ECE2810J
Data Structures and Algorithms

Minimum Spanning Tree

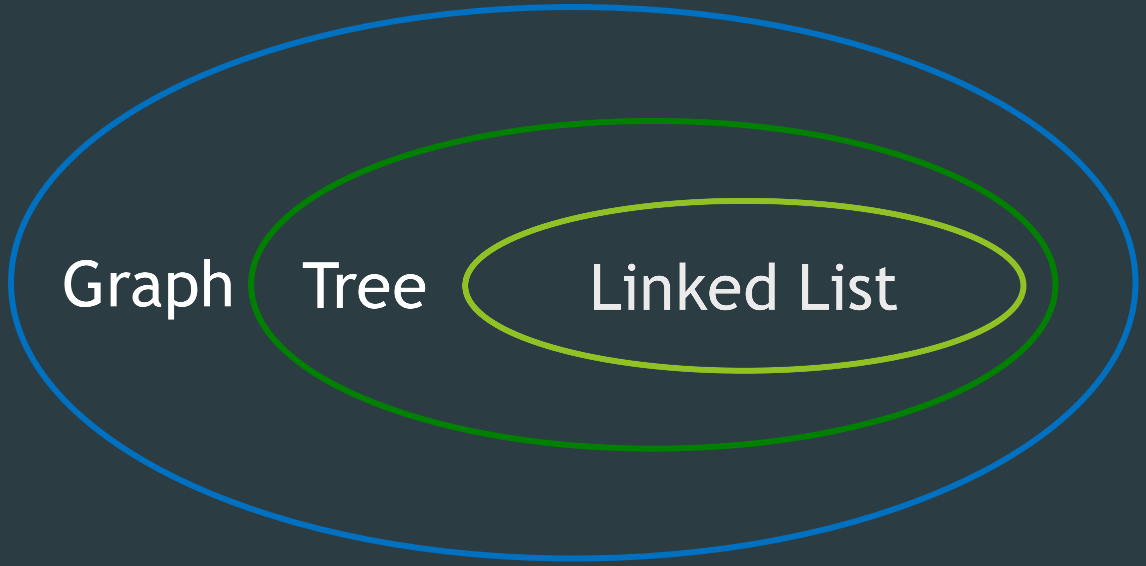
Learning Objectives:

- Know what a minimum spanning tree (MST) is
- Know the Prim's algorithm for finding the MST
- Know how the various choices of the supporting data structures affect the runtime of the Prim's algorithm

Outline

- Minimum Spanning Tree
  - Problem
  - Prim's Algorithm

Linked Lists, Trees, and Graphs

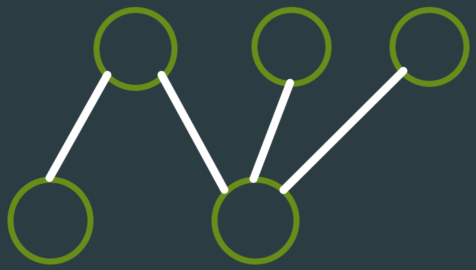


Tree and Graph

- A tree is an acyclic, connected undirected graph.

The tree we see before

However, this is also a tree



Any node can be the root of the tree.

- For a tree, $|E| = |V| - 1$.
- Claim: Any connected graph with N nodes and $N - 1$ edges is a tree.

Subgraph and Spanning Tree

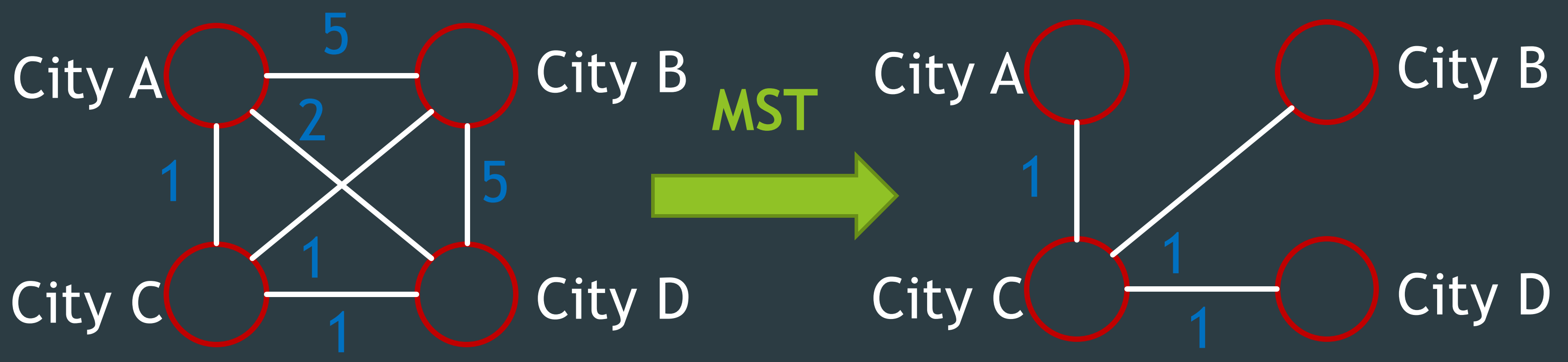
- $G' = (V', E')$ is a subgraph of $G = (V, E)$ if and only if $V' \subseteq V$ and $E' \subseteq E$.
- A spanning tree of a connected undirected graph G is a subgraph of G that
  1. contains all the nodes of G;
  2. is a tree, i.e., connected and acyclic.

