deRham cohomology

basis of concepts about exterior algebra and multilinear algebra

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[1] *Notes when I watch the video of Chen-Yu Chi(Click to watch)*

Contents

Introduction

PART

I

1 Motivation

To simply the form of stokes fomula and construct the beautiful mathematica intuition.

2 Required Background

For *manifolds and cohomologies,* the following packages are required

`vector space, number field, odd(even) permutation, alternation.`

As prerequisites, I assume only that the reader is acquainted with the basic language of mathematics (i.e. essentially *sets and mappings*), and the integers and rational numbers. A more specific description of what is assumed will be summarized below. On a few occasions, we use *determinants* before treating these formally in the text. Most readers will already be acquainted with determinants, and we feel it is better for the organization of the whole book to allow ourselves such minor deviations from a total ordering of the logic involved.[1]

Notice :For the whole note,we adapt the Enistein's summation convention.
According to this convention, when an index variable appears twice in a single term(usually appears to upper and lower indices the same time), it implies summation of that term over all the values of the index.

[1]Please contact me at my email if you have any questions or comments.

Multiplinear Algebra

PART

II

Table 1. Contents for PART II

3 Dual space

Defination 3.1(**dual space**) :Given any vector space V over a field $K(\mathbb{R}, \mathbb{Q}\ or\ \mathbb{C})$, the (algebraic) **dual space** V^* is defined as the set of all linear maps $\varphi : V \rightarrow K$(linear functions). Since linear maps are vector space homomorphisms, the dual space is also sometimes denoted by Hom(V, K). The dual space V^* itself becomes a vector space over F when equipped with an addition and scalar multiplication satisfying:

$$(\phi + \psi)(x) = \phi(x) + \psi(x)$$
$$(a\phi)(x) = a(\phi(x))$$

For short,V^* is just defined as Hom(V, K), whose elements are K-linear function from V to K. Let $\{e_i\}$ be the basis of V. Here is the question, how to find a set of basis of V^*?

It's an easy thing, since define a linear function is define how the function behave on $\{e_i\}$. We can give the defination of dual basis conviensely as follow:
Defination 3.2(**dual basis**) :e_i's dual basis is e^i,who maps $e_j \rightarrow \delta_i^j$,that's to say $e^i(e_j) = \delta_i^j$.
From the above defination, we can see V^* share the same dimesion with V.(Left for the readers who have the interest.)

Let $\{\hat{e_i}\}$ be another basis of V, $\{\hat{e^i}\}$ be another basis of V^*.
Claim3.1 : $\hat{e_l} = a_l^j \cdot e_j \iff a_l^j \cdot \hat{e^l} = e^j$
In the words of matrix,it's:

$$(\hat{e_1}, ..., \hat{e_n}) = (e_1, ..., e_n)A \iff \begin{pmatrix} e^1 \\ \vdots \\ e^n \end{pmatrix} = A \begin{pmatrix} \hat{e^1} \\ \vdots \\ \hat{e^n} \end{pmatrix}$$

Why?(The proof is easy.)

Claim3.1 : $\forall f \in V^*, f$ can be uniquely expressed as $f(e_j)e^j$.

4 Tensor Space

Defination4.1(tensor product of two linear maps):Let$g \in U^*, h \in V^*$.
$g \bigotimes h : U \bigotimes V \rightarrow K,\ (u, v) \rightarrow g(u) \cdot h(v)$
Defination4.2(tensor product of two dual space):Let U,V are vector space over K.
Then $U^* \bigotimes V^* := \{f : U \times V \rightarrow K | fisbilinear\} = \{g \otimes h | g \in U^*, h \in V^*\}$.

Under this defination, we can define $\bigotimes_{k \in \mathbb{Z}} V_k$,where $V_k$is some vector space.
Example: Let $\phi \in U^* \bigotimes V^*, h \in W^*,\ then\ (\phi \otimes h)(u, v, w) = \phi(u, v)h(w)$.
Excersice: Let U,V,W are vector space with finite dimension,and with basis $\{u_i\}, \{v_j\}, \{w_k\}$respectively.Please show $u_i \otimes v_j \otimes w_k$ form a basis of $U^* \bigotimes V^* \bigotimes W^*$.
Propoties: tensor product between functions are multilinear and multiplicative.

5 Form Space

Defination5.1(wegde product of dual space): $\bigwedge^k V^* := \{f : V \times V \cdots V \to K | f\ is\ k-linearandalternating\}$.
Alternating means $f(\cdots, v, \cdots, v, \cdots) = 0\ \forall v \in V$.

