ECE2810J
Data Structures and Algorithms

Shortest Path

Learning Objectives:

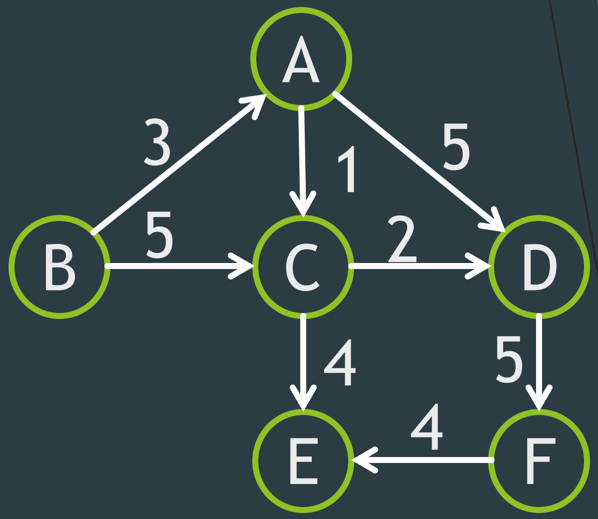
- Know the shortest path problem
- Know Dijkstra’s algorithm and its runtime complexity
- Know the similarity between Prim’s algorithm and Dijkastra’s algorithm

Outline

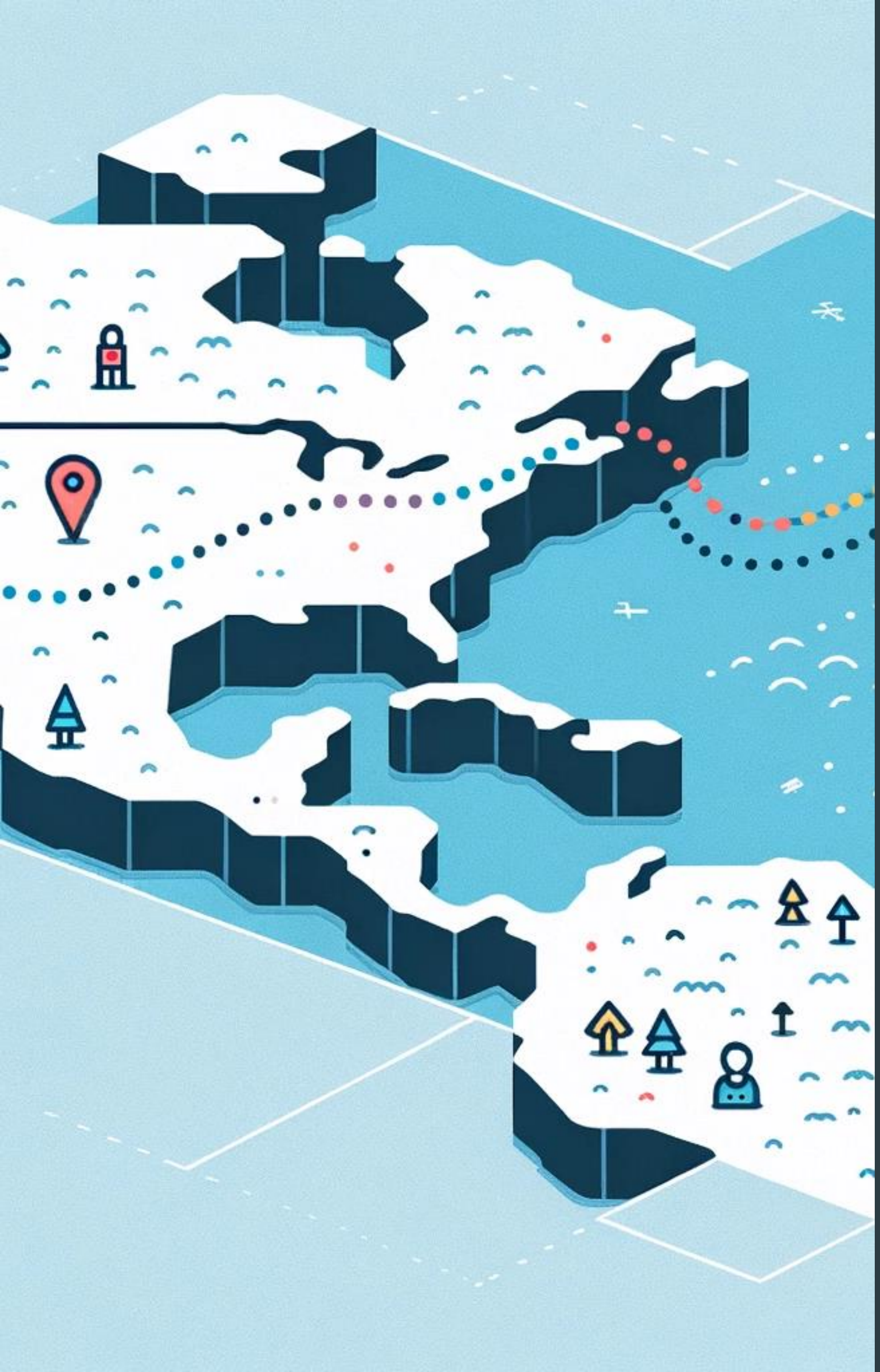
- Shortest Path Problem
  - Unweighted Graph
  - Dijkstra’s Algorithm

Shortest Path Problem
Introduction

- Given a weighted graph $G = (V, E)$, path length is defined as the sum of weights of edges on the path.
  - E.g., length of the path B, C, D, F is 12.
- Shortest path problem: given a weighted graph $G = (V, E)$ and two nodes $s, d \in V$, find the shortest path from s to d.
  - Assume G is a directed graph without parallel edges of the same direction
  - For an undirected graph, we can replace each edge by two edges of the same weight but of different directions.



What is the shortest path from B to F?

