, \tag{8.19}$$

where we have switched to spherical polar coordinates and Θ is the polar angle. Clearly the most easily excited mode is $m = 1$. With a little algebra we may show from (8.19) that instability occurs in this non-axisymmetric way if $B^2 \sin 2\Theta$ increases outward along a spherical surface.

This instability of a toroidal field structure subjected to a large stratification is not, of course, magnetically buoyant in origin; gravitational effects play essentially no rôle except for dictating firmly the very small class of disturbances to which the system is unstable. Criterion (8.19) can alternatively be derived directly from (8.1) (or indeed from (8.8)) by asking in what circumstances instability arises with modes having almost horizontal motions such that $\partial E/\partial h = 0$ and $G \approx 0$.

In the same circumstances $V^2 \gg a^2 \sim gr$ and expressed in the same terms, the criterion for axisymmetric instability (8.16) is that $B^2/\sin^2\Theta$ should increase outward along a spherical surface. A little algebra then shows that non-axisymmetric modes are more easily excited (i.e. by weaker magnetic field gradients) for $\Theta < \frac{1}{3}\pi$ but that axisymmetric modes occur more readily in the equatorial region $\Theta > \frac{1}{3}\pi$.

9. Concluding discussion

I should like to briefly summarize what I believe to be the more significant results of this paper on magnetic buoyancy instability. For simplicity I shall, for the most part, exclude extreme values of the magnetic field gradient from the discussion, envisaging the scale height of B as roughly H or r, as the case may be. It should be borne in mind that the stability criteria quoted will frequently be first (but perfectly satisfactory) approximations to the full (often more unwieldy) inequalities. Statements concerning dependence on z will be framed from the viewpoint of an observer in the northern hemisphere.

No stratification

We have in mind application to convective systems which have established almost adiabatic temperature gradients, and we consider the case of small scale height $H \ll r$, as in the upper layers of the Sun.

In the *non-rotating* case either a decrease of Br with r or of B with z gives non-axisymmetric instability (see (8.3), (8.4)).

In the *rapidly rotating* case a decrease of B/ρ with either r or z gives non-axisymmetric instability in the form of amplifying waves with speed $O(V^2/\Omega H)$ (see (6.1)–(6.3)) provided that ohmic damping is not too heavy, and this requires

$$\mathscr{C}_* \equiv (V^2/2\Omega\eta)(d/H)^2 \tag{9.1}$$

to be greater than about unity. Circumstances are conceivable in which ν and η, though small, are so disparate as to render an axisymmetric mode more easily excited. This would require, very roughly, V^2/Ω^2H^2 to exceed ν/η or η/ν, whichever is the smaller, instability being oscillatory in the first case and monotonic in the second (see (A 5) and (A 11)).

Strong stratification ($V^2 \ll N^2r^2$)

We have in mind here the radiative interiors of stars, and take $H \sim r$ and $\nu \ll \eta \ll \kappa$. Fully three-dimensional adiabatic instabilities are virtually impossible, whether Ω is large or small, and instability modes may avoid the otherwise crushing effect of stratification only by either almost horizontal motions or a multiply-diffusive mechanism of some kind.

In the *non-rotating* case whether axisymmetric or non-axisymmetric modes involving almost horizontal motions occur most readily depends on latitude. If $\Theta > \frac{1}{3}\pi$ instability of the former kind is preferred and sets in when $B^2/\sin^2\Theta$ increases with Θ. If on the other hand $\Theta < \frac{1}{3}\pi$ non-axisymmetric instability is preferred and sets in when $B^2 \sin 2\Theta$ increases outward along a spherical surface. In both cases the mechanism is conceptually quite different from that of magnetic buoyancy, which thrives on a decrease of B in the *vertical* direction.

