Haar Basis Wavelets and Morlet Wavelets

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The **Haar transform**, which is one of the earliest transform functions proposed, was proposed in 1910 by a Hungarian mathematician Alfred Haar. It is found effective as it provides a simple approach for analyzing the local aspects of a signal.

Say we start with an image slice (one dimensional) of size 2^N, so we can write the image as

$$I = \{I(1), I(2), \dots, I(2^N)\} \qquad or \qquad \{I(i); \quad i = 1, \dots, 2^N\} \quad (1)$$

Recursive Process of Decomposing an Image in terms of Sums and Differences

Let us now group the image by consecutive pairs, i.e.,

$$\{I(i); \quad i = 1, \dots, 2^N\} \rightarrow \{[I(1), I(2)], [I(3), I(4)], \dots, [I(2i-1), I(2i)], \dots, [I(2^N), I(2^N - 1)]\} \quad (2)$$

We can apply the following linear transformation for each pair $[I(2i - 1), I(2i)]$, i.e.,

$$\begin{bmatrix} s^1(i) \\ d^1(i) \end{bmatrix} = H_2 \begin{bmatrix} I(2i-1) \\ I(2i) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{bmatrix} I(2i-1) \\ I(2i) \end{bmatrix} \qquad (3)$$

So $s^1(i)$ stores the sum of the two elements in a pair and $d^1(i)$ stores the difference of the two elements in a pair. The upper index "1" refers to this transformation being the first step in a process that follows. Of course, this is invertible, i.e.,

$$\begin{bmatrix} I(2i-1) \\ I(2i) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{bmatrix} s^1(i) \\ d^1(i) \end{bmatrix} \qquad (4)$$

So we can get back the original set of pairs, the original image $\{I(i); \quad i = 1, \dots, 2^N\}$. Thus, we can represent the image I, of size 2^N, as 2^{N-1} pairs of the form

$$I(i) \rightarrow \{[s^1(1), d^1(1)], [s^1(2), d^1(2)], \dots, [s^1(i), d^1(i)], \dots, [s^1(2^{N-1}), d^1(2^{N-1})]\} \quad (5)$$

We can then store the set $D^1 = \{d^1(1), d^1(2) \dots, d^1(i), \dots, d^1(2^{N-1})\}$ of size 2^{N-1} and we decompose further the complementary set

$$S^1 = \{s^1(1), s^1(2) \dots, s^1(i), \dots, s^1(2^{N-1})\} \qquad (6)$$

We interpret S^1 as a new image of half the size of the original one and we represent the set of pairs

$$S^1 = \{[s^1(1), s^1(2)], \ldots, [s^1(2i-1), s^1(2i)], \ldots, [s^1(2^{N-1}-1), s^1(2^{N-1})]\} \quad (7)$$

and we apply the reversible linear transformation H_2 to each pair $[s^1(2i-1), s^1(2i)]$, i.e.,
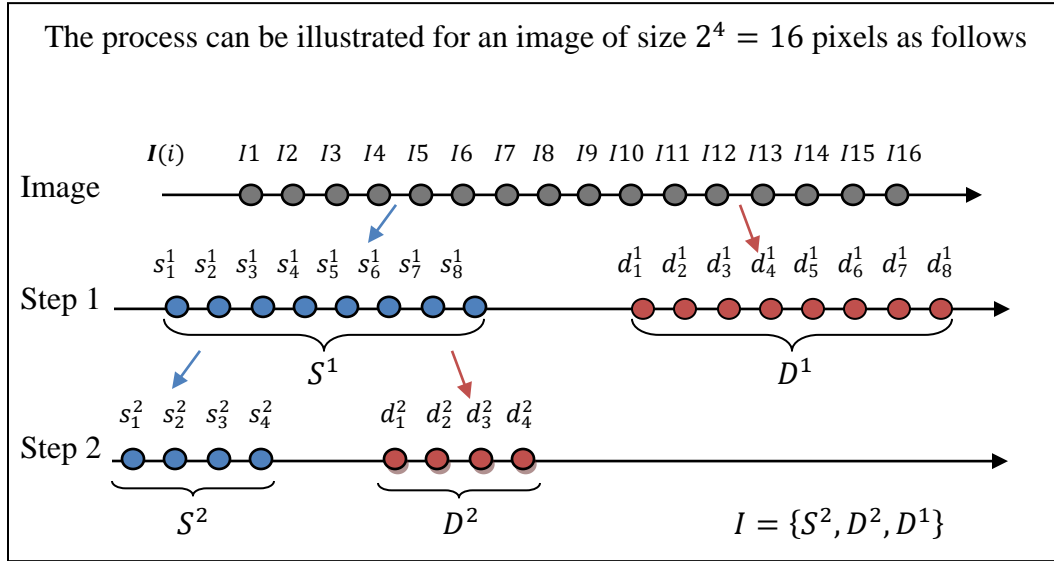
$$\begin{bmatrix} s^2(i) \\ d^2(i) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{bmatrix} s^1(2i-1) \\ s^1(2i) \end{bmatrix} \quad (8)$$

The upper index "2" refers to the second step in the process. We can then represent the "image" S^1, of size 2^{N-1}, as

$$S^1(i) \rightarrow \{[s^2(1), d^2(1)], [s^2(2), d^2(2)], \ldots, [s^2(i), d^2(i)], \ldots, [s^2(2^{N-2}), d^2(2^{N-2})]\} \quad (9)$$

Likewise in the first step, we can store the set $D^2 = \{d^2(1), d^2(2) \ldots, d^2(i), \ldots, d^2(2^{N-2})\}$ of size 2^{N-2} and decompose further the set

$$S^2 = \{s^2(1), s^2(2) \ldots, s^2(i), \ldots, s^2(2^{N-1})\} \quad (10)$$

The process can be illustrated for an image of size $2^4 = 16$ pixels as follows

We now apply the third step of the process on the "image" set S^2, i.e., for each pair $[s^2(2i-1), s^2(2i)]$ we apply the linear transformation

$$\begin{bmatrix} s^3(i) \\ d^3(i) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{bmatrix} s^2(2i-1) \\ s^2(2i) \end{bmatrix} \quad (11)$$

Leading to two sets

$$D^3 = \{d^3(1), d^3(2) \ldots, d^3(i), \ldots, d^3(2^{N-3})\} \; ; \; S^3 = \{s^3(1), s^3(2) \ldots, s^3(i), \ldots, s^3(2^{N-3})\} \quad (12)$$

Up to the third layer of this process we are left with the following new representation for the image

$$\{I(i); \quad i = 1, \ldots, 2^N\} \qquad \Leftrightarrow \qquad \{D^1, D^2, D^3, S^3\} \qquad\qquad (13)$$

where D^1 is of size 2^{N-1}, D^2 of size 2^{N-2}, and D^3, S^3 of sizes 2^{N-3} each. All together, of course, there are 2^N numbers. Every step is reversible, so we can quickly recover the image I from $\{D^1, D^2, D^3, S^3\}$. We could repeat this process up to N steps to obtain $\{D^1, D^2, \ldots, D^i, \ldots, D^N, S^N\}$, where S^N would be one number, representing the sum of all the image values.

The Haar Transformation

Reviewing our process, we started with equation (2), applying a linear transformation H_2 to each pair of data, or

$$\begin{bmatrix} s^1(i) \\ d^1(i) \end{bmatrix} = H_2 \begin{bmatrix} I(2i-1) \\ I(2i) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{bmatrix} I(2i-1) \\ I(2i) \end{bmatrix} \qquad (14)$$

We then described the next step as the same linear transformation, H_2, applied to the "image" S^1 of the first layer, i.e., $\begin{bmatrix} s^2(i) \\ d^2(i) \end{bmatrix} = H_2 \begin{bmatrix} s^1(2i-1) \\ s^1(2i) \end{bmatrix}$. We can represent the first and second layer processes as a linear transformation directly on each four elements of the image (concatenating the two steps and concatenating two pairs of image data), i.e.,

$$\begin{bmatrix} s^2(i) \\ d^2(i) \\ d^1(i) \\ d^1(i+1) \end{bmatrix} = H_4 \begin{bmatrix} I(2^2 i - 3) \\ I(2^2 i - 2) \\ I(2^2 i - 1) \\ I(2^2 i) \end{bmatrix} = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{bmatrix} I(2^2 i - 3) \\ I(2^2 i - 2) \\ I(2^2 i - 1) \\ I(2^2 i) \end{bmatrix} \qquad (15)$$

In this way, we keep simultaneously both layers (both steps) of the representation of the image. Repeating this process towards the third layer, we apply the linear transformation, H_2, to the "image" S^2 of the second layer, i.e., $\begin{bmatrix} s^3(i) \\ d^3(i) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{bmatrix} s^2(2i-1) \\ s^2(2i) \end{bmatrix}$. We can represent this third layer process as a linear transformation directly on each eight elements of the image (concatenating the three steps and concatenating four pairs of image data), i.e.,

$$
\begin{bmatrix} s^3(i) \\ d^3(i) \\ d^2(i) \\ d^2(i+1) \\ d^1(i) \\ d^1(i+1) \\ d^1(i+2) \\ d^1(i+3) \end{bmatrix} = H_8 \begin{bmatrix} I(2^2i-7) \\ I(2^2i-6) \\ I(2^2i-5) \\ I(2^2i-4) \\ I(2^2i-3) \\ I(2^2i-2) \\ I(2^2i-1) \\ I(2^2i) \end{bmatrix}
$$

$$
= \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{bmatrix} I(2^2i-7) \\ I(2^2i-6) \\ I(2^2i-5) \\ I(2^2i-4) \\ I(2^2i-3) \\ I(2^2i-2) \\ I(2^2i-1) \\ I(2^2i) \end{bmatrix} \quad (16)
$$

More generally one can write the linear transformation H_{2i} that is applied to each image subset data of size $2i$, to be

$$
H_{2i} = \begin{bmatrix} H_i \otimes [1 \quad 1] \\ I_i \otimes [1 \quad -1] \end{bmatrix}
$$

where I_i is the identify matrix of size i and $\otimes$ is the Kronecker product. If A is an m x n matrix and B is a p x q matrix, the Kronecker product is given by the mp x nq matrix

$$
A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \vdots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}
$$

The output of applying H_{2^k} to the full set of image windows of size 2^k is a representation of the image in the form $\{D^1, D^2, \dots, D^k, S^k\}$.


The Haar Basis (the function approach)

So far we have represented an image as a vector and the wavelet transformation as a matrix multiplication to the   vector. We now consider the image as a function of the pixels. The wavelets are functions of the pixels. The result of applying the wavelet transform to the image is then described by the set of convolutions of two functions, the image and each of the wavelets. Let us be more precise. Defining the Haar wavelet basis as a mother wavelet and scaling it by σ and translating it by b we have
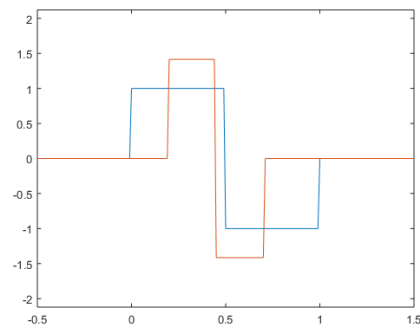
$$\psi(x) = \begin{cases} 1 & 0 < x < \dfrac{1}{2} \\ -1 & \dfrac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \xrightarrow[\text{Scaling and Translating}]{} \qquad \psi_\sigma(x - b) = \frac{1}{\sqrt{\sigma}} \psi\left(\frac{x - b}{\sigma}\right)$$
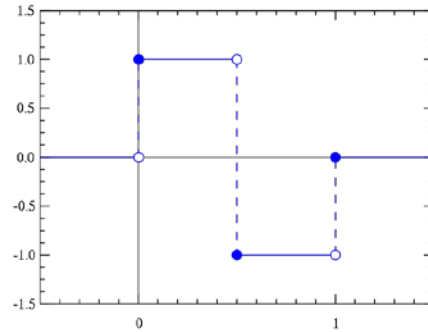
Haar basis $\psi(x)$
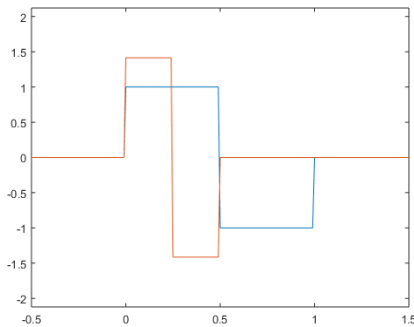
Orange $\psi_\sigma(x - b)$: b=0, σ=0.5   b=0.2, σ=0.5

Two wavelet properties are required: average is zero and normalization

$$0 = \int_{-\infty}^{\infty} \psi_\sigma(x - b)\, dx \quad \text{and} \quad 1 = \int_{-\infty}^{\infty} |\psi_\sigma(x - b)|^2\, dx$$

If $\psi(x)$ satisfy them, so thus $\psi_\sigma(x - b) = \frac{1}{\sqrt{\sigma}} \psi\left(\frac{x-b}{\sigma}\right)$.

Examples with $b=0$, for discrete Wavelets (it is common to assign $\sigma = 2^{-j}$)

$$\sigma = 2^{-1} = {}^1\!/_2 \; \rightarrow \psi_{1/2}(x) = \sqrt{2}\, \psi(2x) = \begin{cases} \sqrt{2} & 0 < 2x < {}^1\!/_2 \;\; or \;\; 0 < x < {}^1\!/_4 \\ -\sqrt{2} & \dfrac{1}{2} \le 2x < 1 \quad or \quad {}^1\!/_4 \le x < {}^1\!/_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\sigma = 2^1 = 2 \quad \rightarrow \quad \psi_2(x) = \frac{1}{\sqrt{2}} \psi\left(\frac{x}{2}\right) = \begin{cases} {}^1\!/\!_{\sqrt{2}} & 0 < \dfrac{x}{2} < \dfrac{1}{2} \quad or \quad 0 < x < 1 \\[2mm] -{}^1\!/\!_{\sqrt{2}} & \dfrac{1}{2} \le \dfrac{x}{2} < 1 \quad or \quad 1 \le x < 2 \\[2mm] 0 & \qquad\qquad otherwise \end{cases}$$

$$\sigma = 2^3 = 8 \quad \rightarrow \quad \psi_2(x) = \frac{1}{\sqrt{8}} \psi\left(\frac{x}{8}\right) = \begin{cases} {}^1\!/\!_{\sqrt{8}} & 0 < \dfrac{x}{8} < \dfrac{1}{2} \quad or \quad 0 < x < 4 \\[2mm] -{}^1\!/\!_{\sqrt{8}} & \dfrac{1}{2} \le \dfrac{x}{8} < 1 \quad or \quad 4 \le x < 8 \\[2mm] 0 & \qquad\qquad otherwise \end{cases}$$

We can write a row associated to "$d^j(i)$" at (14), (15), (16) as the Haar basis convolution at $x=0$, i.e., $\psi_\sigma * I\,(x) = \int_{-\infty}^{\infty} \psi_\sigma(x')\, I(x - x')\, dx'$ for $\sigma = 2^{-j}$, $j = \cdots, -3, -2, -1, 0, 1, 2, 3,$, respectively (and $b = 0$).

The Haar wavelet transform is then the convolution

$$W_\psi f(\sigma, b) = \int_{-\infty}^{\infty} \psi_\sigma{}^*(x - b)\, I(x)\, dx = \frac{1}{\sqrt{\sigma}} \int_{-\infty}^{\infty} \psi^*\left(\frac{x - b}{\sigma}\right) I(x)\, dx$$

or in discrete coordinates

$$W_\psi f(\sigma, b) = \frac{1}{\sqrt{\sigma}} \sum_{n=-\infty}^{\infty} \psi^*\left(\frac{x_n - b}{\sigma}\right) I(x_n)$$

Where $\psi_\sigma{}^*(x - b)$ is the complex conjugate of $\psi_\sigma(x - b)$, and for the discrete case, $\sigma = 2^{-j}$ is called the dyadic dilation and $b = k\, 2^{-j}$ is the dyadic position. So the basis is described by integers j,k.

Complex Numbers

A complex number $z = x + i\,y$ has x and y real values and $i = \sqrt{-1}$. It can be represented in polar coordinates representation as $z = \rho\, e^{i\,\theta}$. The pair (x, y) in Cartesian coordinates or the pair (ρ, θ) in polar coordinates are equivalent with $\rho = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$.
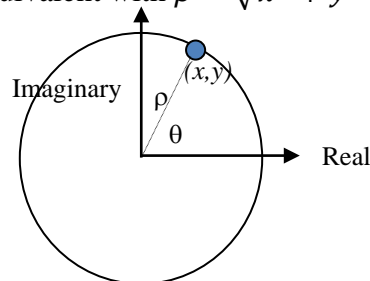
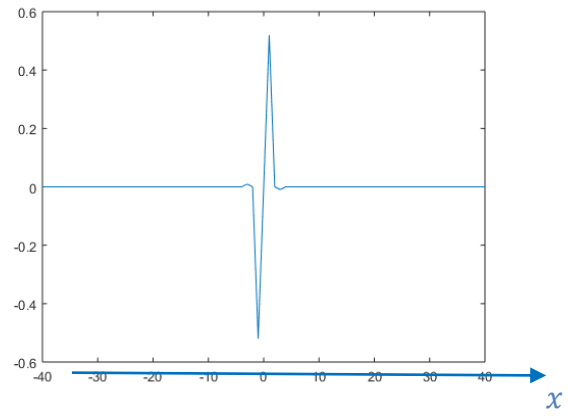## The Morlet Wavelet Basis in 1Dimension

$$\psi^{1D}(x) = C_1\left(e^{i\frac{2\pi}{\xi}x} - C_2\right)e^{-\frac{x^2}{2}}$$

where $e^{i\frac{2\pi}{\xi}x} = cos\left(\frac{2\pi}{\xi}x\right) + i\,sin\left(\frac{2\pi}{\xi}x\right)$ is a complex number, i.e., $e^{i\frac{2\pi}{\xi}x}$ is written in polar coordinates $(\rho,\ \theta)=\left(1,\frac{2\pi}{\xi}x\right)$ while $cos\left(\frac{2\pi}{\xi}x\right) + i\,sin\left(\frac{2\pi}{\xi}x\right)$ is written in Cartesian coordinates $(x,y) = \left(cos\left(\frac{2\pi}{\xi}x\right), sin\left(\frac{2\pi}{\xi}x\right)\right).$
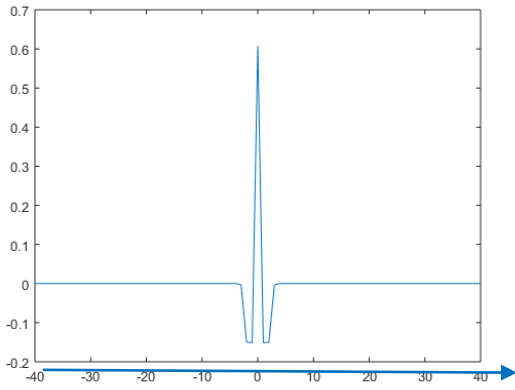
$\xi = 4$ Real Part   Imaginary Part x

Calculating C_2 and C_1:

Two wavelet properties are required : average is zero and normalization

$$0 = \int_{-\infty}^{\infty}\psi(x)\,dx \quad \text{and} \quad 1 = \int_{-\infty}^{\infty}|\psi(x)|^2\,dx$$

which allow us to obtain C_2 and C_1 as follows:

First, we obtain C_2 , real variable, by adjusting it so that

$$0 = \sum_{x}\psi(x) \quad\rightarrow\quad 0 = \sum_{x=1}^{N_x}\left(e^{i\frac{2\pi}{\xi}x} - C_2\right)e^{-\frac{x^2}{2}} \rightarrow C_2 = \frac{\sum_{x=1}^{N_x}e^{i\frac{2\pi}{\xi}x}e^{-\frac{x^2}{2}}}{\sum_{x=1}^{N_x}e^{-\frac{x^2}{2}}}$$

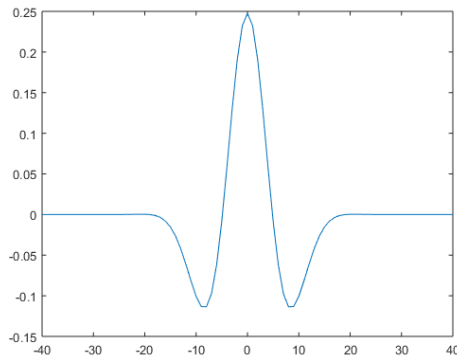and C_1 is adjusted so that

$$1 = \sum_{x=1}^{N_x}\psi(x)\psi^*(x)$$

where $\psi^*(x) = \frac{C_1}{\sigma}\left(e^{-i\frac{2\pi}{\xi}x} - C_2\right)e^{-\frac{x^2}{2}}$ is the conjugate of $\psi(x)$.
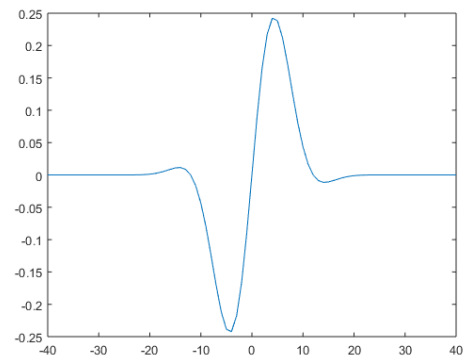
Scaling and Translation

The full orbit of the 1D Morlet wavelets (scaling and translating) is then

$$\psi_\sigma^{1D}(x - x_0) = \frac{1}{\sqrt{\sigma}}\psi_\sigma^{1D}\left(\frac{x - x_0}{\sigma}\right) = \frac{C_1}{\sqrt{\sigma}}\left(e^{i\frac{2\pi}{\xi\sigma}(x-x_0)} - C_2\right)e^{-\frac{(x-x_0)^2}{2\sigma^2}}$$

And we can plot the real and complex functions (e.g. for σ=6, ξ=4 centered in x₀=0 pixels)

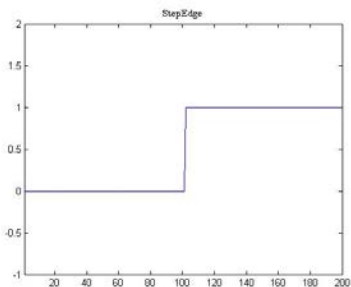Real   σ=6, ξ=4   Imaginary σ=6, ξ=4
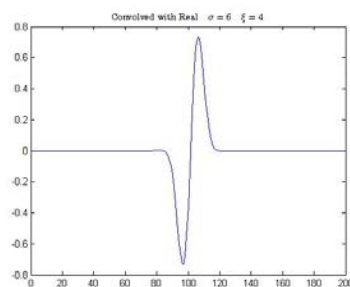
Convolution of Morlet Wavelet Basis in 1Dimension
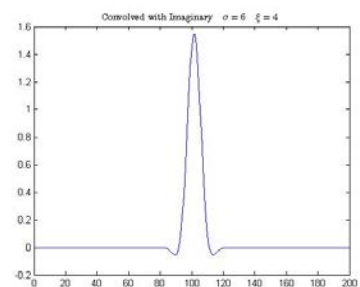
Convolving Morlet with a 1D image, I, of a step edge, i.e.,

Step *Edge*   *Real part of the Convolution*   *Imaginary part*

where $(I * \psi_\sigma^{1D})(y) = \sum_x I(y - x)\, \psi_\sigma^{1D}(x)$ is the complex value output shown above. The "zeros" of the real part help detect edges as well as the maximum of the complex part.

The Morlet Wavelets in 2Dimensions

The Morlet in 2D (two dimensions), assuming $\sigma = \sigma_x = \sigma_y$, is defined as

$$\psi_{\sigma,\theta}(u) = \frac{C_1}{\sigma}\left(e^{i\frac{2\pi}{\xi\sigma}(u\,.\,e_\theta)} - C_2 \right)e^{-\frac{u^2}{2\sigma^2}}$$

where $u = (x, y)$, $e_\theta = (\cos\theta, \sin\theta)$ and $e^{i\frac{2\pi}{\xi}(u\,.e_\theta)} = \cos\left[\frac{2\pi}{\xi\sigma}(u\,.\,e_\theta)\right] + i\,\sin\left[\frac{2\pi}{\xi\sigma}(u\,.\,e_\theta)\right]$. So $u^2 = x^2 + y^2$.

Let us work with $\xi = 4$ (just one peak), so

$$\psi_{\sigma,\theta}(u) = \frac{C_1}{\sigma}\left(e^{i\frac{\pi}{2\sigma}(u\,.\,e_\theta)} - C_2 \right)e^{-\frac{u^2}{2\sigma^2}}$$

Two wavelet properties are required : average is zero and normalization

$$0 = \int_{-\infty}^{\infty} \psi_\sigma(x - b)\, dx \quad \text{and} \quad 1 = \int_{-\infty}^{\infty} |\psi_\sigma(x - b)|^2\, dx$$

Which allow us to obtain C_2 and C_1 as follows:

First, we obtain C_2 , real variable, by adjusting it so that the sum of ψ over all pixels is zero, i.e.,

$$0 = \sum_x \sum_y \psi(\vec{u}) \quad \to \quad 0$$

$$= \sum_{x=1}^{N_x}\sum_{y=1}^{N_y}\left(e^{i\frac{\pi}{2\sigma}(u\,.e_\theta)} - C_2 \right)e^{-\frac{u^2}{2\sigma^2}} \to C_2 = \frac{\sum_{x=1}^{N_x}\sum_{y=1}^{N_y} e^{i\frac{\pi}{2\sigma}(u\,.e_\theta)}e^{-\frac{u^2}{2\sigma^2}}}{\sum_{x=1}^{N_x}\sum_{y=1}^{N_y} e^{-\frac{u^2}{2\sigma^2}}}$$

and C_1 is adjusted so that

$$1 = \sum_{x=1}^{N_x}\sum_{y=1}^{N_y} \psi(\vec{u})\psi^*(\vec{u})$$

where $\psi^*(\vec{u}) = \frac{C_1}{\sigma}\left(e^{-i\frac{\pi}{2\sigma}(u\,.e_\theta)} - C_2 \right)e^{-\frac{u^2}{2\sigma^2}}$ is the conjugate of $\psi(\vec{u})$. So

$$Z = \sum_{x=1}^{N_x} \sum_{y=1}^{N_y} \left(e^{i \frac{\pi}{2\sigma}(u.e_\theta)} - C_2 \right) e^{-\frac{u^2}{2\sigma^2}} \left(e^{-i \frac{\pi}{2\sigma}(u.e_\theta)} - C_2 \right) e^{-\frac{u^2}{2\sigma^2}} \qquad \rightarrow \qquad C_1 = \frac{1}{\sqrt{Z}}$$

Thus,

$$Z = \sum_{x=1}^{N_x} \sum_{y=1}^{N_y} \left(1 - C_2 \left(e^{-i \frac{\pi}{2\sigma}(u.e_\theta)} + e^{i \frac{\pi}{2\sigma}(u.e_\theta)} \right) + C_2^2 \right) e^{-\frac{u^2}{\sigma^2}}$$

$$Z = \sum_{x=1}^{N_x} \sum_{y=1}^{N_y} \left(1 - 2 C_2 \cos\left(\frac{\pi}{2\sigma}(u.e_\theta) \right) + C_2^2 \right) e^{-\frac{u^2}{\sigma^2}} \qquad \rightarrow \qquad C_1 = \frac{1}{\sqrt{Z}}$$