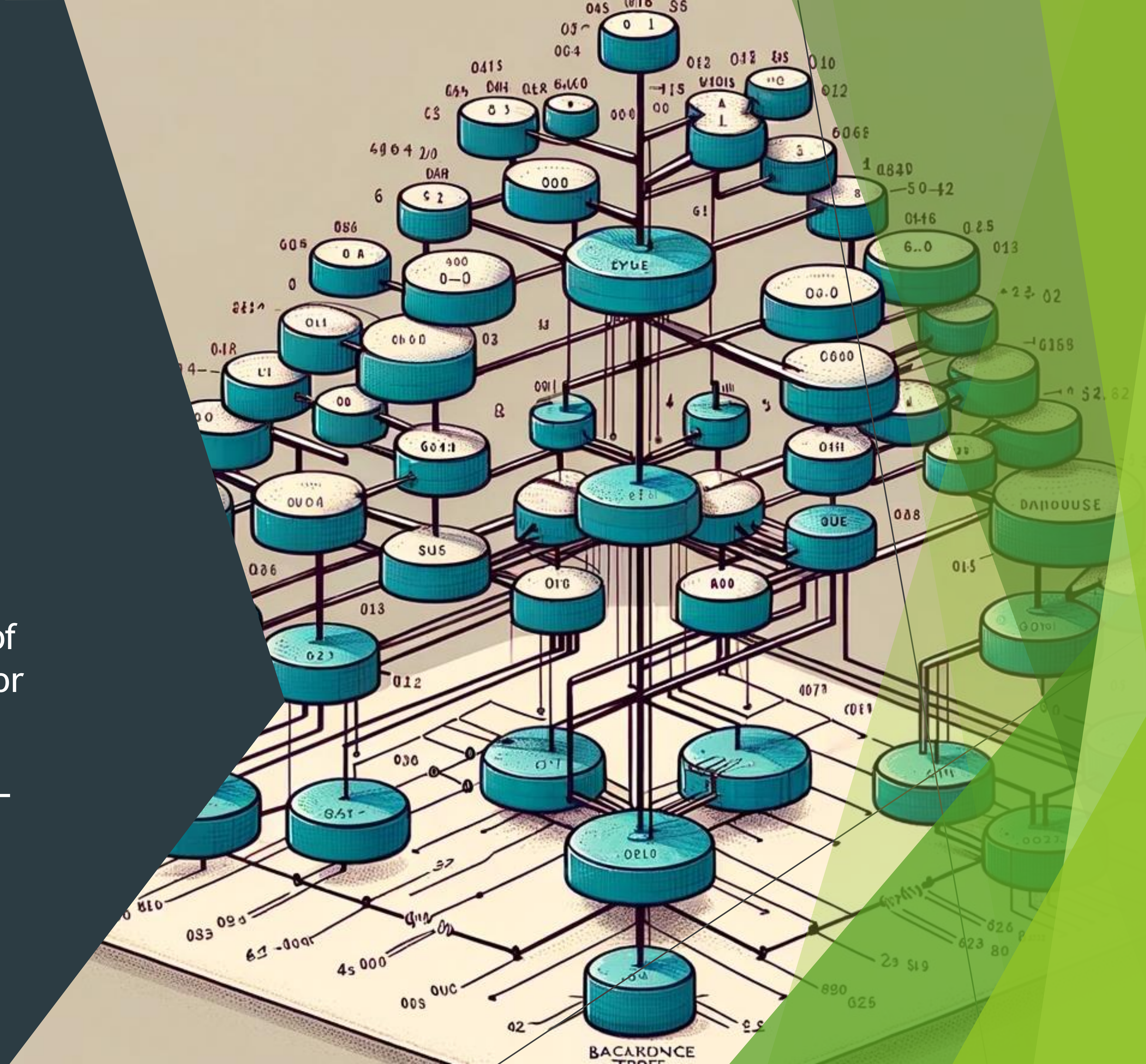
ECE2810J
Data Structures and Algorithms

AVL Trees

Learning Objectives:

- Know the general balanced condition for a balanced search tree
- Know the balance condition of an AVL tree and balance factor
- Know the four types of rotation operations for an AVL tree and how to apply them during insertion



Outline

- Balanced Search Trees
  - AVL Trees
- AVL Tree Insertion
- Supporting Data Members and Functions of AVL Tree

Motivation

- Given n nodes, the average case time complexities for search, insertion, and removal on BST are all $O(\log n)$.
- However, the worst case time complexities are still $O(n)$.
  - The reason is that a tree could become "unbalanced" after a number of insertions and removals.
- We want to maintain the tree as a "balanced" tree.

Balanced Search Trees

- What are the requirements to call a tree a balanced tree?

- Would you require a tree to be perfect/complete to call it balanced?
  - No! They are too restrictive.

Balanced Search Trees

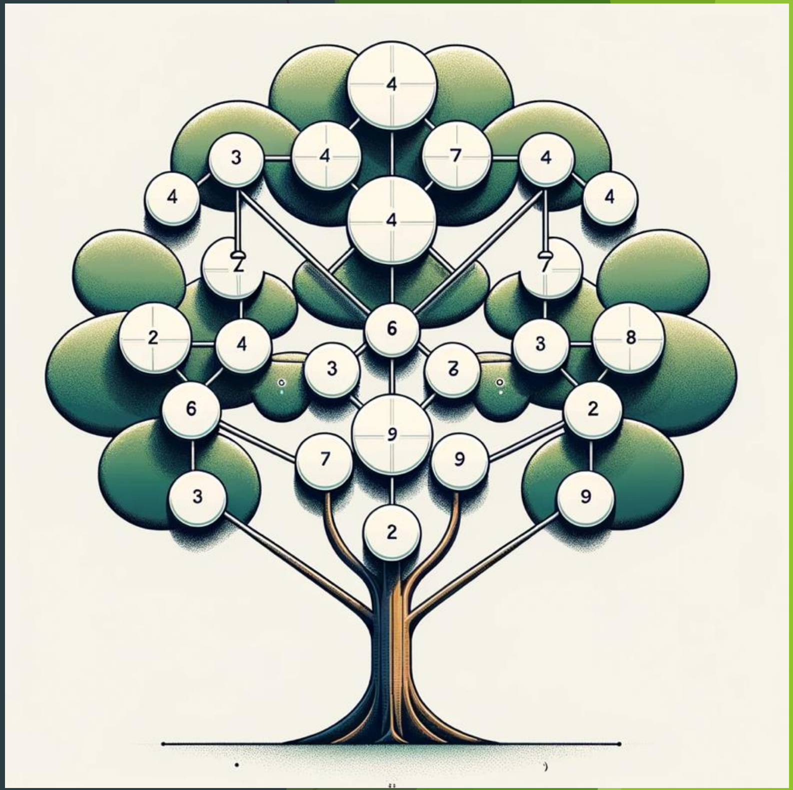
- We need another definition of "balanced condition."
- We want the definition to satisfy the following two criteria:
  1. Height of a tree of n nodes = $O(\log n)$.
  2. Balance condition can be maintained efficiently: $O(\log n)$ time to rebalance a tree.
- Several balanced search trees, each with its own balance condition
  - AVL trees
  - 2-3 trees
  - red-black trees

AVL Trees

- Adelson-Velsky and Landis' trees
  - AVL tree is a binary search tree.

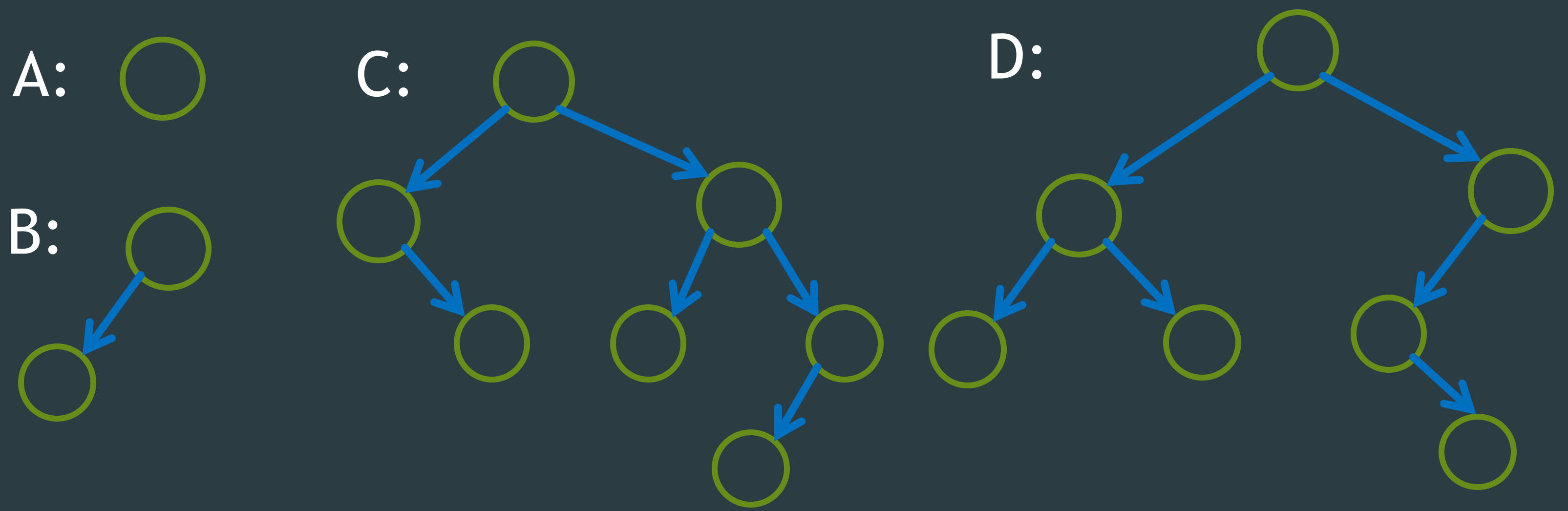
- AVL trees' balance condition:
  - An empty tree is AVL balanced.
  - A non-empty binary tree is AVL balanced if

1. Both its left and right subtrees are AVL balanced, and
2. The height of left and right subtrees differ by at most 1.



Which of the Following Trees Are AVL Balanced?

- Select all the AVL balanced trees.



AVL trees' balance condition:

- An empty tree is **AVL balanced**.
- A non-empty binary tree is **AVL balanced** if
  1. Both its left and right subtrees are AVL balanced, and
  2. The height of left and right subtrees differ by **at most 1**.

Properties of AVL Trees

- The height h of an AVL balanced tree with n internal nodes satisfies
$$\log_2(n+1) - 1 \le h \le 1.44 \log_2(n+2)$$

- AVL trees satisfies the general "balanced condition" 1:
  - The height of a tree of n nodes is $O(\log n)$.
  - Search is guaranteed to always be $O(\log n)$ time!
- We will also show that AVL trees satisfy the general "balance condition" 2:
  - Balance condition can be maintained efficiently.

Height of AVL Tree Is O(log n)

- Let N_h denote the minimum number of nodes in an AVL tree of height h

- $N_h = N_{h-1} + N_{h-2} + 1$
- $N_{h-1} = N_{h-2} + N_{h-3} + 1$

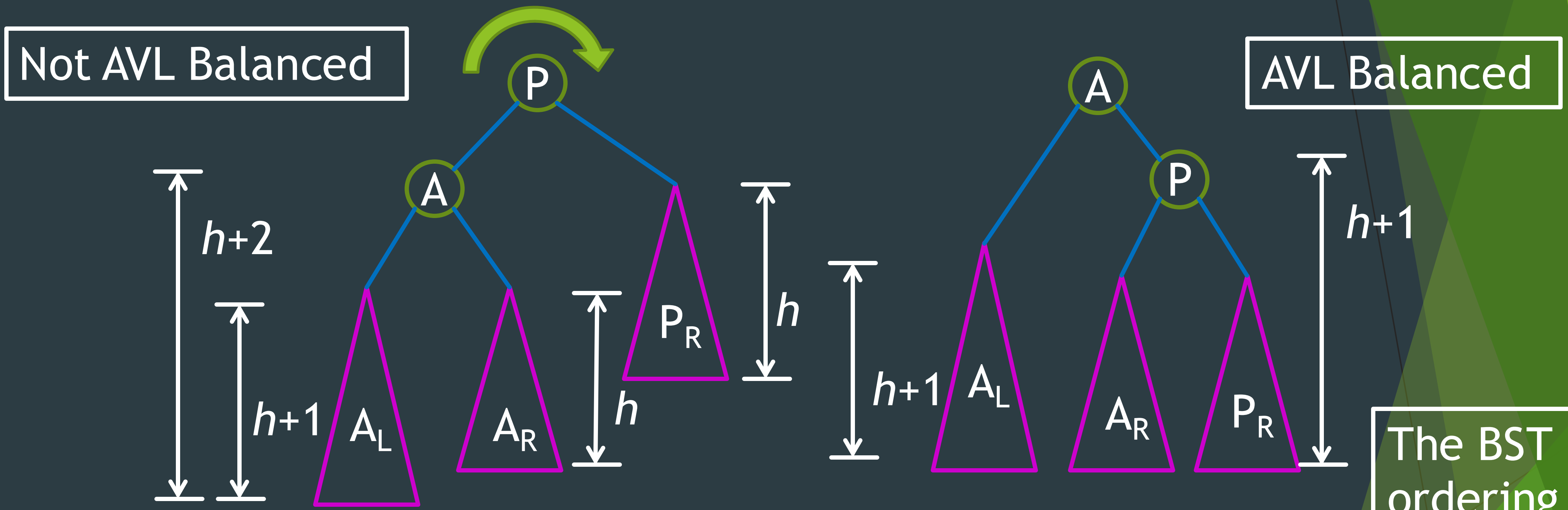
- $N_h > 2N_{h-2}$
- $N_h > 4N_{h-4}$
- $N_h > 8N_{h-6}$
- $N_h > 2^{h/2}$

- $Log_2(N_h) > h$

AVL Trees Operations

- Search, insertion, and removal all work exactly the same as with BST.
- However, after each insertion or removal, we must check whether the tree is still AVL balanced.
  - If not, we need to "re-balance" the tree.

Re-Balance the Tree via Rotation

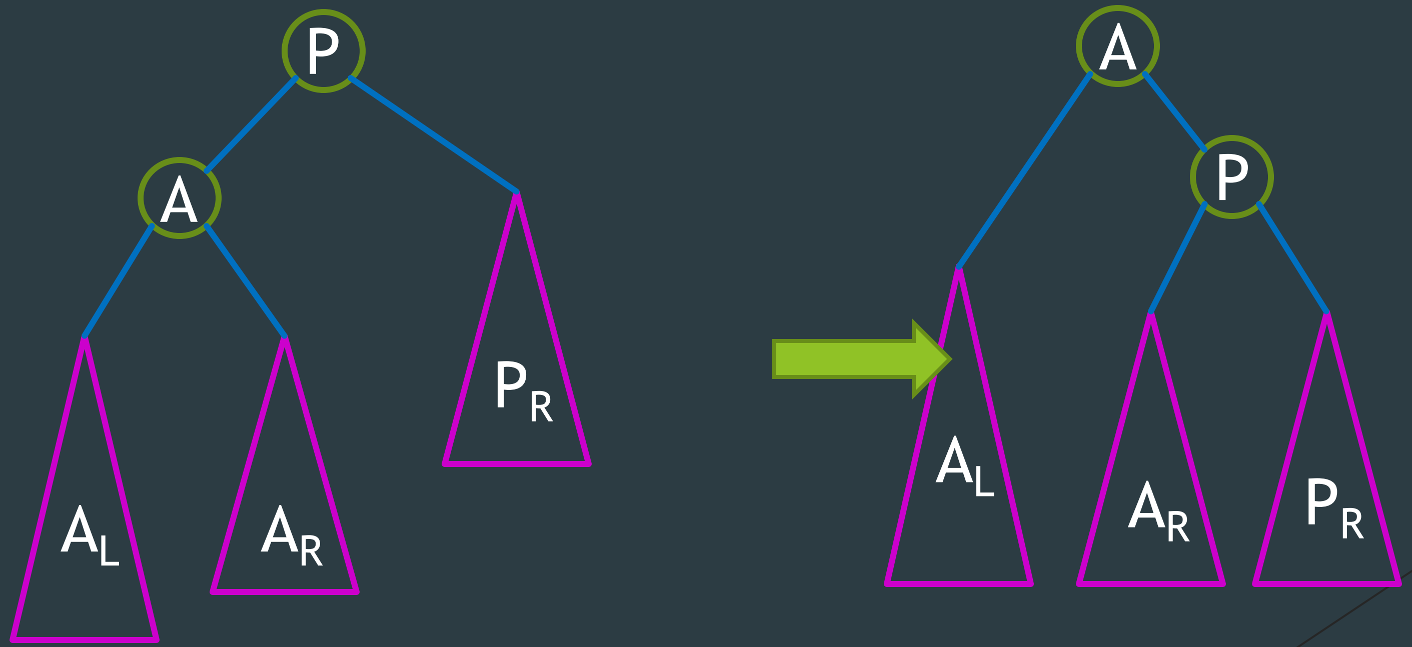


The BST ordering on keys are preserved.

- The rotation operation:
  - Interchange the role of a parent and one of its children, while still preserving the BST ordering on the keys.

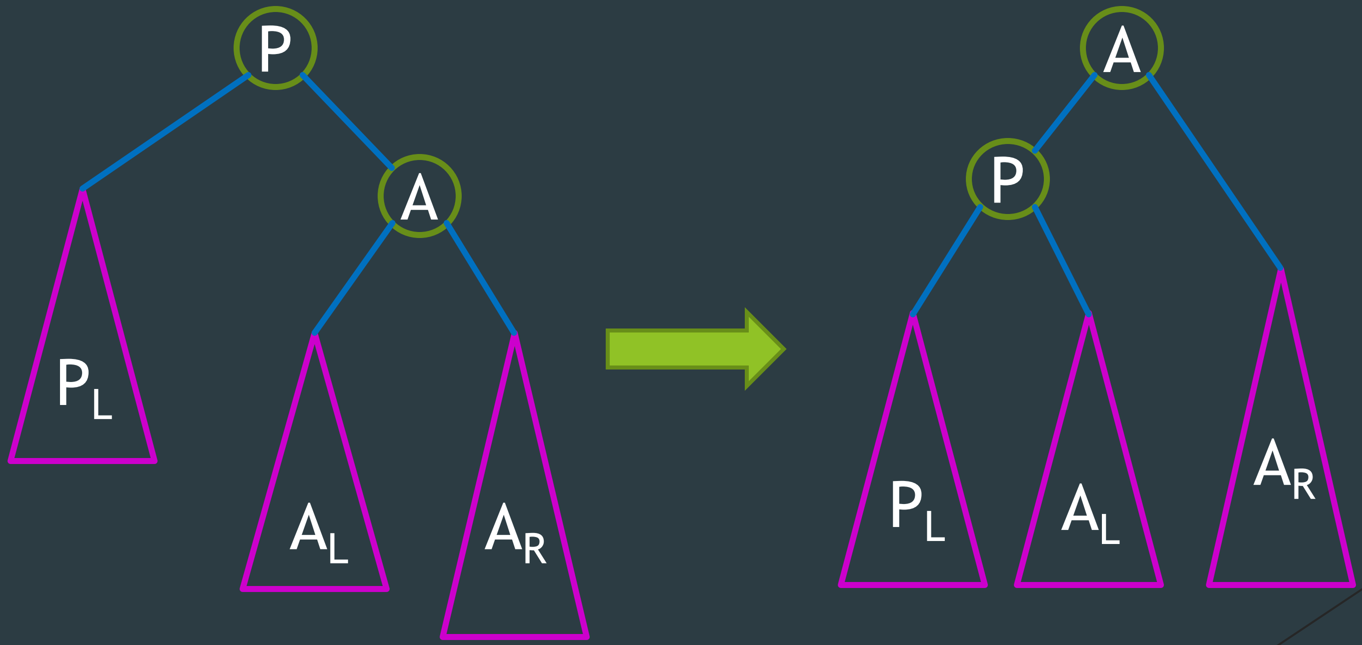
Right Rotation

1. The right link of the **left child** becomes the left link of the **parent**.
2. **Parent** becomes right child of the **old left child**.



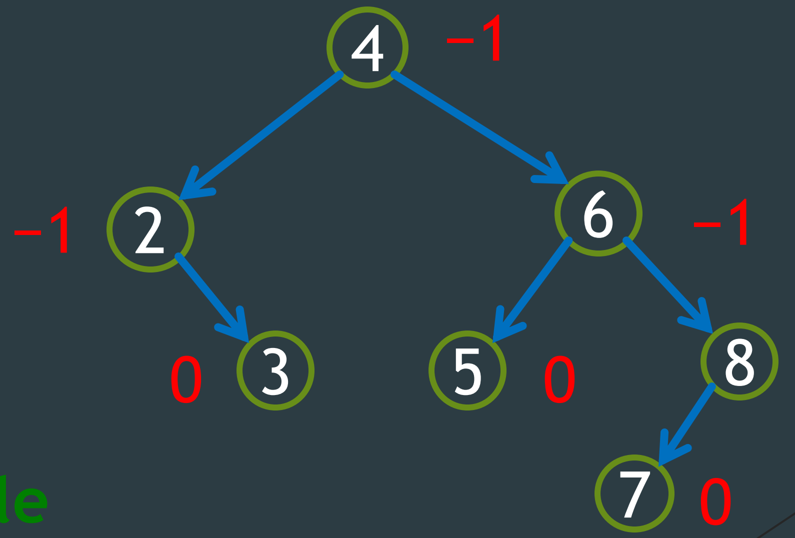
Left Rotation

- The left link of the right child becomes the right link of the parent.
- Parent becomes left child of the old right child.



Balance Factor

- Let T_l and T_r be the left and right subtrees of a tree rooted at node T.
- Let h_l be the height of T_l and h_r be the height of T_r.
- Define the balance factor (B_T) of node T as
$$B_T = h_l - h_r$$

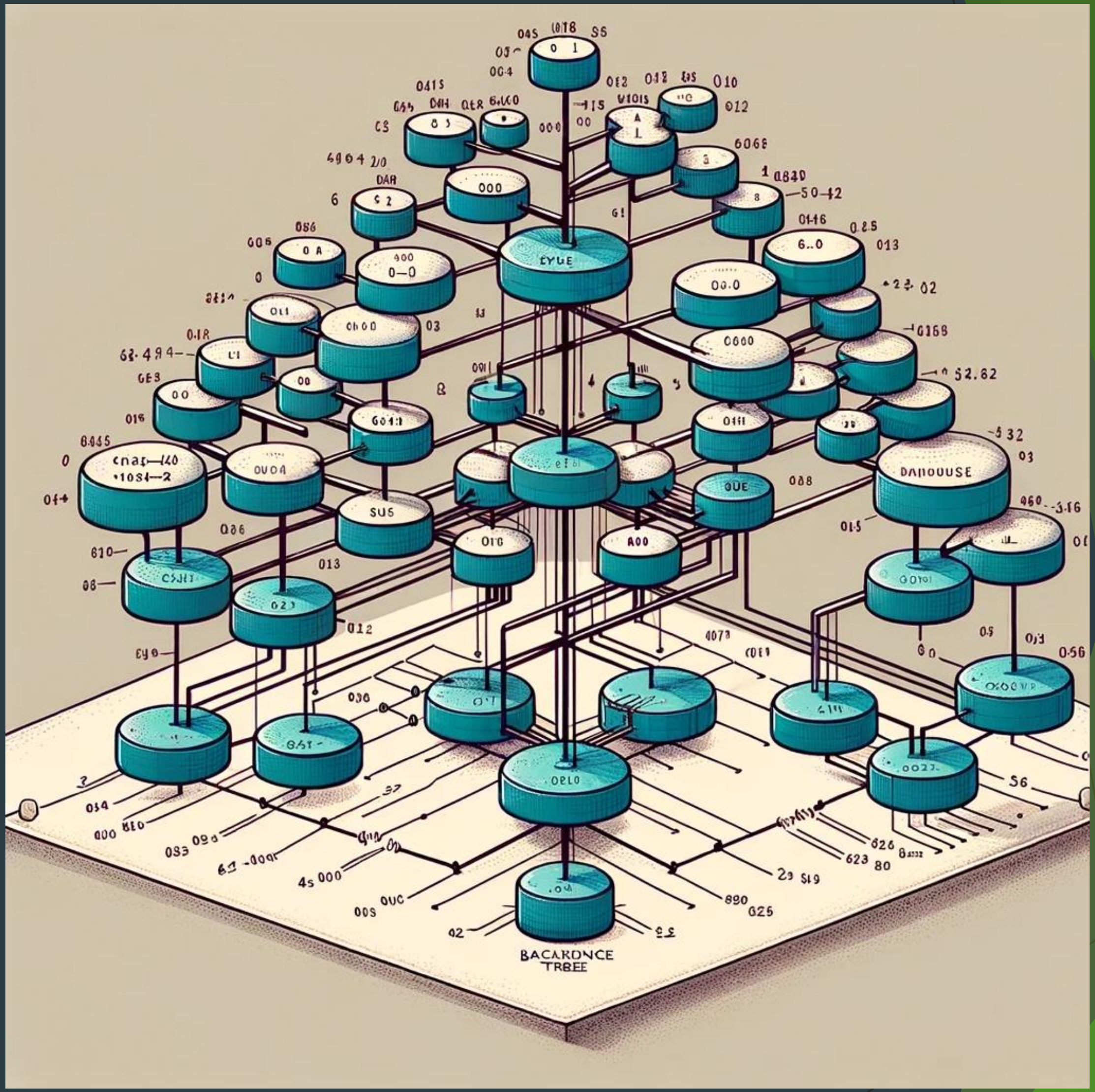


- AVL tree's balance condition:
  - For every node T in the tree, $|B_T| \leq 1$.

Balance Factor Example

Outline

- Balanced Search Trees
  - AVL Trees
- AVL Tree Insertion
- Supporting Data Members and Functions of AVL Tree



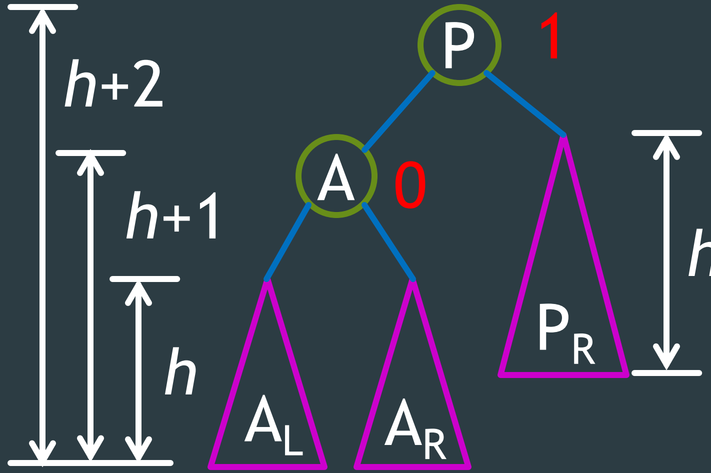
Insertion

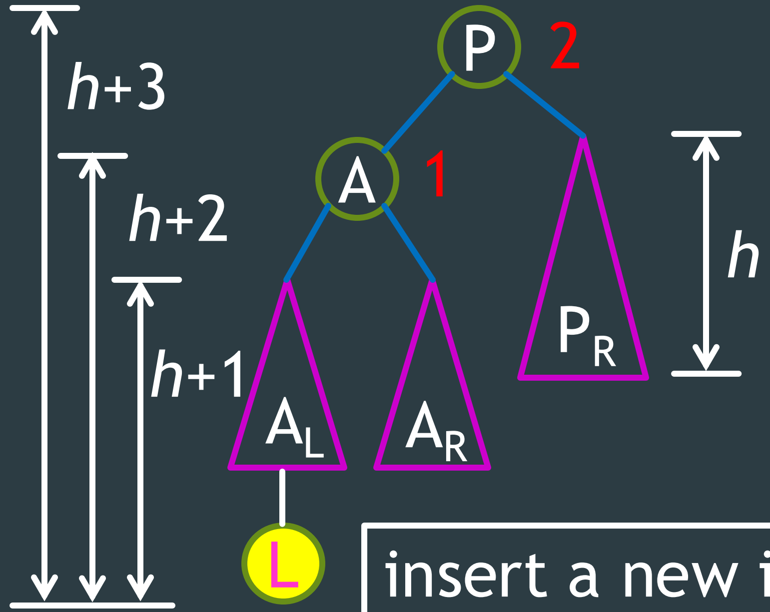
- Inserting an item in a tree affects potentially the heights of all of the nodes along the access path, i.e., the path from the root to that leaf.
- When an item is inserted in a tree, the height of any node on the access path may increase by one.
- To ensure the resulting tree is still AVL balanced, the heights of all the nodes along the access path must be recomputed and the AVL balance condition must be checked.
  - Sometimes, increasing the height by one does not violate the AVL balance condition.
  - In other cases, the AVL balance condition is violated.
  - We will fix the first unbalanced node in the access path from the leaf.

Breaking AVL Balance Condition

Left-Left Insertion

P is the **first unbalanced node** in the access path from the leaf.



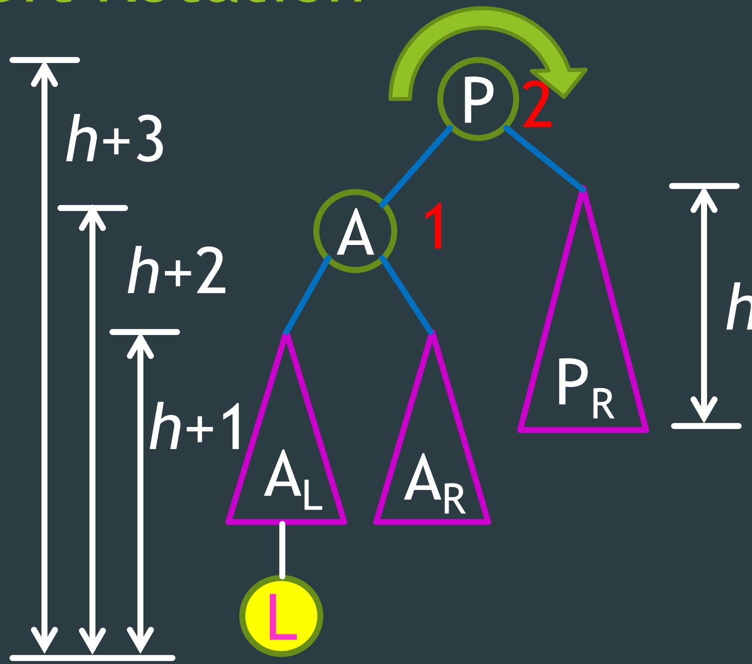


insert a new item

Left-left insertion: the first two edges in the insertion path from node P both go to the left.

Restoring AVL Balance Condition

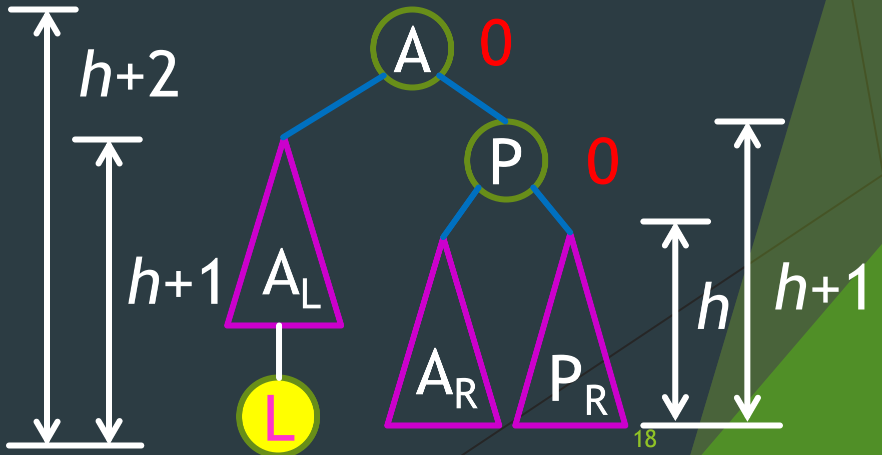
Left-Left Rotation



How to restore AVL balance?

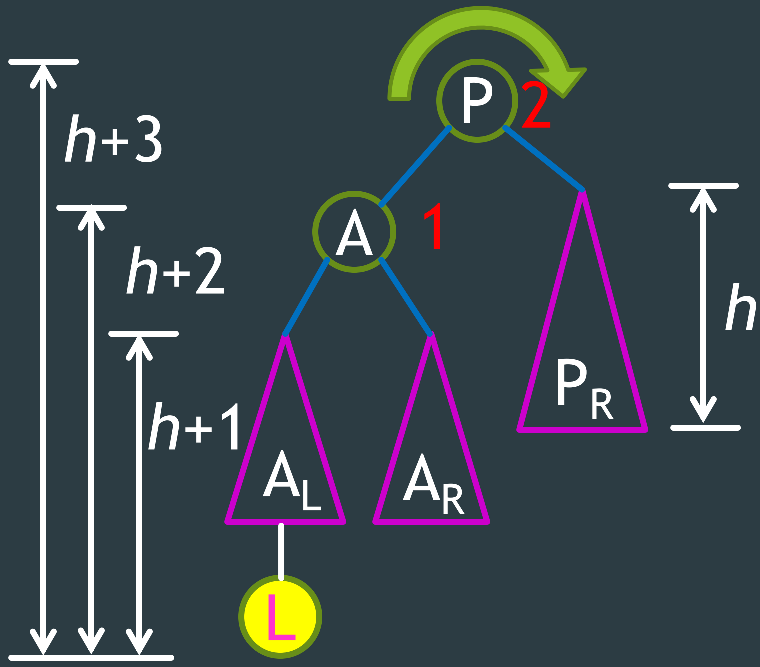
Do a right rotation at node P.

The rotation is also called **left-left (LL) rotation**.



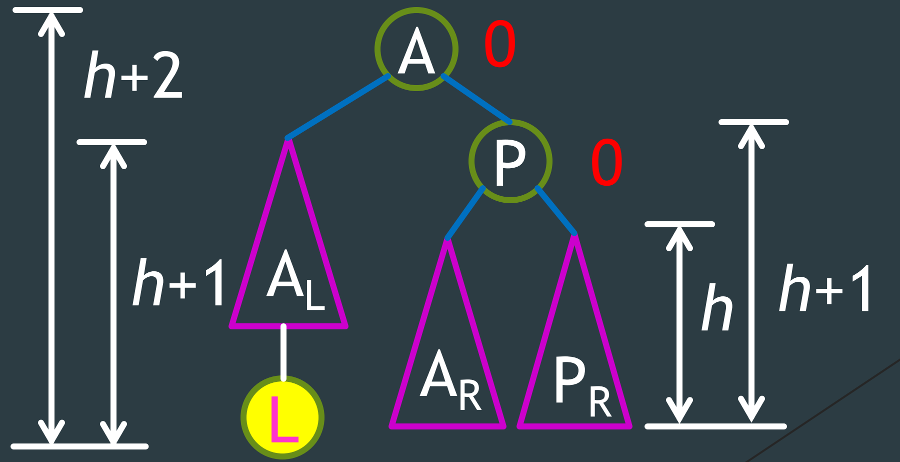
Restoring AVL Balance Condition

Left-Left Rotation



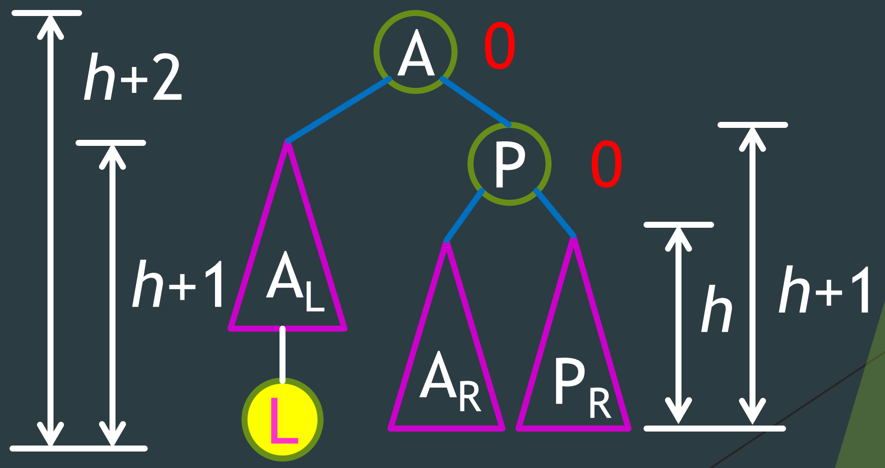
An **LL rotation** is called for when the node becomes unbalanced with a **positive** balance factor and the left subtree of the node also has a **positive** balance factor.

The rotation is also called **left-left (LL) rotation**.



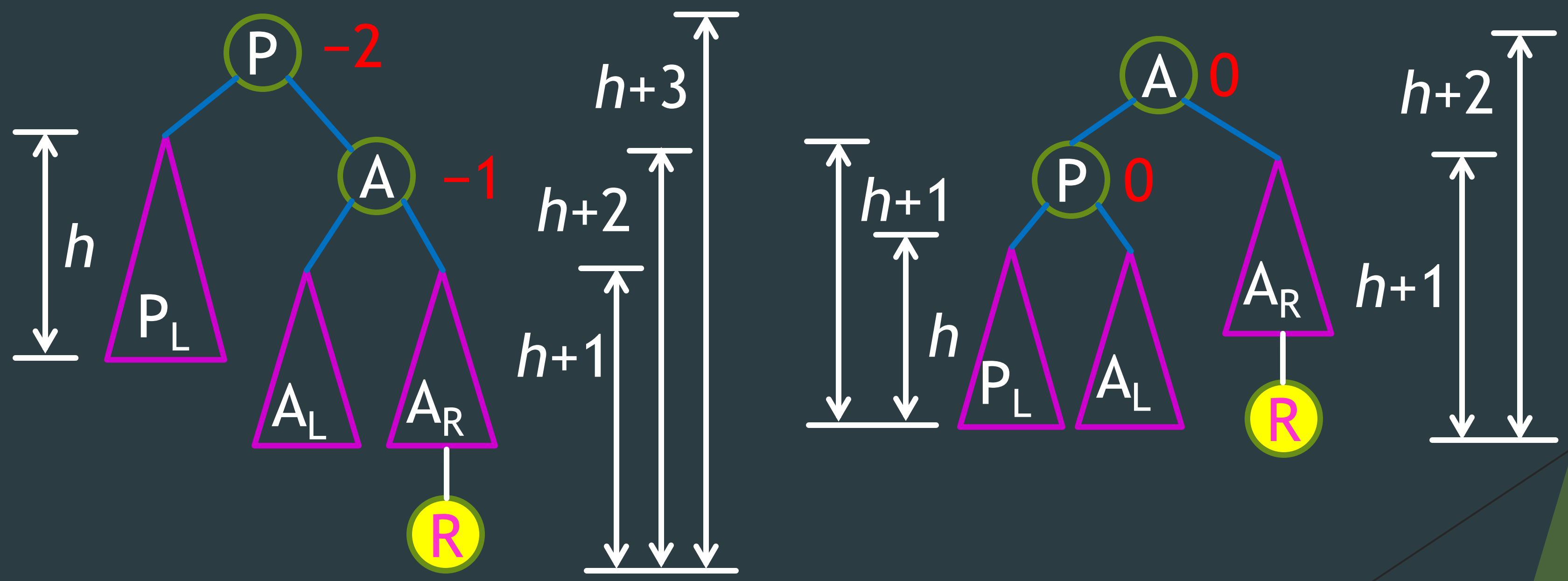
Properties of Left-Left Rotation

- The ordering property of BST is kept.
- Both nodes A and P have balance factor of 0.
- The height of the tree after the rotation is the same as the height of the tree before insertion.



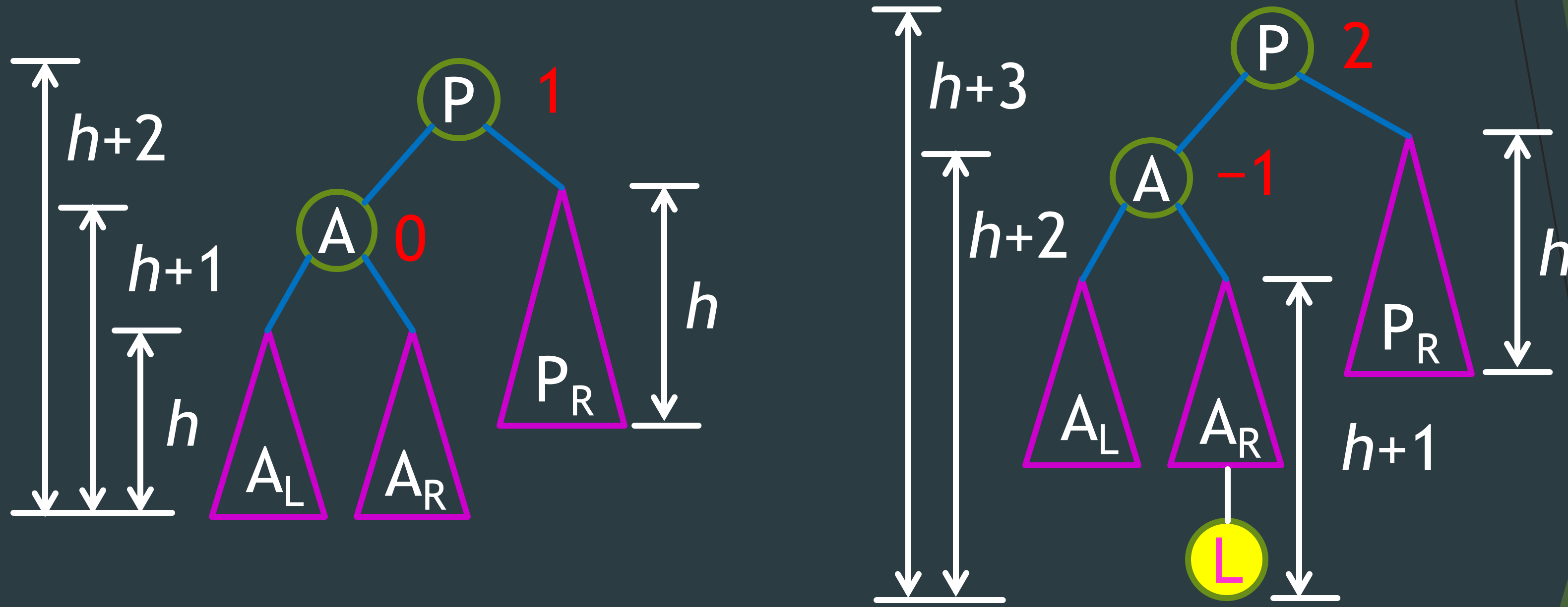
Right-Right (RR) Rotation

- Symmetric to left-left rotation.
- An RR rotation is called for when the node becomes unbalanced with a negative balance factor and the right subtree of the node also has a negative balance factor.



Breaking AVL Balance Condition

Left-Right Insertion

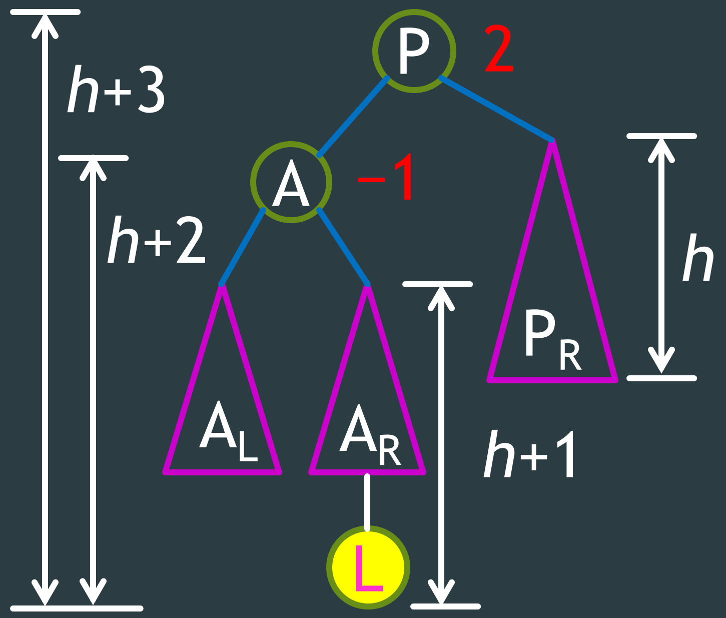


insert a new item

Left-right insertion: the first edge in the insertion path goes to the left and the second edge goes to the right.

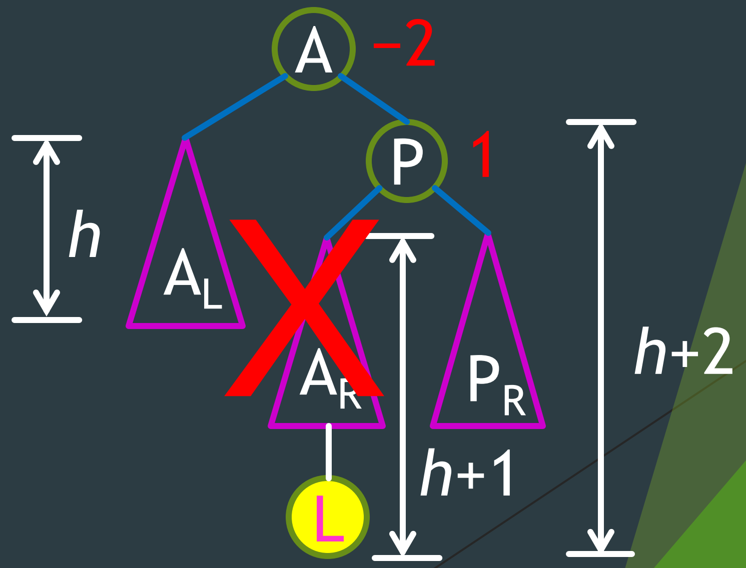
Restoring AVL Balance Condition

Left-Right Insertion

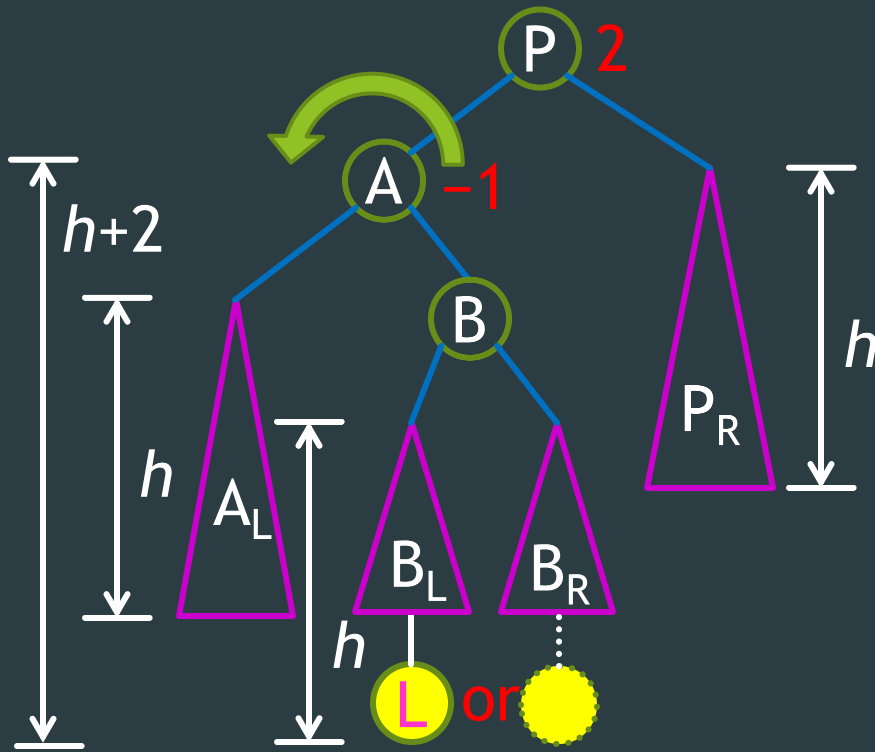


How to restore AVL balance?