An alternative mechanism is the relaxation of conventionally buoyant restoring forces by fast thermal diffusion, as invoked by Gilman (1970) and Cadez (1974). This is effective provided

$$D \equiv \frac{\kappa}{\eta}\frac{V^2}{\gamma N^2 r^2} \tag{9.2}$$

is greater than about unity, but not otherwise. This parameter is totally analogous to the parameter α^{-1} that arises in Goldreich–Schubert instability (see (1.6)). It is a measure of the ratio between the extent to which thermal diffusion κ weakens the stabilizing effect of stratification N^2 and the extent to which the magnetic diffusivity η weakens the destabilizing effect of the magnetic field gradient. When $D \gtrsim 1$ magnetic instability occurs for a modest variation of B with r or z, but whether axisymmetric or non-axisymmetric modes are most easily excited depends in a complicated way on the particular configuration involved† (see (8.9), (8.10), (A 1), (A 2)). Thus curvature effects render Schubert's (1967) work on axisymmetric magnetic instabilities relevant to stellar interiors in a way that it is not at higher levels, where H is small and non-axisymmetric magnetic buoyancy is preferred (see (1.13) and (1.14)). The parameter D arises in both axisymmetric and non-axisymmetric cases, and if it is substantially smaller than unity buoyancy forces are only weakly relaxed by thermal diffusion and a large magnetic field gradient is needed for instability:

$$-\left(\frac{g_r}{a^2}-\frac{2}{r}\right)V^2\frac{\partial}{\partial r}\ln B > \frac{\eta}{\kappa}\gamma N_r^2 \quad \text{or} \quad -\frac{g_z}{a^2}V^2\frac{\partial}{\partial z}\ln B > \frac{\eta}{\kappa}\gamma N_z^2 \tag{9.3}$$

(see (8.9), (8.10), (A 1), (A 2)).

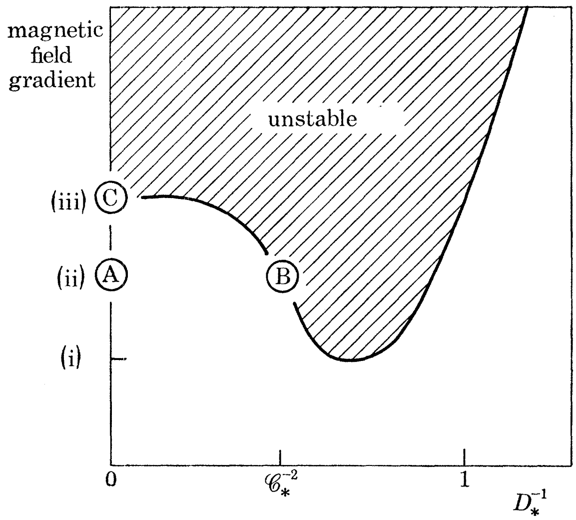

FIGURE 1. Magnetic instability in a rapidly rotating and strongly-stratified fluid. The magnitude of the field gradient (i.e. dimension of system ÷ magnetic field e-folding height) is here plotted against D_*^{-1}, a suitable measure of the stratification (proportional to N^2). The small parameter $\mathscr{C}_*^{-2}$ is proportional to Ω^2. The effects of stratification are virtually annulled by thermal diffusion until N^2 is large enough that $D_*^{-1} \sim \mathscr{C}_*^{-2}$. An increase of stratification beyond this value at first has a very significant *destabilizing* effect on the system, but as N^2 becomes larger still buoyancy forces play their more traditional stabilizing rôle. The points A, B and C locate previous studies, discussed in the text. The marked values of D_*^{-1} along the horizontal axis denote orders of magnitude only.

In the *rapidly rotating* case non-axisymmetric instability involving almost horizontal motions occurs if $B^2\,(\tan\Theta\sin^2\Theta)^{-1}$ increases outward along a spherical surface. It takes the form of westward-propagating waves with speed $O(V^2/\Omega r)$ (see (6.10)), but only occurs if the ohmic damping is not too large, i.e. if $\mathscr{C} \gtrsim 1$ (see (6.12)). Two alternative instabilities both invoke thermal diffusion to diminish the effectiveness of the stratification. The first is *axisymmetric*, and somewhat novel in that it calls for the simultaneous operation of two quite different doubly diffusive mechanisms. This stems from the fact that one of the stabilizing agents (N^2) has associated with it a large diffusivity ($\kappa \gg \eta$) while the other (Ω^2) has a small diffusivity ($\nu \ll \eta$) associated

† For an isothermal basic state, to take one example, non-axisymmetric instability is preferred for $r > 2H$ and requires a decrease of B with r or z. This is a magnetic buoyancy instability and the result coincides with Gilman's in the 'plane layer' limit $H \to 0$. Axisymmetric instability, on the other hand, is preferred for $H < r < 2H$, while for $r < H$ non-axisymmetric instability is preferred again (!), neither of these two being of a magnetically buoyant type.

with it. Instability occurs when the gradient of F exceeds the square-mean-root of the values it would have needed to overcome (*in quite different ways*) the effect of each constraint separately:

$$-\left(\frac{g_r}{a^2}-\frac{2}{r}\right)V^2\frac{\mathrm{d}F}{\mathrm{d}r} > \left[\left(\frac{\eta}{\kappa}\gamma N_r^2\right)^{\frac{1}{2}}+\left(8\frac{\nu}{\eta}\Omega^2\right)^{\frac{1}{2}}\right]^2 \tag{9.4}$$

(see (A 20)). Growth is on a dynamical time scale, albeit one reduced by multiply-diffusive effects (see (A 26)). The second type of instability is *non*-axisymmetric and it is the azimuthal pressure gradients and Lorentz forces implied by this non-axisymmetry, rather than the action of viscosity (as in the multiply-diffusive instability above), that breaks the otherwise thoroughly stabilizing constraint that individual fluid rings initially symmetric about the rotation axis should conserve their angular momentum.

This instability may be conveniently discussed in a qualitative way by reference to figure 1, while the relevant instability criteria may be written:

$$\left.\begin{aligned} -\left(\frac{g_r}{a^2}-\frac{2}{r}\right)r^2\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(\frac{B}{\rho r}\right) &> 1+\left(\frac{r}{a}\frac{\mathrm{d}a}{\mathrm{d}r}\right)^2, \quad \text{for} \quad \mathscr{C}^2 \ll D;\\ -\left(\frac{g_r}{a^2}-\frac{2}{r}\right)r^2\frac{\mathrm{d}}{\mathrm{d}r}\ln(Br) &> 1, \quad \text{for} \quad 1 \ll D \ll \mathscr{C}^2;\\ -\left(\frac{g_r}{a^2}-\frac{2}{r}\right)r^2\frac{\mathrm{d}}{\mathrm{d}r}\ln B &> D^{-1}, \quad \text{for} \quad D \ll 1. \end{aligned}\right\} \tag{9.5}$$

If regarded very schematically, figure 1 covers a broad range of situations provided the *magnitude* of the magnetic field gradient (by which we mean inverse magnetic field scale height, suitably non-dimensionalized by the scale of the system) is regarded as increasing monotonically in some way up the vertical axis. The gradient must, of course, be of the appropriate sign for instability. Inside the critical radius instability is promoted by B increasing with r, and the mechanism is an essentially 'incompressible' one (cf. Acheson 1972). Outside the critical radius a decrease of B with r promotes instability and the mechanism is that of magnetic buoyancy, for which compressibility effects are crucial. In addition, figure 1 holds good even if the instability mechanism is a weak *differential rotation* with the angular velocity Ω decreasing slightly with r (see (7.11)) and the magnetic field acting essentially as a catalyst, in which case the magnitude of the Ω-gradient would be regarded as increasing up the vertical axis.

Alternatively we may consider the special case of the uniformly-rotating and isothermal plane layer, for which magnetic buoyancy is the only instability mechanism. The criteria corresponding to (9.5) are

$$\left.\begin{aligned} -H\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(\frac{B}{\rho}\right) &> \frac{3\pi^2\surd 3}{\mathscr{C}_*}, \quad \text{for} \quad \mathscr{C}_*^2 \ll D_*;\\ -H\frac{\mathrm{d}}{\mathrm{d}r}\ln B &> 6\surd 3\pi^2\frac{D_*^{\frac{1}{2}}}{\mathscr{C}_*}+\frac{1}{D_*}, \quad \text{for} \quad 1 \ll D_* \ll \mathscr{C}_*^2;\\ -H\frac{\mathrm{d}}{\mathrm{d}r}\ln B &> D_*^{-1}, \quad \text{for} \quad D_* \ll 1. \end{aligned}\right\} \tag{9.6}$$

In this case the points (i), (ii) and (iii) in figure 1 may be specifically identified with the field distributions $B=$ constant, $B \propto \rho^{\frac{1}{2}}$ and $B \propto \rho$ respectively. Along the horizontal axis is plotted a suitable measure of the stratification (or inverse measure of the thermal diffusivity κ), namely the reciprocal of the dimensionless parameter

$$D_* \equiv \frac{\kappa}{\eta}\frac{V^2}{\gamma N^2 H^2}. \tag{9.7}$$

We emphasize the following restrictions which we placed on the analysis in §7 leading to figure 1: (i) $V^2/H^2 \ll \Omega^2 \ll N^2$, (ii) $\mathscr{C}_* \gg 1$, (iii) $\kappa/\eta \gg N^2/\Omega^2$.

Gilman's (1970) system is at point A and is stable at the rapid rotation speeds we are currently considering. Acheson & Gibbons (1978) essentially travelled up *vertically* from A (i.e. keeping $\kappa = \infty$) and demonstrated instability, for a rather steeper magnetic field gradient than Gilman considered, at C. As we decrease κ from its infinite value the stability criterion is hardly affected until D_* drops to $O(\mathscr{C}_*^2)$, when a significant *decrease* in the critical magnetic field gradient results. At point B this destabilizing effect of the finite thermal diffusivity is sufficient to make Gilman's constant-Alfvén-speed atmosphere unstable, as Roberts & Stewartson (1977) discovered by (effectively) travelling along the *horizontal* line from A to B. An even more puzzling feature of this portion of the stability diagram is that *increasing the 'statically stable' stratification tends to destabilize the fluid*, although Roberts (1978) and Roberts & Loper (1978) have noted similar curious behaviour in a related incompressible system. As we reduce κ further there comes a point, with $1 \ll D_* \ll \mathscr{C}_*^2$, at which these destabilizing effects are most potent, and for the plane layer model only a tiny decrease of B with height is sufficient for instability when they are working at their best. This point is at $D_* \sim \mathscr{C}_*^{\frac{2}{3}}$ in the plane layer case, but at $D \sim \mathscr{C}^{\frac{1}{2}}$ in cylindrical geometry, the difference arising because the wavenumber component along field lines is quantized in the latter case but not in the former. As κ is reduced still further the more easily understood stabilizing effects of buoyancy begin to assert themselves. Thus, when D_* has dropped to $O(1)$ the critical magnetic field gradient is back to something like its $D_* = \infty$ value, and as κ continues to decrease this critical gradient rapidly increases as κ^{-1} until the breakdown of both the low-frequency and fast-thermal-diffusion approximations takes the system out of the range of validity of the present theory.

I wish to thank Professor P. H. Roberts for helpful discussions, and I am very grateful to him and Professor K. Stewartson, F.R.S., for sending me their paper on magnetic buoyancy instability well in advance of publication, for this stimulated the investigations reported in §7. I am also grateful to the C.E.G.B. for a Research Fellowship during the period in which this work was carried out.

Appendix. The axisymmetric diffusive instability of toroidal magnetic fields in a rotating gas

By D. J. Acheson and M. P. Gibbons

The question we are asking here is whether, when $\nu \ll \eta \ll \kappa$, a toroidal magnetic field can give rise to axisymmetric instability by some multiply-diffusive mechanism when, according to the diffusionless theory in §4, rotation and stratification act jointly to suppress it. It turns out to be essential to first examine how the magnetic field might overcome the rotation and stratification when acting *separately*, for quite different mechanisms are involved in the two cases.

Magnetic instability of a stratified gas

In the absence of rotation we find from (4.13) and (4.14) that instability occurs if either of the following criteria are satisfied:

$$-\left(g_r - \frac{2a^2}{r}\right)\frac{V^2}{a^2}\frac{\partial F}{\partial r} > \frac{\eta}{\kappa} g_r \frac{\partial E}{\partial r}, \tag{A 1}$$

$$-g_z \frac{V^2}{a^2}\frac{\partial F}{\partial z} > \frac{\eta}{\kappa} g_z \frac{\partial E}{\partial z}. \tag{A 2}$$

We are interested in the case $\eta \ll \kappa$, and comparison with the diffusionless criteria (4.3) and (4.4) shows how the stabilizing effect of the entropy gradient is here reduced by thermal diffusion. Other stability characteristics are much the same as in the diffusionless case; a decrease of F with r promotes instability outside a critical radius, while an increase of F with r promotes instability within it. The critical radius is a factor γ smaller than in the diffusionless case. All these results have been obtained by Schubert (1968). The essential requirement for instability, given $g \sim a^2/r$ and that the field has the correct kind of gradient, is

$$\frac{V^2}{a^2} \gg \frac{\eta}{\kappa}. \tag{A 3}$$

Even if (A 3) is not satisfied we note from (4.15) that instability will still occur if

$$g_z\left(\frac{\partial E}{\partial r}\frac{\partial F}{\partial z}-\frac{\partial E}{\partial z}\frac{\partial F}{\partial r}\right) < 0, \tag{A 4}$$

i.e. if the magnetic flux *increases* outward along a surface of constant specific entropy, the associated motions being such that the term $G\,\partial E/\partial h$ in (4.12) is zero, or perhaps even negative, for reasons mentioned in other contexts in §4.

Magnetic instability of a rotating gas

In the absence of any stratification we note that it makes no qualitative difference whether we consider perturbations that are virtually adiabatic ($\omega \gg \kappa s^2$) or good heat exchangers ($\omega \ll \kappa s^2$); reference to (4.1) shows that one must simply use the appropriate speed of sound, adiabatic or isothermal, in each case. We shall frame our results in the latter terms. According to (4.13) diffusive magnetic instability may occur despite a stabilizing angular momentum distribution ($\partial R/\partial r > 0$) if

$$-\left(g_r-\frac{2a^2}{r}\right)\frac{V^2}{a^2}\frac{\partial F}{\partial r} > \frac{\eta}{\nu}\Omega^2 r\frac{\partial R}{\partial r}. \tag{A 5}$$

If η were less than ν this would be entirely analogous to the preceding instability and to the reverse situation of Goldreich–Schubert instability in §4, and its physical explanation would follow similar lines. We are interested, however, in η *greater* than ν, and an interesting state of affairs then arises if we compare (A 5) with the criterion we would have obtained with $\nu = \eta = 0$ but with $\omega \ll \kappa s^2$, or κ formally infinite. Hopefully the reader will accept, in view of the above remarks, that this can be obtained by knocking out the entropy gradient term in (4.3) and replacing γa^2 by a^2, whence:

$$-\left(g_r-\frac{2a^2}{r}\right)\frac{V^2}{a^2}\frac{\partial F}{\partial r} > \left(1+\frac{V^2}{a^2}\right)\Omega^2 r\frac{\partial R}{\partial r}, \tag{A 6}$$

and evidently if

$$\frac{\eta}{\nu} > 1+\frac{V^2}{a^2} \equiv j \tag{A 7}$$

the diffusive instability criterion (A 5) is less easily satisfied than the corresponding diffusionless one (A 6)!

The source of this behaviour is clear, at least in a qualitative way. In the previous example, as in Goldreich–Schubert instability, the diffusivity associated with the driving mechanism was the smaller of the two involved. We are now trying to drive the instability with the mechanism having the *higher* diffusivity, and given the nature of the physical explanation of Goldreich–Schubert and related 'salt-finger' instabilities it is no surprise that instability is not now occurring in the same way. As with other double-diffusive problems of this kind (see, for example, Turner 1973) what happens is that if (A 7) is satisfied the system *still* goes unstable for a weaker magnetic flux gradient than indicated by (A 6), but it occurs in the form of *slowly growing oscillations*, as we now show.

Take (4.1), set $\kappa = \infty$ and $\nabla E = 0$, and divide by $j\omega + \mathrm{i}\eta s^2$. Equate the real and imaginary parts of the resulting equation separately to zero, regarding $\omega \neq 0$ as real. Eliminate F between the two to obtain

$$\omega^2 + \nu^2 s^4 = \left(\frac{1-j\nu/\eta}{1+j\nu/\eta}\right)\frac{n^2}{s^2}\Omega^2 r\frac{\partial R}{\partial h}, \tag{A 8}$$

as an expression for the frequency of oscillation ω, and eliminate R between the two to obtain an expression for the marginally stable magnetic flux gradient:

$$-\left(\frac{G}{a^2}-\frac{2}{r}\right)V^2\frac{\partial F}{\partial h} = 2\frac{\nu}{\eta}\frac{s^2}{n^2}\frac{(j^2\omega^2+\eta^2 s^4)}{(1-j\nu/\eta)}. \tag{A 9}$$

We note at once that ω, as given by (A 8), can only be real and represent an oscillation if (A 7) is satisfied. In other circumstances, such that (A 5) would have been a 'satisfactory' diffusive instability inequality, the oscillatory instability we are currently pursuing would be impossible.

It is clear that if the diffusivities ν and η are very small the terms $\nu^2 s^4$ in (A 8) and $\eta^2 s^4$ in (A 9) may be neglected for all but the shortest disturbance scales of interest. Substituting for ω^2 in (A 9) we then find that instability will occur essentially for any mode with wavenumbers l and n such that

$$-\left(\frac{G}{a^2}-\frac{2}{r}\right)V^2\frac{\partial F}{\partial h} > \frac{2j^2\nu/\eta}{1+j\nu/\eta}\left(\Omega^2 r\frac{\partial R}{\partial h}\right). \tag{A 10}$$

By applying the same procedure as on previous occasions in this paper we conclude that the system is unstable if any of the following criteria are satisfied:

$$-\left(\frac{g_r}{a^2}-\frac{2}{r}\right)V^2\frac{\partial F}{\partial r} > \frac{2j^2\nu/\eta}{1+j\nu/\eta}\left(\Omega^2 r\frac{\partial R}{\partial r}\right), \tag{A 11}$$

$$g_z\,\partial F/\partial z < 0, \tag{A 12}$$

$$g_z\left(\frac{\partial R}{\partial r}\frac{\partial F}{\partial z}-\frac{\partial R}{\partial z}\frac{\partial F}{\partial r}\right) < 0. \tag{A 13}$$

By comparing (A 11) with (4.3) it is easy to see that when the former is satisfied instability occurs in its present oscillatory fashion for weaker magnetic flux gradients than required according to the diffusionless theory. Indeed, if $j \sim 1$ but $\nu \ll \eta$ the stabilizing effect of rotation is greatly reduced, as (A 11) shows. Thus, if ν and η are widely different (no matter which is the larger) the stabilizing effect of rotation is considerably relaxed, one way or the other.

A triply diffusive magnetic instability

In practice, of course, both rotation and stratification may be acting in concert to hinder magnetic buoyancy instability, and an interesting situation then arises because with $\nu \ll \eta \ll \kappa$ one of the stabilizing influences has a higher diffusivity than that of the driving mechanism, while the other has a lower diffusivity. To overcome the stable stratification the fluid wants to go unstable in a monotonic way, while as we have seen in the subsection above it needs to go unstable in an oscillatory manner to overcome the stabilizing effect of rotation. The method by which it resolves this dilemma, essentially by compromise, is rather interesting.

First we have to turn back again to (4.1) and proceed, in the first instance, on the assumption that $|\omega| \ll \kappa s^2$ at least, for otherwise it is difficult to envisage how the stabilizing effects of stratification could possibly be relaxed by multiply-diffusive effects. By much the same procedure as outlined in the previous subsection we thus obtain an expression, at marginal stability, for the frequency of oscillation

$$\omega^2+\nu^2 s^4 = \frac{(1-j\nu/\eta)\,\Omega^2 r\,\partial R/\partial r}{(1+j\nu/\eta)\,s^2/n^2+(g_r+2V^2/r)\,(\eta\kappa s^4)^{-1}\,\partial E/\partial r} \tag{A 14}$$

and for the critical magnetic flux gradient:

$$-\left(\frac{g_r}{a^2}-\frac{2}{r}\right)V^2\left(1+j\frac{\nu}{\eta}\right)\frac{\partial F}{\partial r} = \frac{\eta}{\kappa}\frac{\partial E}{\partial r}\left[g_r\left(1+\frac{j\nu}{\eta}\right)+\left(g_r+\frac{2V^2}{r}\right)\left(\frac{j\omega^2}{\eta^2 s^4}-\frac{\nu}{\eta}\right)\right] + 2\frac{\nu}{\eta}\,\Omega^2 r\frac{\partial R}{\partial r}\left(\frac{j^2\omega^2+\eta^2 s^4}{\omega^2+\nu^2 s^4}\right). \tag{A 15}$$

We are here restricting attention for simplicity to the equatorial plane, so that $g_z = 0$. In other respects, as may be checked, (A 14) and (A 15) are simply extensions of (A 8) and (A 9) to include the effect of an entropy gradient, but within the 'large' thermal diffusion approximation $|\omega| \ll \kappa s^2$.

One or two further simplifications can be made. We have been thinking in terms of magnetic buoyancy instability, for which $g_r > 2a^2/r$, and we see that according to the diffusionless criterion (4.3) this will only be strongly suppressed by the stratification if

$$V^2/a^2 \ll 1, \quad \text{and thus} \quad V^2/r \ll g_r, \tag{A 16}$$

and similarly it is only strongly suppressed by the effects of rotation if

$$V^2/a^2 \ll \Omega^2 r/g. \tag{A 17}$$

(Since $\Omega^2 r \ll g$ (A 17) in fact implies the first of (A 16).) We therefore make these approximations in addition to that of $\nu/\eta \ll 1$. Furthermore, it is clear by inspection of the final term of (A 15) that there is no hope of a multiply-diffusive reduction of the stabilizing effect of rotation unless $\omega \gg \nu s^2$, for otherwise that term is of order $(\eta/\nu)\,\Omega^2 r$. With these additional approximations, (A 14) and (A 15) become

$$\omega^2 = \frac{\Omega^2 r\,\partial R/\partial r}{s^2/n^2+g_r(\kappa\eta s^4)^{-1}\,\partial E/\partial r} \tag{A 18}$$

and

$$-\left(\frac{g_r}{a^2}-\frac{2}{r}\right)V^2\frac{\partial F}{\partial r} = \frac{\eta}{\kappa}g_r\frac{\partial E}{\partial r}\left(1+\frac{\omega^2}{\eta^2 s^4}\right)+2\frac{\nu}{\eta}\,\Omega^2 r\frac{\partial R}{\partial r}\left(1+\frac{\eta^2 s^4}{\omega^2}\right) \tag{A 19}$$

respectively. We may minimize the right hand side of (A 19) with respect to $\omega^2/\eta^2 s^4$ to obtain the critical magnetic flux gradient:

$$-\left(\frac{g_r}{a^2}-\frac{2}{r}\right)V^2\frac{\partial F}{\partial r} = \left[\left(\frac{\eta}{\kappa}g_r\frac{\partial E}{\partial r}\right)^{\frac{1}{2}}+\left(2\frac{\nu}{\eta}\,\Omega^2 r\frac{\partial R}{\partial r}\right)^{\frac{1}{2}}\right]^2, \tag{A 20}$$

and this occurs for any pair of wavenumbers (l, n) satisfying

$$\left(\frac{s^6}{n^2}+\frac{g_r}{\eta\kappa}\frac{\partial E}{\partial r}\right)^2 = \frac{(\Omega^2 r\,\partial R/\partial r)\,(g_r\,\partial E/\partial r)}{2\kappa\nu\eta^2}. \tag{A 21}$$