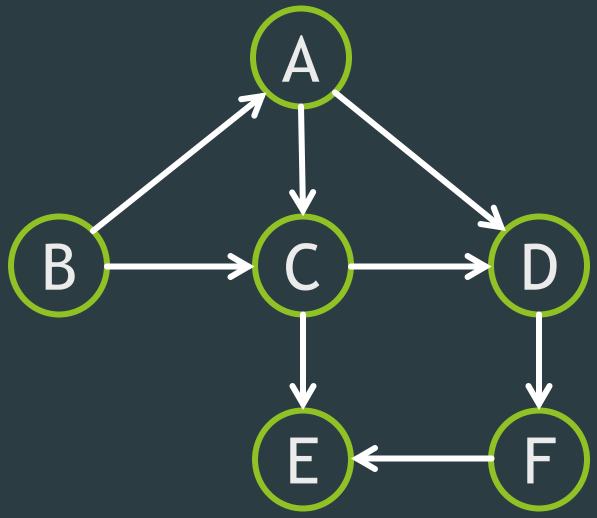


Shortest Path Problem

- The starting node on the path is the **source** node and the ending node is the **destination** node.
- The previous problem is a **single source single destination** problem.
- What we will solve is a **single source all destinations** problem: Given $G = (V, E)$ and a node $s \in V$, find the shortest path from s to **every other** node in G.
  - Single source single destination problem can be solved by solving the single source all destinations problem.
  - However, single source single destination problem is not much easier than the single source all destinations problem.

Shortest Path Problem
A Simple Version: Unweighted Graphs

- For an unweighted graph, path length is defined as the number of edges on the path.
- How do you obtain the shortest path between a pair of nodes?

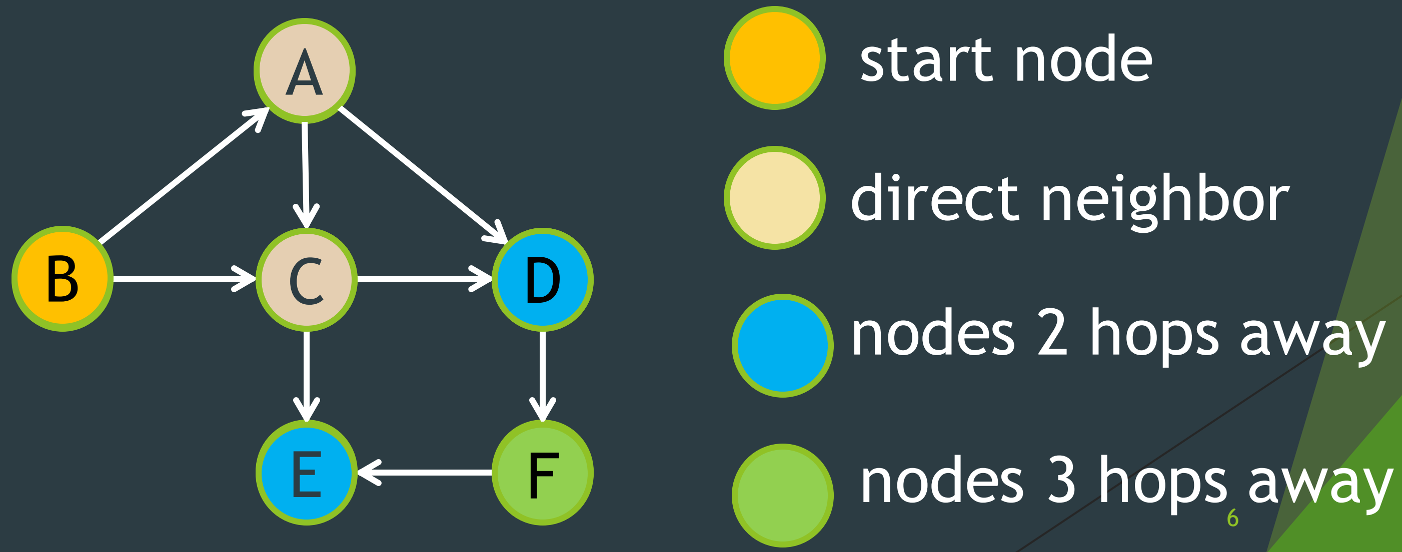


What is the shortest path from B to F?

Using breadth-first search!

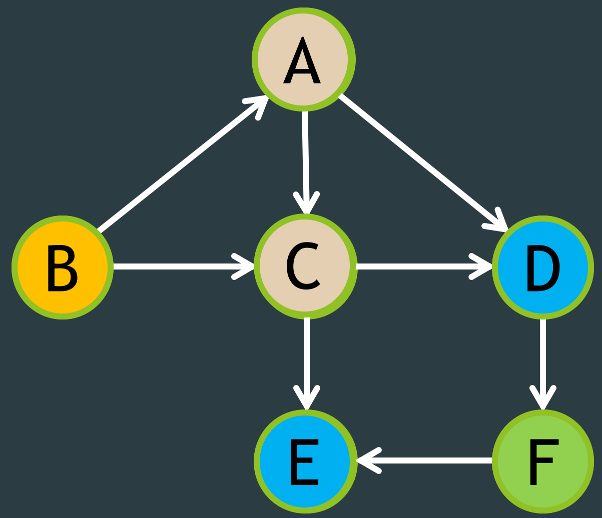
Shortest Path for Unweighted Graphs

- Recall breadth-first search (BFS): Given a start node, visit all directly connected neighbors first, then nodes 2 hops away, 3 hops away, and so on.
  - This is exactly what we want!
  - When the node visited is the destination node, we stop.
  - When the queue becomes empty, there is no path between the two nodes.



Shortest Path for Unweighted Graphs

- Additional bookkeeping
  - Store the distance.
  - Store the **predecessor** on the shortest path, i.e., the previous node on the path.



start node

direct neighbor

nodes 2 hops away

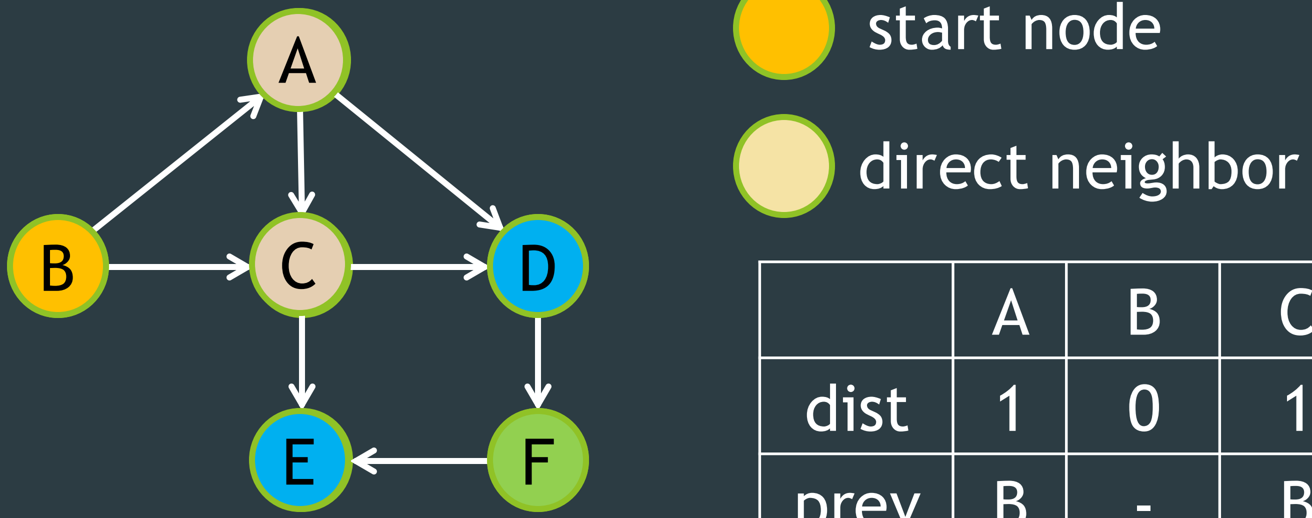
nodes 3 hops away

	A	B	C	D	E	F
dist	1	0	1	2	2	3
pred	B	-	B	A	C	D

Shortest Path for Unweighted Graphs

- We can obtain the shortest path by backtracking.
  - E.g., shortest path from B to F

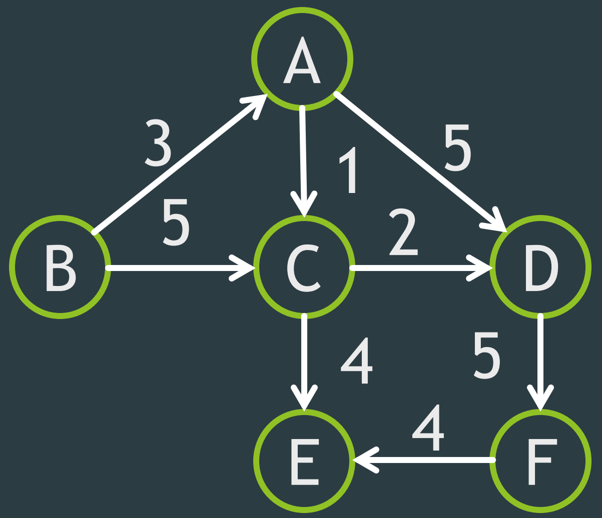
B→A→D→F



	A	B	C	D	E	F
dist	1	0	1	2	2	3
prev	B	-	B	A	C	D

Shortest Path for Weighted Graphs

- The problem becomes more difficult when edges have different weights.
  - Breadth-first search won't work!
  - What is the shortest path from B to F?



- If the weights are **non-negative**, then we can apply **Dijkstra's Algorithm** (more details & examples from Ve203)
  - Works only when all weights are non-negative
  - A greedy algorithm for solving single source all destinations shortest path problem

Dijkstra's Algorithm

- Keep distance estimate $D(v)$ and predecessor $P(v)$ for each node v.
  - Predecessor: the previous node on the shortest path.

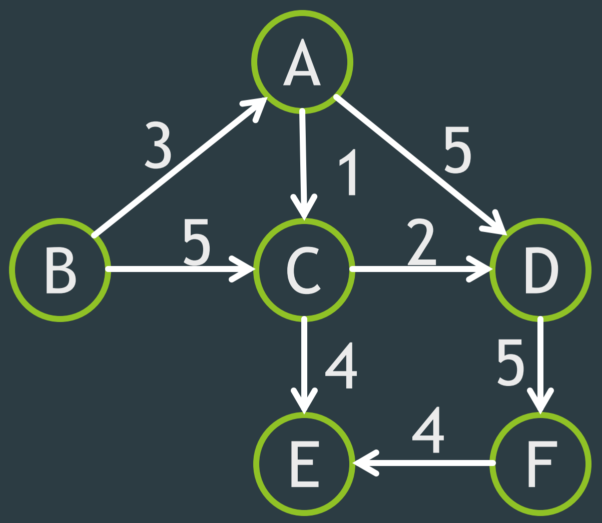
1. Initially, $D(s) = 0$; $D(v)$ for other nodes is $+\infty$; $P(v)$ is unknown.
2. Store all the nodes in a set R.
3. While R is not empty
   1. Choose node v in R such that $D(v)$ is the smallest. Remove v from the set R.
   2. Declare that v's shortest distance is known, which is $D(v)$.
   3. For each of v's neighbors u that is still in R, update distance estimate $D(u)$ and predecessor $P(u)$.

Updating

- For each of v's neighbors u that is **still in** R, if $D(v) + w(v, u) < D(u)$, then update $D(u) = D(v) + w(v, u)$ and the predecessor $P(u) = v$.
  - I.e., update $D(u)$ if the path going through v is shorter than the best path so far to u.

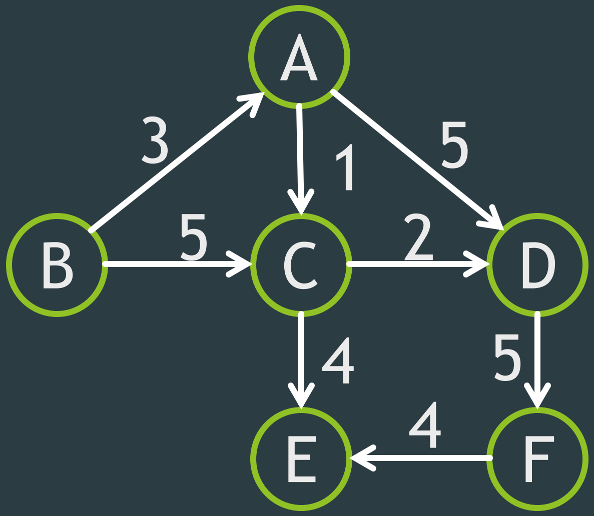
Example : Shortest path A->E

Explored	Unexplored
	A_∞ B_∞ C_∞ D_∞ E_∞ F_∞



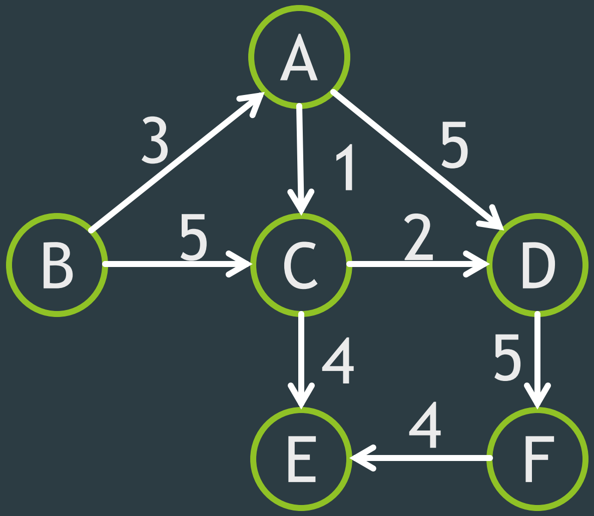
Example: Shortest path A->E

Explored	Unexplored
	A_0^A B_∞ C_∞ D_∞ E_∞ F_∞



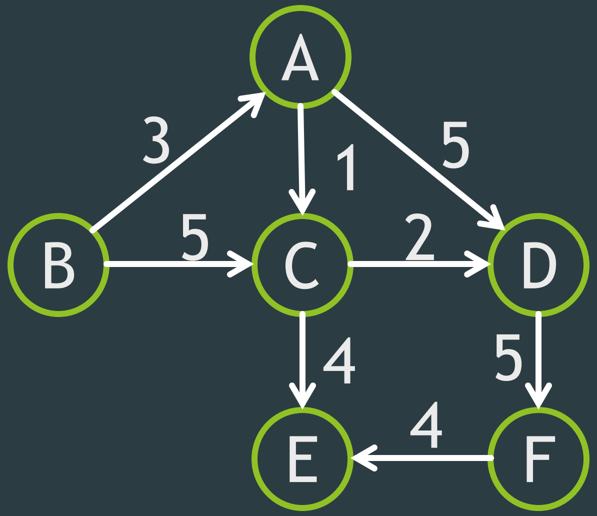
Example : Shortest path A->E

Explored	Unexplored
A_0^A	B_∞ C_∞ D_∞ E_∞ F_∞



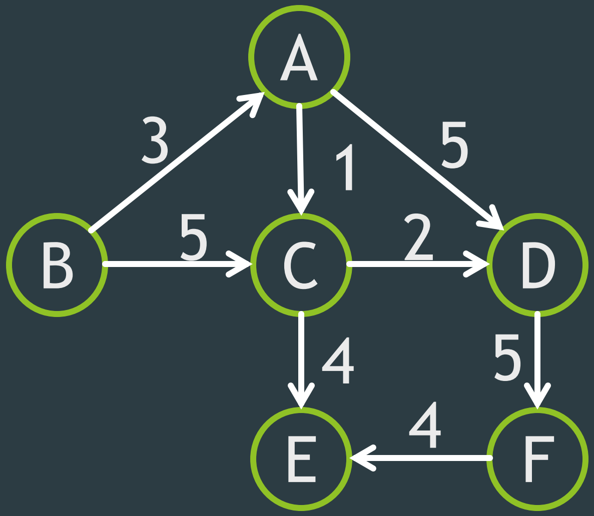
Example : Shortest path A->E

Explored	Unexplored
A_0^A	B_∞ C_1^A D_5^A E_∞ F_∞



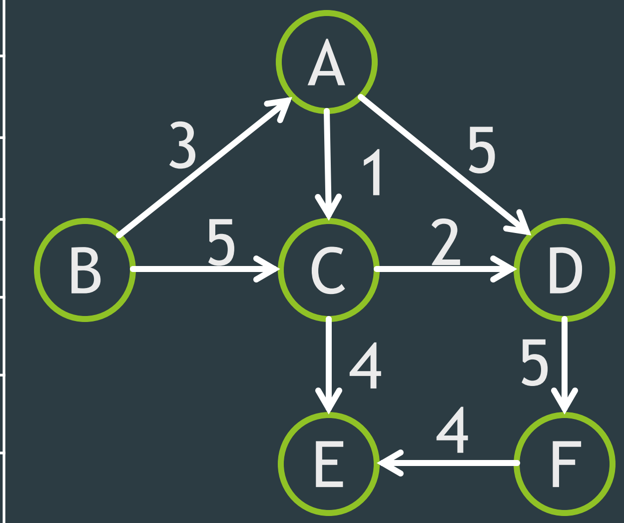
Example: Shortest path A->E

Explored	Unexplored
A_0^A	B_∞ C_1^A D_5^A E_∞ F_∞
A_0^A C_1^A	



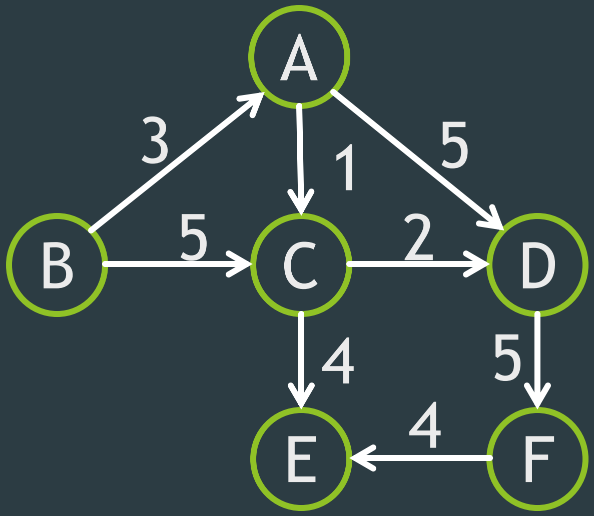
Example: Shortest path A->E

Explored	Unexplored
A_0^A	B_∞ C_1^A D_5^A E_∞ F_∞
A_0^A C_1^A	B_∞ D_3^C E_5^C F_∞



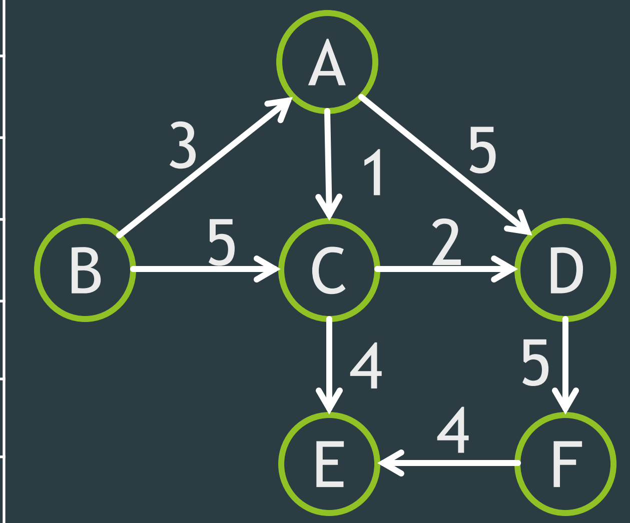
Example: Shortest path A->E

Explored	Unexplored
A_0^A	B_∞ C_1^A D_5^A E_∞ F_∞
A_0^A C_1^A	B_∞ D_3^C E_5^C F_∞
A_0^A C_1^A D_3^C	



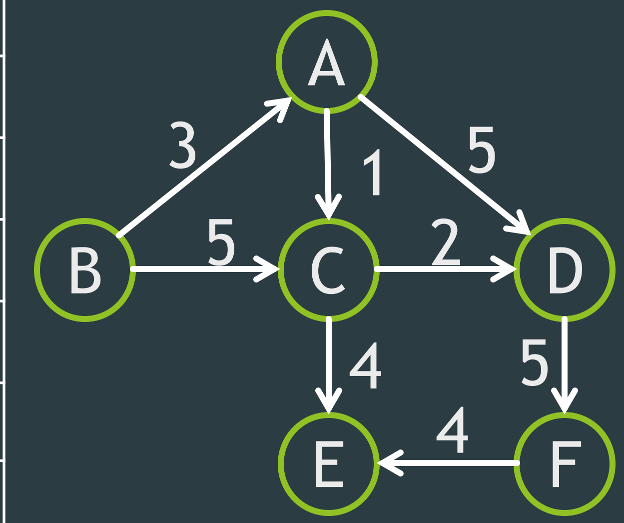
Example: Shortest path A->E

Explored	Unexplored
A_0^A	B_∞ C_1^A D_5^A E_∞ F_∞
A_0^A C_1^A	B_∞ D_3^C E_5^C F_∞
A_0^A C_1^A D_3^C	B_∞ E_5^C F_8^D



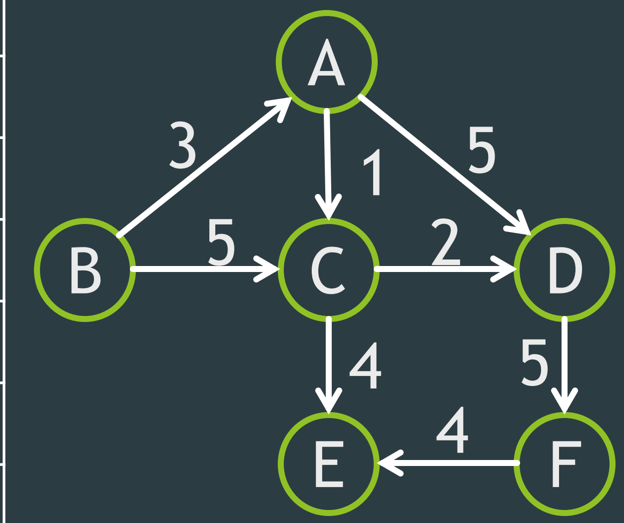
Example: Shortest path A->E

Explored	Unexplored
A_0^A	B_∞ C_1^A D_5^A E_∞ F_∞
A_0^A C_1^A	B_∞ D_3^C E_5^C F_∞
A_0^A C_1^A D_3^C	B_∞ E_5^C F_8^D
A_0^A C_1^A D_3^C E_5^C	B_∞ F_8^D



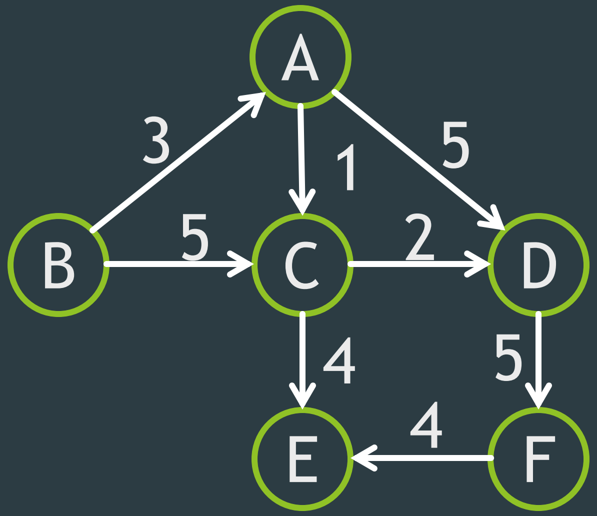
Example: Shortest path A->E

Explored	Unexplored
A_0^A	B_∞ C_1^A D_5^A E_∞ F_∞
A_0^A C_1^A	B_∞ D_3^C E_5^C F_∞
A_0^A C_1^A D_3^C	B_∞ E_5^C F_8^D
A_0^A C_1^A D_3^C E_5^C	B_∞ F_8^D
A_0^A C_1^A D_3^C E_5^C F_8^D	B_∞



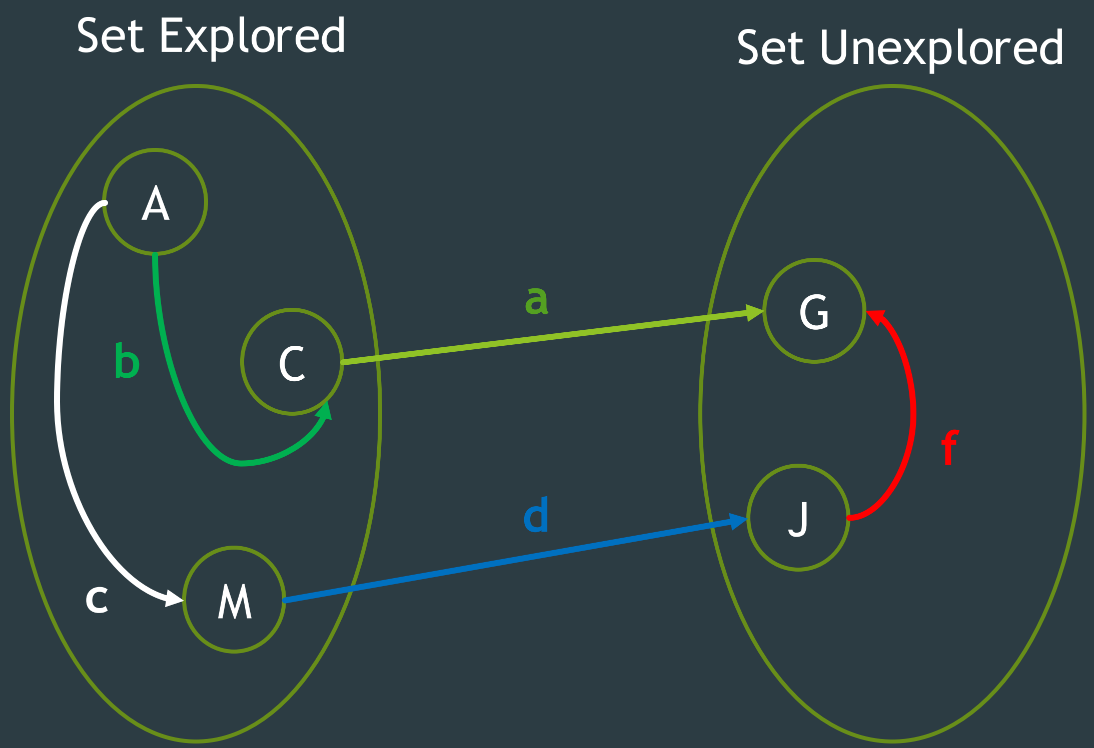
Example: Shortest path A->E

Explored	Unexplored
A_0^A	B_∞ C_1^A D_5^A E_∞ F_∞
A_0^A C_1^A	B_∞ D_3^C E_5^C F_∞
A_0^A C_1^A D_3^C	B_∞ E_5^C F_8^D
A_0^A C_1^A D_3^C E_5^C	B_∞ F_8^D
A_0^A C_1^A D_3^C E_5^C F_8^D	B_∞
A_0^A C_1^A D_3^C E_5^C F_8^D B_∞	



Why Does it Work?

- Proof by induction
  - All nodes in set Explored already have shortest paths
  - Node in set Unexplored with the smallest distance has the shortest path



b is already the shortest distance to C

b+a is already the smallest distance to G through C

b+a ≤ c+d

b+a ≤ c+d+f

Dijkstra’s Algorithm v.s. Prim’s Algorithm

- Dijkstra’s algorithm is similar to Prim’s algorithm
  - Prim’s algorithm: grow the set of nodes we add to the MST.
  - Dijkstra’s algorithm: grow the set of nodes to which we know the shortest path.

Dijkstra's Algorithm
Time Complexity

- Number of times to find the smallest $D(v)$: $|V|$.
  - Each cost?

Linear scan: $O(|V|)$; Binary heap: $O(\log|V|)$;
Fibonacci heap: $O(\log|V|)$

- Total number of times to update the neighbors: $|E|$.
  - Since each neighbor of each node could be potentially updated.
  - Each cost?

Linear scan: $O(1)$; Binary heap: $O(\log|V|)$;
Fibonacci heap: $O(1)$

- Total time complexity
  - Linear scan: $O(|E| + |V|^2) = O(|V|^2)$
  - Binary heap: $O(|V|\log|V| + |E|\log|V|)$
  - Fibonacci heap: $O(|V|\log|V| + |E|)$

Recent Advances of Dijkstra’s Algorithm

Universal Optimality of Dijkstra via Beyond-Worst-Case Heaps*

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Abstract

This paper proves that Dijkstra’s shortest-path algorithm is universally optimal in both its running time and number of comparisons when combined with a sufficiently efficient heap data structure.

Universal optimality is a powerful beyond-worst-case performance guarantee for graph algorithms that informally states that a single algorithm performs as well as possible for every single graph topology. We give the first application of this notion to any sequential algorithm.

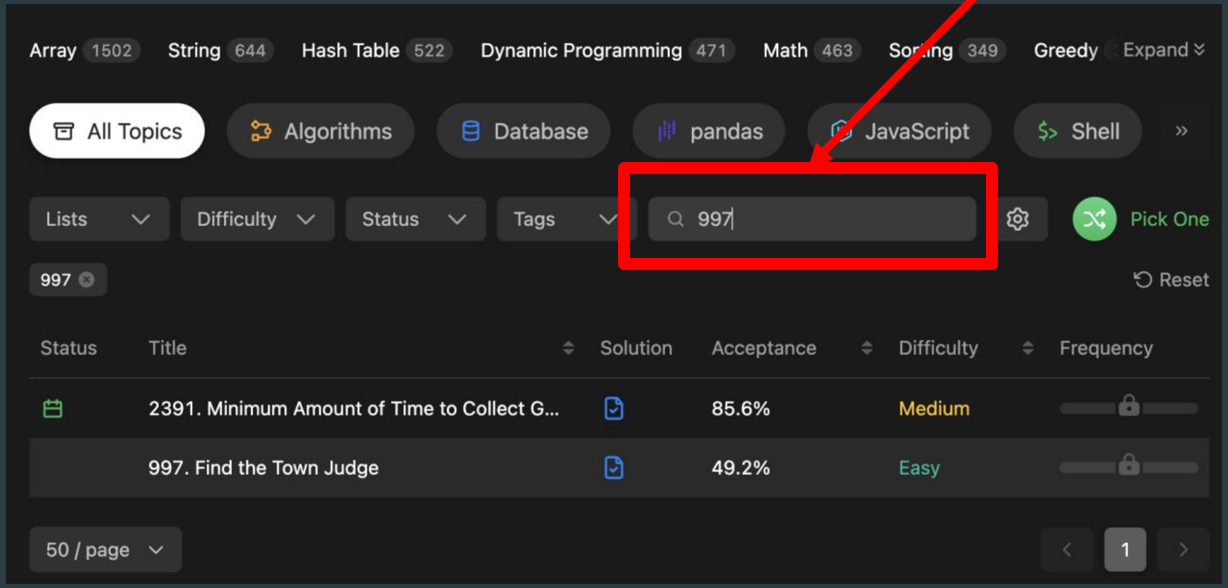
We design a new heap data structure with a working-set property guaranteeing that the heap takes advantage of locality in heap operations. Our heap matches the optimal (worst-case) bounds of Fibonacci heaps but also provides the beyond-worst-case guarantee that the cost of extracting the minimum element is merely logarithmic in the number of elements inserted after it instead of logarithmic in the number of all elements in the heap. This makes the extraction of recently added elements cheaper.

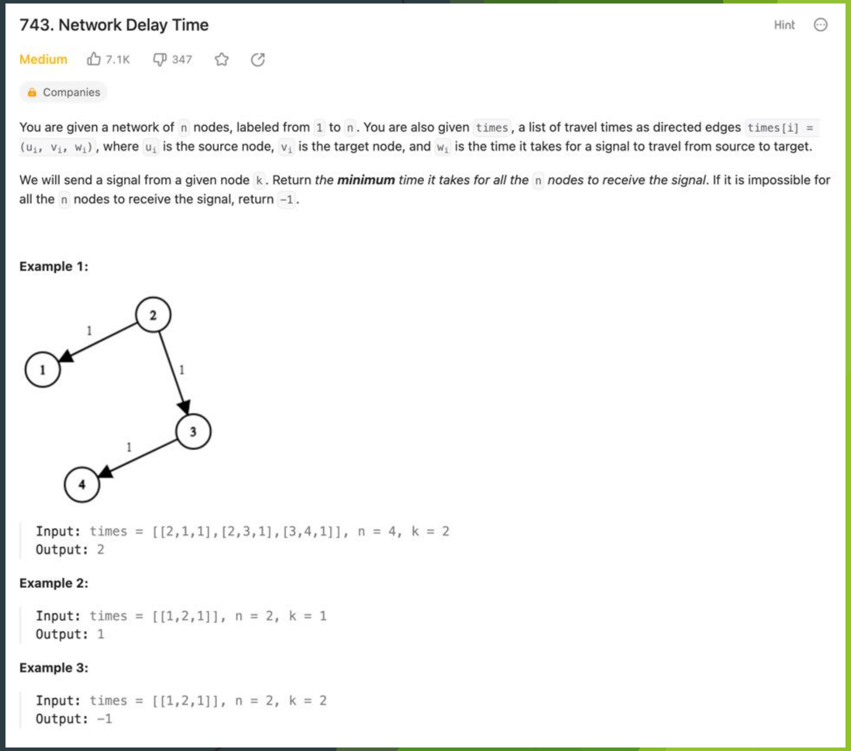
We prove that our working-set property guarantees universal optimality for the problem of ordering vertices by their distance from the source vertex: The sequence of heap operations generated by any run of Dijkstra’s algorithm on a fixed graph possesses enough locality that one can couple the number of comparisons performed by any heap with our working-set bound to the minimum number of comparisons required to solve the distance ordering problem on this graph for a worst-case choice of arc lengths.

Exercise 1

743. Network Delay Time

Search





743. Network Delay Time

Hint

Medium 7.1K 347

Companies

You are given a network of `n` nodes, labeled from `1` to `n`. You are also given `times`, a list of travel times as directed edges `times[i] = (ui, vi, wi)`, where `ui` is the source node, `vi` is the target node, and `wi` is the time it takes for a signal to travel from source to target.

We will send a signal from a given node `k`. Return *the **minimum** time it takes for all the `n` nodes to receive the signal*. If it is impossible for all the `n` nodes to receive the signal, return `-1`.

Example 1:

```
Input: times = [[2,1,1],[2,3,1],[3,4,1]], n = 4, k = 2
Output: 2
```

Example 2:

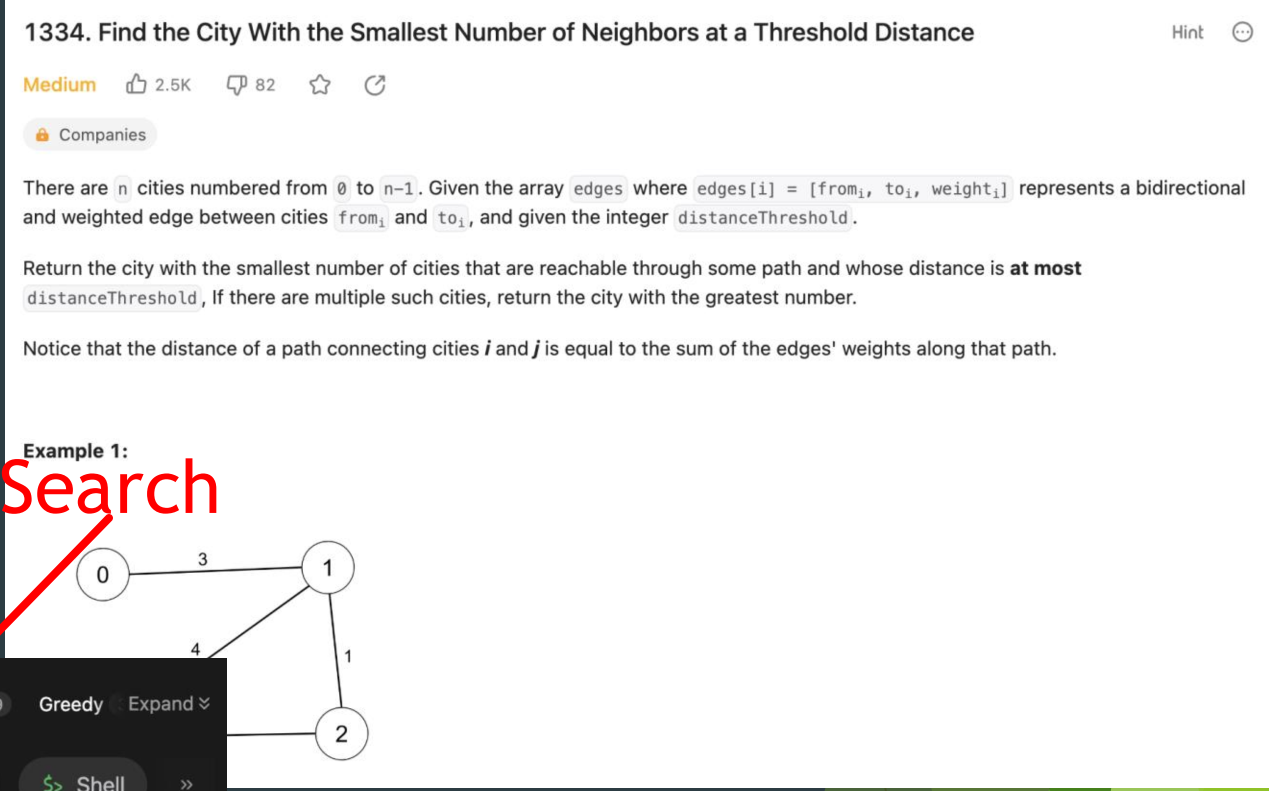
```
Input: times = [[1,2,1]], n = 2, k = 1
Output: 1
```

Example 3:

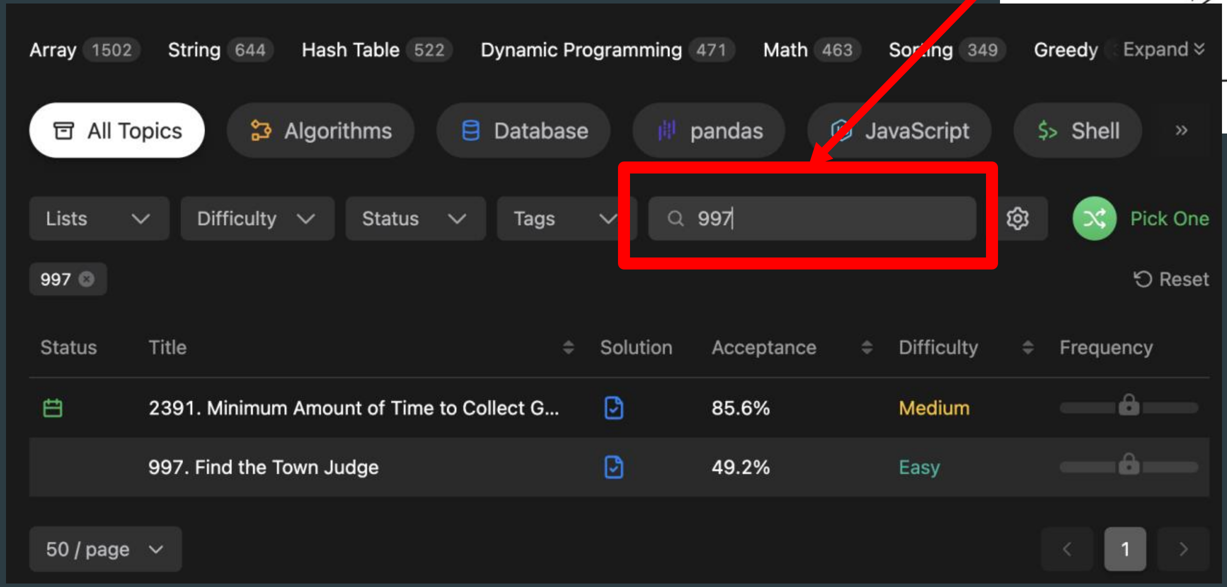
```
Input: times = [[1,2,1]], n = 2, k = 2
Output: -1
```

Exercise 2

1334. Find the City With the Smallest Number of Neighbors at a Threshold Distance



Search



Exercise 3 Graph

684. Redundant Connection