deRham cohomology

basis of concepts about exterior algebra and multilinear algebra

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[1] *Notes when I watch the video of Chen-Yu Chi(Click to watch)*

Contents

PART

I

Introduction

1 Motivation

To simply the form of stokes fomula and construct the beautiful mathematica intuition.

2 Required Background

For *manifolds and cohomologies,* the following packages are required

```
vector space, number field, odd(even) permutation, alternation.
```

As prerequisites, I assume only that the reader is acquainted with the basic language of mathematics (i.e. essentially *sets and mappings*), and the integers and rational numbers. A more specific description of what is assumed will be summarized below. On a few occasions, we use *determinants* before treating these formally in the text. Most readers will already be acquainted with determinants, and we feel it is better for the organization of the whole book to allow ourselves such minor deviations from a total ordering of the logic involved.[1]

Notice :For the whole note,we adapt the Enistein's summation convention.
According to this convention, when an index variable appears twice in a single term(usually appears to upper and lower indices the same time), it implies summation of that term over all the values of the index.

[1]Please contact me at my email if you have any questions or comments.

PART

II

Multiplinear Algebra

Table 1. Contents for PART II

3 Dual space

Defination 3.1(dual space) :Given any vector space V over a field $K(\mathbb{R}, \mathbb{Q}$ *or* $\mathbb{C})$, the (algebraic) **dual space** V^* is defined as the set of all linear maps $\varphi : V \to K$(linear functions). Since linear maps are vector space homomorphisms, the dual space is also sometimes denoted by Hom(V, K). The dual space V^* itself becomes a vector space over F when equipped with an addition and scalar multiplication satisfying:

$$(\phi + \psi)(x) = \phi(x) + \psi(x)$$
$$(a\phi)(x) = a(\phi(x))$$

For short,V^* is just defined as Hom(V, K), whose elements are K-linear function from V to K. Let $\{e_i\}$ be the basis of V. Here is the question, how to find a set of basis of V^*?

It's an easy thing, since define a linear function is define how the function behave on $\{e_i\}$. We can give the defination of dual basis conviensely as follow:
Defination 3.2(dual basis) :e_i's dual basis is e^i,who maps $e_j \to \delta_i^j$,that's to say $e^i(e_j) = \delta_i^j$.
From the above defination, we can see V^* share the same dimesion with V.(Left for the readers who have the interest.)

Let $\{\hat{e_i}\}$ be another basis of V, $\{\hat{e^i}\}$ be another basis of V^*.
Claim3.1 : $\hat{e_l} = a_l^j \cdot e_j \iff a_l^j \cdot \hat{e^l} = e^j$
In the words of matrix,it's:

$$(\hat{e_1}, ..., \hat{e_n}) = (e_1, ..., e_n)A \iff \begin{pmatrix} e^1 \\ \vdots \\ e^n \end{pmatrix} = A \begin{pmatrix} \hat{e^1} \\ \vdots \\ \hat{e^n} \end{pmatrix}$$

Why?(The proof is easy.)

Claim3.1 : $\forall f \in V^*, f$ can be uniquely expressed as $f(e_j)e^j$.

4 Tensor Space

Defination4.1(tensor product of two linear maps):Let$g \in U^*, h \in V^*$.
$g \bigotimes h : U \bigotimes V \to K,\ (u, v) \to g(u) \cdot h(v)$
Defination4.2(tensor product of two dual space):Let U,V are vector space over K.
Then $U^* \bigotimes V^* := \{f : U \times V \to K | fisbilinear\} = \{g \otimes h | g \in U^*, h \in V^*\}$.

Under this defination, we can define $\bigotimes_{k \in \mathbb{Z}} V_k$,where $V_k$is some vector space.
Example: Let $\phi \in U^* \bigotimes V^*, h \in W^*,\ then\ (\phi \otimes h)(u, v, w) = \phi(u, v)h(w)$.
Excersice: Let U,V,W are vector space with finite dimension,and with basis $\{u_i\}, \{v_j\}, \{w_k\}$respectively.Please show $u_i \otimes v_j \otimes w_k$ form a basis of $U^* \bigotimes V^* \bigotimes W^*$.
Propoties: tensor product between functions are multilinear and multiplicative.

5 Form Space

Defination5.1(wegde product of dual space): $\bigwedge^k V^* := \{f : V \times V \cdots V \to K | f \ is \ k-linearandalternating\}$.
Alternating means $f(\cdots, v, \cdots, v, \cdots) = 0 \ \forall v \in V$.