Minimum Spanning Tree (MST)

- Given a weighted, connected, undirected graph $G = (V, E)$, a **minimum spanning tree** T of G is a spanning tree of G whose sum of all edge weights is the minimal.

Application of MST

- A government planning a freeway system to connect all the cities.



- A power company planning where to lay down high-voltage power lines.

Minimum Spanning Tree
Algorithms

- Main idea: greedily select edges one by one and add to a growing sub-graph.

- Two standard algorithms:
  - Prim’s algorithm
  - Kruskal’s algorithm

Outline

- Minimum Spanning Tree
  - Problem
  - Prim's Algorithm

Prim's Algorithm

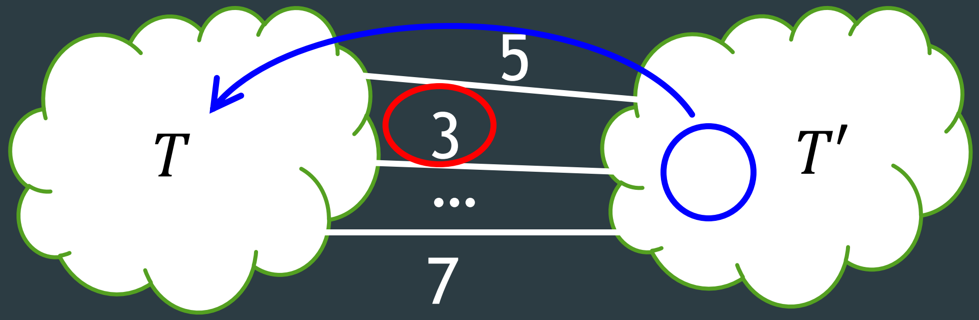
- Separate V into two sets:
  - T: the set of nodes that have been added to the MST.
  - T': those nodes that have not been added to the MST, i.e., $T' = V - T$.

- Prim's algorithm initially sets T ={s}, where s is an arbitrarily picked **node**, and T' = V − $\{s\}$. The algorithm moves one node from T' to T in each iteration. After the last iteration, $T = V$ and we have constructed the MST.

Prim's Algorithm
Basic Version

1. Arbitrarily pick one node s; set $T = \{s\}$ and $T' = V - \{s\}$.

2. While $T' \neq \emptyset$
   - Select an edge with the **smallest weight** that connects between a node in T and a node in T'. Suppose the edge connects with node v in T'. Move v from T' to T.



Selecting the Smallest Edge and Node

- For each node $v \in T'$, keep a measure $D(v)$, storing the "current" smallest weight over all edges that connect v to a node in T.
  - Will be updated later.

- To choose the edge with the smallest weight that connects between a node in T and a node in T', we pick the node $v \in T'$ with the smallest $D(v)$.
  - If edge (u, v) gives the smallest $D(v)$, then (u, v) is the edge with the smallest weight across set T and T'.

Updating *v*'s Neighbor

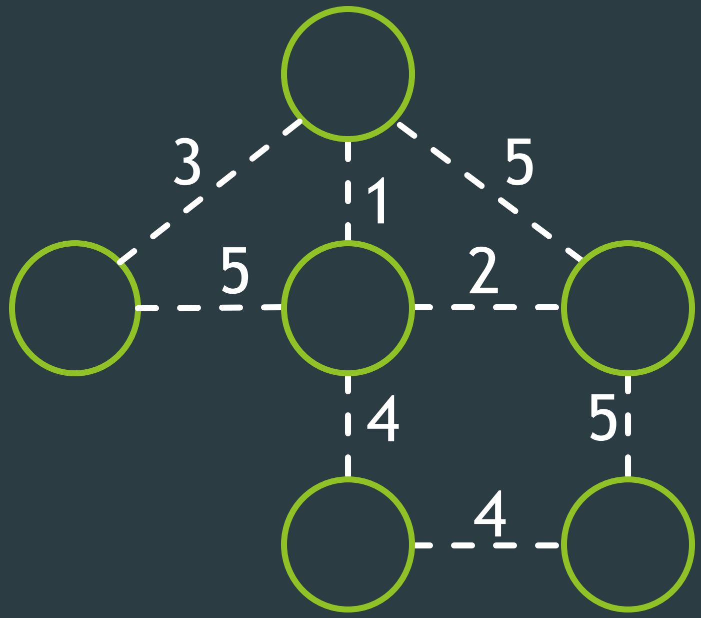
- If we move a node v from T' to T, then for each of v's neighbor u that is still in T', we update its $D(u)$ as follows:
  - If $D(u) > w(v,u)$, then let $D(u) = w(v,u)$.
  - I.e., update $D(u)$ if the weight of edge (v,u) is smaller than the weight of any other edge that connects a node in T to u.