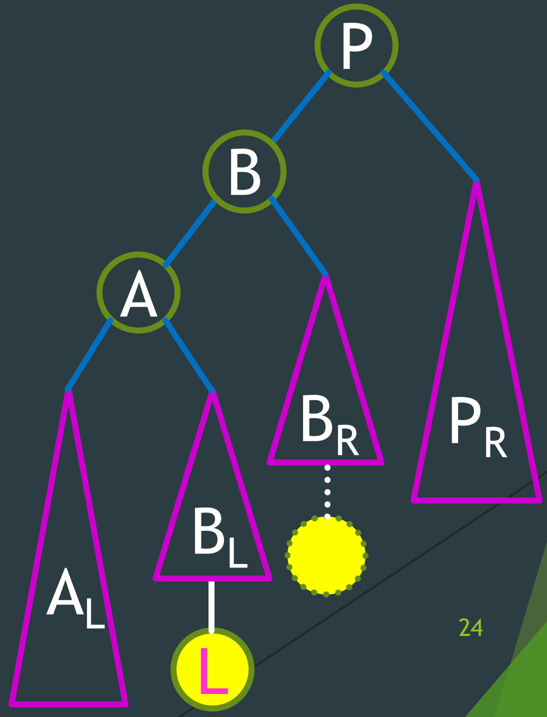
A right rotation at node P does not work!



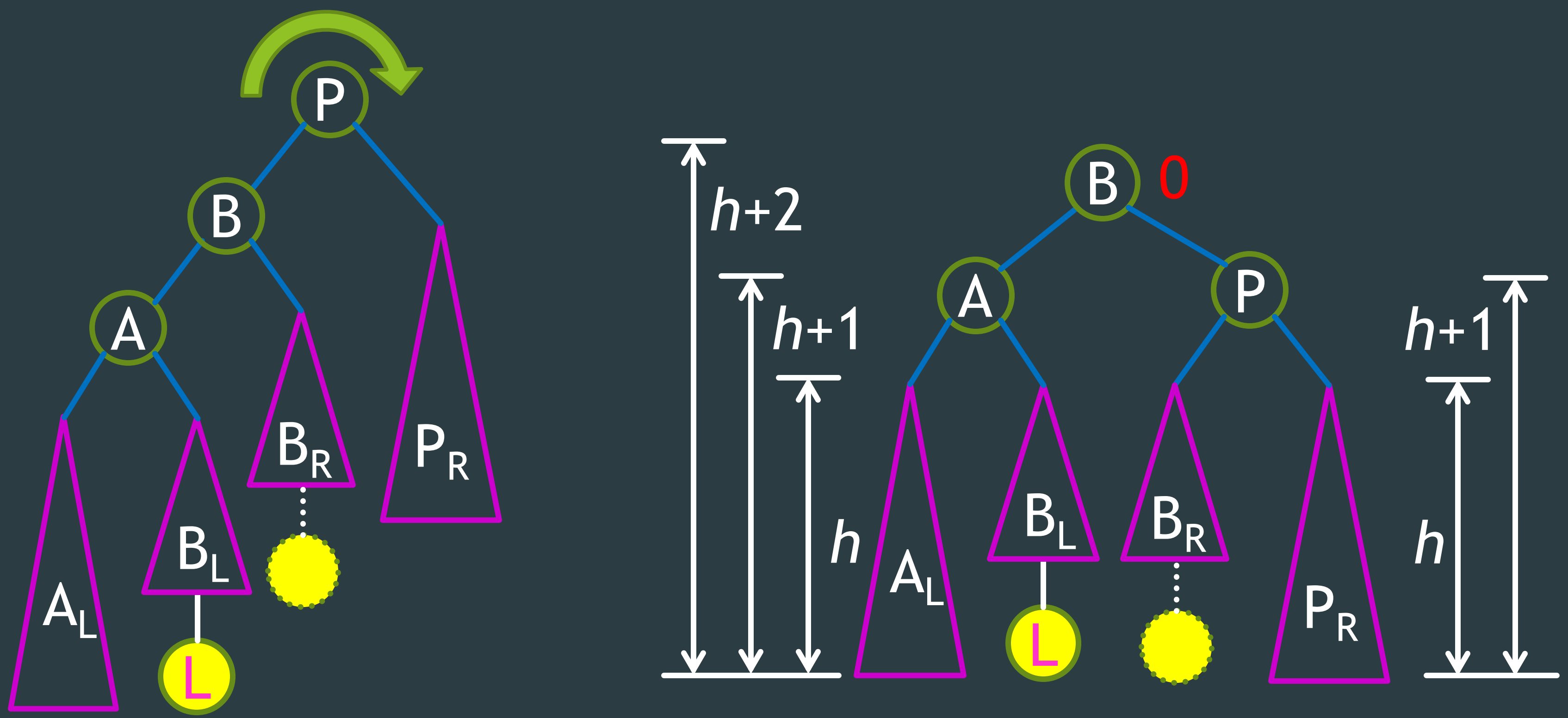
Left-Right (LR) Rotation



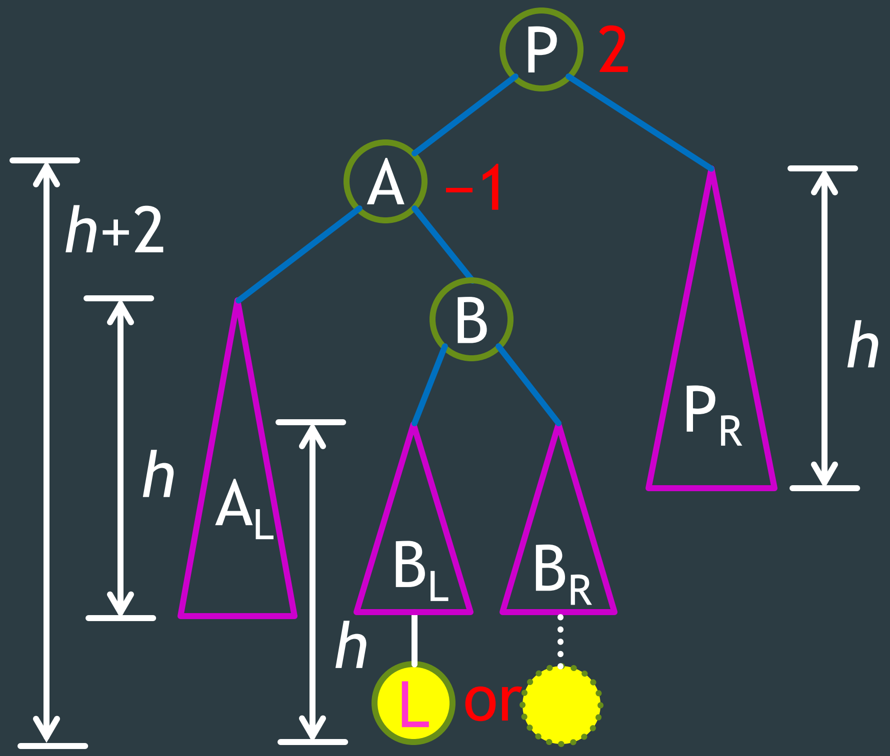
A double rotation to re-balance:
Do a left rotation on node A;
then a right rotation on node P
(next slide).



Left-Right (LR) Rotation



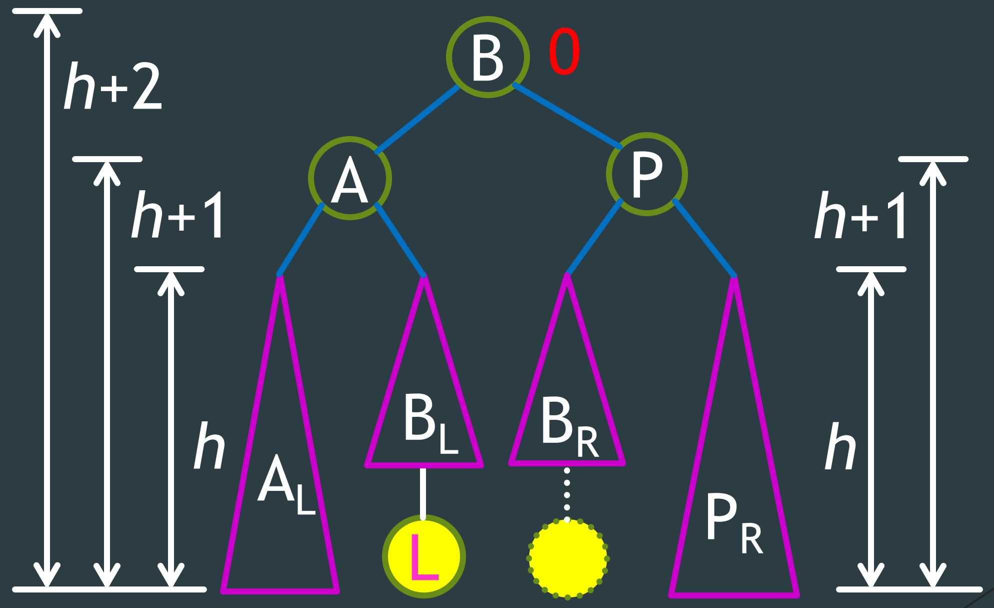
Left-Right (LR) Rotation



An **LR rotation** is called for when the node becomes unbalanced with a **positive** balance factor but the left subtree of the node has a **negative** balance factor.

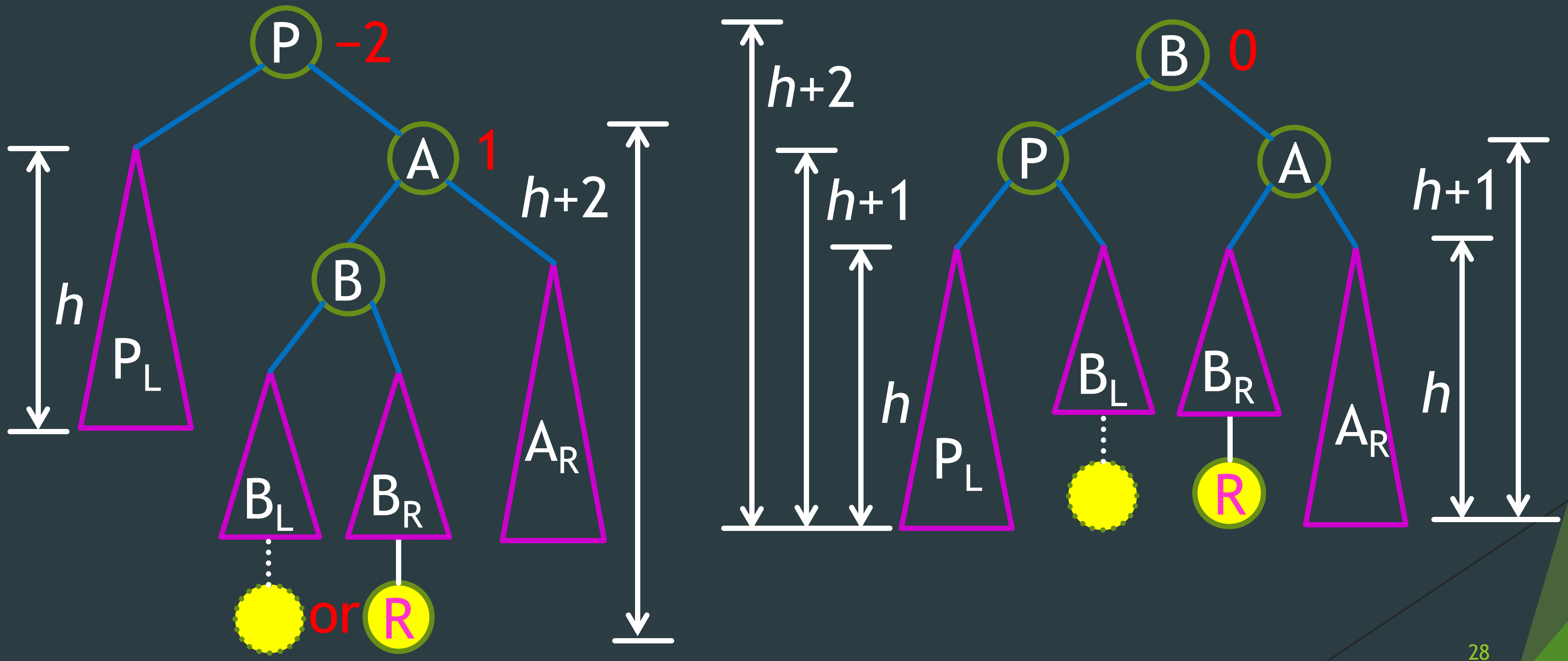
Properties of Left-Right Rotation

- The ordering property of BST is kept.
- Node B has a balance factor of 0.
- The height of the tree after the rotation is the same as the height of the tree before insertion.



Right-Left (RL) Rotation

- Symmetric to left-right rotation; also a double rotation.
- An RL rotation is called for when the node becomes unbalanced with a negative balance factor but the right subtree of the node has a positive balance factor.



Rotation Summary

- When an AVL tree becomes unbalanced, there are four cases to consider depending on the **direction** of the first two edges on the insertion path from the <u>unbalanced node</u>:
  - Left-left
  - Right-right
  - Left-right
  - Right-left

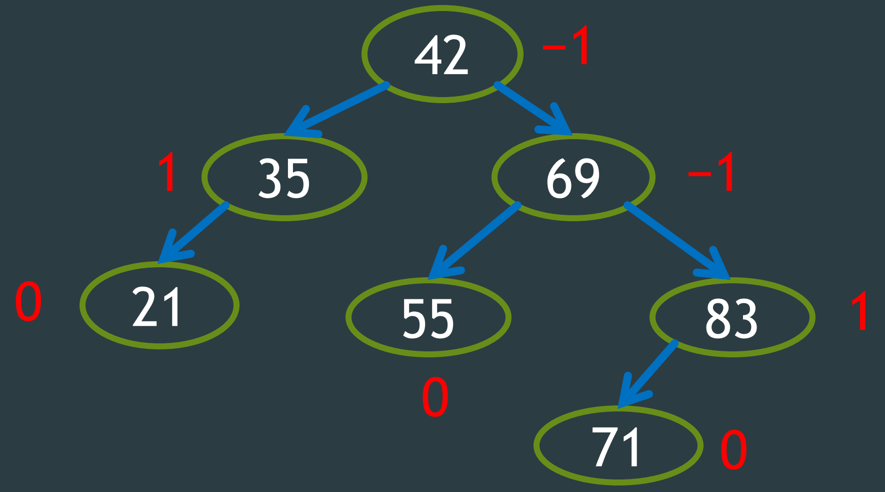
LL Rotation
RR Rotation
single rotation

LR Rotation
RL Rotation
double rotation

Note: We fix **the first unbalanced node** in the access path **from the leaf**.

Exercises

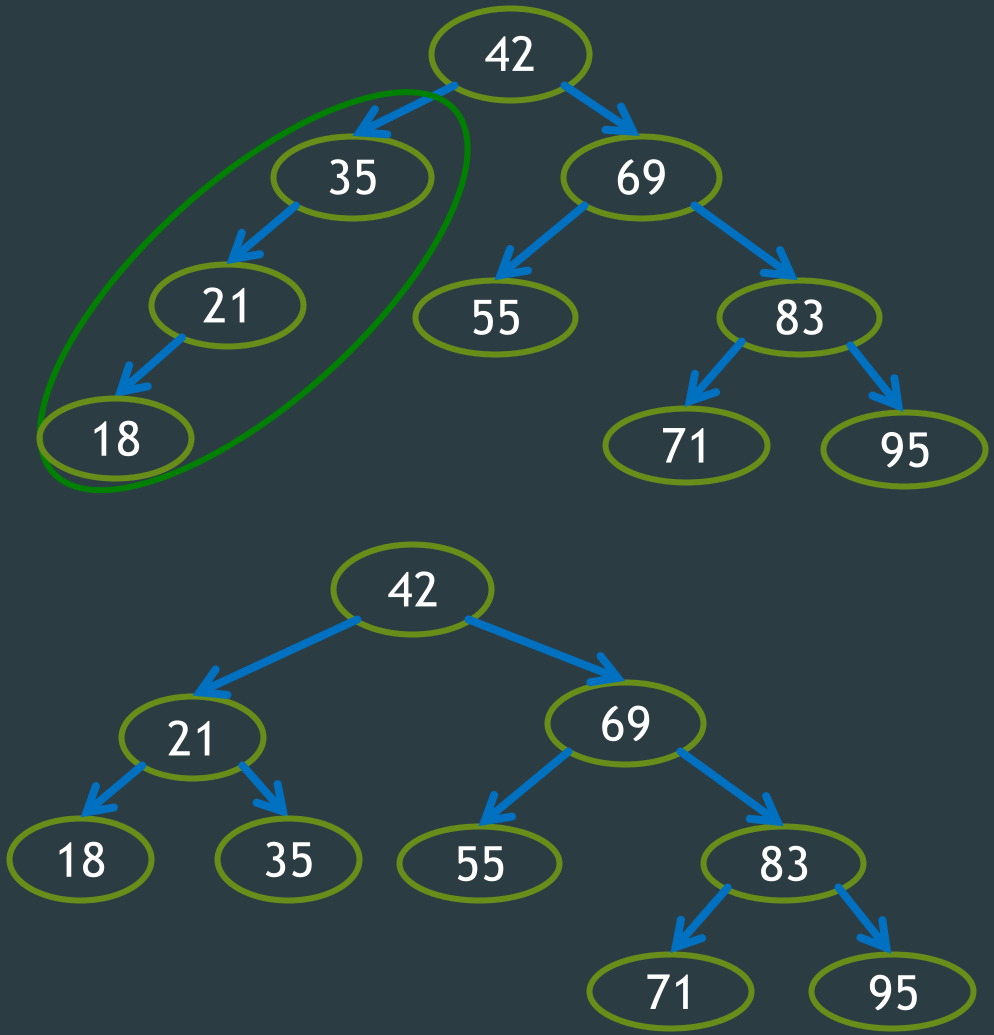
- Insert into an empty BST: 42, 35, 69, 21, 55, 83, 71.
  - Compute the balance factors.
  - Is the tree AVL balanced?



- Insert 95, 18, 75?

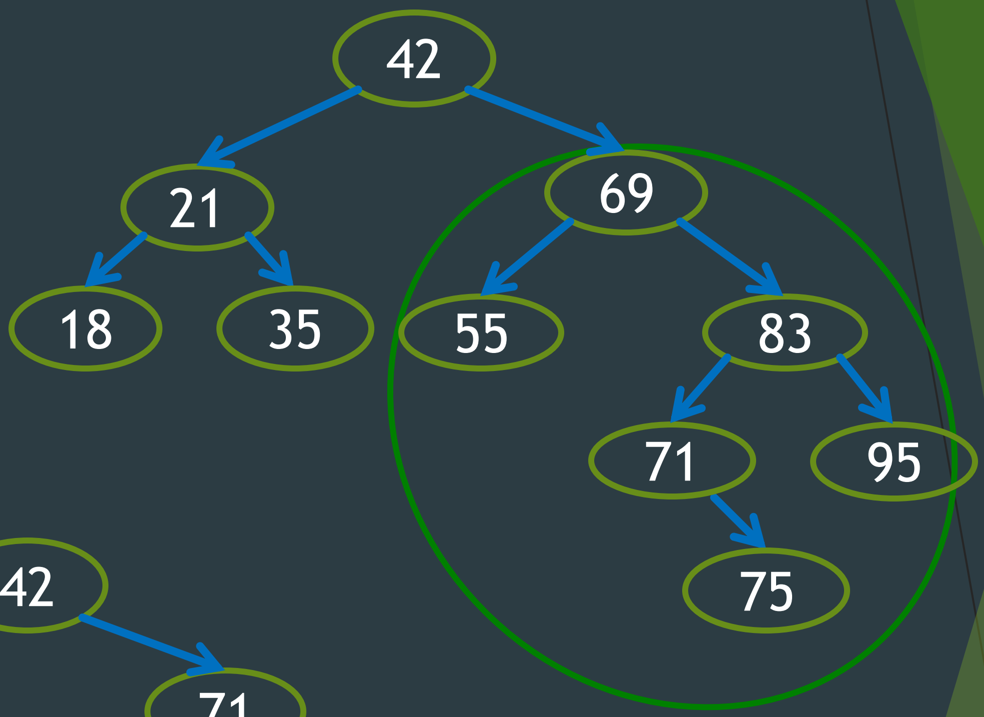
Exercises

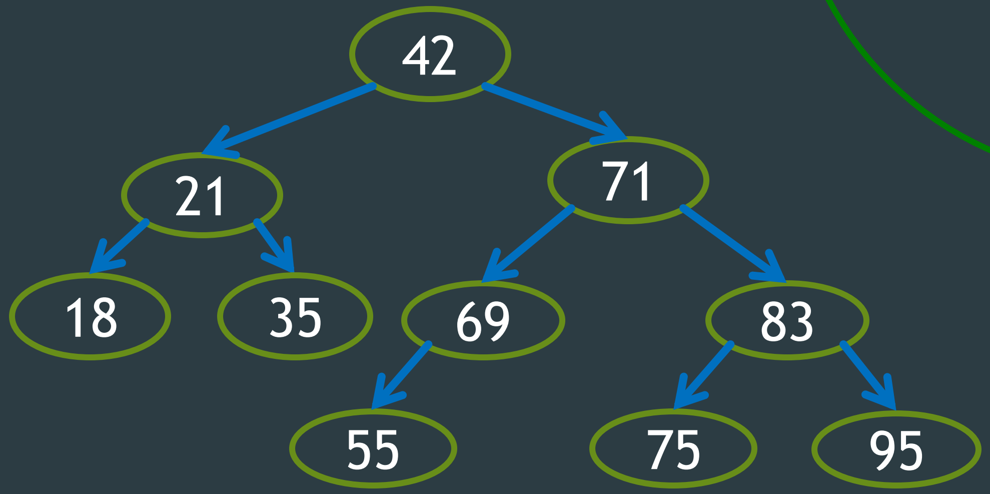
- Insert 95, 18



Exercises

- Insert 75





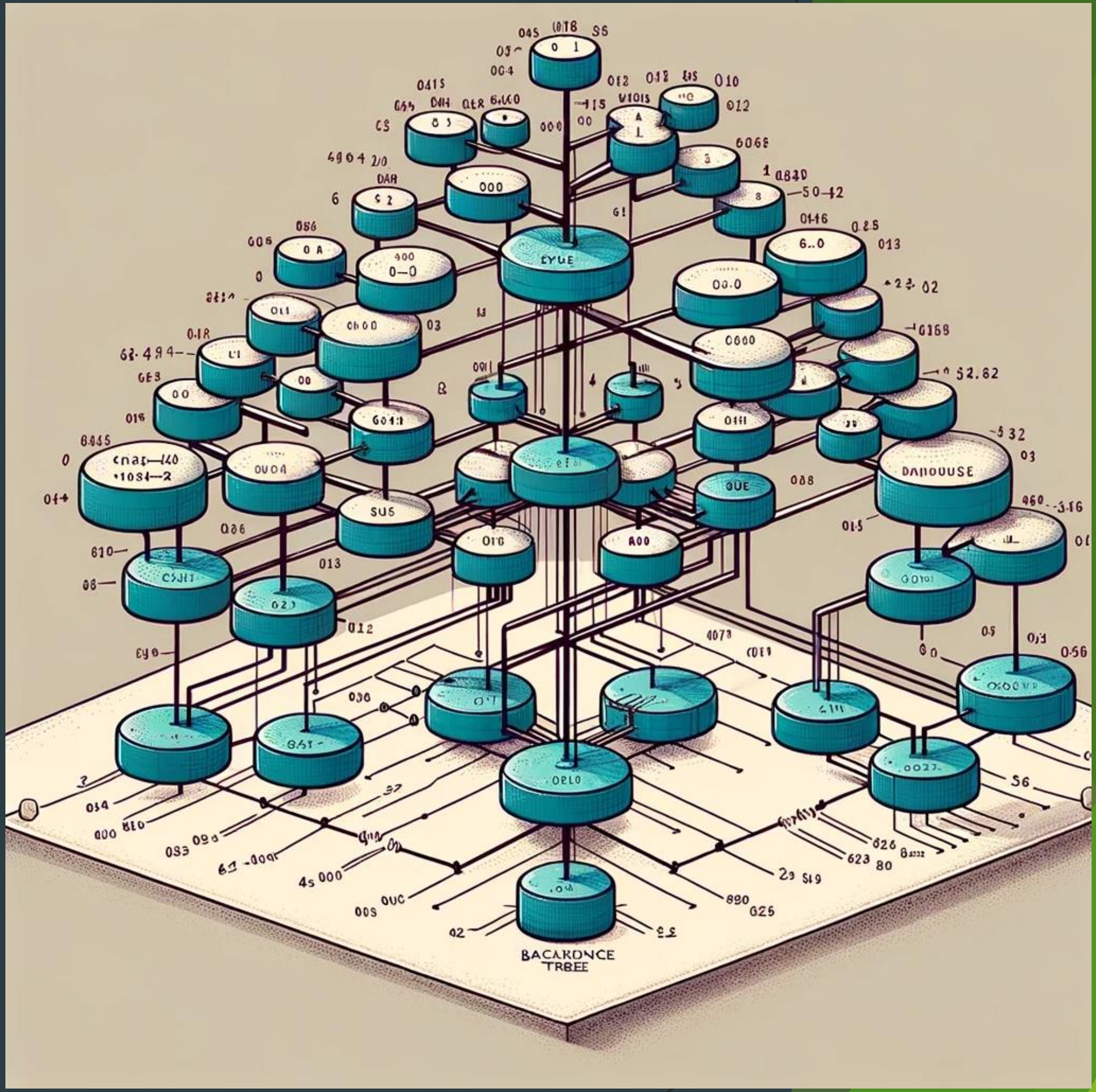
The Number of Rotations Required

- When an AVL tree becomes unbalanced after an insertion, **exactly one** single or double rotation is required to balance the tree.
  - Before the insertion, the tree is balanced.
  - Only nodes on the access path of the insertion can be unbalanced. All other nodes are balanced.
  - We rotate at the first unbalanced node **from the leaf**.
  - By the properties of rotation, the height of the node after rotation is the same as that before insertion.
  - All ancestors of that node on the access path should now be balanced.

Animation

Outline

- Balanced Search Trees
  - AVL Trees
- AVL Tree Insertion
- Supporting Data Members and Functions of AVL Tree



AVL Trees

Supporting Data Members and Functions

```
struct node {
  Item item;
  int height;
  node *left;
  node *right;
};
```

```
int Height(node *n) {
  if(!n) return -1;
  return n->height;
}

void AdjustHeight(node *n) {
  if(!n) return;
  n->height = max( Height(n->left),
    Height(n->right) ) + 1;
}

int BalFactor(node *n) {
  if(!n) return 0;
  return (Height(n->left) -
    Height(n->right));
}
```

AVL Trees

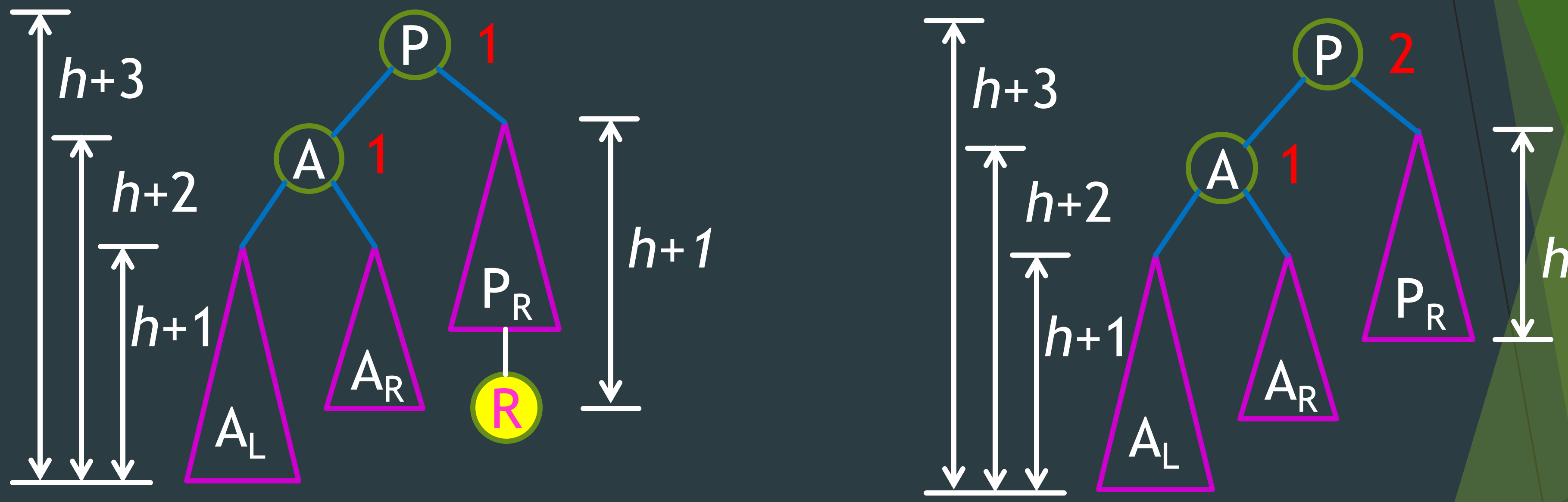
Supporting Functions

```
void LLRotation(node *&n);
void RRRotation(node *&n);
void LRRotation(node *&n);
void RLRotation(node *&n);


void Balance(node *&n) {
  if(BalFactor(n) > 1) {
    if(BalFactor(n->left) > 0) LLRotation(n);
    else LRRotation(n);
  }
  else if(BalFactor(n) < -1) {
    if(BalFactor(n->right) < 0) RRRotation(n);
    else RLRotation(n);
  }
}
```

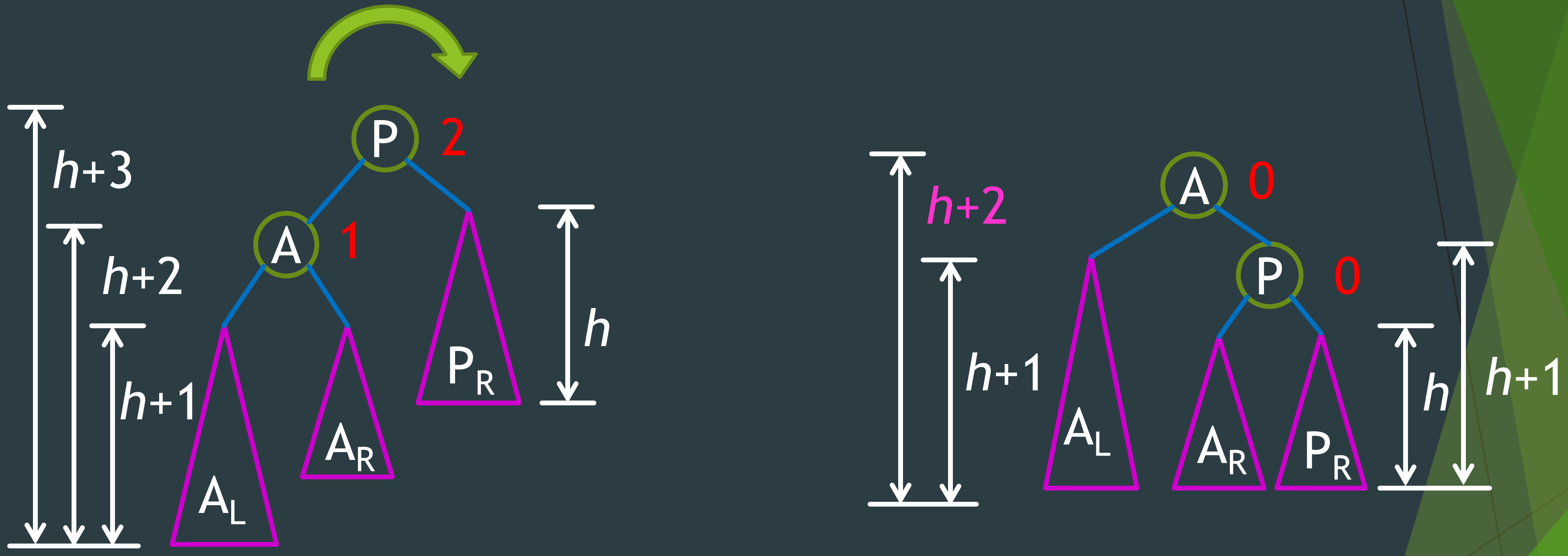
AVL Trees
Changes to Insertion

```
void insert(node *&root, Item item)
{
  if(root == NULL) {
    root = new node(item);
    return;
  }
  if(item.key < root->item.key)
    insert(root->left, item);
  else if(item.key > root->item.key)
    insert(root->right, item);

  Balance(root);
  AdjustHeight(root);
}
```