Practice: Please show that

1. f is skew-symmetric.
2. Let σ be a permutation of 1,2,...,n.Define $\sigma f : (v_1, v_2, ..., v_n) \mapsto f(v_{\sigma(1)}, ..., v_{\sigma(n)})$. Then σ is a autmorphism of$\bigotimes^k V^*$.
3. Let $f \in \bigotimes^k V^*$,define $Aut(f)(v_1, \cdots, v_n) = \frac{1}{k!}\sum_{\sigma \in S_n}(-1)^\sigma f(v_{\sigma(1)}, ..., v_{\sigma(n)})$. Then Alt is a isomorphism between $\bigotimes^k V^*$ and $\bigwedge^k V^*$
4. If$f \in \bigotimes^k V^*$,then$f \in \bigwedge^k V^* \iff Alt(f) = f$.
5. Define $\widehat{Alt} = k!Alt$,then $\widehat{Alt}(\phi_1 \otimes ...\phi_k)(v_1, ..., v_k) = det(\phi_i(v_j)), where \ \phi_i \in V^*$.
6. How to define wedge product between elements in $\bigwedge^k V^*$and$\bigwedge^l V^*$?

NotesTeX	**rocks!**

Table 2. Margintable

Excersices:

1. AltAlt=Alt
2. Let $\phi \in \bigotimes^k V^*, \psi \in \bigotimes^l V^*$,then show that $Alt(\psi) \to Alt(\phi \otimes \psi) = 0$.

NotesTeX	**rocks!**

Table 3. Margintable

5.1 Basis of Form Space

The following is still waiting for fill.

6 TEX Shortcuts

NotesTeX comes built in with a minimal set of keyboard shortcuts for a few special characters. All of these shortcuts can be found in `NotesTeX.sty` just under

```
% ------------------------------------------------------------------------
%           User Created Commands
% ------------------------------------------------------------------------
...
```

If one has their own macros,[1] then simply add it under this area.

[1] Most people have their own shortcuts for commonly used mathematics, such as derivatives or integrals. For those looking for physics shortcuts, the excellent `physics` package (automatically included in *NotesTeX*) has possible everything that one can imagine.

6.1 Available Shortcuts

To keep things light, *NotesTeX* offers the following simplifications/shortcuts to the mathematics dictionary.

1. Vector Bold Math: The shortcut `{\bf ·}` $\longrightarrow$ `\bf·` creates boldface text. For example, `{\bf a}` becomes `\bfa` producing the output $\mathbf{a}$. This works for both upper and lower case alphabet.

2. `Mathfrak:` The `\mathfrak` environment is quite similar to the Vector Bold Math in the shortcut argument, `\mathfrak{·}` $\longrightarrow$ `\mf·`. This works for both upper and lower case producing

$$\mathfrak{abcdefghijklmnopqrstuvwxyz}$$
$$\mathfrak{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$$

3. `Mathbb`: All `\mathbb{·}` objects are turned into `\bb{·}`. This only works for uppercase alphabet.

$$\mathbb{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$$

4. `Mathcal:` All `\mathcal{·}` objects are turned into `\mc{·}`. This only works for uppercase alphabet.

$$\mathcal{ABCDEFGHIJKLMNOPQRSTUVWXYZ}$$

7 amsthm Environments

amsthm environments are defined as usual being enclosed by \begin{environment}⋯ \end{environment} and most have been modified ostensibly from the original amsthm presets. Primarily, most environments, with the exception of the exercise environment, are now integrated with the wonderful tcolorbox package. Note that the counting for theorems and lemmas is distinct from the counting for definitions. Also note that the breakable for tcolorbox allows these environments to span multiple pages.

Definition 7.1. The definition environment and the associated tcolorbox are provided by the following code in NotesTeX.sty:

```
\tcolorboxenvironment{definition}{
  boxrule=0pt,
  boxsep=0pt,
  colback={White!90!Cerulean},
  enhanced jigsaw,
  borderline west={2pt}{0pt}{Cerulean},
  sharp corners,
  before skip=10pt,
  after skip=10pt,
  breakable,
}
```

Theorem 7.1. The theorem environment and the associated tcolorbox are provided by the following code in NotesTeX.sty:

```
\tcolorboxenvironment{theorem}{
  boxrule=0pt,
  boxsep=0pt,
  colback={White!90!Dandelion},
  enhanced jigsaw,
  borderline west={2pt}{0pt}{Dandelion},
  sharp corners,
  before skip=10pt,
  after skip=10pt,
  breakable,
}
```