Prim's Algorithm
Full Version

- We keep $P(v)$ for each node v: $(P(v), v)$ is the edge chosen in the MST.

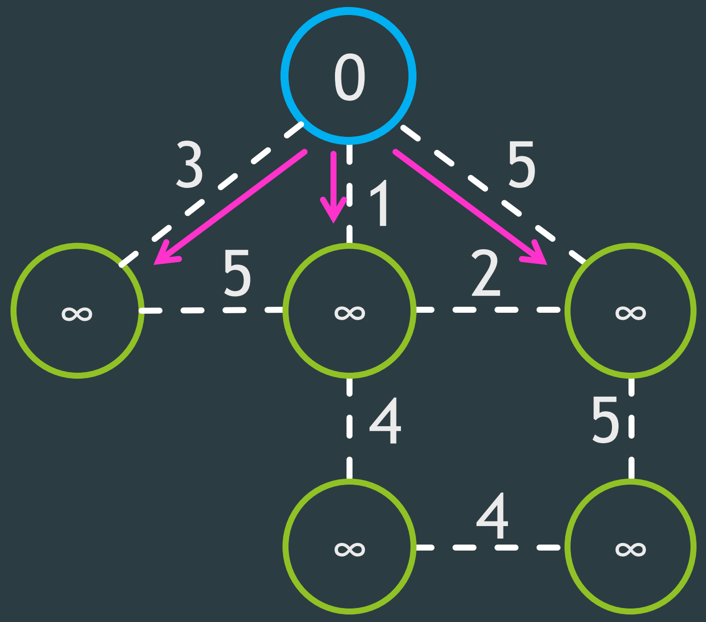
1. Arbitrarily pick one node s. Set $D(s) = 0$. For any other node v, set $D(v)$ as infinite and $P(v)$ as unknown.
2. Set $T' = V$.
3. While $T' \neq \emptyset$
    1. Choose node v in T' such that $D(v)$ is the smallest. Remove v from the set T'.
    2. For each of v's neighbors u that is still in T',
       if $D(u) > w(v, u)$, then update $D(u)$ as $w(v, u)$ and $P(u)$ as v.

Prim's Algorithm
Example

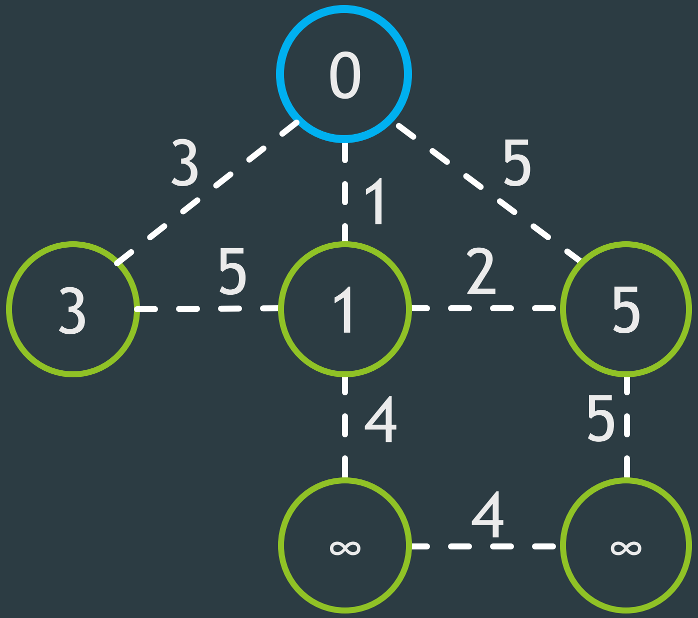


Prim's Algorithm

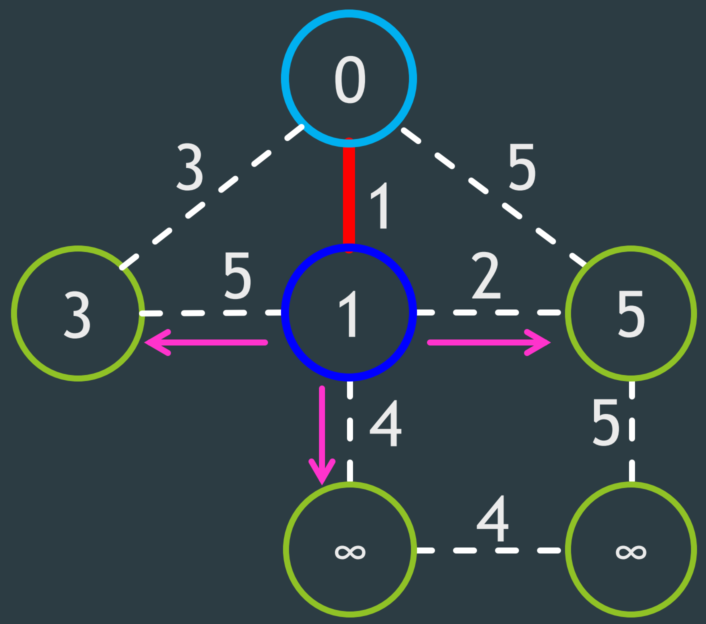
Example



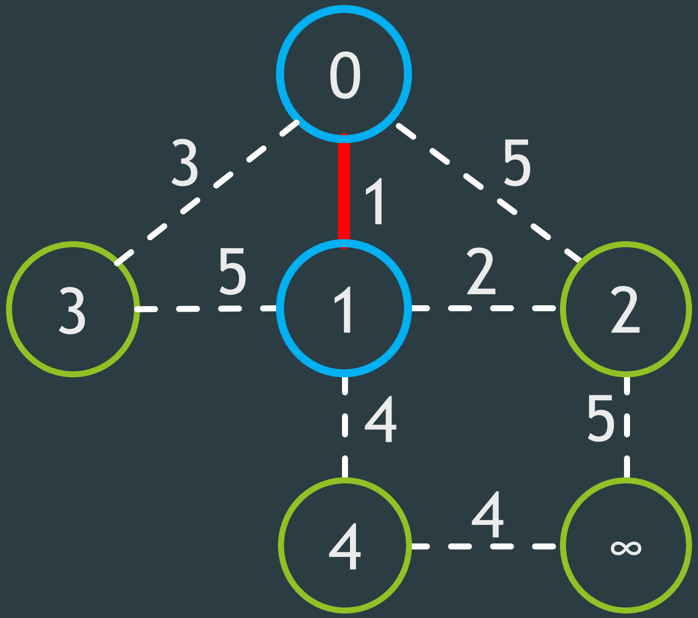
Prim’s Algorithm
Example



Prim’s Algorithm
Example



Prim’s Algorithm
Example



Prim's Algorithm
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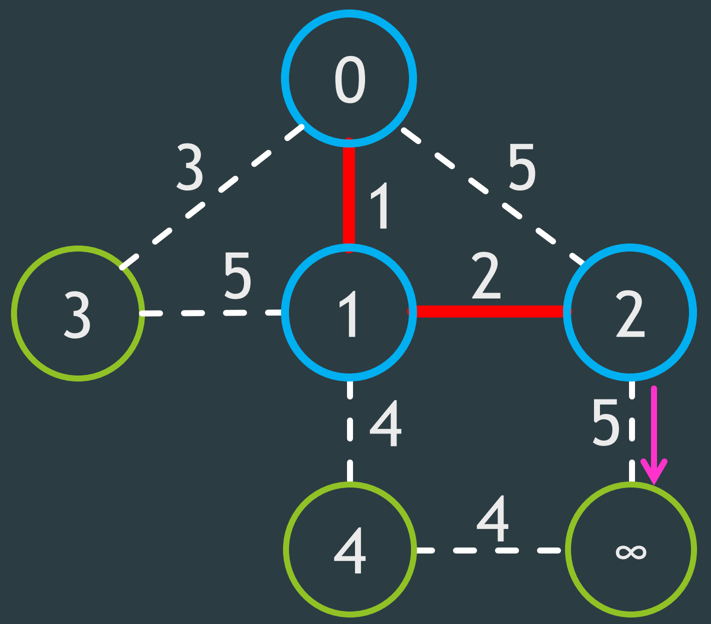
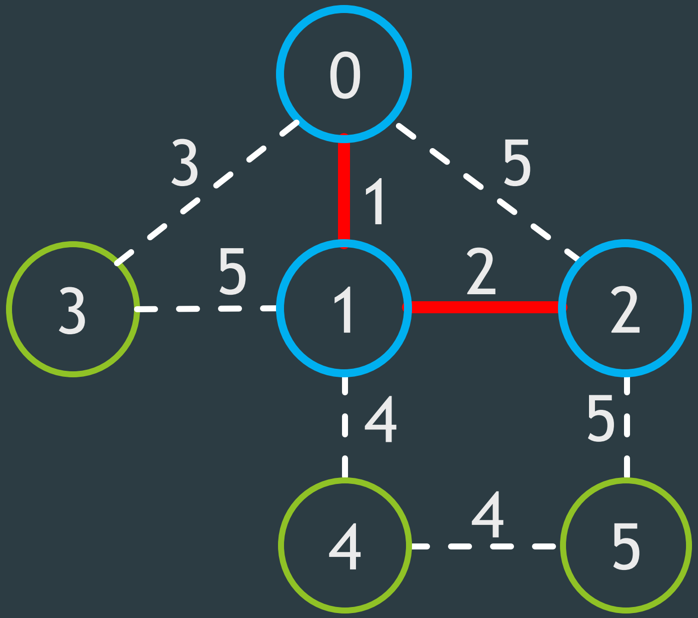
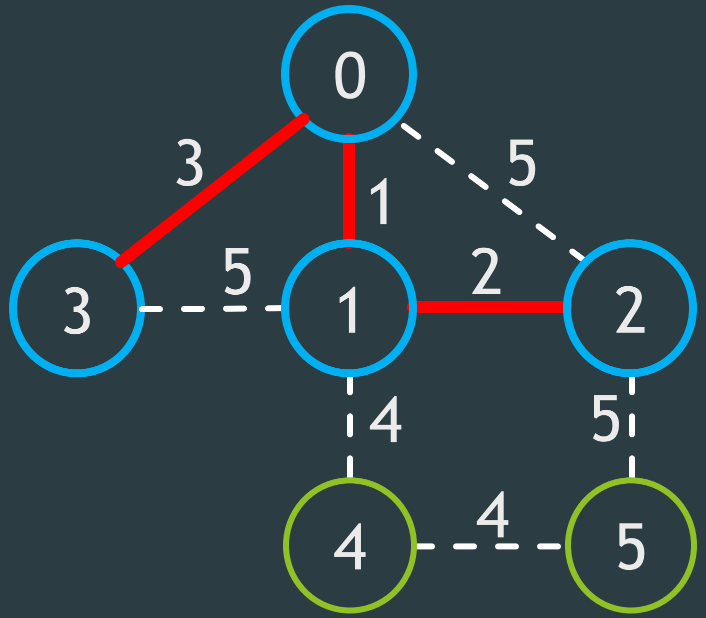


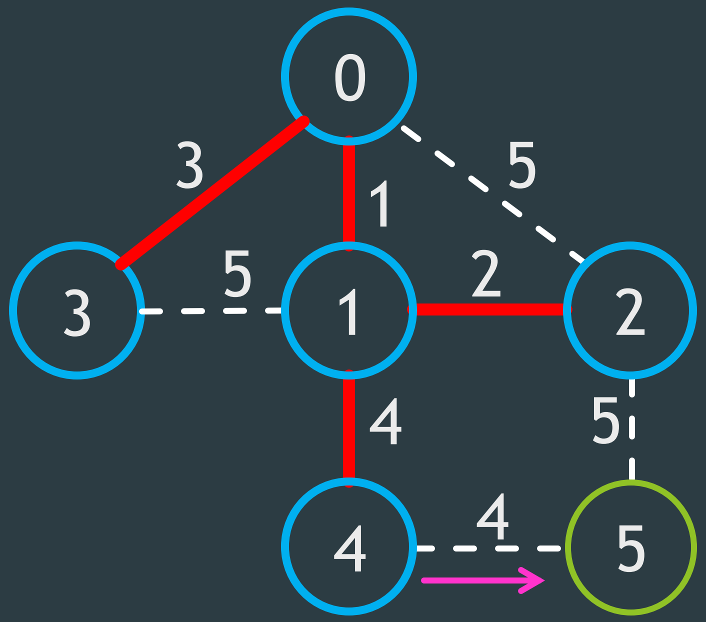
Prim's Algorithm
Example



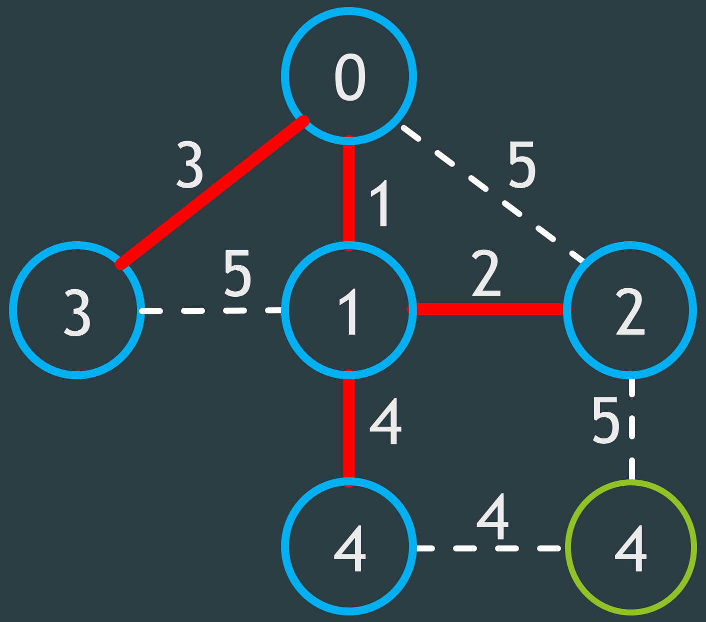
Prim’s Algorithm
Example



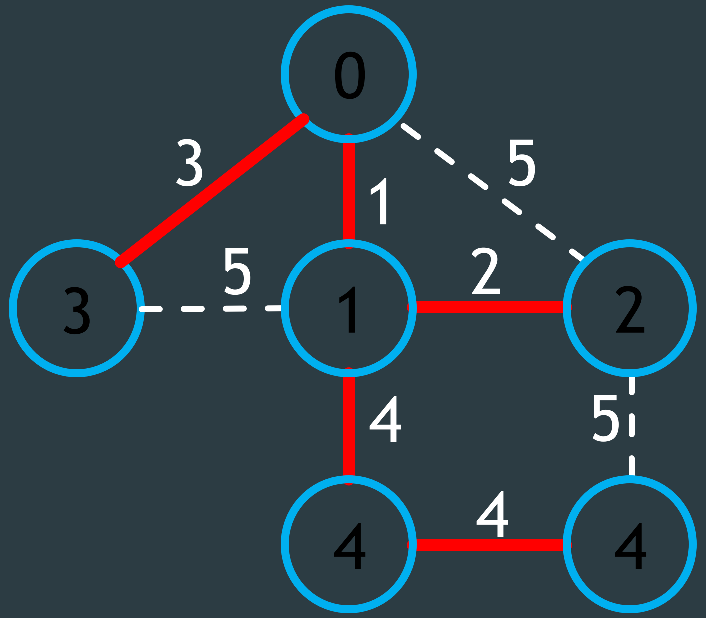
Prim’s Algorithm
Example



Prim's Algorithm
Example

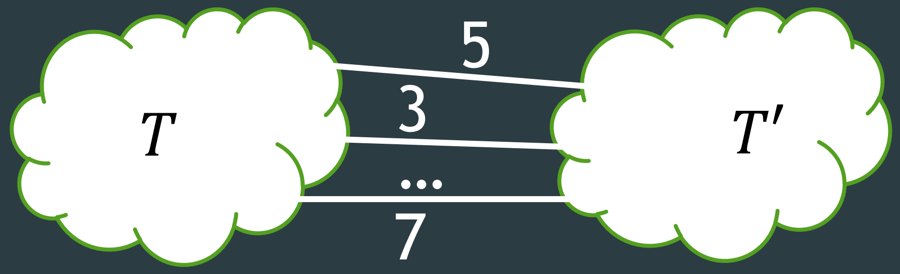


Prim’s Algorithm
Example



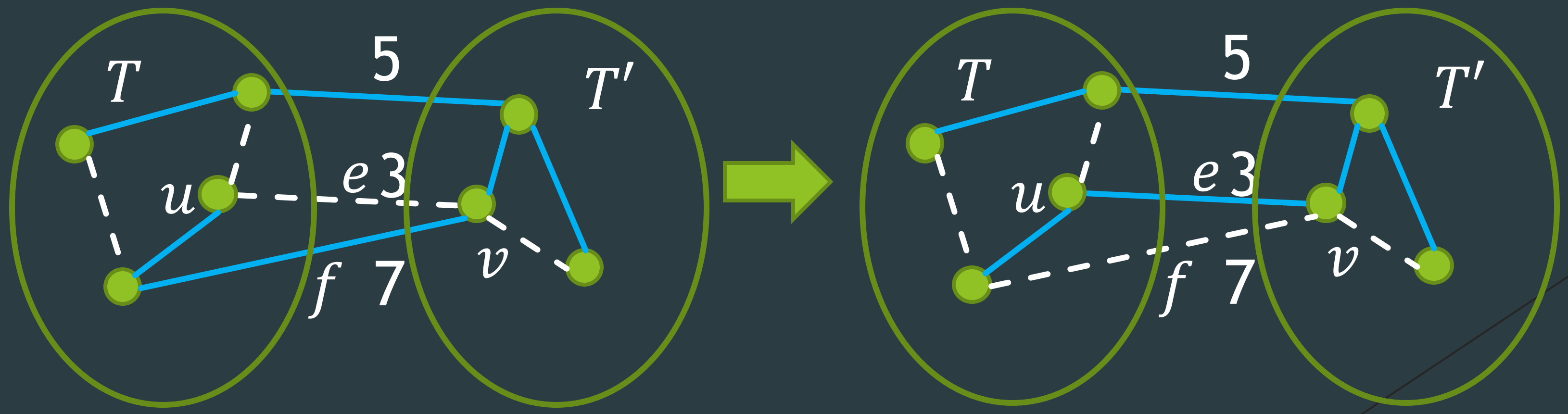
Prim's Algorithm
Justification

- Claim: the obtained subgraph is a tree
- Proof:
  - The nodes in set T are connected (can be shown by induction)
  - Furthermore, $|V| = |E| + 1$
  - Claim: Any **connected** graph with N nodes and $N - 1$ edges is a tree



Prim's Algorithm Justification

- Claim: the obtained subgraph is an MST
- Proof by contradiction:
  - Assume the MST does not contain the cheapest edge e between T and T'
  - Assume $e = (u, v)$. Its weight is w
  - In the MST, there exists a unique path between u and v. On this path, there is an edge f across T and T'. Its weight $> w$
  - We replace f by e in original MST.
  - The new graph is a tree with smaller sum of edge weights



Prim's Algorithm
Time Complexity

1. Arbitrarily pick one node s. Set $D(s) = 0$. For any other node v, set $D(v)$ as infinite and $P(v)$ as unknown.
2. Set $T' = V$.
3. While $T' \neq \emptyset$
   1. Choose node v in T' such that $D(v)$ is the smallest. Remove v from the set T'.
   2. For each of v's neighbors u that is still in T',
      if $D(u) > w(v,u)$, then update $D(u)$ as $w(v,u)$ and $P(u)$ as v.

What is the time complexity of Prim's algorithm?

Prim's Algorithm
Time Complexity

- Method 1: linear scan the set T' to find the smallest $D(v)$.
- Number of times to find the smallest $D(v)$: $|V|$.
  - Each cost: $O(|V|)$.
- Maximal number of times to update the neighbors: $|E|$.
  - Since each neighbor of each node could be **potentially** updated.
  - Each cost: $O(1)$.
- Total running time is $O(|E| + |V|^2) = O(|V|^2)$.

Prim's Algorithm
Time Complexity

- Method 2: use a binary heap to store $D(v)$'s.
- Number of times to extract the smallest $D(v)$: $|V|$.
  - Each cost: $O(\log |V|)$.

- Maximal number of times to update the neighbors: $|E|$.
  - Each cost is $O(\log |V|)$, since after updating $D(v)$, we should percolate up new $D(v)$ into right location of binary heap.

- Total running time is $O(|V| \log|V| + |E| \log|V|) = O\big((|V| + |E|) \log|V|\big)$.

Prim's Algorithm
Time Complexity

- Method 3: use a Fibonacci heap to store $D(v)$'s.
- Number of times to extract the smallest $D(v)$: $|V|$.
  - Each cost: $O(\log |V|)$.
- Maximal number of times to update the neighbors: $|E|$.
  - Each cost is $O(1)$ (decreaseKey operation; amortized time).
- Total running time is $O(|V| \log|V| + |E|)$.

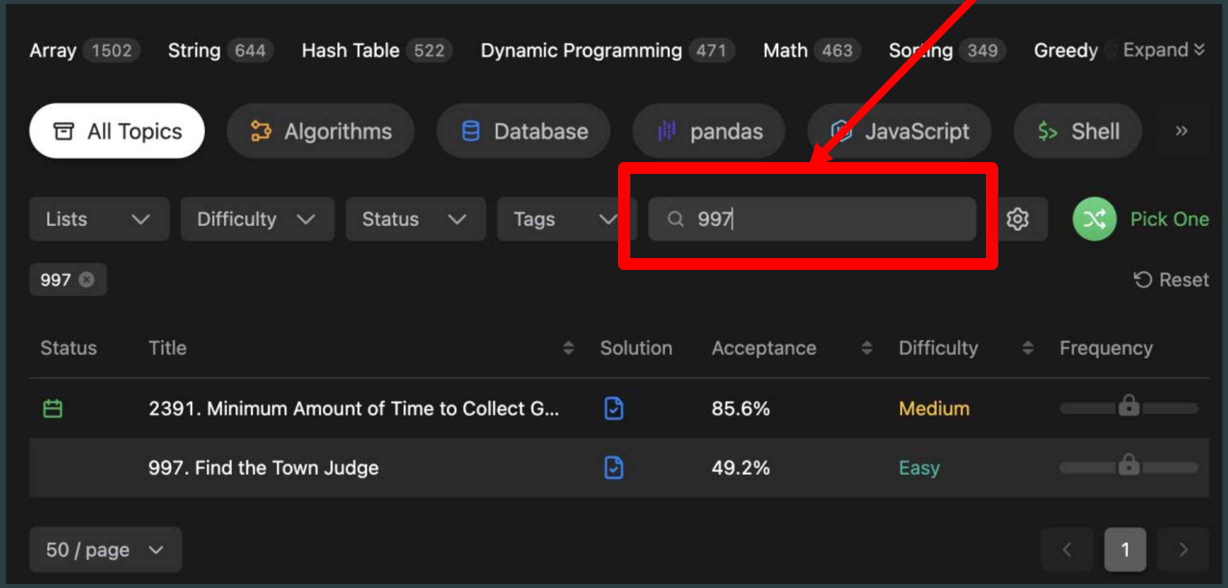
Prim's Algorithm
Time Complexity

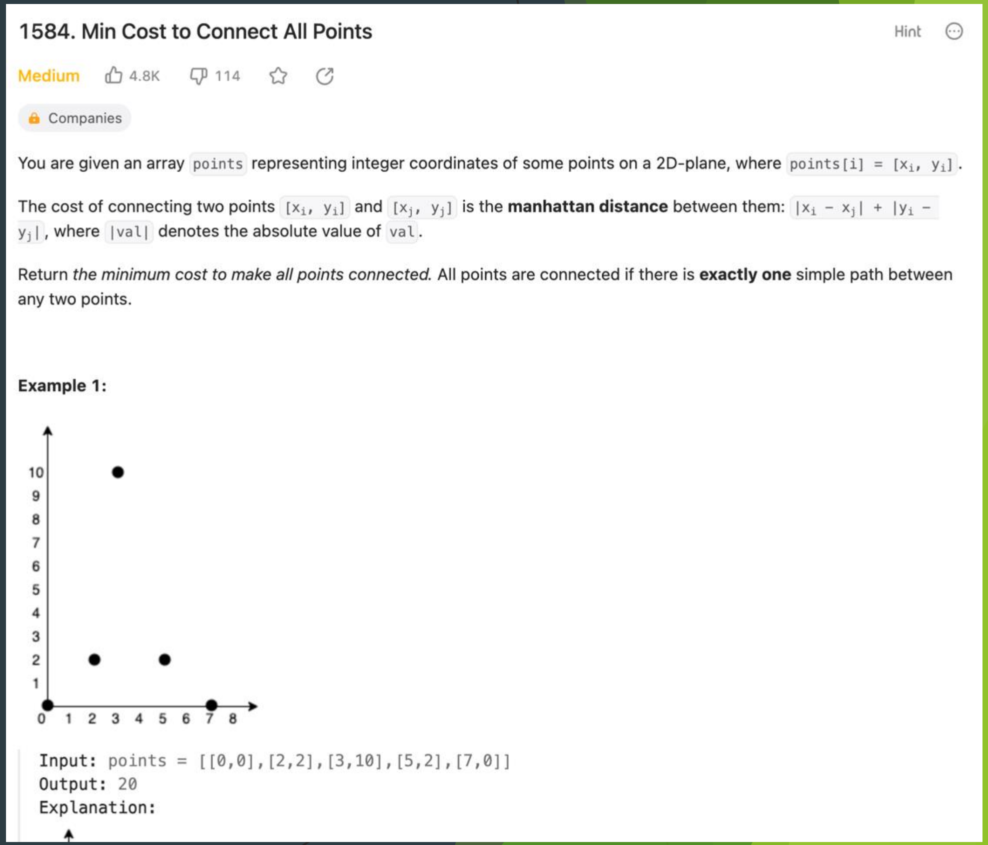
- Method 1: linear scan the set T' to find the smallest $D(v)$
  - Total runtime: $O(|V|^2)$
- Method 2: use a binary heap to store $D(v)$'s
  - Total runtime: $O((|V| + |E|) \log |V|)$
- Method 3: use a Fibonacci heap to store $D(v)$'s
  - Total runtime: $O(|V| \log|V| + |E|)$