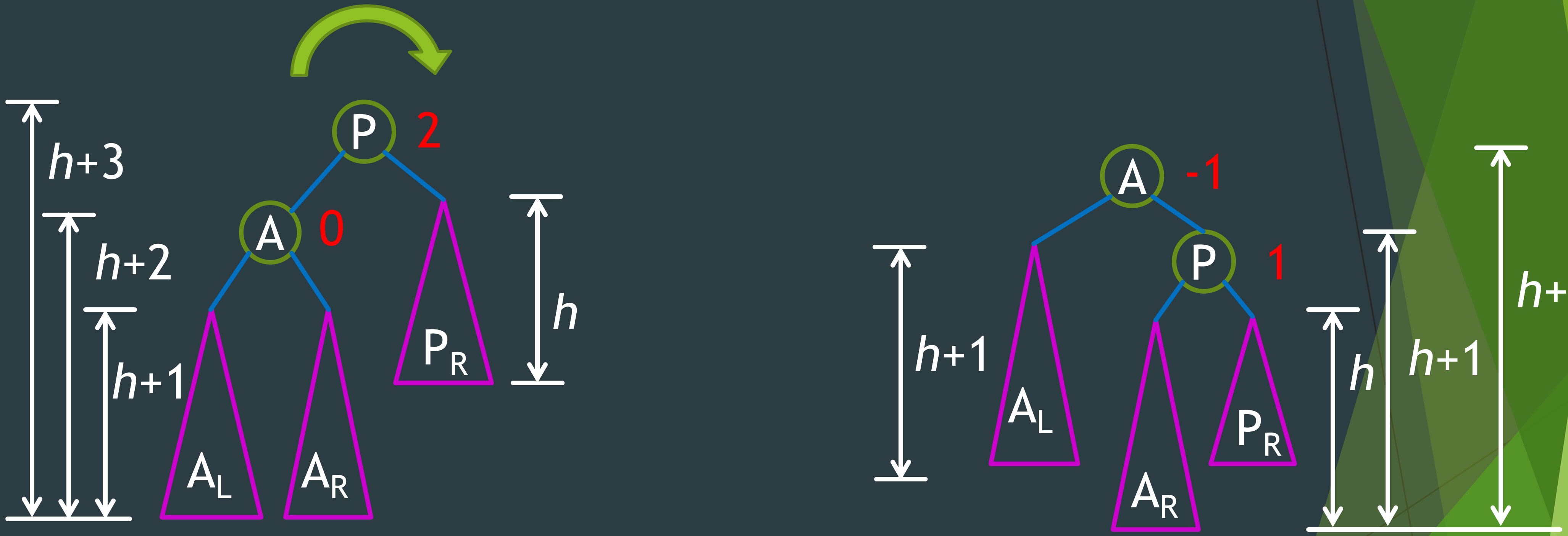
Imbalance after Removal



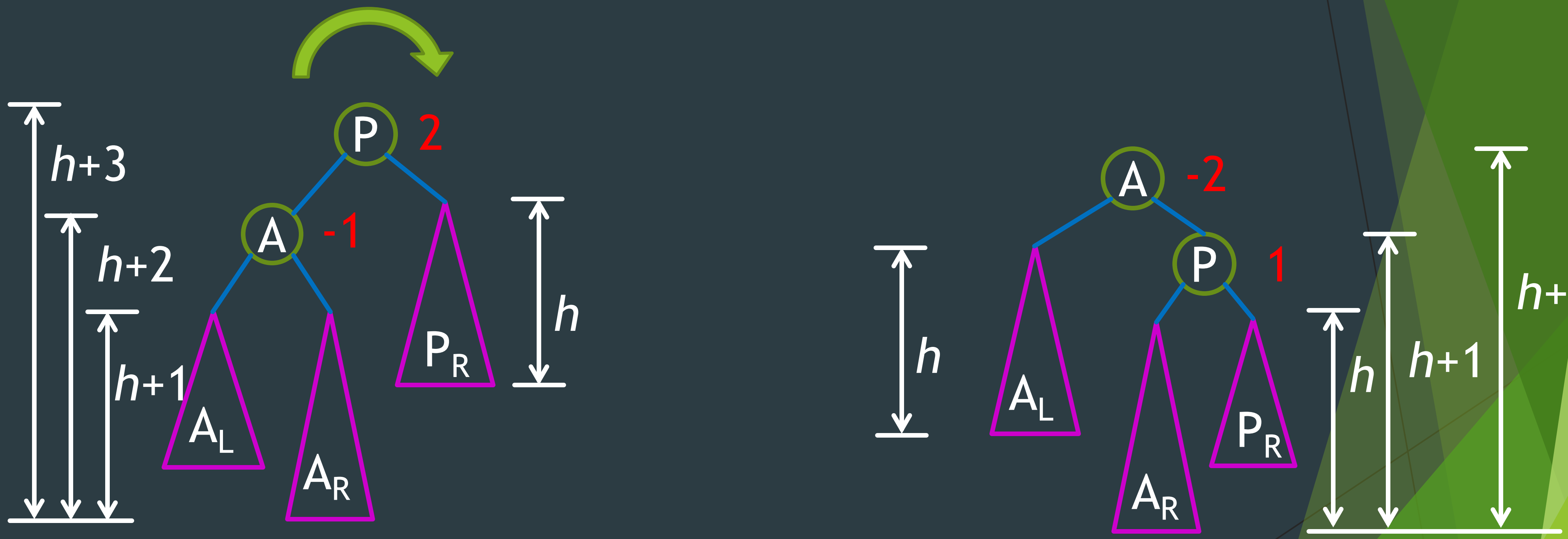
Imbalance after Removal: Case 1



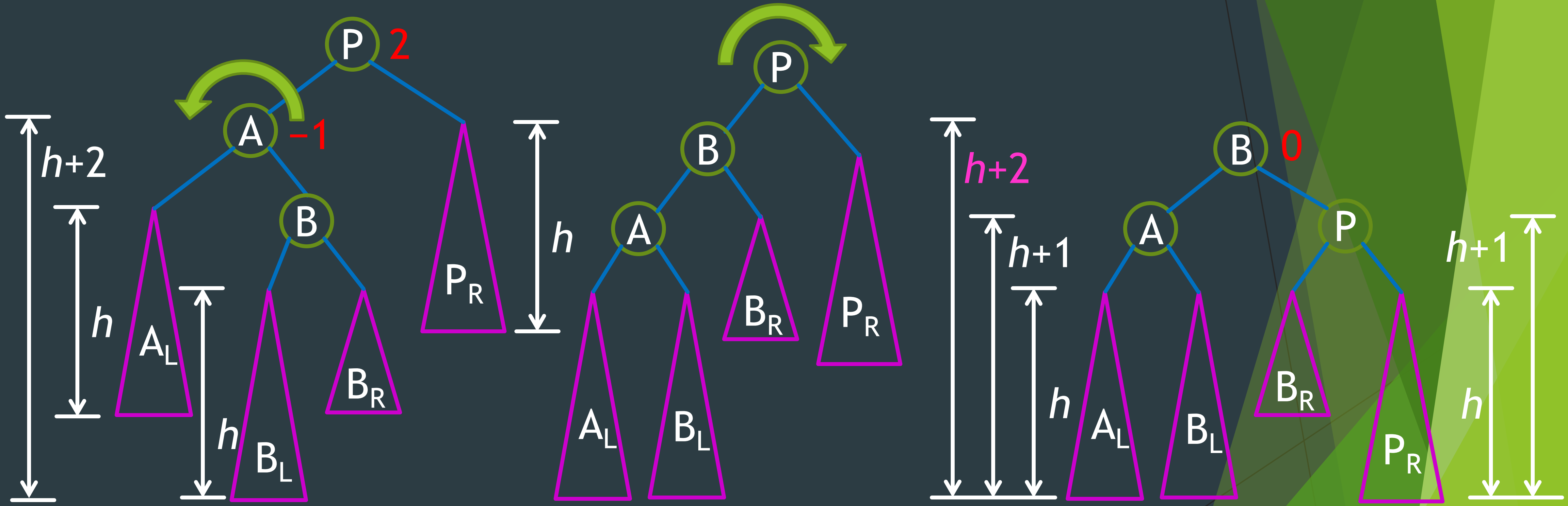
Imbalance after Removal: Case 2



Imbalance after Removal: Case 3



Case 3



Removal

- First remove node as with BST
- Then update the balance factors of those ancestors in the access path and rebalance as needed.
- **Difference from insertion: a single rotation might not completely fix all AVL imbalance**
- Time Complexity: O(log n)
  - Why?
  - Only rebalance along the ancestor path

Summary of AVL Tree

- Search: O(log n)
- Insert: O(log n)
- Delete: O(log n)

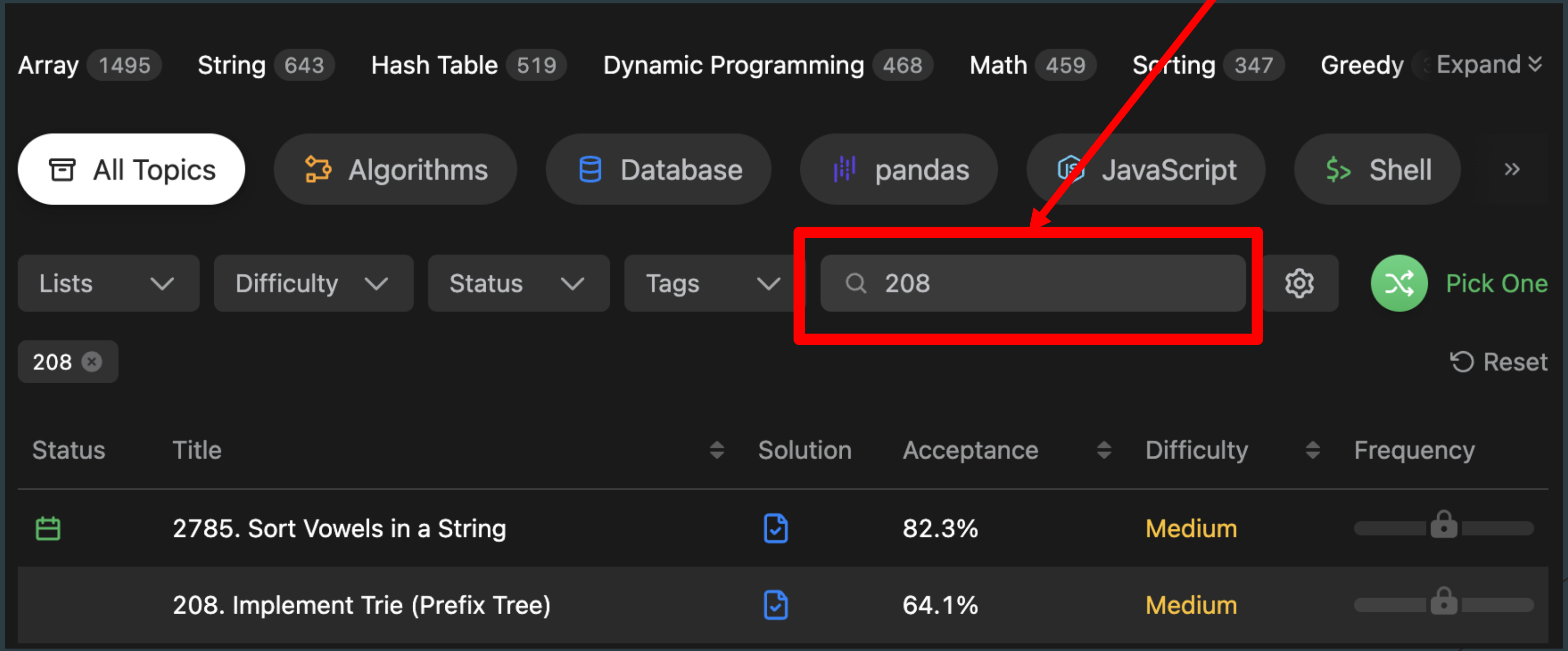
Exercise 1

Canvas -> Exercise -> AVL Trees

Exercise 2

Problem 14. Longest Common Prefix

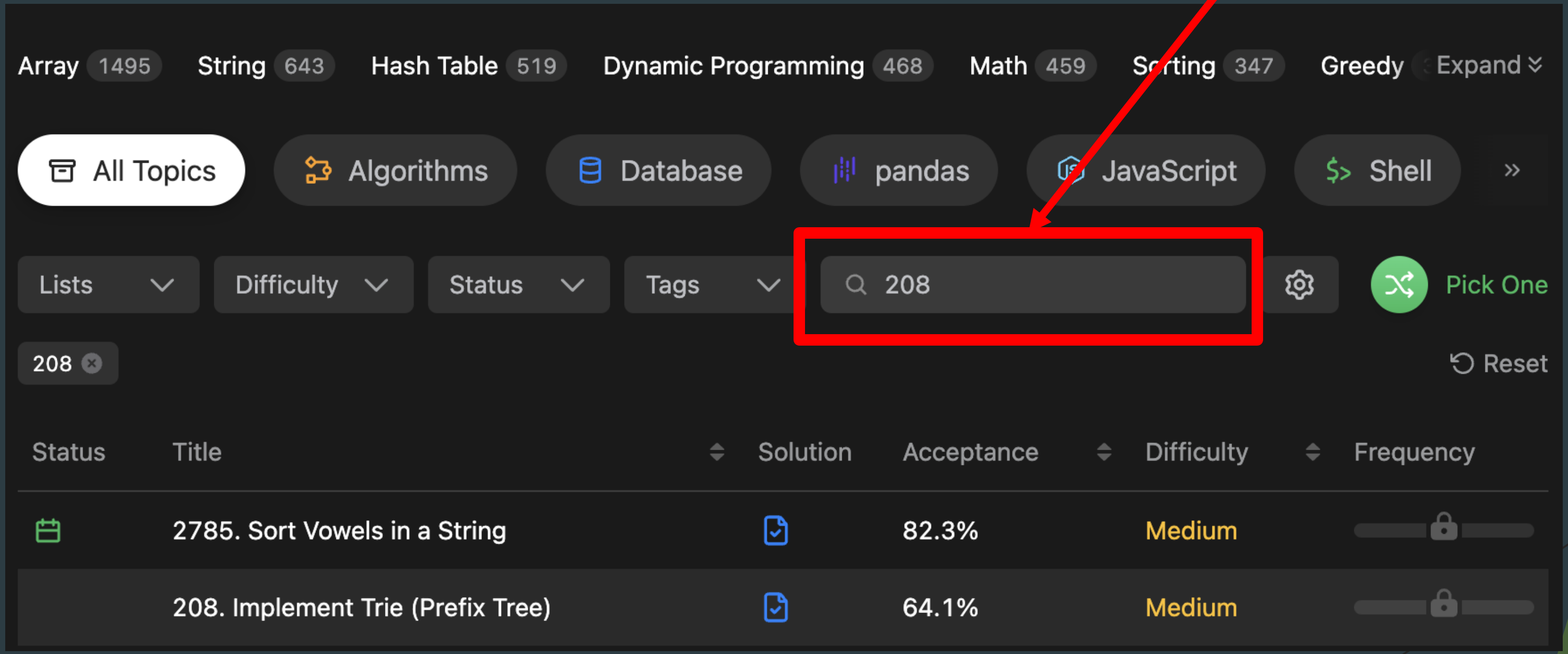
Search



Exercise 3

Problem 720. Longest Word in Dictionary

Search



Exercise 4

Problem 692. Top K Frequent Words

Search