ECE2810J
Data Structures and Algorithms

Asymptotic Algorithm Analysis

Learning Objective:

- Understand best, worst, and average cases
- Understand Big-Oh, Big-Omega, Big-Theta notations
- Know how to analyze time complexity of a program

Outline

- Asymptotic Analysis: Big-Oh
- Relatives of Big-Oh
- Analyzing Time Complexity of Programs

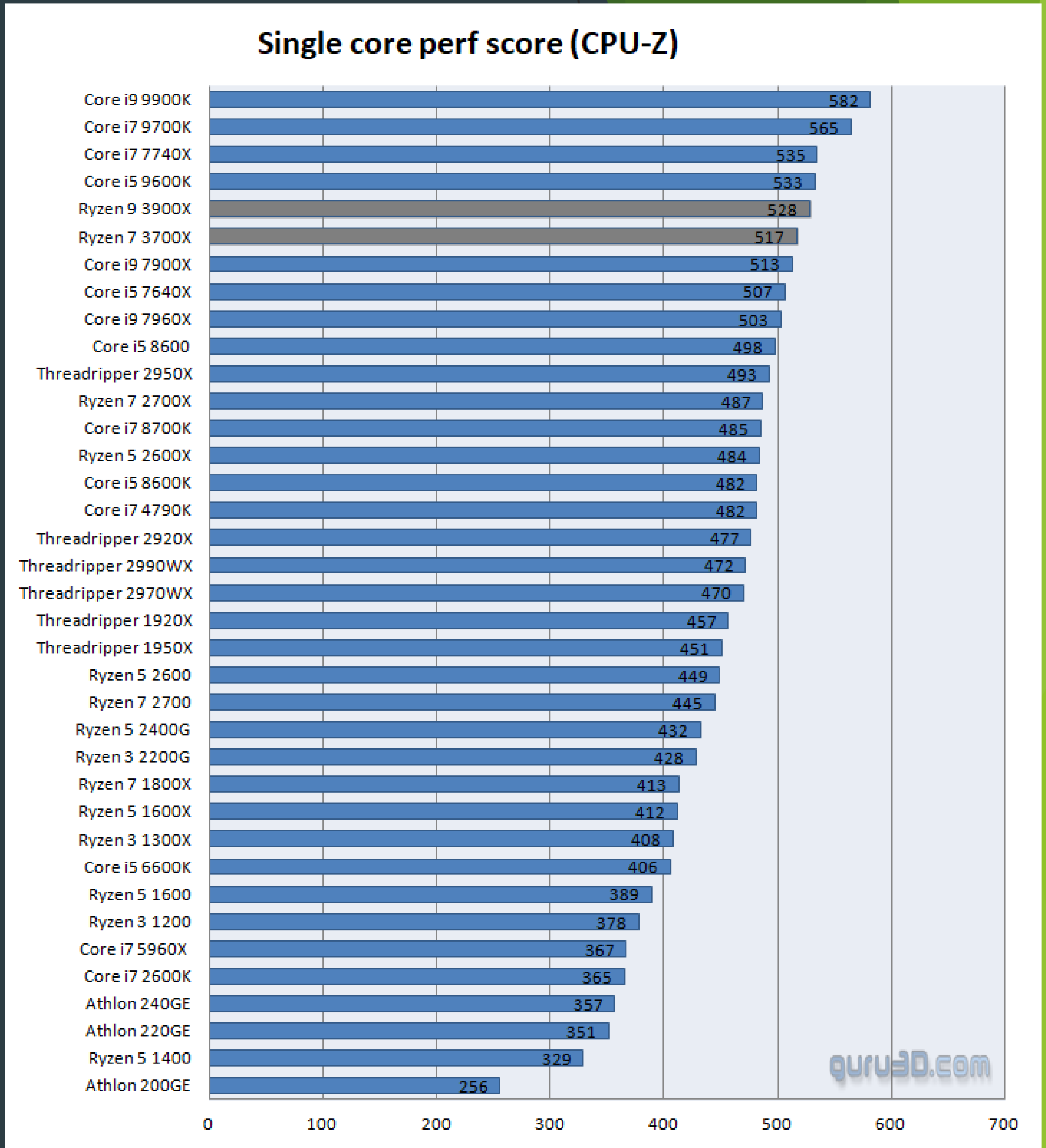
How to Measure Efficiency?

- Empirical comparison: run programs
  - Use the **wall-clock** time to measure the runtime
  - Empirical comparison could be tricky.
  - It depends on
    - Compiler
    - Machine (CPU speed, memory, etc.)
    - CPU load
    - Machine model
      - CPU
      - GPU

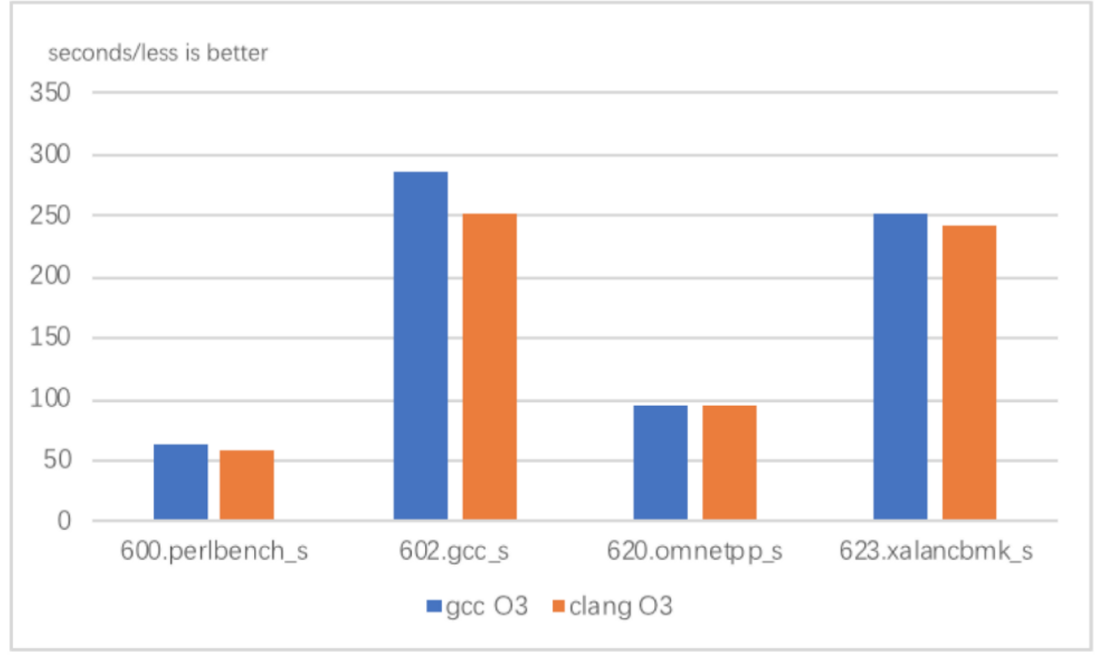
- Asymptotic Algorithm Analysis
  - For most algorithms, running time depends on the “size” of the input.
  - Running time is expressed as $T(n)$ for some function T on input size n.

Empirical Comparison

CPU benchmark



GCC vs LLVM Clang

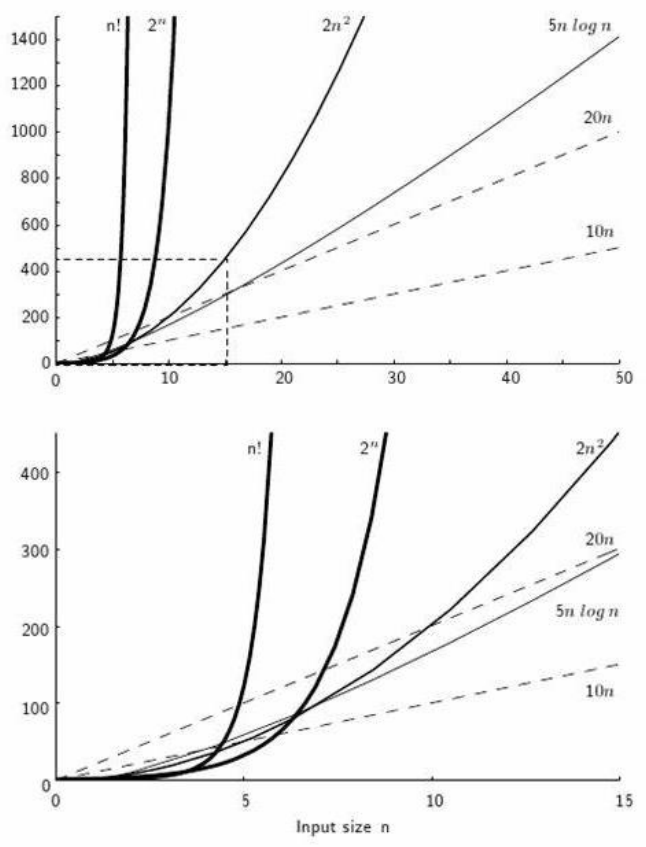


Compiler Execution Time

Why Asymptotic?

- 3 Algorithms: A, B and C
  - $T_A(n) = 2^n$
  - $T_B(n) = 1024*n$
  - $T_C(n) = 4096*n$
- When does $T_A > T_B$
  - $n > 13$
- When does $T_A > T_C$
  - $n > 15$
- $T_X(n) = c*n$
  - Will $T_A > T_X$ at some point?
- Asymptotic analysis ➔ what happens with a very large input size?

```
void fun(int n) {
    int i, j, k, count = 0;
    for(i = n/2; i <= n; i++)
        for(j = 1; j+n/2 <= n; j++)
            for(k = 1; k <= n; k = k*2)
                count++;
}
```



Asymptotic Analysis

Evaluate the time to run your function as the input size grows

Input Dependency: Example

- Summing an array of n elements

```
// REQUIRES: a is an array of size n
// EFFECTS: return the sum
int sum(int a[], unsigned int n) {
  int result = 0;
  for(unsigned int i = 0; i < n; i++)
    result += a[i];
  return result;
}
```

- The runtime is roughly cn, where c is some constant.
- With n fixed, any array has roughly the **same** runtime.

Best, Worst, Average Cases

The speed with regard to a parametrized size

- In the example of summing an array, all inputs of a given size take the same time to run.
- However, in some other cases, this is not true, i.e., not all inputs of a given size take the same time to run.
- Example: linear search

```
// REQUIRES: a is an array of size n
// EFFECTS: return the index of the element
// equals key. If no such element, return n.
int search(int a[], unsigned int n, int key) {
  for(unsigned int i = 0; i < n; i++)
    if(a[i] == key) return i;
  return n;
}
```

Which Statements Are True for Linear Search?

Complete the following statements:

- The best case occurs when **`key`** is the first element in the array.
- In the worst case, we need to do **`n`** comparisons with **`key`**.
- When does worst case happen? When **`key`** is not in the array.
- Suppose **`key`** is uniformly located in the array. Then, on average, the number of comparisons with **`key`** is **`n/2`**.

```
// REQUIRES: a is an array of size n
// EFFECTS: return the index of the element
// equals key. If no such element, return n.
int search(int a[], unsigned int n, int key) {
  for(unsigned int i = 0; i < n; i++)
    if(a[i] == key) return i;
  return n;
}
```

Best, Worst, Average Cases

- **Best case**: **least** number of steps required, corresponding to the ideal input

- **Worst case**: **most** number of steps required, corresponding to the most difficult input

- **Average case**: **average** number of steps required
  - What is “average”?
  - Often defined as “over purely random inputs”

Is the Following Statement Wrong?

"The best case for my algorithm is $n = 1$ (only a single input) because that is the fastest."

- Wrong!
- Best case is a **special input** case of a **[defined size n]** that is cheapest among all input cases of size n
  - **The input size is fixed during the analysis!**

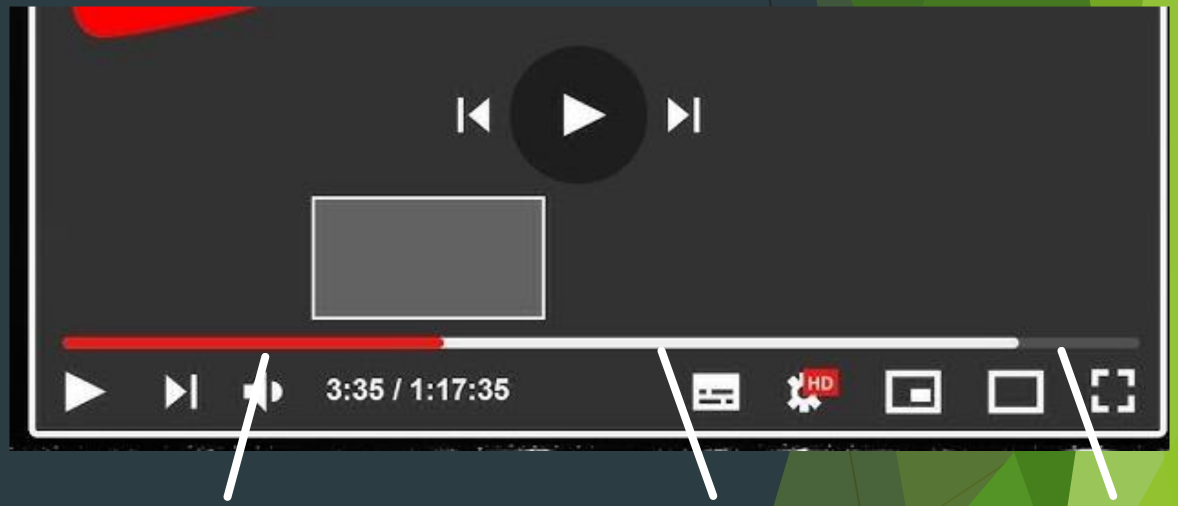
Which Case to Evaluate an Algo?

- The average case or the worst case are the most common
- While average time appears to be the fairest measure, it may be **difficult** to determine
  - Sometime, it requires domain knowledge, e.g., the distribution of inputs
- Worst case is pessimistic, but it **gives an upper bound**
  - Bonus: worst case usually easier to analyze

Average or Worst? Reality Check

- Whichever is the most advantageous
  - Quicksort is usually quite fast
- Fibonacci Heap is quite cumbersome but it **always** scales well
  - Very important in Quality of Service (QoS) analysis

What happens if video playing speed > buffer speed?



Average or Worst? Reality Check

- Whichever is the most advantageous
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- Fibonacci Heap is quite cumbersome but it **always** scales well
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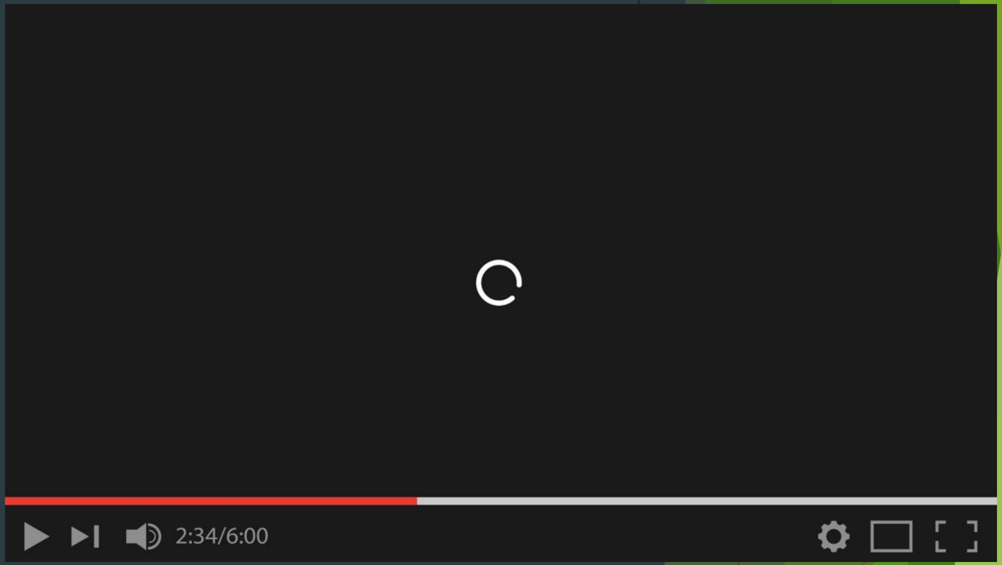


How to Analyze Complexity of Algorithm?