- Which one is the best?
  - Answer: Fibonacci heap.
  - For sparse graphs, i.e., $|E| \approx \Theta(|V|)$, using binary heap has same runtime complexity as Fibonacci heap. The runtime complexity is $O(|V| \log|V|)$

Exercise 1

1584. Min Cost to Connect All Points

Search

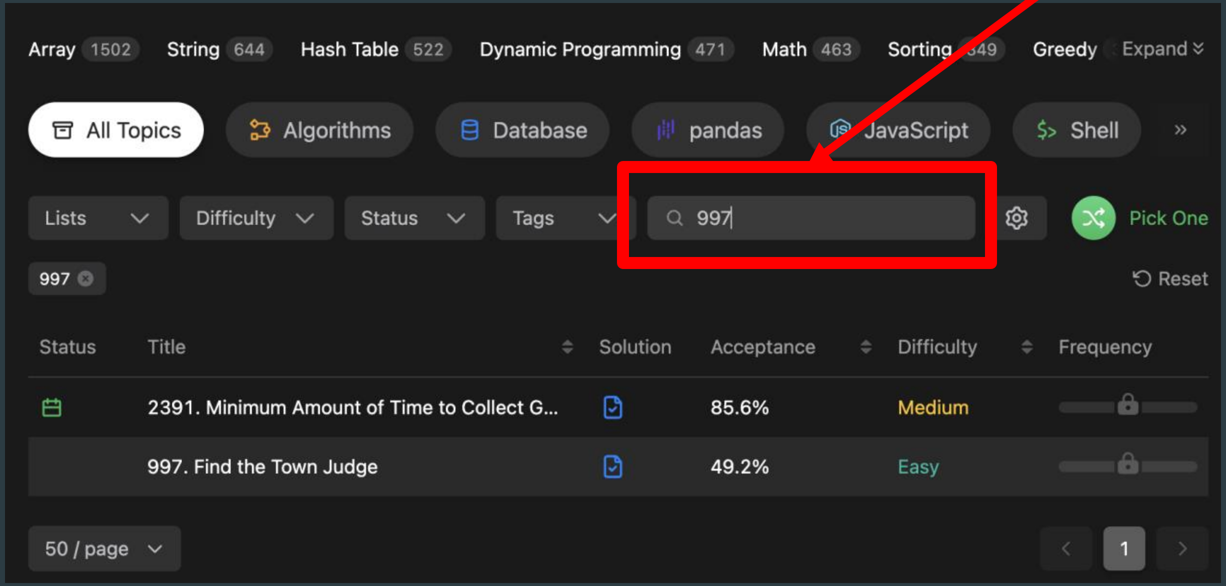




Exercise 2

1489. Find Critical and Pseudo-Critical Edges in Minimum Spanning Tree

Search



Given a weighted undirected connected graph with `n` vertices numbered from `0` to `n - 1`, and an array `edges` where `edges[i] = [`a_i`, `b_i`, weight`$_i$`]` represents a bidirectional and weighted edge between nodes a_i and b_i. A minimum spanning tree (MST) is a subset of the graph's edges that connects all vertices without cycles and with the minimum possible total edge weight.

Find *all the critical and pseudo-critical edges in the given graph's minimum spanning tree (MST)*. An MST edge whose deletion from the graph would cause the MST weight to increase is called a *critical edge*. On the other hand, a pseudo-critical edge is that which can appear in some MSTs but not all.

Note that you can return the indices of the edges in any order.

Example 1: