ECE2810J
Data Structures and Algorithms

Average-Case Time Complexity of BST

Learning Objectives:

- Know the average-case time complexity of search, insertion, and removal operations for a binary search tree

Which Statements Are Correct?

- Suppose the depth (height) of a binary search tree is h. Consider the time complexity for a successful search.

  A. In the worst case, the complexity is $O(h)$

  B. In the average case, the complexity is $O(h)$

- Suppose the number of nodes of a binary search tree is n. Consider the time complexity for a successful search.

  C. In the worst case, the complexity is $O(n)$

  D. In the worst case, the complexity is $O(\log n)$

How about average-case time complexity for a successful search in terms of the number of nodes n?

Average Case Analysis

- If the successful search reaches a node at depth d, the number of nodes visited is $d + 1$.
  - The complexity is $\Theta(d)$.

- Assume that it is equally likely for the object of the search to appear in any node of the search tree. The average complexity is $\Theta(\bar{d})$
  - $\bar{d}$ is the average depth of the nodes in a given tree

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i$$

Internal Path Length

- $\sum_{i=1}^{n} d_i$ is called internal path length.
- To get the average case complexity, we need to get the average of $\sum_{i=1}^{n} d_i$ for all trees of n nodes.
- Define the average internal path length of a tree containing n nodes as $I(n)$.
  - $I(1) = 0$.
- For a tree of n nodes, suppose it has l nodes in its left subtree.
  - The number of nodes in its right subtree is $n - 1 - l$.
  - The total internal path length for such a tree is
$$T(n; l) = I(l) + I(n - 1 - l) + n - 1$$
- $\boxed{I(n) \text{ is average of } T(n; l) \text{ over } l = 0, 1, \ldots, n - 1.}$

Internal Path Length

- Assume all insertion sequences of n keys $k_1 < \cdots < k_n$ are equally likely.
  - The first key inserted being any k_l are equally likely.
- Note: If first key inserted is k_{l+1}, the left subtree has l nodes.
- Claim: All left subtree sizes are equally likely.
- Therefore, we have

$$I(n) = \frac{1}{n}\sum_{l=0}^{n-1} T(n;l)$$

$$= \frac{1}{n}\sum_{l=0}^{n-1} [I(l) + I(n-1-l) + n - 1]$$

$$= \frac{2}{n}\sum_{l=0}^{n-1} I(l) + (n-1)$$

Solving the Recursion

$$I(n) = \frac{2}{n}\sum_{l=0}^{n-1} I(l) + (n-1)$$

replace n
with $n-1$

$$I(n-1) = \frac{2}{n-1}\sum_{l=0}^{n-2} I(l) + (n-2)$$

$$\sum_{l=0}^{n-2} I(l) = \frac{(n-1)[I(n-1)-(n-2)]}{2}$$

Solving the Recursion

$$I(n) = \frac{2}{n}\sum_{l=0}^{n-1} I(l) + (n-1) \qquad \sum_{l=0}^{n-2} I(l) = \frac{(n-1)[I(n-1)-(n-2)]}{2}$$

$$I(n) = \frac{n+1}{n} I(n-1) + \frac{2(n-1)}{n}$$

$$\frac{I(n)}{n+1} = \frac{I(n-1)}{n} + \frac{2(n-1)}{n(n+1)} \quad \leq \frac{I(n-1)}{n} + \frac{2}{n}$$

Solving the Recursion

$$\frac{I(n)}{n+1} \leq \frac{I(n-1)}{n} + \frac{2}{n}$$

$$\frac{I(n)}{n+1} \leq \frac{2}{n} + \frac{2}{n-1} + \frac{2}{n-2} + \cdots + \frac{2}{2} + \frac{I(1)}{2}$$

$$\boxed{I(1) = 0}$$

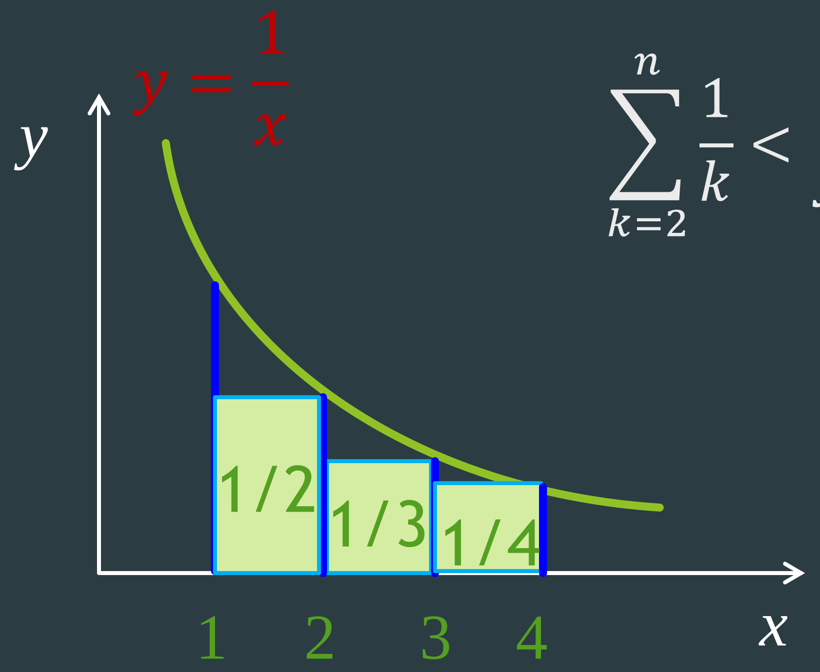
$$\frac{I(n)}{n+1} \leq 2\sum_{k=2}^{n} \frac{1}{k}$$

Note: $\sum_{k=2}^{n} \frac{1}{k} < \ln n$

Proof of the Claim

$$\sum_{k=2}^{n} \frac{1}{k} < \ln n$$

- Claim:



$$\sum_{k=2}^{n} \frac{1}{k} < \int_{1}^{n} \frac{1}{x} dx = \ln n$$

Average Case Analysis Conclusion

- What we get so far: $$\frac{I(n)}{n+1} \le 2\sum_{k=2}^{n}\frac{1}{k} < 2\, ln\, n$$

- Thus, we have
$$I(n) = O(n\log n)$$

- Thus, the average complexity for a successful search is
$$\Theta\left(\frac{1}{n}I(n)\right) = O(\log n)$$

Average Case Time Complexity

- It can also be shown that given n nodes, the average-case time complexity for an unsuccessful search is $O(\log n)$.
- Given n nodes, the average-case time complexities for search, insertion, and removal are all $O(\log n)$.
  - Insertion and removal include “search”.

	Search	Insert	Remove
Linked List	$O(n)$	$O(n)$	$O(n)$
Sorted Array	$O(\log n)$	$O(n)$	$O(n)$
Hash Table	$O(1)$	$O(1)$	$O(1)$
BST	$O(\log n)$	$O(\log n)$	$O(\log n)$

So, why we use BST, not hash table?

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Binary Search Tree Additional Operations

Learning Objectives:

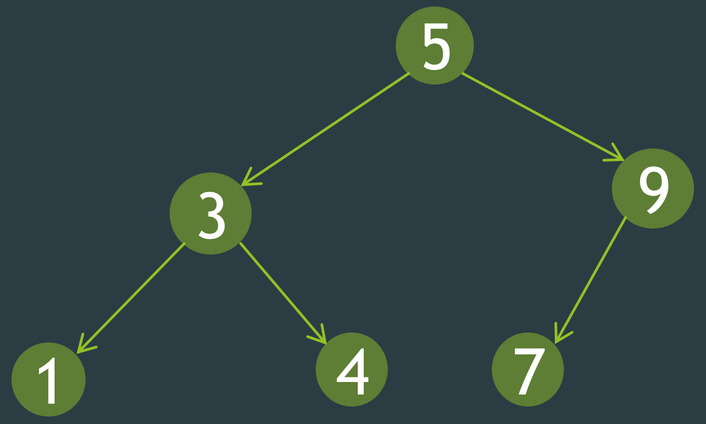
- Know some additional efficient operations of binary search tree
- Know how these operations are implemented and their time complexity

Why BST?