- Guiding Principle #1: Ignore constant factors.
  - Justification:
  1. Way easier
  2. Constants depend on architecture, compiler, etc
  3. Lose very little predictive power (as we will see)

- Guiding Principle #2: Focus on running time for **large input size n**
  - Justification: **scaling** is very important!
  - Thus, we will compare the runtime of two algorithms when n is very large
    - E.g., $1000 \log_2 n$ is "better" than $0.001n$

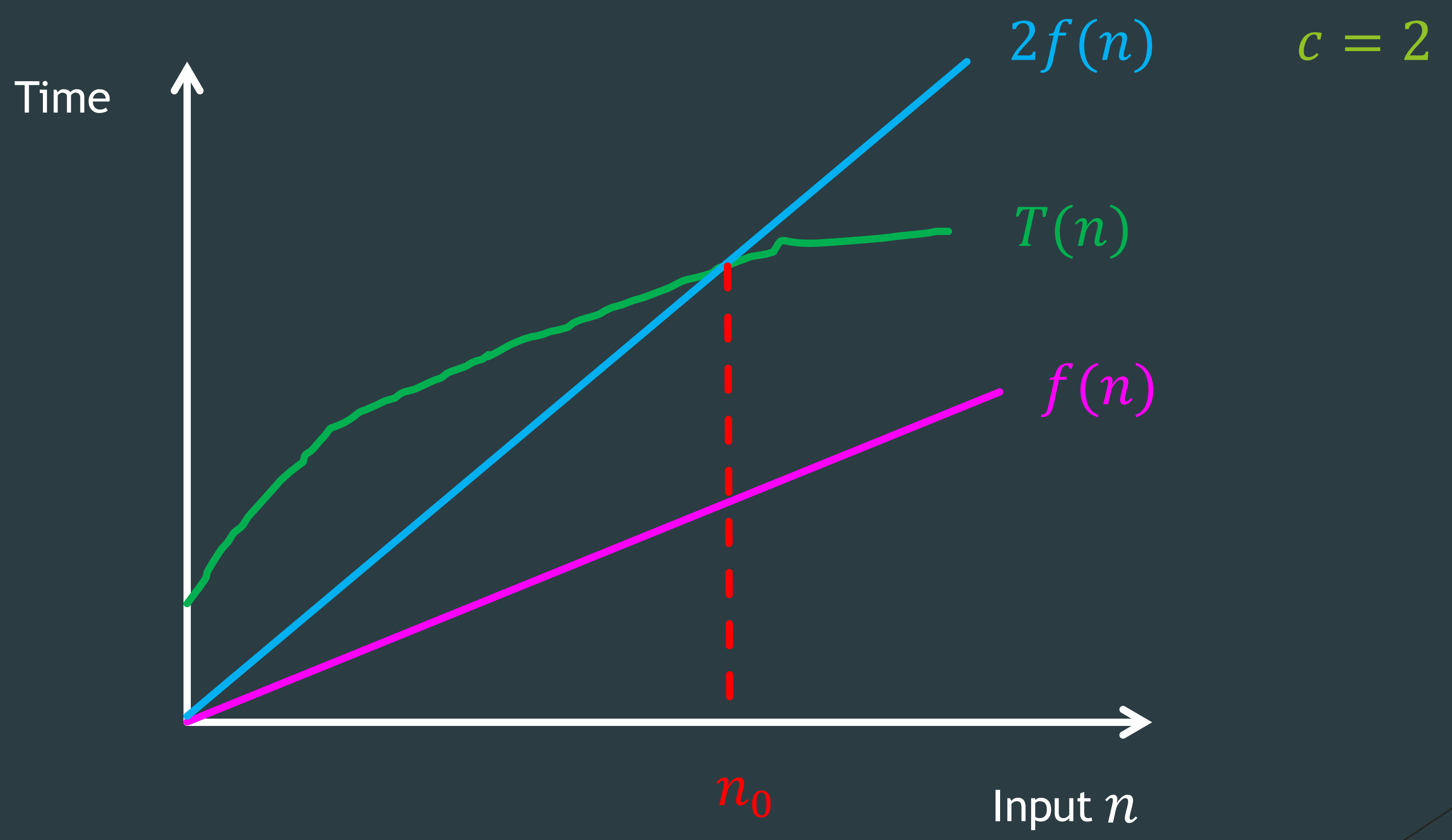
Asymptotic Analysis: Big-Oh

- Definition: A non-negatively valued function, $T(n)$, is in the **set** $O(f(n))$ if there **exist** two positive constants c and n_0 such that $T(n) \leq cf(n)$ **for all** $n > n_0$

- Usage: The algorithm is in $O(n^2)$ in best/average/worst case
  - E.g., quicksort has an average-case complexity of $O(n\ logn)$
  - E.g., quicksort has a worst-case complexity of $O(n^2)$

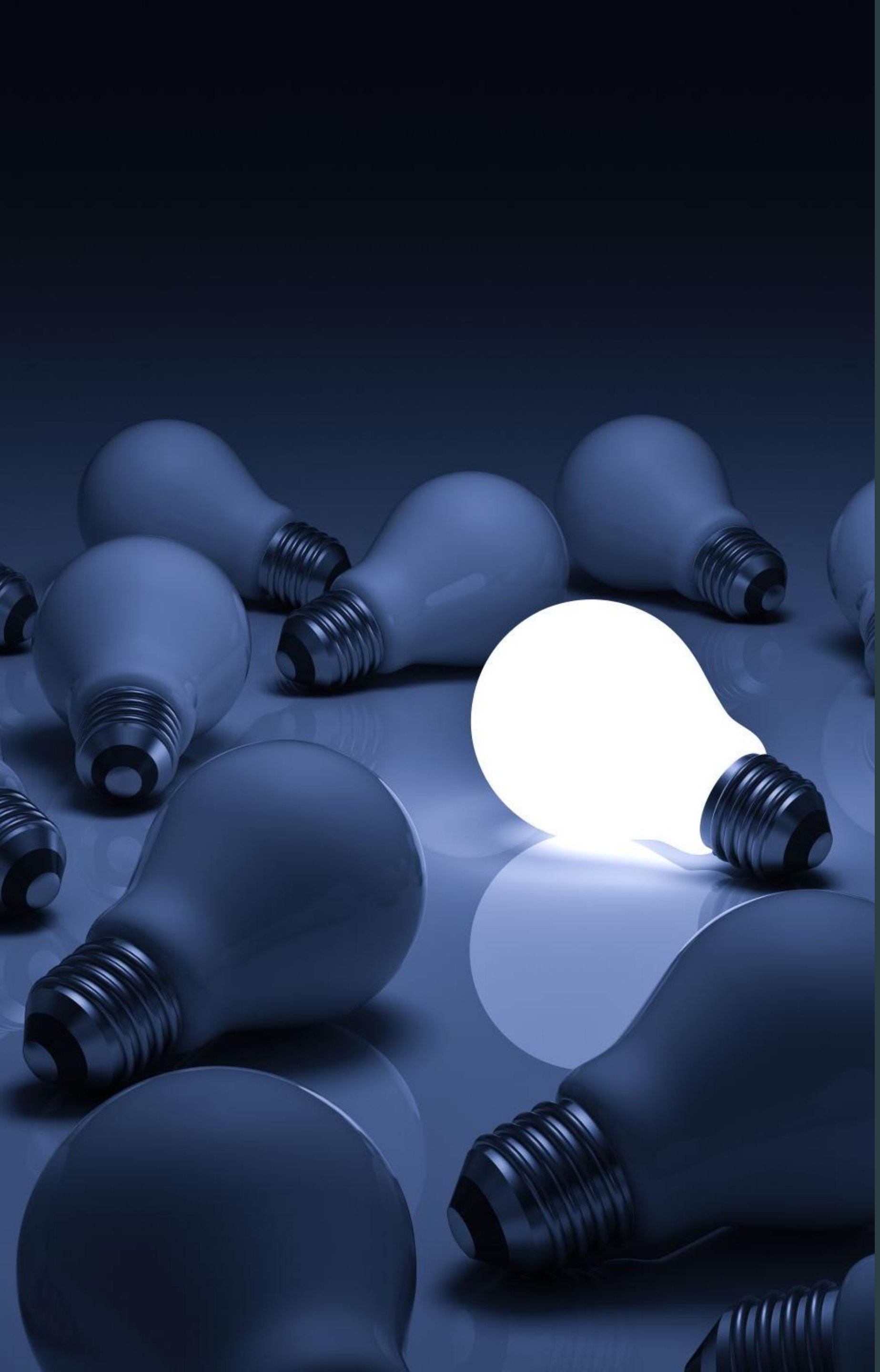
- Meaning: For all data sets big enough (i.e.,$n > n_0$), the algorithm always executes in **less than** $cf(n)$ steps in best/ average/worst case

Graph Visualization of Big-Oh



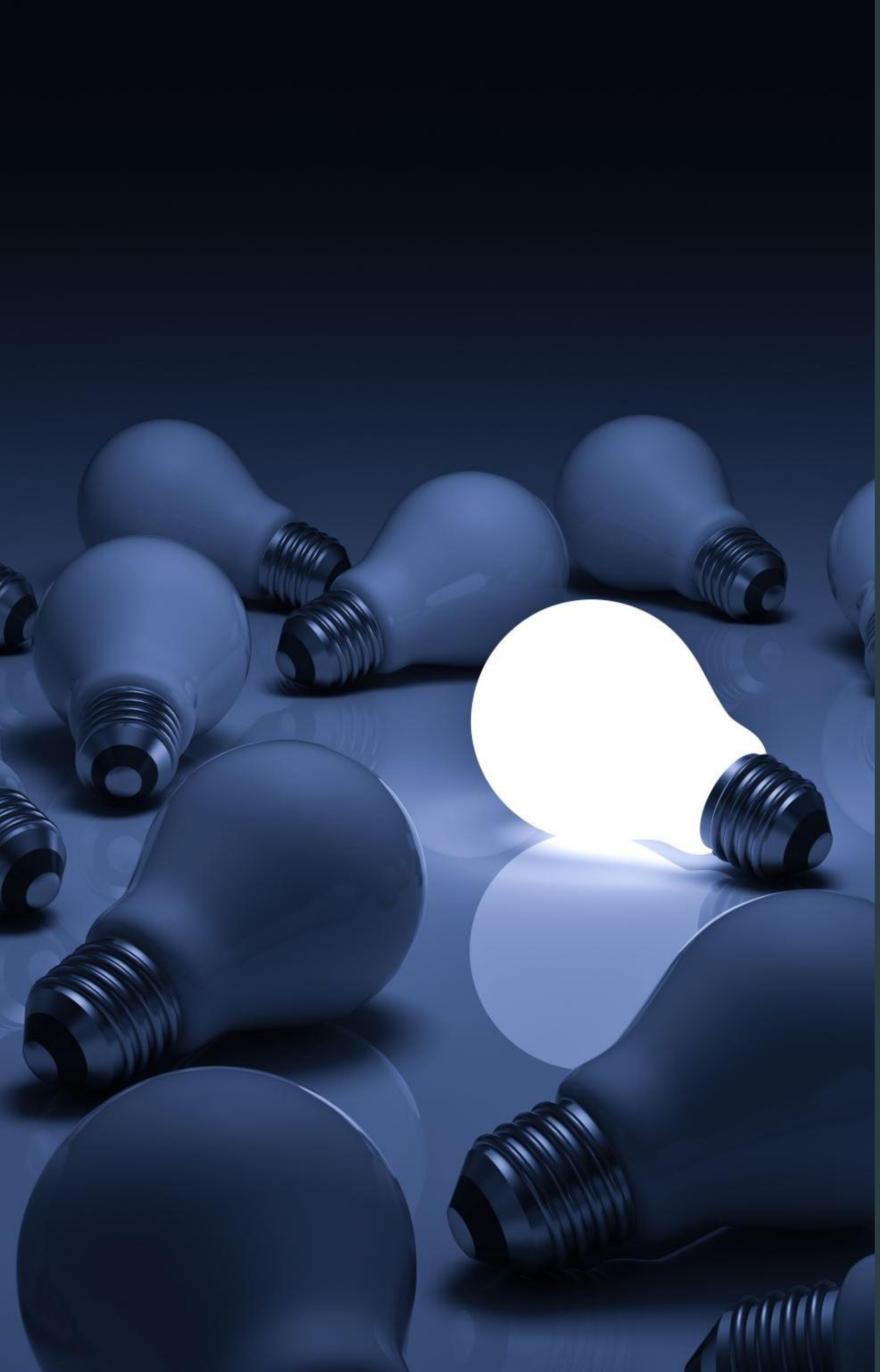
Big-Oh Notation

- Strictly speaking, we say that $T(n)$ is in $O(f(n))$, i.e.,
  $T(n) \in O(f(n))$
- However, for convenience, people also write:
  $T(n) = O(f(n))$
- Notice that the "=" here might not communicative
  $2n = O(n^2)$
  $n^2 \neq O(2n)$



Tricks to compute Big Oh

1. Find the fastest growing term
2. Ignore the coefficient



Tricks to compute Big Oh

1. Find the fastest growing term

2. Ignore the coefficient

Important!

Big-Oh Example 0

- $$T(n) = a + b$$
  $$= c$$
  $$= O(1)$$

Tricks:

1. Find the fastest growing term
2. Ignore the coefficient

Big-Oh Example 1

- $T(n) = an + b$

  $= O(n)$

Tricks:

1. Find the fastest growing term
2. Ignore the coefficient

Big-Oh Example 2

- $T(n) = cn^2 + dn + e$

  $= O(n^2)$

Tricks:

1. Find the fastest growing term
2. Ignore the coefficient

Big-Oh Example 3

- Claim: If $T(n) = a_k n^k + \cdots + a_1 n + a_0$, then
$$T(n) = O(n^k)$$

- Proof:
  - Need to pick constants c and n_0 so that for any $n > n_0$, $T(n) \leq c \cdot n^k$.
  - Choose $n_0 = 1$ and $c = |a_k| + \cdots + |a_1| + |a_0|$
  - Only need to show that for any $n > n_0$, $T(n) \leq cn^k$.
  - Anyone?

Big-Oh Example 4

- Claim: $2^{n+10} = O(2^n)$

- Proof:
  - Need to pick constants c and n_0 so that for any $n > n_0$,
    $$2^{n+10} \le c \cdot 2^n \qquad (*)$$
  - We note $2^{n+10} = 1024 \cdot 2^n$.
  - So if we choose $c = 1024$ and $n_0 = 1$, then (*) holds.

Big-Oh Notation

- Big-oh notation indicates an **upper bound**.
- Example: If $T(n) = 3n^2$ then $T(n)$ is in $O(n^2)$.
- However, we can also say $T(n)$ is in $O(n^3)$ or $O(n^4)$ (why?).

- We seek the **tightest** upper bound:
  - While $T(n) = 3n^2$ is in $O(n^3)$, we prefer $O(n^2)$.
  - In CS research, tightening the upper bound is a common focus:
    - E.g., prove that the avg. complexity is $O(n\ logn)$ rather than $O(n^2)$

True or False?

- Consider the following statements, are they true:
  - $3n = O(2n)$?
  - $3^n = O(2^n)$?
  - $n^3 = O(n^2)$?
  - $\log_2 n = \log_3 n$?

True or False?

- Consider the following statements, are they true:
  - $3n = O(2n)$?
  - $3^n = O(2^n)$?
  - $n^3 = O(n^2)$?
  - $\log_2 n = O(\log_3 n)$?

A Sufficient Condition of Big-Oh

$$\text{If } \lim_{n\to\infty} \frac{f(n)}{g(n)} = c < \infty, \text{ then } f(n) \text{ is } O(g(n))$$

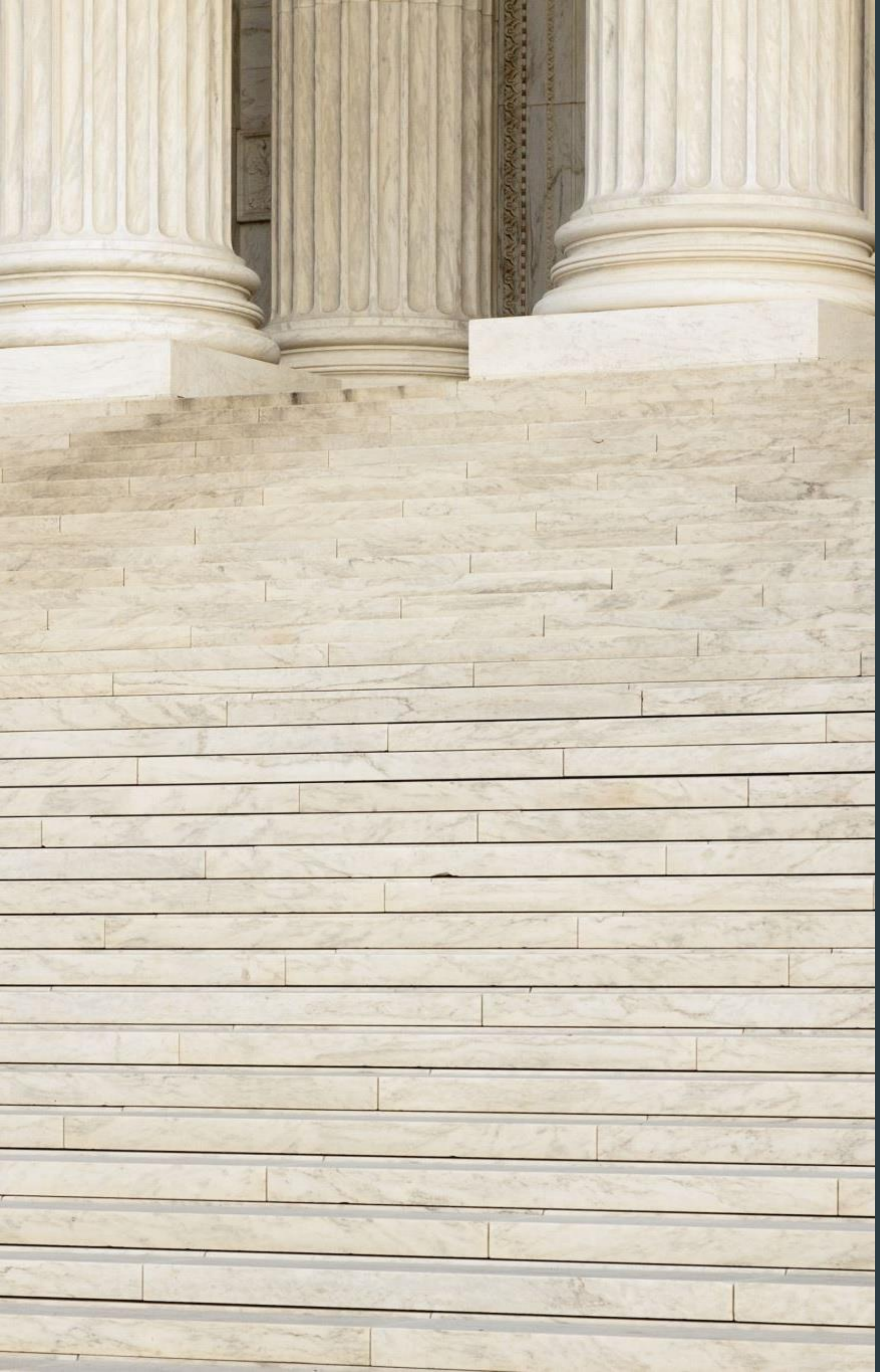
- Why?

  There exists a n_0, for $\mathrm{n} > n_0$, $f(n) < \mathrm{c} \cdot g(n)$

- With this theorem, we can easily prove that
- $$T(n) = \boxed{c_1 n^2} + \boxed{c_2 n} \text{ is } O(n^2)$$
  - Proof: $\lim_{n\to\infty} \frac{c_1 n^2 + c_2 n}{n^2} = c_1 < \infty$

Rules of Big-Oh

- **Rule 1**: If $f(n) = O\big(g(n)\big)$, then $cf(n) = O(g(n))$
  - Example: $3n^2 = O(n^2)$
- **Rule 2**: If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$
  - Then $f_1(n) + f_2(n) = \max\{O\big(g_1(n)\big), O(g_2(n))\}$
  - Example: $n^3 + 2n^2 = \max\{O(n^3), O(n^2)\} = O(n^3)$



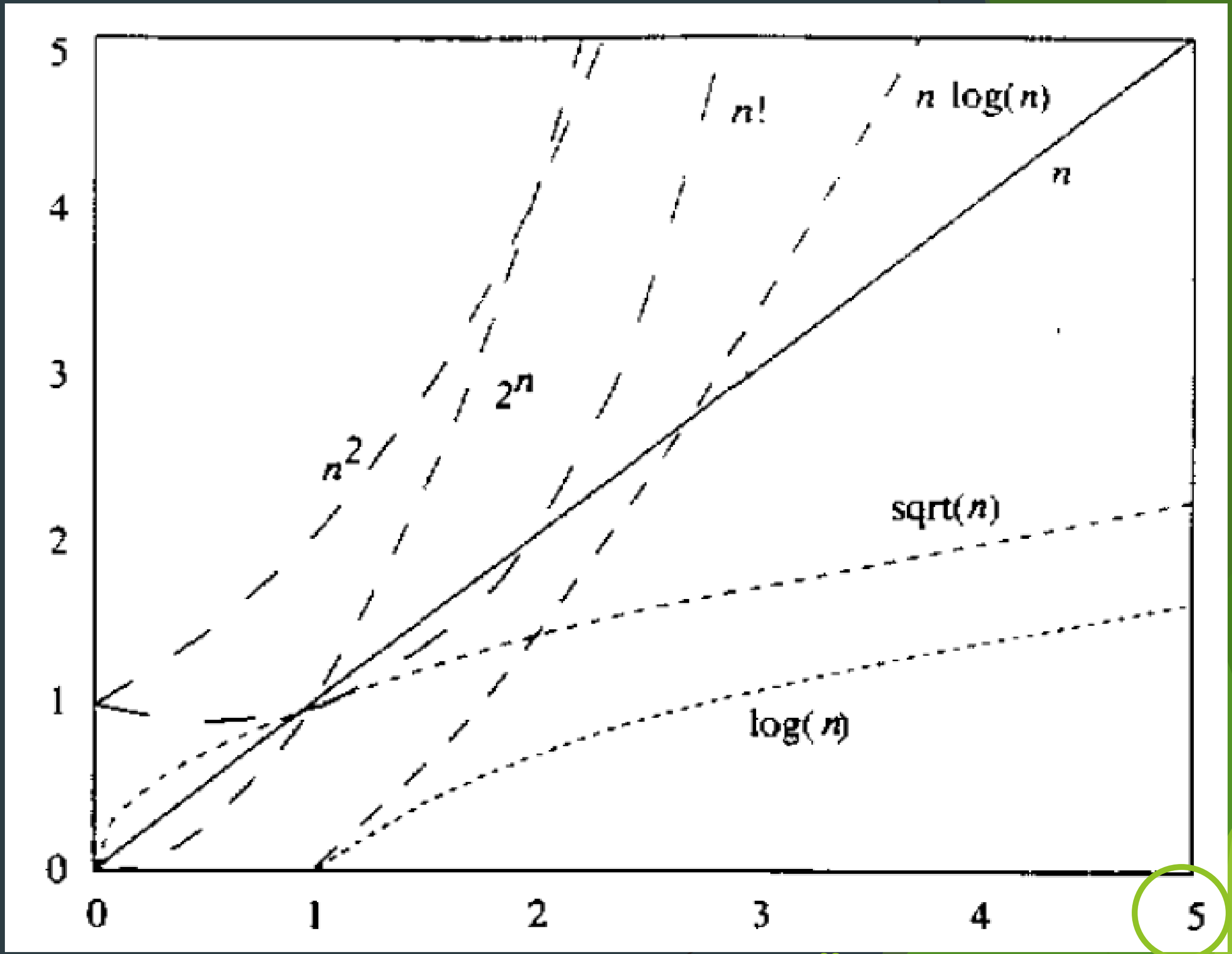
Rules of Big-Oh

- **Rule 3:** If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$

- **Rule 4:** If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$

Common Functions and Their Growth Rates

- constant: 1
- logarithmic: $\log n$
  refers to $\log_2 n$
- square root: $\sqrt{n}$
- linear: n
- loglinear: $n \log n$
- quadratic: n^2
- cubic: n^3
- general polynomial: n^k
  $k \geq 1$
- exponential: $a^n, a > 1$
- factorial: $n!$



Common Functions and Their Growth Rates

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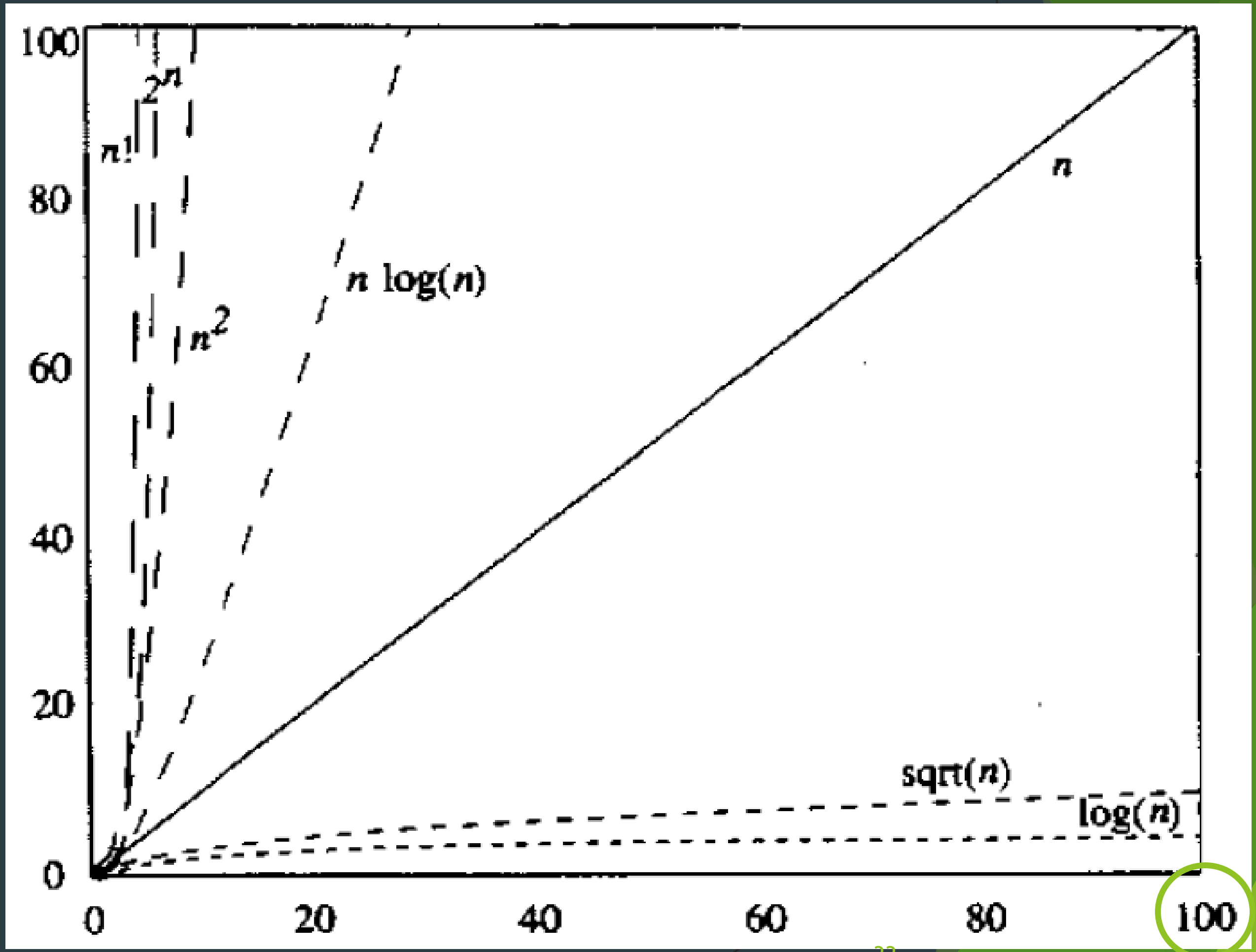


Code Exercise 0

What is the Big O notations of the following code, and why?

python Copy code

```python
def print_first_element(arr):
    print(arr[0])
```

Code Exercise 0

What is the Big O notations of the following code, and why?

```python
def print_first_element(arr):
    # This code always accesses the first element of the array,
    # which takes a constant amount of time, regardless of the array's size
    print(arr[0])
```

Constant Time (O(1))

Code Exercise 1

What is the Big O notations of the following code, and why?

python Copy code

```python
def find_max(arr):
    max_val = arr[0]
    for num in arr:
        if num > max_val:
            max_val = num
    return max_val
```

Code Exercise 1

What is the Big O notations of the following code, and why?

python Copy code

```python
def find_max(arr):
    max_val = arr[0]
    for num in arr:
        # This loop iterates through the entire array once, making it
        # directly proportional to the size of the array (n).
        if num > max_val:
            max_val = num
    return max_val
```

Linear Time (O(n))

Code Exercise 2

What is the Big O notations of the following code, and why?

python   Copy code

```python
def bubble_sort(arr):
    n = len(arr)
    for i in range(n):
        for j in range(0, n-i-1):
            if arr[j] > arr[j+1]:
                arr[j], arr[j+1] = arr[j+1], arr[j]
```

Code Exercise 2

What is the Big O notations of the following code, and why?

python Copy code

```python
def bubble_sort(arr):
    n = len(arr)

    for i in range(n):
        for j in range(0, n-i-1):
            # This code uses nested loops, resulting in
            # a time complexity of O(n^2) as it compares
            # and swaps elements within the array.
            if arr[j] > arr[j+1]:
                arr[j], arr[j+1] = arr[j+1], arr[j]
```

Quadratic Time (O(n^2))

Code Exercise 3

What is the Big O notations of the following code, and why?

python Copy code

```python
def binary_search(arr, target):
    left, right = 0, len(arr) - 1
    while left <= right:
        mid = (left + right) // 2
        if arr[mid] == target:
            return mid
        elif arr[mid] < target:
            left = mid + 1
        else:
            right = mid - 1
    return -1
```

Code Exercise 3

What is the Big O notations of the following code, and why?

python   Copy code

```python
def binary_search(arr, target):
    left, right = 0, len(arr) - 1
    while left <= right:
        mid = (left + right) // 2
        # Binary search repeatedly divides the search space in half,
        # leading to a logarithmic time complexity (O(log n)).
        if arr[mid] == target:
            return mid
        elif arr[mid] < target:
            left = mid + 1
        else:
            right = mid - 1
    return -1
```

Recommended Resources

CS Dojo (~30 mins)

NeetCode (~20 mins)

python   Copy code

```python
def merge_sort(arr):
    if len(arr) > 1:
        mid = len(arr) // 2
        left_half = arr[:mid]
        right_half = arr[mid:]

        merge_sort(left_half)
        merge_sort(right_half)

        i = j = k = 0

        while i < len(left_half) and j < len(right_half):
            if left_half[i] < right_half[j]:
                arr[k] = left_half[i]
                i += 1
            else:
                arr[k] = right_half[j]
                j += 1
            k += 1

        while i < len(left_half):
            arr[k] = left_half[i]
            i += 1
            k += 1

        while j < len(right_half):
            arr[k] = right_half[j]
            j += 1
            k += 1
```

Code Exercise 4

What is the Big O notations of the following code, and why?

Keyword:

- Master theorem (from course MATH2030J Discrete Mathematics)

Code Exercise 4

```python
def merge_sort(arr):
    # Check if the array has more than one element.
    if len(arr) > 1:
        # Divide the array into two halves.
        mid = len(arr) // 2
        left_half = arr[:mid]
        right_half = arr[mid:]

        # Recursive splitting and sorting.
        # This part contributes to O(n log n) time complexity
        # because it divides the array into halves in a
        # logarithmic manner (O(log n)).
        merge_sort(left_half)
        merge_sort(right_half)

        i = j = k = 0
```

```python
        # Merge the sorted halves.
        while i < len(left_half) and j < len(right_half):
            if left_half[i] < right_half[j]:
                arr[k] = left_half[i]
                i += 1
            else:
                arr[k] = right_half[j]
                j += 1
            k += 1

        # Copy any remaining elements from the left and right halves.
        while i < len(left_half):
            arr[k] = left_half[i]
            i += 1
            k += 1

        while j < len(right_half):
            arr[k] = right_half[j]
            j += 1
            k += 1

# The overall time complexity of merge_sort is O(n log n)
# due to the recursive splitting (O(log n)) and merging (O(n))
# of sorted halves.
```

Master theorem

$T(n) = 2*T(n/2) + O(n)$ → **Linearithmic Time (O(n log n))**

Code Exercise 5

What is the Big O notations of the following code, and why?

python   Copy code

```python
def fibonacci_recursive(n):
    if n <= 1:
        return n
    else:
        return fibonacci_recursive(n-1) + fibonacci_recursive(n-2)
```

The Fibonacci Sequence:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,144,233,377 …

Code Exercise 5

What is the Big O notations of the following code, and why?

python Copy code

```python
def fibonacci_recursive(n):
    if n <= 1:
        return n
    else:
        # The recursive Fibonacci algorithm makes two recursive calls
        # for each value of n, leading to exponential time
        # complexity (O(2^n)).
        return fibonacci_recursive(n-1) + fibonacci_recursive(n-2)
```

Exponential Time (O(2^n))

Exercise

- EZ Question: what is the complexity of the code below?

```
// REQUIRES: a is an array of size n
// EFFECTS: return the index of the element
// equals key. If no such element, return n.
int search(int a[], unsigned int n, int key) {
  for(unsigned int i = 0; i < n; i++)
    if(a[i] == key) return i;
  return n;
}
```

A Few Results about Common Functions

- For a polynomial in n of the form $f(n) = a_m n^m + a_{m-1} n^{m-1} + \cdots + a_1 n + a_0$
  where $a_m > 0$, we have $f(n) = O(n^m)$.

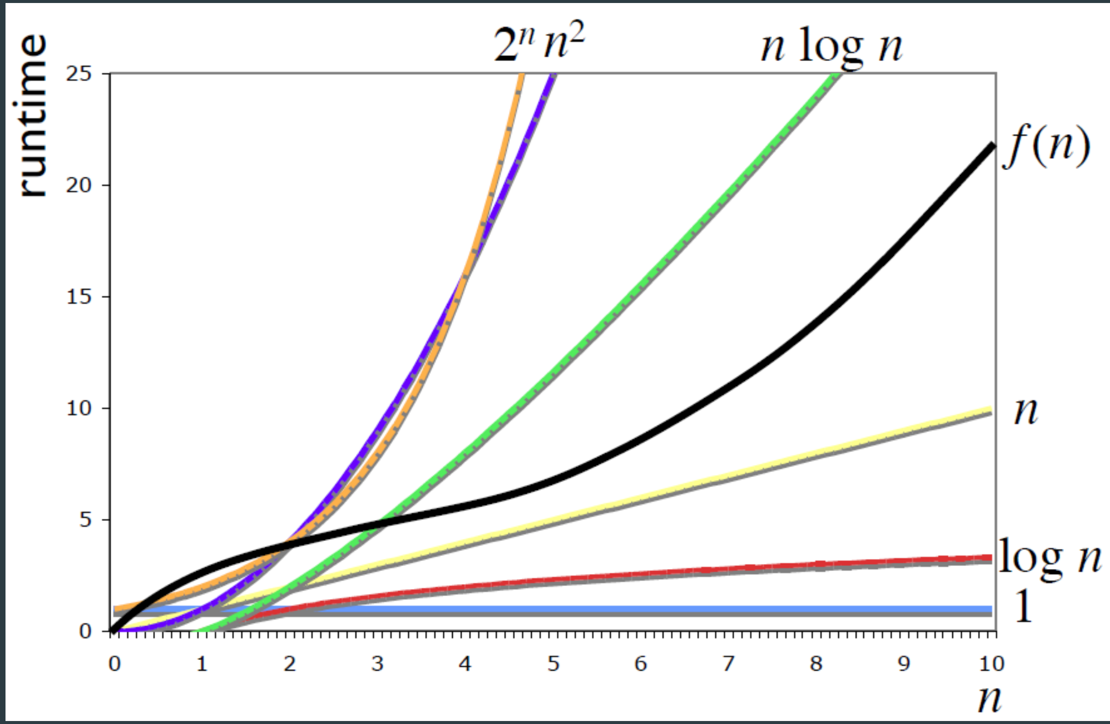
- For every integer $k \geq 1, log_k\, n = O(n)$.
  - Tightest bound: $log_k\, n = O(log\, n)$

- For every integer $k \geq 1$, $n^k = O(2^n)$.
  - Tightest bound: $n^k = O(n^k)$.

How Fast Is Your Code?

- Let $f(n) = 0.5n + nlog_2n$ be the complexity of your code, how fast would you advertise it as?

A. $O(\log n)$ **B.** $O(n \log n)$
C. $O(n)$ **D.** $O(n^2)$



$f(n) = O(g(n))$; You want to pick a $g(n)$ that is as close to $f(n)$ as possible.

What Is a “Fast” Algorithm?

Fast algorithm ≈ worst-case/average-case running time grows slowly with input size

It scales well!

- Usually as close to linear ($O(n)$) as possible.
  - Going sublinear (e.g., $O(\log n)$) is usually very hard! But still possible!
  - Which algorithm has a $O(\log n)$ complexity?

Outline

Asymptotic Analysis: Big-Oh

Relatives of Big-Oh

Analyzing Time Complexity of Programs

Relative of Big-Oh: Big-Omega

- Definition: For $T(n)$ a non-negatively valued function, $T(n)$ is in the **set** $\Omega(g(n))$ if there **exist** two positive constants c and n_0 such that $T(n) \geq cg(n)$ **for all** $n > n_0$

- Meaning: For all data sets big enough (i.e., $n > n_0$), the algorithm always requires **more than** $cg(n)$ steps

- Big-omega gives a lower bound
- We usually want the greatest lower bound

Big-Omega Example

- Consider $T(n) = c_1 n^2 + c_2 n$, where c_1 and c_2 are positive
- What is the big-omega notation for $T(n)$?

- Solution:
  - $c_1 n^2 + c_2 n \geq c_1 n^2$ for all $n > 1$
  - $T(n) \geq c n^2$ for $c = c_1$ and $n_0 = 1$
  - Therefore, $T(n)$ is in $\Omega(n^2)$ by the definition

Rules of Big-Omega

- **Rule 1**: If $f(n) = \Omega\big(g(n)\big)$, then $cf(n) = ?$
- **Rule 2**: If $f_1(n) = \Omega(g_1(n))$ and $f_2(n) = \Omega(g_2(n))$
  - Then $f_1(n) + f_2(n) = ?$

Rules of Big-Omega

- **Rule 3**: If $f_1(n) = \Omega(g_1(n))$ and $f_2(n) = \Omega(g_2(n))$, then $f_1(n) \cdot f_2(n) = ?$

- **Rule 4**: If $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$, then $f(n) = ?$

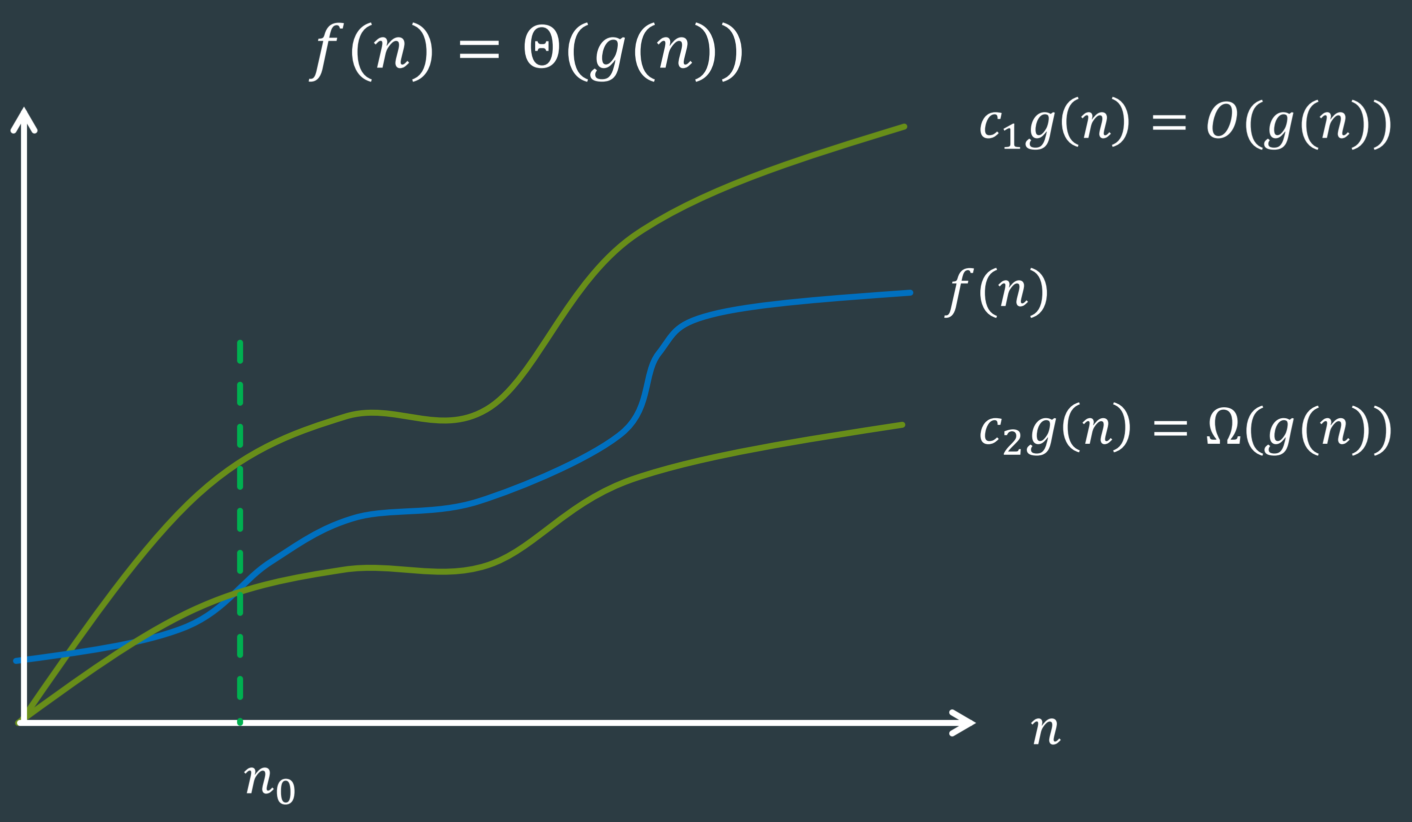
Theta Notation

- When big-oh and big-omega are the same, we indicate this by using big-theta (Θ) notation.

- Definition: $T(n)$ is said to be in the set $\Theta(g(n))$ if it is in $O(g(n))$ and it is in $\Omega(g(n))$.
  - In other words, there **exist** three positive constants c_1, c_2, and n_0 such that $c_1 g(n) \leq T(n) \leq c_2 g(n)$ **for all** $n > n_0$.

- What is the Θ of $\mathrm{T(n)} = c_1 n^2 + c_2 n$?
  - $\Theta\big(T(n)\big) = n^2$

Theta Notation



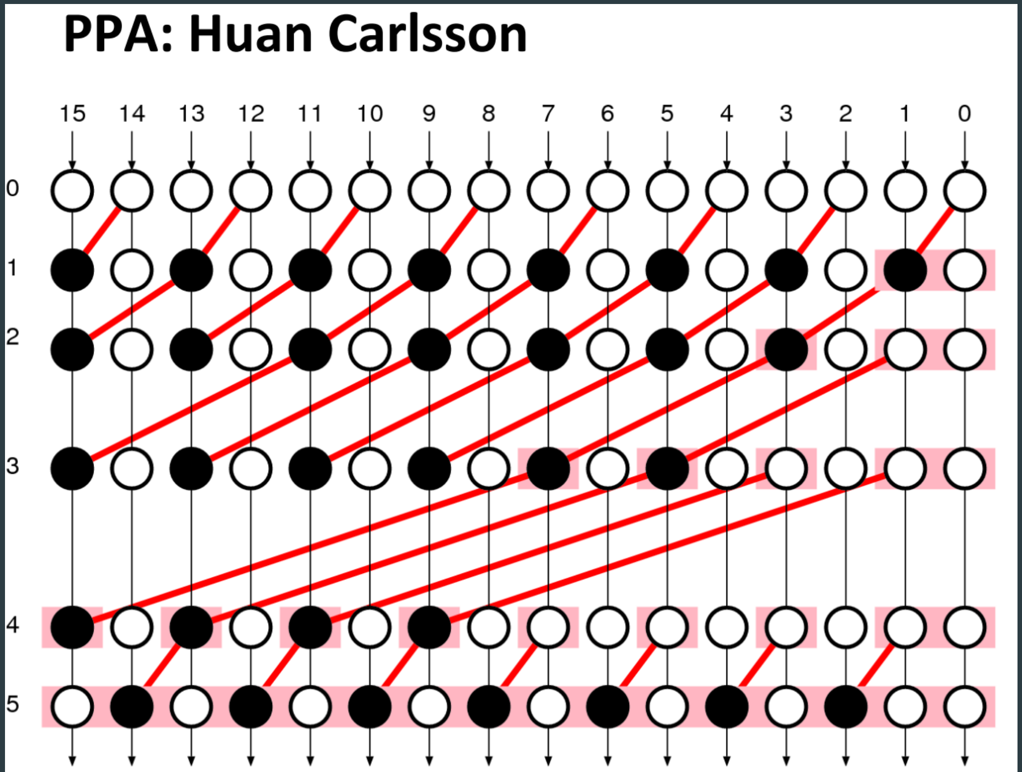
- Question: Does $f(n) = \Theta\big(g(n)\big)$ indicate $g(n) = \Theta(f(n))$?

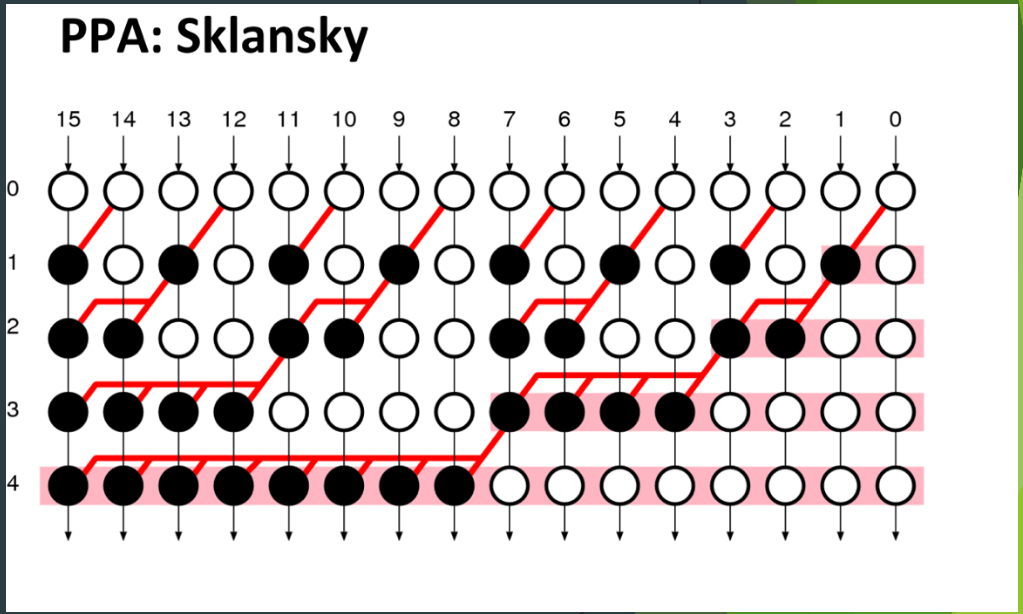
Outline

- Asymptotic Analysis: Big-Oh
- Relatives of Big-Oh
- Analyzing Time Complexity of Programs

Analyzing Time Complexity of Programs

- For atomic statement, such as assignment or addition, its complexity is $\Theta(1)$
  - Addition is atomic?
    - Conceptually yes, but in reality… it's complicated!





Analyzing Time Complexity of Programs

- For branch statement, such as if-else statement and switch statement, its complexity is that of the most expensive Boolean expression plus that of the most expensive branch.
  - What is an addition of two complexity statements?

```
if(Boolean_Expression_1) {Statement_1}
else if (Boolean_Expression_2) {Statement_2}
…
else if (Boolean_Expression_n) {Statement _n}
else {Statement_For_All_Other_Possibilities}
```

- Why?

Analyzing Time Complexity of Programs

- For subroutine call, its complexity is that of the subroutine

- For loops, such as while and for loop, its complexity is related the number of operations required in the loop

Time Complexity Example One

- What is the time complexity of the following code?

```
 sum = 0;
for(i = 1; i <= n; i++)
  sum += i;
```

- The entire time complexity is $\Theta(n)$

Time Complexity Example Two

- What is the time complexity of the following code?

```
sum = 0;
for(i = 1; i <= n; i++)
  for(j = 1; j <= i; j++)
    sum++;
```

- Note that the statements
  ```
  j <= i;
  j++;
  sum++;
  ```
  all occur (roughly) $1 + 2 + \cdots + n = n(n+1)/2$ times.
- The time complexity is $\Theta(n^2)$.

Time Complexity Example Three

- What is the time complexity of the following code?

```
sum = 0;
for(i = 1; i <= n; i *= 2)
   for(j = 1; j <= n; j++)
      sum++;
```

- The outer loop occurs $\log n$ times
- The statements `sum++ / j<=n / j++` occur $n \log n$ times
- The time complexity is $\Theta(n \log n)$

What Is the Time Complexity of the Following Code?

- Choose the correct answer.

```
sum = 0;
for(i = 1; i <= n; i *= 2)
  for(j = 1; j <= i; j++)
    sum++;
```

A. $\Theta(\log n)$ B. $\Theta(n \log n)$
C. $\Theta(n)$ D. $\Theta(n^2)$

1+2+4+8+…+2^{log n} \approx. 2n-1

Multiple Parameters

- Example: Compute the rank ordering for all C (i.e., 256) pixel values in a picture of P (i.e., 64×64) pixels.

$\Theta(C)$

```
for(i=0; i<C; i++)     // Initialize count
  count[i] = 0;
```

$\Theta(P)$

```
for(i=0; i<P; i++)     // Look at all pixels
  count[value[i]]++;  // Increment count
```

$\Theta(C \log C)$

```
sort(count);           // Sort pixel counts
```

- The time complexity is $\Theta(P + C\log C)$
- One general application is to analyze graph algorithm (#nodes and #edges)

Space/Time Trade-off Principle

- One can often reduce time if one is willing to sacrifice space, or vice versa

- Example: factorial
  - Iterative method: Get “n!” using a for-loop
  - This requires $\Theta(1)$ memory space and $\Theta(n)$ runtime
  - Table lookup method: Pre-compute the factorials for $1,2,\cdots,N$ and store all the results in an array
  - This requires $\Theta(n)$ memory space and $\Theta(1)$ runtime (fetching from an array)

That is All for today!

Questions?