Lemma 7.2. The lemma environment and the associated tcolorbox are provided by the following code in NotesTeX.sty:

```
\tcolorboxenvironment{lemma}{
  boxrule=0pt,
  boxsep=0pt,
  blanker,
  borderline west={2pt}{0pt}{Red},
  before skip=10pt,
  after skip=10pt,
  sharp corners,
  left=12pt,
```

```
  right=12pt,
  breakable,
}
```

Proof. The `proof` environment and the associated `tcolorbox` are provided by the following code in `NotesTeX.sty`:

```
\tcolorboxenvironment{proof}{
  boxrule=0pt,
  boxsep=0pt,
  blanker,
  borderline west={2pt}{0pt}{NavyBlue!80!white},
  before skip=10pt,
  after skip=10pt,
  left=12pt,
  right=12pt,
  breakable,
}
```

□

Example. The `example` environment and the associated `tcolorbox` are provided by the following code in `NotesTeX.sty`:

```
\tcolorboxenvironment{example}{
  boxrule=0pt,
  boxsep=0pt,
  blanker,
  borderline west={2pt}{0pt}{Black},
  sharp corners,
  before skip=10pt,
  after skip=10pt,
  left=12pt,
  right=12pt,
  breakable,
}
```

Remark. The `remark` environment and the associated `tcolorbox` are provided by the following code in `NotesTeX.sty`:

```
\tcolorboxenvironment{remark}{
  boxrule=0pt,
  boxsep=0pt,
  blanker,
  borderline west={2pt}{0pt}{Green},
  before skip=10pt,
  after skip=10pt,
  left=12pt,
  right=12pt,
  breakable,
}
```

Exercise. The exercise environment remains unchanged from the `amsthm` presets.

7.1 tcolorbox Environment and Known Issues

There is one issue with this however. Since we are using a tcolorbox, this proof environment is incompatible with \sn and \sidenote, as it results in a **Float(s) Error**. However, this environment is compatible with \mn and \marginnote thankfully.[2]

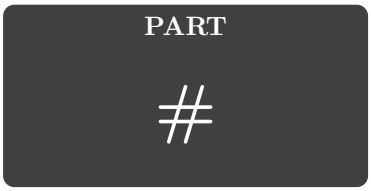
The breakable should allow the proof environment to span multiple pages. If one wishes to change the color, simply modify the line which states borderline west={1pt}{0pt}{blue}. The first numeric value dictates the width of the line, the second dictates how close it is away from the *left* margin, while the last argument obviously dictates the color. This code could also be used to change any of the other amsthm environments.

Not a major one but frustrating nonetheless.

[2] *As one can see right here.*

8 The Part Environment

In the original Jhep format, the \part environment is not special and is set to the default given by the article class. In *NotesTeX*, the part environment produces the following image. Furthermore the code responsible is noted below.




```
\titleformat{\part}[hang]{\thispagestyle{plain}}\Huge\bfseries}{\marginnote{
\begin{tcolorbox}
[width=\marginparwidth,height=\marginparwidth/2,colback=black!75!white,
   colframe=black!75!white,center title,fonttitle=\bfseries\normalsize,title=PART,
   text fill]

  \begin{center}
   {\color{white}\thepart}
  \end{center}

\end{tcolorbox}
}[-1.25in]}{0pt}{\Huge\bfseries}
```

This combines the titlesec and the tcolorbox packages, placing the title of the \part on the left hand side, and the \part number in the margin. It is recommended that one do not mess with this, other than changing the colors given by colback and colframe.

9 Fullpage Environment

The `fullpage` environment is defined by

```
\begin{fullpage}
...
\end{fullpage}
```

with the with of the `fullpage` environment given by `\textwidth+\marginparsep+\marginparwidth`.

There are some clear benefits of having use of the full page at times. Suppose that one wants to place a figure that cannot fit into the margins, or if an equation is quite long and it bleeds into the margin, then the `fullpage` environment can both clearly separate these from the surrounding text and allot for the dimensions without hassle. The code in `NotesTeX.sty` that is responsible for the `fullpage` environment is given by[3]

[3] *This is also an example of why the* `fullpage` *environment is nice.*

```
\newenvironment{fullpage}{
{\smallskip\noindent
\begin{minipage}{\textwidth+\marginparwidth+\marginparsep}\hrule\smallskip\smallskip}
{\smallskip\smallskip\hrule\end{minipage}\vspace{.1in}
}
```

Remark. If one do not like the lines at the beginning and end of the `fullpage` environment, simply remove all the `\hrule` that is in the code. Similarly, it is possible to change the vertical spacing after the `fullpage` is over, by modifying the `\vspace{}` argument.

