Worked Exercises for Linear Diophantine Equations

Exercise 1. Solve the linear Diophantine equation: $7x - 9y = 3$.

Exercise 2. Find all integers x and y such that: $2173x + 2491y = 53$.

Exercise 3. Find all integers x and y such that: $2173x + 2491y = 159$.

Exercise 4. Show there is no integers solution to: $2173x + 2491y = 210$.

Exercise 5. Solve the linear Diophantine equation: $858x + 253y = 33$.

Exercise 6. Find all integer solutions to: $258x + 147y = 369$.

Exercise 7. Show there are no integers solution to: $155x + 45y = 7$.

Exercise 8. Solve the linear Diophantine equation: $60x + 33y = 9$.

Solutions

Exercise 1. Solve the linear Diophantine equation: $7x - 9y = 3$.

Solution. We find a particular solution of the given equation. Such a solution exists because $\gcd(7,9) = 1$ and 3 is divisible by 1. One solution, found *by inspection*, of the given equation is

$$x = 3,\ y = 2.$$

We obtain all integer solutions of the given equation: $x = 3 + 9k$, $y = 2 + 7k$, for k an integer.
NOTE: we can use any variable name for k here, such as t.

Exercise 2: Find integers x and y such that $2173x + 2491y = 53$.

Solution: From the Euclidean Algorithm:

$$\begin{gathered} 2491 = 1 \cdot 2173 + 318 \\ 2173 = 6 \cdot 318 + 265 \\ 318 = 1 \cdot 265 + 53 \\ 265 = 5 \cdot 53 + 0 \end{gathered}$$

Thus, $\gcd(2173, 2491) = 53$, which divides $c = 53$, so there is a solution to this linear Diophantine equation.

Working the Euclidean Algorithm backwards we find:

$$53 = 318 - 1 - 265$$
$$= 318 - 1(2173 - 6 \cdot 318)$$
$$= 7 \cdot 318 - 1 \cdot 2173$$
$$= 7(2491 - 1 \cdot 2173) - 1 \cdot 2173$$
$$= 7(2491) - 8(2173)$$

Therefore, a particular solution is $x = -8$ and $y = 7$. The general solution is

$x = -8 + (2491/53)k = -8 + 47k$
$y = 7 - (2173/53)k = 7 – 41k$, where k is any integer

Exercise 3: Find all integers x and y such that $2173x + 2491y = 159$.

Solution: Since gcd(2173, 2491) = 53 and 53 divides 159, we know that there is a solution to this linear Diophantine equation.

We note that gcd(2173; 2491) = 53 from the previous exercise. Multiplying both sides by 3 we obtain 159. We know from Exercise 2 that

$53= 7(2491) - 8(2173)$

So one solution of $2173x + 2491y = 159$ is $x = -24$, $y = 21$. The general solution is therefore

$x = -24 + (2491/53)k = -24 + 47k$,
$y = 21 – (2173/53)k = 21 – 41k$, where k is any integer.

Exercise 4: Show there is no integer solution to: $2173x + 2491y = 210$.

Solution: In Exercise 2, we found that gcd(2173, 2491) = 53, and since 53 does not divide 210, there is no integer solution.

Exercise 5. Solve the linear Diophantine equation $858 x + 253 y = 33$.

Solution. We apply the Euclidean algorithm to find gcd(858, 253). We have that

$$858 = 253 \cdot 3 + 99$$
$$253 = 99 \cdot 2 + 55$$
$$99 = 55 \cdot 1 + 44$$
$$55 = 44 \cdot 1 + 11$$
$$44 = 11 \cdot 4$$

So gcd(858, 253) = 11. Since 11 | 33, the equation has solutions.

Working the Euclidean algorithm backwards, we find that

$$
\begin{aligned}
11 &= 55 - 44 \cdot 1 \\
&= 55 - 1 \cdot (99 - 55 \cdot 1) \\
&= 2 \cdot 55 - 99 \\
&= 2 \cdot (253 - 99 \cdot 2) - 99 \\
&= 2 \cdot 253 - 5 \cdot 99 \\
&= 2 \cdot 253 - 5 \cdot (858 - 253 \cdot 3) \\
&= 17 \cdot 253 - 5 \cdot 858.
\end{aligned}
$$

In our equation, the right hand side is $c = 33$, so we have (after multiplying by 3):

$$33 = 51 \cdot 253 - 15 \cdot 858.$$

Thus, a particular solution is $x = -15$, $y = 51$.

The general solution of $858x + 253y = 33$ is therefore: $x = -15 - 253k$, $y = 51 + 858k$, where k is an integer.

Exercise 6. Find all integer solutions to: $258x + 147y = 369$.

Solution. We use the Euclidean Algorithm to find gcd(147,258):

$$
\begin{aligned}
258 &= 147 \cdot 1 + 111 \\
147 &= 111 \cdot 1 + 36 \\
111 &= 36 \cdot 3 + 3 \\
36 &= 3 \cdot 12.
\end{aligned}
$$

So gcd(147,258)=3. Since $3 | 369$, the equation has integer solutions. Working the Euclidean algorithm backwards, we have that:

$$
\begin{aligned}
3 &= 111 - 3 \cdot 36 \\
&= 111 - 3(147 - 111) = 4 \cdot 111) - 3 \cdot 147) \\
&= 4(258 - 147) - 3 \cdot 147 \\
&= 4 \cdot 258 - 7 \cdot 147.
\end{aligned}
$$

We take $258 \cdot 4 + 147 \cdot (-7) = 3$, and multiply through by 123 as $3 \cdot 123 = 369$.

So one solution is $x = 492$ and $y = -861$. All other solutions will have the form

$X = 492 - (147/3)k = 492 - 49k$
$y = -861 + (258/3)k = 86k - 861$, where k is an integer.

Exercise 7. Show there are no integers solution to: $155x + 45y = 7$.

Solution. Check that $\gcd(155,45) = 5$. As 5 does not divide 7, the equation has no integer solution.

Exercise 8. Solve the linear Diophantine equation: $60x + 33y = 9$.

Solution. By the Euclidean algorithm we find:

$$
\begin{gathered}
60 = 1\cdot 33 + 27 \; 33 \\
= 1\cdot 27 + 6 \; 27 \\
= 4\cdot 6 + 3 \; 6 \\
= 2\cdot 3 + 0.
\end{gathered}
$$

We see the last nonzero remainder is 3 so $\gcd(60, 33) = 3$. As $3|9$, the equation has integer solutions. Reversing the steps, we find:

$$
\begin{aligned}
3 &= 27 - 4\cdot 6 \\
&= 27 - 4\cdot(33 - 27) \\
&= 5\cdot 27 - 4\cdot 33 \\
&= 5\cdot(60 - 33) - 4\cdot 33 \\
&= 5\cdot 60 - 9 \cdot 33.
\end{aligned}
$$

One solution is then $x = 15$ and $y = 27$ (notice we multiplied by 3). All the solutions are given by

$x = 15 + (33/3)k = 15 + 11k$,
$y = -27 - (60/3)k = -27 - 20k$, where k is an integer.