On substituting (A 21) into (A 18) we can find *a posteriori* the conditions under which our original assumption of $\nu s^2 \ll \omega \ll \kappa s^2$ is valid, and when combined with the condition that s^6/n^2 be positive in (A 21) these are essentially

$$\frac{\nu}{\kappa} \lesssim \frac{\Omega^2 r}{g} \ll \left(\frac{\kappa}{\nu}\right)\left(\frac{\kappa}{\eta}\right)^2. \tag{A 22}$$

The smallness of the diffusivities means that the total wavenumber s must, according to (A 21), be large at the onset of instability, and when

$$\frac{\nu}{\kappa} \ll \frac{\Omega^2 r}{g}, \tag{A 23}$$

the second term in brackets in (A 21) is negligible. We note, without presenting the details of this particular calculation, that the expression for the growth rate near marginal stability then takes the form

$$q = \left[1+\left(\frac{\eta^2}{2\nu\kappa}\frac{g}{\Omega^2 r}\frac{\partial E/\partial r}{\partial R/\partial r}\right)^{\frac{1}{2}}\right]\left[\frac{\partial F/\partial r}{(\partial F/\partial r)_{\text{c}}}-1\right]\nu s^2. \tag{A 24}$$

The modes that grow most rapidly have a total wavenumber s as large as possible subject to (A 21) and thus take the form, as in the Goldreich–Schubert (1967) instability, of long narrow cells normal to the rotation axis with a large axial wavenumber n ($\gg l$) given approximately by

$$n^8 \approx \frac{(\Omega^2 r\,\partial R/\partial r)\,(g_r\,\partial E/\partial r)}{2\kappa\nu\eta^2}. \tag{A 25}$$

Substituting into (A 24) we find that when $\partial F/\partial r$ exceeds its critical value by a small fraction δ:

$$q_{\max} \sim \delta\left(\frac{\nu}{\kappa}\frac{g_r\,\partial E/\partial r}{\Omega^2 r\,\partial R/\partial r}\right)^{\frac{1}{4}} \max\left\{\left(\frac{\nu}{\eta}\right)^{\frac{1}{2}}\left(r\frac{\partial R}{\partial r}\right)^{\frac{1}{2}}\Omega,\ \left(\frac{\eta}{\kappa}\right)^{\frac{1}{2}}\left(r\frac{\partial E}{\partial r}\right)^{\frac{1}{2}}\left(\frac{g}{r}\right)^{\frac{1}{2}}\right\} \tag{A 26}$$

so that growth takes place on a *dynamical* time scale, albeit one lengthened by multiply-diffusive effects.

Compare (A 20) with the diffusionless criterion (4.3) in the case $V^2/a^2 \ll 1$:

$$-\left(\frac{g_r}{\gamma a^2}-\frac{2}{r}\right)V^2\frac{\partial F}{\partial r} > \frac{g_r}{\gamma}\frac{\partial E}{\partial r}+\Omega^2 r\frac{\partial R}{\partial r}. \tag{A 27}$$

Despite the parameter ordering $\nu \ll \eta \ll \kappa$, the system can evidently find a way of reducing the effectiveness of the stratification by a factor $O(\eta/\kappa)$ *at the same time* as reducing the effectiveness of the rotation by $O(\nu/\eta)$. It manages this 'triply diffusive' instability essentially as follows. The stabilizing effect of rotation is vulnerable to being broken by overstability, so to achieve this the fluid partakes in an essentially *inertial oscillation* (A 18), but one with a *carefully chosen short wavelength*. This wavelength is not so short that viscous effects damp out the oscillations ($\nu/\Omega \ll \lambda^2$) but it is short enough for rapid heat diffusion ($\lambda^2 \ll \kappa/\Omega$) to annul most of the stabilizing influence of the stratification.

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