A major benefit of having a `fullpage` is the ability to use `multicols` to its fullest extent. For example, these empty sentences are an example of how effective the `multicols` package can be inside of the `fullpage` environment. This would be especially useful for formatting exercises in multiple columns and it makes the text distinct from the rest of the `fullpage` environment. The author has run out of things to say.

9.1 Known Issues with Fullpage

There are a few issues with the `fullpage` however.

Remark. Since the `fullpage` environment uses a `minipage`, and minipages do not work over multiple pages, one will need a new `fullpage` per page.

Remark. Also, please note `marginnote`, or `sidenotes` cannot be used in this environment. Footnotes, on the other hand, can be used but are really ugly.

Remark. If the `twoside` option is enabled in the `documentclass` header, then the `fullpage` is known to bleed out beyond the margin.

PART

III

Advanced Page Formatting

For those wanting to adjust the margin sizes, or the `fancyhdr` layout, there are a few comments that could be made here.

10 Page Dimensions

NotesTeX relies on the `geometry` package to set its dimensions. The associated code is the deceptively simple chunk of code given by

```
\geometry{paperheight=845pt,paperwidth=597pt,
          marginparsep=.02\paperwidth,marginparwidth=.23\paperwidth,
          inner=.05\paperwidth,voffset=-1in,headheight=.02\paperheight,
          headsep=.03\paperheight,footskip=20pt,
          textheight=.84\paperheight,textwidth=.64\paperwidth}
```

Ignoring most of the arguments, the `\paperheight` and `\paperwidth` are set to be the standard 8×11 inches but in `pt` format instead. All other options, with the exception of `\voffset`, inherit fractions of `\paperheight` and `\paperwidth`, the most important being `\marginparwidth`. Increasing `\marginparwidth` causes the margin to bleed off of the right side of the page. In order to increase, one **must** decrease the `\textwidth` accordingly.

11 `Fancyhdr` Layout

As mentioned before, `fancyhdr` is overridden on the title page, the contents page, and the `\part` page, and sets the header for all other pages through the code

```
\pagestyle{fancy}%
\newlength{\offset}%
\setlength{\offset}{\marginparwidth + \marginparsep}%
\renewcommand{\sectionmark}[1]{\markboth{#1}{}}%
\renewcommand{\subsectionmark}[1]{\markright{#1}{}}%

\fancypagestyle{fancynotes}{%
  \fancyhf{}%
  \fancyheadoffset[rh]{\offset}%
  \renewcommand{\headrulewidth}{0pt}%
  \fancyhead[L]{\textsc{\leftmark}}%
  \fancyhead[R]{\footnotesize \textit{\rightmark}~~~~ \thepage}%
}%
```

The header style is set so that it spans the width of the entire page as opposed to just the `\textwidth` through the line `\fancyheadoffset[rh]{\myoddoffset}`. The `\sectionmark` and `\subsectionmark` are set up so that the `section` appears on the left and `subsections` appear on the right along with the page number, and this is given in the last two lines of code.

12 Alternative Language Integration: Persian

For languages which flow from right to left, such as Persian, we have developed a way[4] to use *NotesTeX* and `xePersian`. A complied example can be found in the Persian folder.

Remark. This only works with the XeLaTeX engine.

Remark. We have only tested the XB Niloofar font. In principle, any Persian font could work.

Remark. One must load the `xePersian` package just before `begin{document}`.

The `twoside` option must be enabled in the `documentclass`. A minimal working example is given below.

```
\documentclass[10pt,twoside]{article}

\usepackage{lipsum}
\usepackage{NoTeX_Persian}

\usepackage{xepersian}
\settextfont{XB Niloofar}
\setlatintextfont{Times New Roman}

\begin{document}
...
```

As we have noted before, the `fullpage` environment will fail here since it is incompatible with the `twoside` option. Also the title page is shifted by the `twoside` option, see here for a workaround. However, the `sidenote` and `marginnote` packages work well with the `twoside` option.

[4] *We are still exploring issues with integration of Persian.*