Practice: Please show that

1. f is skew-symmetric.
2. Let σ be a permutation of 1,2,...,n.Define $\sigma f : (v_1, v_2, ..., v_n) \mapsto f(v_{\sigma(1)}, ..., v_{\sigma(n)})$. Then σ is a autmorphism of$\bigotimes^k V^*$.
3. Let $f \in \bigotimes^k V^*$,define $Aut(f)(v_1, \cdots, v_n) = \frac{1}{k!} \sum_{\sigma \in S_n} (-1)^\sigma f(v_{\sigma(1)}, ..., v_{\sigma(n)})$. Then Alt is a isomorphism between $\bigotimes^k V^*$ and $\bigwedge^k V^*$
4. If$f \in \bigotimes^k V^*$,then$f \in \bigwedge^k V^* \iff Alt(f) = f$.
5. Define $\widehat{Alt} = k!Alt$,then $\widehat{Alt}(\phi_1 \otimes ...\phi_k)(v_1, ..., v_k) = det(\phi_i(v_j)), where\ \phi_i \in V^*$.
6. How to define wedge product between elements in $\bigwedge^k V^*$and$\bigwedge^l V^*$?

NotesTeX	**rocks!**

Table 2. Margintable

Excersices:

1. AltAlt=Alt
2. Let $\phi \in \bigotimes^k V^*, \psi \in \bigotimes^l V^*$,then show that $Alt(\psi) \to Alt(\phi \otimes \psi) = 0$.

5.1 Basis of Form Space

The following is still waiting for fill.

6 Manifolds

Definition 6.1. *Defination6.1(Topology)* Given a set A, a `topology`$\mathcal{T}$ of A is a subset of power set of A,which satisified:

1. $\emptyset, A\ are \in \mathcal{T}$.
2. For any index set$I, \bigcup_{i \in I} X_i \in \mathcal{T}, where\ X_i \in A$.
3. $\bigcap_{i=0}^{n} X_i \in \mathcal{T}$

[1] *Example:* Given $A = \mathbb{R}^2$,the trivial topology $\mathcal{T}$ of A is all the open set of A.We call $(\mathcal{T}, A)$a topology space, usually write A for short.

[1] Given a topology $\mathcal{T}$ of A means define an open set structure on A.

Defination6.2(open map) An open map is a function between two topology spaces which maps open sets to open sets.
Defination6.3(Homeomorphism) A function $f : X \to Y$ between two topology spaces is a **homeomorphism**[2] if it has the following properties:

1. f is a bijection(one-to-one and onto),
2. f is continious,
3. the inverse function f^-1 is continuous (f is an open mapping).

[2] A continuous deformation between a coffee mug and a donut (torus) illustrating that they are homeomorphic.

A homeomorphism is sometimes called a bicontinuous function. If such a function exists, X and Y are homeomorphic. A self-homeomorphism is a homeomorphism from a topological space onto itself. "Being homeomorphic" is an equivalence relation on topological spaces. Its equivalence classes are called homeomorphism classes.

Defination6.4(manifold) A manifold is a topological space that locally resembles Euclidean space near each point. More precisely, each point of an n-dimensional manifold has a neighbourhood that is homeomorphic to the Euclidean space of dimension n.
Example The circle is the simplest example of a topological manifold. Topology ignores bending, so a small piece of a circle is treated exactly the same as a small piece of a line.

6.1 Tangent space

*Informal description:*In differential geometry, one can attach to every point x of a differentiable manifold a tangent spacea real vector space that intuitively contains the possible directions in which one can tangentially pass through x. The elements of the tangent space at x are called the tangent vectors at x. This is a generalization of the notion of a bound vector in a Euclidean space. The dimension of the tangent space at every point of a connected manifold is the same as that of the manifold itself.

Formal description:

Definition 6.2. Suppose that M is a C^k manifold and that$x \in M$. Pick a coordinate chart $\varphi : U \to \mathbf{R}^n$, where U is an open subset of M containing x. Suppose further that two curves $\gamma_1, \gamma_2 : (-1, 1) \to M$with $\gamma_1(0) = x = \gamma_2(0)$ are given such that both $\varphi \circ \gamma_1, \varphi \circ \gamma_2 : (-1, 1) \to \mathbf{R}^n$ are differentiable in the ordinary sense (we call these differentiable curves initialized at x). Then $\gamma_1$and γ_2 are said to be equivalent at 0

> if and only if the derivatives of $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ at 0 coincide.
> This defines an equivalence relation on the set of all differentiable curves initialized at x, and equivalence classes of such curves are known as tangent vectors of M at x. The equivalence class of any such curve γ is denoted by $\gamma'(0)$. The tangent space of M at x, denoted by T_xM, is then defined as the set of all tangent vectors at x; it does not depend on the choice of coordinate chart $\varphi : U \to \mathbf{R}^n$.

7 Diff. form | Tensor fields

$M : (C)^\infty\ mfd.$ if dim m.
Tangent bundle of M: $TM := \bigcup_{p\in M} T_pM$
Cotangent bundle of M: $T^*M := \bigcup_{p\in M} T_p^*M$
The k-th tensor field: $\bigotimes^k T^*M := \bigcup_{p\in M} \bigotimes^k T_p^*M$
The k-th form field: $\bigwedge^k T^*M := \bigcup_{p\in M} \bigwedge^k T_p^*M$