Other Operations Supported by BST	Average-Case Time Complexity
Output in Sorted Order	$O(n)$
Get Min/Max	$O(\log n)$
Get Predecessor/Successor	$O(\log n)$
Rank Search	$O(\log n)$
Range Search	$O(n)$

<u>Note</u>: Hash table does not support efficient implementation of the above methods.

Output in Sorted Order

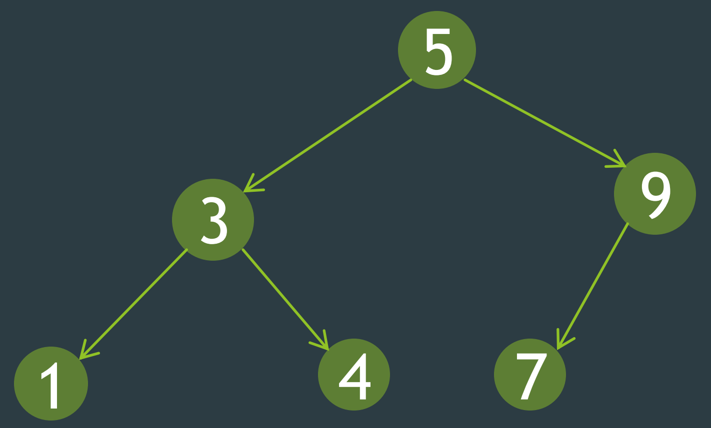


- Output: 1, 3, 4, 5, 7, 9
- How?
  - In-order depth-first traversal.

– Visit the left subtree
– Visit the node
– Visit the right subtree

- Time complexity: $O(n)$.

Get Min/Max

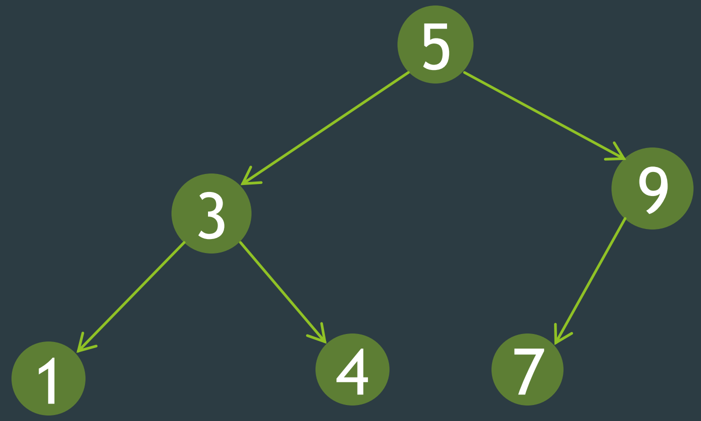


- To get min (max) key of the tree:
  - Start at root.
  - Follow left child pointer (right for max) until you cannot go anymore.
  - Return the last key found.
- Time complexity?

$O(\text{height})$. On average: $O(\log n)$.

Get Predecessor/Successor

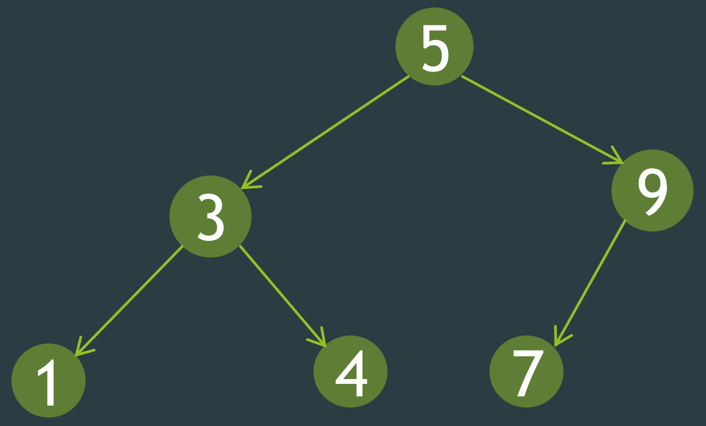
- Given a node in the BST, get its predecessor/successor.
  - Predecessor: the node with the largest key that is smaller than the current key.
  - Successor: the node with the smallest key that is larger than the current key.
  - Predecessor/Successor is in the sense of in-order depth-first traversal.



What's predecessor of key 5?

What's successor of key 5?

Get Predecessor of a Node



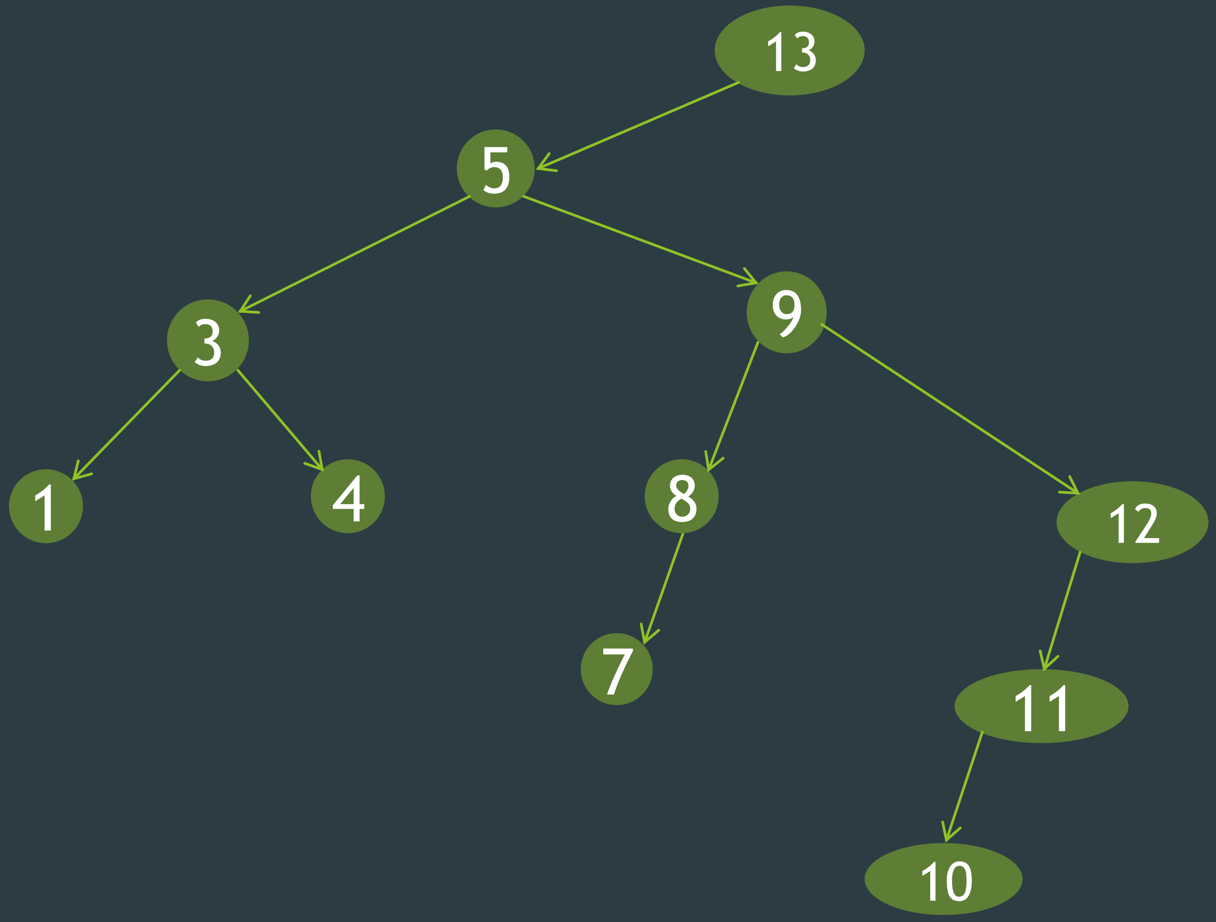
What's predecessor of key 5?

What's predecessor of key 7?

- Easy case: left subtree of the node is nonempty…
  - … return max key in left subtree.
- Otherwise: left subtree is emtpy …
  - … follow parent pointers until you get to a key less than the current key.
  - Equivalent: its first <u>left</u> ancestor.
- Time complexity?

$O(\text{height})$. On average: $O(\log n)$.

What's the Predecessor of 10?

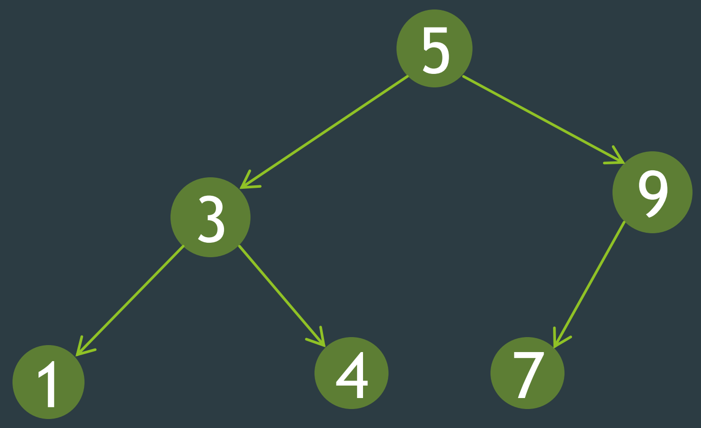


Exercise 1 & 2

- Exercise 1: Implement your own **predecessor find** function in BST
- Exercise 2: Implement your own **successor find** function in BST

Rank Search

- **Rank**: the index of the key in the **ascending order**.
  - We assume that the smallest key has rank 0.
- **Rank search**: get the key with rank k (i.e., the k-th smallest key).
  - Hash table does not support efficient rank search.
  - How to do rank search with a BST?
    - Simple solution: keep counting during an in-order depth-first traversal.



What's the average-case time complexity?

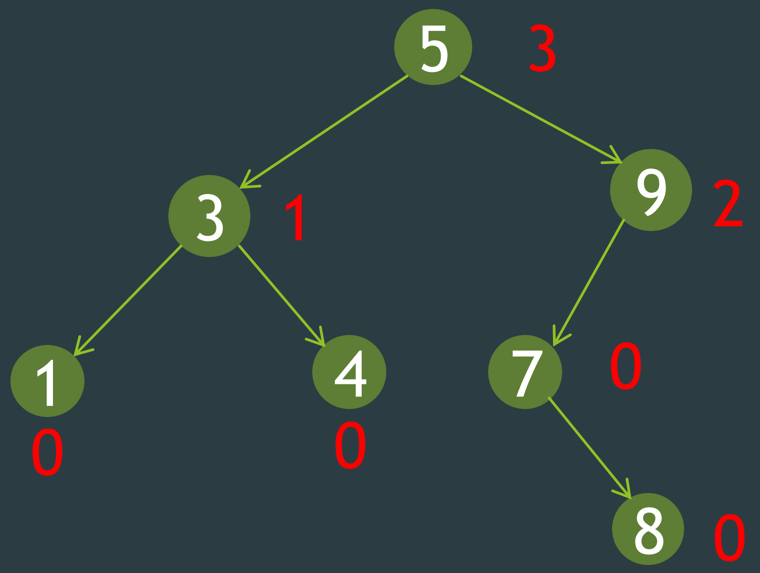
Can we do better?

BST with `leftSize`

- Each node has an additional field **`leftSize`**, indicating the number of nodes in its left subtree.

```
struct node {
  Item item;

  int leftSize;
  node *left;
  node *right;
};
```



Which Statements Are Correct?

- Suppose we modify the basic BST to implement a BST with leftsize. Select all the correct statements.

  A. The search method should be updated.

  B. The insertion method should be updated, but not for the removal method.

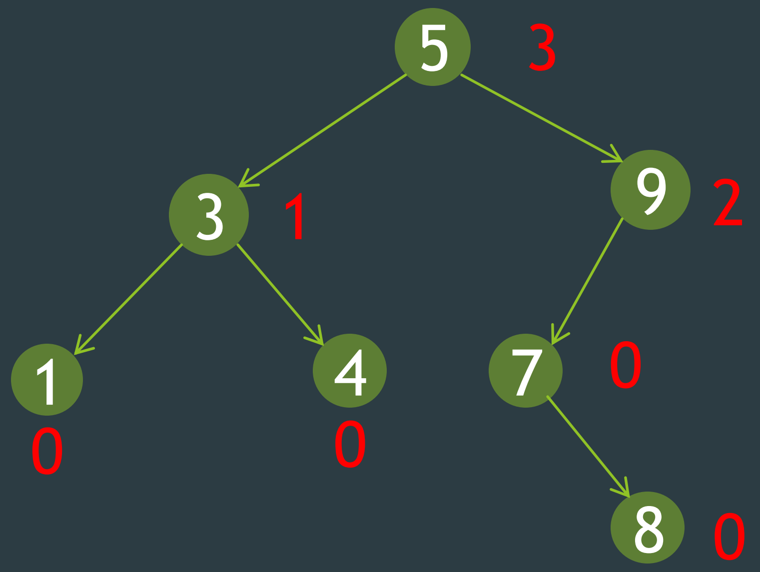
  C. The removal method should be updated, but not for the insertion method.

  D. Both the insertion and removal methods should be updated.

Rank Search

- Can we increase the efficiency of rank search with a BST with **`leftSize`**?

- What is the node with
  - rank = 3?
  - rank = 2?
  - rank = 5?



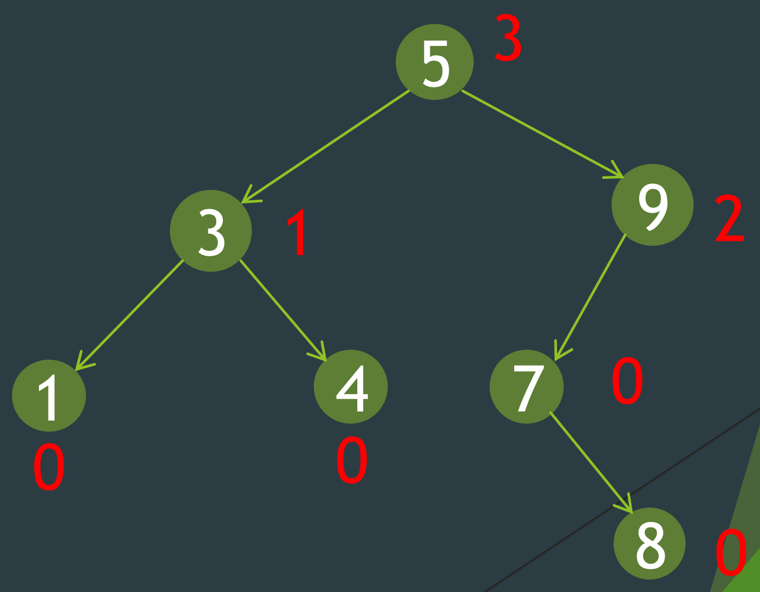
- Observation: **`x.leftSize`** = the rank of **`x`** in the tree rooted at **`x`**.
  - The rank of node 9 is 2 in the tree rooted at node 9.

Rank Search

```
node *rankSearch(node *root, int rank) {
  if(root == NULL) return NULL;
  if(rank == root->leftSize) return root;
  if(rank < root->leftSize)
    return rankSearch(root->left, rank);
  else
    return rankSearch(root->right, rank - 1 - root->leftSize);
}
```

The number of nodes including the current `root` and its left subtree.

What will `rankSearch(root,5)` return?



Rank Search Example

```
node *rankSearch(node *root, int rank) {
  if(root == NULL) return NULL;
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  if(rank < root->leftSize)
    return rankSearch(root->left, rank);
  else
    return rankSearch(root->right, rank - 1 - root->leftSize);
}
```

What will `rankSearch(root,5)` return?

Rank Search

```
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}
```
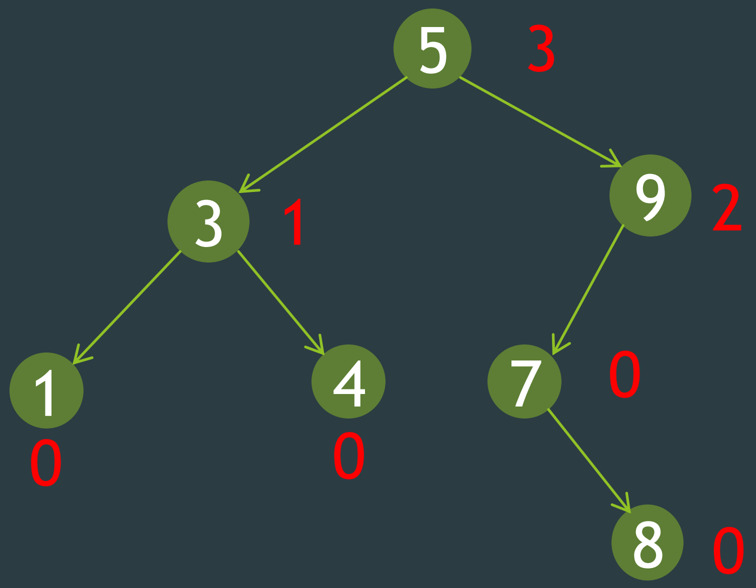
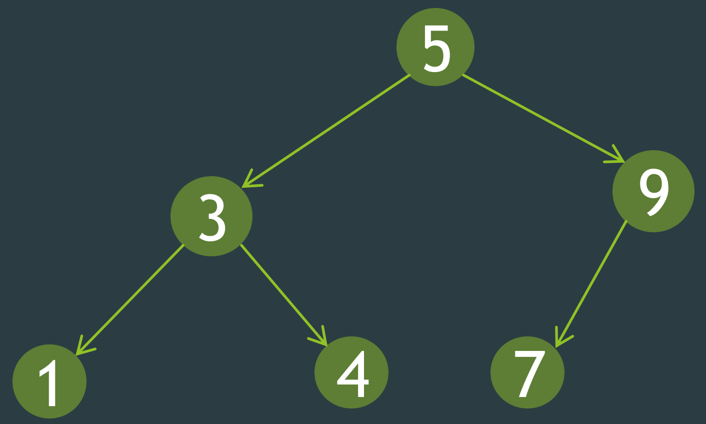



Time complexity?

$O(\text{height})$. On average: $O(\log n)$.

Range Search

- Instead of finding an exact match, find all items whose keys fall between a range of values, inclusive, in sorted order
  - E.g., between 4 and 8, inclusive.



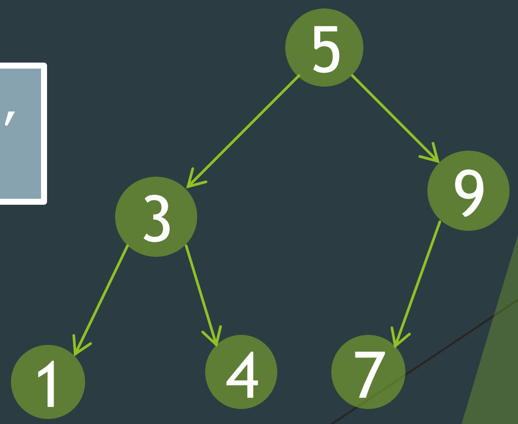
- Example applications:
  - Buy ticket for travel between certain dates.

How could you implement range search?

Range Search: Algorithm

1. Compute range of left subtree.
   - If search range covers all or part of left subtree, search left. (recursive call)
2. If root is in search range, add root to results.
3. Compute range of right subtree.
   - If search range covers all or part of right subtree, search right. (recursive call)
4. Return results.

```
void rangeSearch(node *root, Key searchRange[],
  Key treeRange[], List results)
```

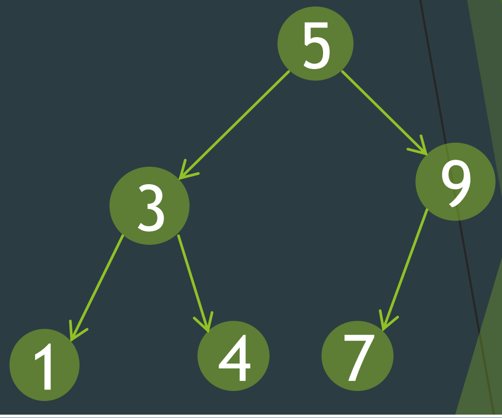


Range Search
Example

rangeSearch('5', [4,8], (-∞,+ ∞), results)

searchRange treeRange

Call rangeSearch('3', [4,8],(- ∞ ,5),results)



Does (-∞,5) overlap [4,8]? Yes

Does (-∞,3) overlap [4,8]? No

Is 3 in [4,8]? No

Does (3,5) overlap [4,8]? Yes

Is 4 in [4,8]? Yes

results ← 4

Is 5 in [4,8]? Yes

results ← 5

Does (5,+ ∞) overlap [4,8]? Yes

Call rangeSearch('9', [4,8],(5,+ ∞),results)

Does (5,9) overlap [4,8]? Yes

Is 7 in [4,8]? Yes

results ← 7

Is 9 in [4,8]? No

Does (9,+ ∞) overlap [4,8]? No

results: 4,5,7

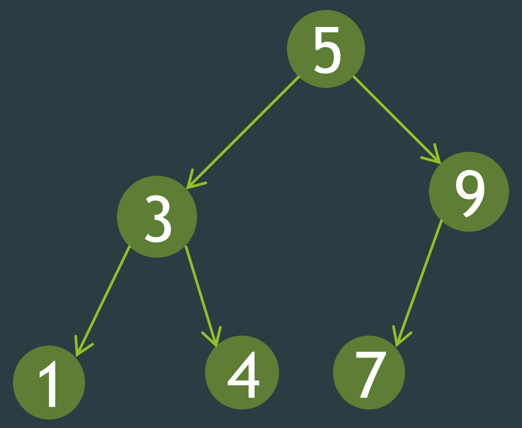
Note: results are in order

Range Search
Supported Functions

- If node is in the search range, add node to the **`results`** list.
- Compute subtree's range:
  - Replace upper bound of left subtree by node's key
  - Replace lower bound of right subtree by node's key
- If search range covers all or part of subtree, search subtree.
  - Recursive calls

Range Search

1. Compute range of left subtree.
   - If search range covers all or part of left subtree, search left. (recursive call)
2. If root is in search range, add root to results.
3. Compute range of right subtree.
   - If search range covers all or part of right subtree, search right. (recursive call)
4. Return results.



Time complexity? $O(n)$

Exercise 3 & 4

- Exercise 3: Implement your own rank search function in BST
- Exercise 4: Implement your own range search function in BST

Exercise 4 LeetCode 230

230. Kth Smallest Element in a BST

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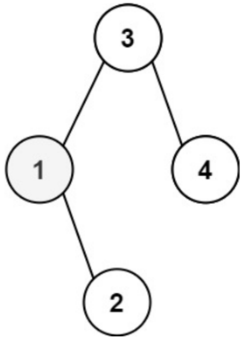
Given the `root` of a binary search tree, and an integer `k`, return *the* k^{th} *smallest value (**1-indexed**) of all the values of the nodes in the tree.*

Example 1:




```
Input: root = [3,1,4,null,2], k = 1
Output: 1
```

Example 2: