Signals and Systems

Lectures # 13 and # 14

Laplace Transform

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Topics of the lecture:

- **Definition and Importance.**
- **Region of Convergence (ROC) and its Properties.**
- **Inverse Laplace Transform.**
- **Laplace Transform Properties.**

➢ Definition and Importance.

consider a continuous-time LTI system with impulse response $h(t)$. For an input $x(t)$, we can determine the output through the use of the convolution integral, so that with $x(t) = e^{st}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}\,d\tau.$$

Expressing $e^{s(t-\tau)}$ as $e^{st}e^{-s\tau}$, and noting that e^{st} can be moved outside the integral, we see that eq. becomes

$$y(t) = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau.$$

Assuming that the integral on the right-hand side of eq. converges, the response to e^{st} is of the form

$$y(t) = H(s)e^{st},$$

where $H(s)$ is a complex constant whose value depends on s and which is related to the system impulse response by

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau.$$

A signal for which the system output is a (possibly complex) constant times the input is referred to as an *eigenfunction* of the system, and the amplitude factor is referred to as the system's *eigenvalue*.

When $s = j\omega$, eq. becomes

$$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt,}$$

of a general signal $x(t)$ which corresponds to the *Fourier transform* of $x(t)$; that is,

$$X(s)|_{s=j\omega} = \mathfrak{F}\{x(t)\}.$$

For general values of the complex variable s, it is referred to as the *Laplace transform*

The Laplace transform of a general signal $x(t)$ is defined as

$$\boxed{X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt,}$$

➢ Definition and Importance.

Definition:

The Laplace Transform of a continuous-time signal $x(t)$ is defined as:

$$X(s) \overset{\Delta}{=} \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

Where *s* is a complex variable and in general *s = σ + jw*

Importance:

- The Laplace Transform provides a good tool for easier analysis of continuous-time systems even if it may be unstable using useful algebraic properties extending the Fourier Transform tools to analyze more complex systems.

- The Laplace Transform convert many difficult or daunting Time-Domain operations such as : Convolution, Time-Shifts, Differentiation, and Integration into algebraic operations {+ , - , x , / }.

➢ Definition and Importance.

Example 1: compute the Laplace transform of the signal $x(t) = e^{-at}u(t)$

$$X(s) \overset{\Delta}{=} \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

$$X(s) = \int_{-\infty}^{+\infty} e^{-at}.u(t).e^{-st}\,dt = \int_{0}^{+\infty} e^{-(a+s)t}\,dt$$

$$X(s) = \left.\frac{e^{-(a+s)t}}{-(a+s)}\right|_{0}^{+\infty} = \frac{-1}{a+s}\left\{e^{-\infty} - e^{0}\right\} = \frac{1}{a+s}\;; \quad a + \mathrm{Re}\{s\} > 0$$

$$e^{-(a+s)\infty} \quad \textit{as } \underline{\underline{s}} \textit{ is generally complex} = e^{-\left(a+\left[\mathrm{Re}\{s\}+j\,\mathrm{Im}\{s\}\right]\right)\infty}$$

$$= e^{-\left[a+\mathrm{Re}\{s\}\right]\infty}.e^{-j\,\mathrm{Im}\{s\}\infty} \quad \textit{to converge} \quad \Rightarrow \quad a + \mathrm{Re}\{s\} > 0$$

as $e^{j\theta}$ *never goes to* ∞ *as it is an object rotates in a unit circle*

Remember that any complex number $z=|z|e^{jArg\{z\}}$ doesn't go to ∞ unless it's magnitude |z| goes

$$\Rightarrow \quad X(s) = \frac{1}{a+s}\;; \quad \mathrm{Re}\{s\} > -a$$

➢ Definition and Importance.

Example 2: compute the Laplace transform of the signal $x(t) = -e^{-at}u(-t)$

$$X(s) \stackrel{\Delta}{=} \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

$$X(s) = -\int_{-\infty}^{+\infty} e^{-at}.u(-t).e^{-st}dt = -\int_{-\infty}^{0} e^{-(a+s)t}dt$$

$$X(s) = -\left.\frac{e^{-(a+s)t}}{-(a+s)}\right|_{-\infty}^{0} = \frac{1}{a+s}\left\{e^{0} - e^{(a+s)(+\infty)}\right\} = \frac{1}{a+s}\ ; \quad a + \mathrm{Re}\{s\} < 0$$

$$e^{(a+s)\infty} \quad \text{as } \underline{\underline{s}} \text{ is generally complex} = e^{(a+[\mathrm{Re}\{s\}+j\,\mathrm{Im}\{s\}])\infty}$$

$$= e^{[a+\mathrm{Re}\{s\}]\infty}.e^{j\,\mathrm{Im}\{s\}\infty} \quad \text{to converge} \quad \Rightarrow \quad a + \mathrm{Re}\{s\} < 0$$

as $e^{j\theta}$ *never goes to* ∞ *as it is an object rotates in a unit circle*

Remember that any complex number $z=|z|e^{jArg\{z\}}$ doesn't go to ∞ unless it's magnitude |z| goes

$$\Rightarrow X(s) = \frac{1}{a+s}\ ; \quad \mathrm{Re}\{s\} < -a$$

➢ Region of Convergence (ROC) and its Properties.

From Example 1:

$$x(t) = e^{-at}u(t) \quad \Rightarrow \quad X(s) = \frac{1}{a+s} \quad ; \quad \mathrm{Re}\{s\} > -a$$

From Example 2:

$$x(t) = -e^{-at}u(-t) \Rightarrow \quad X(s) = \frac{1}{a+s} \quad ; \quad \mathrm{Re}\{s\} < -a$$

Note : **Same Algebraic expression** **BUT** **different set of *S* to converge**

Then: to specify/determine the Laplace Transform both algebraic expression and Region Of Convergence (ROC) are required.

With ROC is the set of values of *S* for which the algebraic expression converges.

➢ Region of Convergence (ROC) and its Properties.

A convenient way to display the ROC is shown in Figure 1. The variable s is a complex number, and in the figure we display the complex plane, generally referred to as the s-plane, associated with this complex variable. The coordinate axes are $\Re e\{s\}$ along the horizontal axis and $\Im m\{s\}$ along the vertical axis. The horizontal and vertical axes are sometimes referred to as the σ-axis and the $j\omega$-axis, respectively. The shaded region in Figure 1(a) represents the set of points in the s-plane corresponding to the region of convergence for Example 1. The shaded region in Figure 1(b) indicates the region of convergence for Example 2.

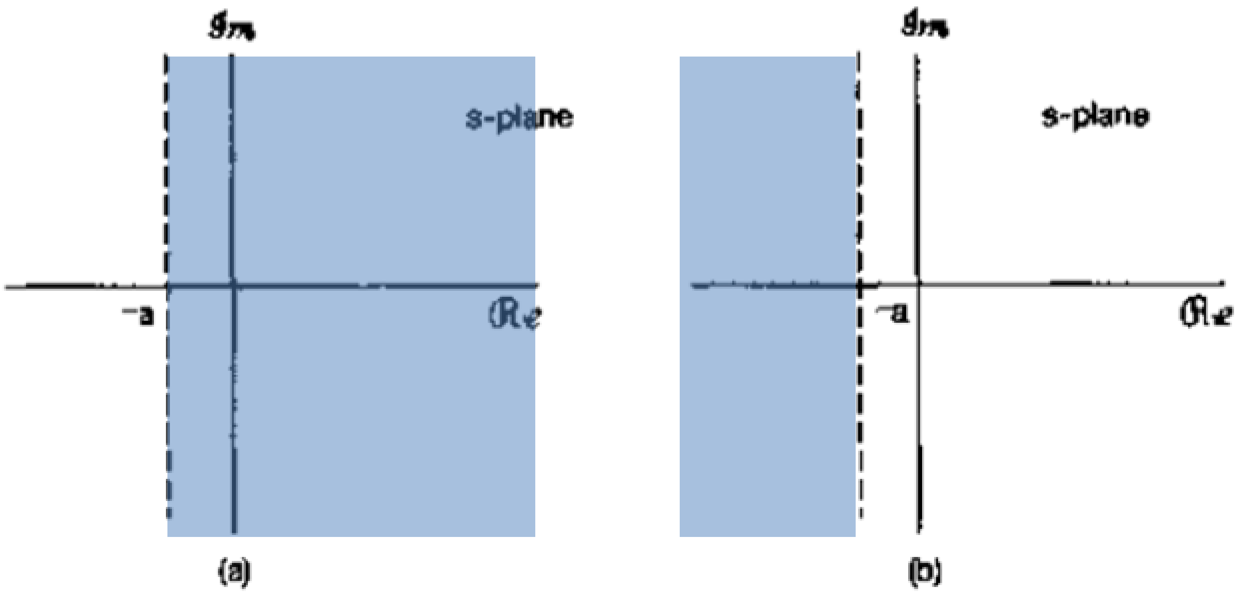


Figure 1 (a) ROC for Example 1; (b) ROC for Example 2.

➢ Region of Convergence (ROC) and its Properties.

Example 3: compute the Laplace transform of the signal

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}\,u(t)$$

From Example 1: $x(t) = e^{-at}u(t) \Rightarrow \quad X(s) = \dfrac{1}{a+s} \;; \quad \mathrm{Re}\{s\} > -a$

$$e^{-2t}u(t) \overset{Laplace}{\Longrightarrow} \frac{1}{2+s} \;; \quad \mathrm{Re}\{s\} > -2$$

$$e^{-t}\,u(t) \overset{Laplace}{\Longrightarrow} \frac{1}{1+s} \;; \quad \mathrm{Re}\{s\} > -1$$

$$x(t) = 3e^{-2t}\,u(t) - 2e^{-t}\,u(t) \overset{Laplace}{\Longrightarrow} X(s) = \frac{3}{2+s} + \frac{-2}{1+s}$$

In order to determine the *ROC*, we need the set values of s for which both algebraic expressions converge simultaneously. i.e. the intersection of ROC1 and ROC2

$$\Rightarrow X(s) = \frac{3}{2+s} + \frac{-2}{1+s} = \frac{s-1}{(s+1)(s+2)} = \frac{s-1}{s^2+3s+2} \;; \quad \mathrm{Re}\{s\} > -1$$

➢ Region of Convergence (ROC) and its Properties.

Example 4: compute the Laplace transform of the signal

$$x(t) = e^{-2t}u(t) + e^{-t}\left(\cos 3t\right)u(t)$$

$$x(t) = e^{-2t}u(t) + e^{-t}\left(\frac{e^{j(3t)} + e^{j(-3t)}}{2}\right)u(t)$$

$$x(t) = e^{-2t}u(t) + \frac{1}{2}e^{-(1-3j)t}\,u(t) + \frac{1}{2}e^{-(1+3j)t}\,u(t)$$

$$e^{-2t}u(t) \quad \overset{Laplace}{\Longrightarrow} \quad \frac{1}{2+s} \quad ; \quad \mathrm{Re}\{s\} > -2$$

$$e^{-(1-3j)t}\,u(t) \quad \overset{Laplace}{\Longrightarrow} \quad \frac{1}{(1-3j)+s} \quad ; \quad \mathrm{Re}\{s\} > -1$$

$$e^{-(1+3j)t}\,u(t) \quad \overset{Laplace}{\Longrightarrow} \quad \frac{1}{(1+3j)+s} \quad ; \quad \mathrm{Re}\{s\} > -1$$

$$\Rightarrow X(s) = \frac{1}{2+s} + \frac{1}{2}\frac{1}{(s+1)-3j} + \frac{1}{2}\frac{1}{(s+1)+3j} = \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}\ ; \quad \mathrm{Re}\{s\} > -1$$

➢ Region of Convergence (ROC) and its Properties.

Laplace Transform ($\mathscr{L}$) is rational for linear combination of real and complex exponential signals, also for LTI systems specified in terms of linear constant-coefficient differential equations.

Rational $\mathscr{L}$ means $\Rightarrow \quad X(s) = \dfrac{Polynomial\ in\ S}{Polynomial\ in\ S} = \dfrac{N(s)}{D(s)}$

For rational $\mathscr{L}$: Roots of *N(s)* are called Zeros and Roots of *D(s)* are called Poles

A convenient way of describing the $\mathscr{L}$ is to mark the locations of Zeros and Poles in S-Plane and indicating the *ROC* , then this plot is called *Pole-Zero plot*.

Rational

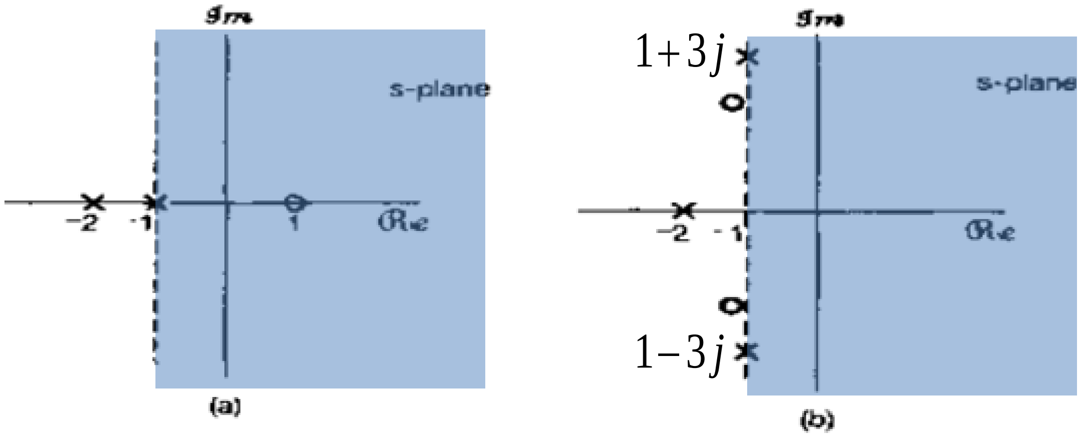
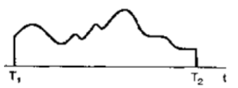


Figure s-plane representation of the Laplace transforms for (a) Example 3 and (b) Example 4. Each × in these figures marks the location of a pole of the corresponding Laplace transform—i.e., a root of the denominator. Similarly, each ○ marks a zero—i.e., a root of the the numerator. The shaded regions indicate the ROCs.

➢ Region of Convergence (ROC) and its Properties.

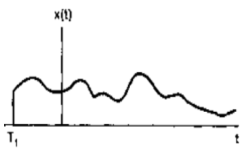
Property 1: The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane.

Property 2: For rational Laplace transforms, the ROC does not contain any poles.



Property 3: If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

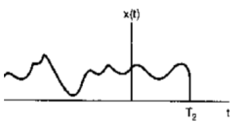
$$\int_{T_1}^{T_2} x(t)e^{-st}dt < \infty$$



Property 4: If $x(t)$ is right sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\Re e\{s\} > \sigma_0$ will also be in the ROC.

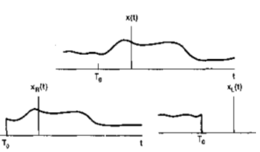
$$\int_{T_1}^{+\infty} x(t)e^{-st}dt < \infty$$

Keep the exponential decaying
remember sign of t, i.e. st>0



Property 5: If $x(t)$ is left sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\Re e\{s\} < \sigma_0$ will also be in the ROC.

$$\int_{-\infty}^{T_1} x(t)e^{-st}dt < \infty$$



Property 6: If $x(t)$ is two sided, and if the line $\Re e\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\Re e\{s\} = \sigma_0$.

intersection

A *two-sided* signal is a signal that is of infinite extent for both $t > 0$ and $t < 0$.

Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

➢ Region of Convergence (ROC) and its Properties.

1 $\boxed{\text{The ROC of } X(s) \text{ consists of strips parallel to the } j\omega\text{-axis in the } s\text{-plane.}}$

Explanation:- as $X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} x(t)\, e^{-\sigma t} e^{-j\omega t} dt$

as $s = \sigma + j\omega$

as $x(t)$ does not go to infinity [i.e. absolutely integrable

The only possibly variable that control the convergence is σ which is the $Re\{s\}$ whatever ω is.

if value σ_0 in the ROC then the line $\sigma_0 - j\infty$ to $\sigma_0 + j\infty$ will be included in the ROC {may be except at some poles} $\Rightarrow$ there will be line segments parallel to $j\omega$-axis that are in the ROC at least

2 $\boxed{\text{For rational Laplace Transform; the ROC does not contain any pole}}$

Explanation:- the pole makes $X(s) = \infty$ i.e. not converge

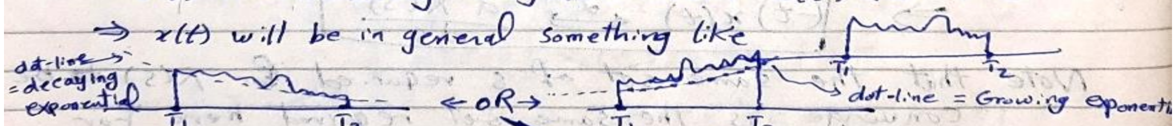
$\therefore$ ROC by definition cannot contain poles

3 $\boxed{\text{if } x(t) \text{ is of finite-duration and is absolutely integrable then the ROC is the entire } s\text{-plane}}$

Explanation:- $x(t)$ is finite-duration means finite-domain

$x(t)$ is absolutely integrable means $x(t) < \infty \ \forall t$

$\Rightarrow x(t)$ will be in general something like



$\int_{T_1}^{T_2} |x(t)|\, e^{-\sigma t}\, dt$

σ +ve $\equiv$ decaying exponential

$\therefore \ < e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)|\, dt$

as $x(t)$ is absolutely integrable $\Rightarrow \int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$

and $e^{-\sigma T_1} < \infty$ as σ +ve

$< \infty$

Except if T_1 is −ve and $\sigma = +\infty$ needs check

$\int_{T_1}^{T_2} |x(t)|\, e^{-\sigma t}\, dt$

σ −ve $\equiv$ growing exponential

$\therefore \ < e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)|\, dt$

as $x(t)$ is absolutely integrable $\Rightarrow \int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$

and $e^{-\sigma T_2} < \infty$ as $\sigma T_2 \rightarrow -\infty$

$< \infty$

Except if T_2 is +ve and $\sigma = -\infty$ needs check

➢ Region of Convergence (ROC) and its Properties.

if $\sigma = 0 \Rightarrow \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt = \int_{T_1}^{T_2} |x(t)| dt < \infty$

as $x(t)$ is absolutely integrable

Note:- if $x(t)$ is in general complex $\Rightarrow x(t) = |x(t)| e^{j\angle x(t)}$

then the only parameter that controls the converges is $|x(t)|$

$\therefore \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < \infty$ Except if $(-\sigma t) = \infty$ needs check or if $\sigma t = -\infty$

$\boxed{4}$ if $x(t)$ is right-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC then ALL values of s for which $Re\{s\} > \sigma_0$ will also be in the ROC

Explanation:- right-sided means the domain of non-zero of $x(t)$ will start from a value $t = T$ and extends to $t = +\infty$

$$\therefore \int_{-\infty}^{+\infty} |x(t)| e^{-st} dt = \int_{T}^{+\infty} |x(t)| e^{-st} dt$$

if $\sigma_0 \in ROC \Rightarrow \int_{T}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$

and if $\sigma_1 > \sigma_0$ then σ_1 is more positivity than σ_0 $\Rightarrow \boxed{\sigma_1 - \sigma_0 = +ve}$

$$\therefore \int_{T}^{+\infty} |x(t)| e^{-\sigma_1 t} dt = \int_{T}^{+\infty} |x(t)| e^{-\sigma_0 t} \cdot e^{-(\sigma_1 - \sigma_0)t} dt$$

then the expression $-(\sigma_1 - \sigma_0)t$ decreases as t increases

$$\therefore \int_{T}^{+\infty} |x(t)| e^{-\sigma_1 t} dt \leq e^{-(\sigma_1 - \sigma_0)T} \int_{T}^{+\infty} |x(t)| e^{-\sigma_0 t} dt$$

as $(\sigma_1 - \sigma_0)T < \infty$ and as $\int_{T}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$

then $\int_{T}^{+\infty} |x(t)| e^{-\sigma_1 t} dt < \infty$

$\Rightarrow \therefore \sigma_1 \in ROC$

$\boxed{5}$ if $x(t)$ is left-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC Then ALL values of s for which $Re\{s\} < \sigma_0$, will also be in the ROC

Explanation:- left-sided means $\int_{-\infty}^{+\infty} x(t) e^{-st} dt = \int_{-\infty}^{T} x(t) e^{-st} dt$

if $\sigma_0 \in ROC \Rightarrow \int_{-\infty}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$

if $\sigma_1 < \sigma_0$ then σ_1 is more negativity than σ_0 $\Rightarrow \sigma_0 - \sigma_1 = +ve$ OR $\Rightarrow \boxed{\sigma_1 - \sigma_0 = -ve}$

$$\therefore \int_{-\infty}^{T} |x(t)| e^{-\sigma_1 t} dt = \int_{-\infty}^{T} |x(t)| e^{-\sigma_0 t} \cdot e^{-(\sigma_1 - \sigma_0)t} dt$$

$$\leq e^{-(\sigma_1 - \sigma_0)T} \int_{-\infty}^{T} |x(t)| e^{-\sigma_0 t} dt < \infty$$

as $-(\sigma_1 - \sigma_0)T < \infty$ and $\int_{-\infty}^{T} |x(t)| e^{-\sigma_0 t} dt < \infty$ as σ_0 is in ROC

$\Rightarrow \therefore \sigma_1$ also $\in ROC$

➢ Region of Convergence (ROC) and its Properties.

6 if $x(t)$ is two-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC. Then the ROC will consist of a strip in s-plane that includes line $Re\{s\} = \sigma_0$

Explanation:

The two-sided signal is a signal that is of infinite extent for both $t > 0$ and $t < 0$

then $x(t)$ can be divided into two sub-signals.

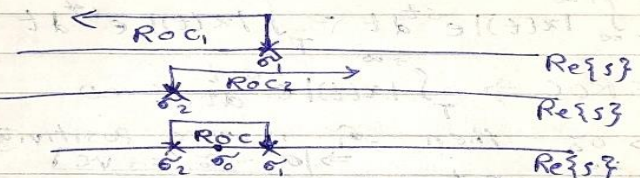
one $x_1(t)$ that is left-sided with ROC_1 left-half plane

and one $x_2(t)$ that is right-sided with ROC_2 is right-half plane

i.e. ROC_1 comes from $-\infty$ and bounded by a pole at σ_1

and ROC_2 comes from specific pole at σ_2 and goes to $+\infty$.

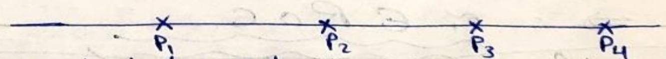
and as $X(s)$ exist then $ROC = ROC_1 \cap ROC_2$



Then ROC will be strip bounded by two poles that includes σ_0

7 if the Laplace transform $X(s)$ is rational the its ROC is bounded by poles or extends to infinity, no poles are contained in the ROC

Explanation: if $X(s)$ is rational $\frac{N(s)}{D(s)}$ and there are poles



and, as ROC never includes poles

then ROC will stopped by a pole or goes to infinity

stopped here ≡ bounded

8 if $X(s) = \frac{N(s)}{D(s)}$; then if $x(t)$ is right-sided then ROC right to the rightmost pole

And if $x(t)$ is left-sided then ROC left to the leftmost pole

it is a conclusion of previous properties of ROC

➢ Inverse Laplace Transform.

the Laplace transform of a signal $x(t)$ is

$$X(\sigma + j\omega) = \mathfrak{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt$$

for values of $s = \sigma + j\omega$ in the ROC. We can invert this relationship using the inverse Fourier transform as

$$x(t)e^{-\sigma t} = \mathfrak{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega,$$

or, multiplying both sides by $e^{\sigma t}$, we obtain

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}\,d\omega.$$

That is, we can recover $x(t)$ from its Laplace transform evaluated for a set of values of $s = \sigma + j\omega$ in the ROC, with σ fixed and ω varying from $-\infty$ to $+\infty$. We can highlight this and gain additional insight into recovering $x(t)$ from $X(s)$ by changing the variable of integration in eq. from ω to s and using the fact that σ is constant, so that $ds = j\,d\omega$. The result is the basic inverse Laplace transform equation:

$$\boxed{x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds.}$$

➢ Inverse Laplace Transform.

Definition:

The Inverse Laplace Transform is defined as:

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

Where *s* is a complex variable and in general *s = σ + jw*

The formal evaluation of the integral for a general *X(s)* requires the use of contour integration in the complex plane, a topic that we will not consider here.

The basic procedure for computing the Laplace inverse is to decompose the Laplace Transform algebraic expression using the partial fraction decomposition into a linear combination of lower order terms.

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i}$$

Then, use the known pairs to get the Inverse Laplace Transform.

➢ Inverse Laplace Transform.

Partial Fraction Decomposition: http://www.mathsisfun.com/algebra/partial-fractions.html

- Start with a **Proper** Rational Expressions (if not do division first)

  Proper: the degree of the top is less than the degree of the bottom. The **degree** is the largest **exponent** the variable has.

- Factor the bottom into: **But don't factor it into complex numbers**
  - linear factors
  - or "irreducible" quadratic factors

  A quadratic is a second-order polynomial, and if you have one write its factor as ➔ $\dfrac{B_1x + C_1}{\text{(Your Quadratic)}}$

- Write out a partial fraction for each factor (and every exponent of each)

  Sometimes you may get a factor with an exponent, like **$(x-2)^3$** ...
  You need a partial fraction for each exponent from 1 up

  $$\frac{1}{(x-2)^3} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3}$$

- Multiply the whole equation by the bottom
- Solve for the coefficients by
  - substituting zeros of the bottom
  - making a system of linear equations (of each power) and solving
- Write out your answer!

Inverse Laplace Transform.

When trying to factor, follow these steps:

- "Factor out" any common terms
- See if it fits any of the identities, plus any more you may know
- Keep going till you can't factor any more

Remember these Identities

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^2 + 2ab + b^2 = (a+b)(a+b)$$

$$a^2 - 2ab + b^2 = (a-b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2-ab+b^2)$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$a^3+3a^2b+3ab^2+b^3 = (a+b)^3$$

$$a^3-3a^2b+3ab^2-b^3 = (a-b)^3$$

➢ Inverse Laplace Transform.

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

➢ Inverse Laplace Transform.

Let $$X(s) = \frac{1}{(s+1)(s+2)}, \qquad \Re e\{s\} > -1.$$

To obtain the inverse Laplace transform, we first perform a partial-fraction expansion to obtain

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

As discussed in the appendix, we can evaluate the coefficients A and B by multiplying both sides of eq. by $(s+1)(s+2)$ and then equating coefficients of equal powers of s on both sides. Alternatively, we can use the relation

$$A = [(s+1)X(s)]|_{s=-1} = 1,$$
$$B = [(s+2)X(s)]|_{s=-2} = -1.$$

Thus, the partial-fraction expansion for $X(s)$ is $$X(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

Since the ROC is to the right of both poles, the same is true for each of the individual terms, Consequently, from Property 8 in the preceding section, we know that each of these terms corresponds to a right-sided signal. The inverse transform of the individual terms in eq. can then be obtained by reference to Example 1:

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, \qquad \Re e\{s\} > -1,$$
$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2}, \qquad \Re e\{s\} > -2.$$

We thus obtain

$$[e^{-t} - e^{-2t}]u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)(s+2)}, \qquad \Re e\{s\} > -1.$$

➢ Inverse Laplace Transform.

Let us now suppose that the algebraic expression for $X(s)$ is again given ,
but that the ROC is now the left-half plane $\Re e\{s\} < -2$. The partial-fraction expansion for $X(s)$ relates only to the algebraic expression, so eq. is still valid. With this new ROC, however, the ROC is to the *left* of both poles and thus, the same must be true for each of the two terms in the equation. That is, the ROC for the term corresponding to the pole at $s = -1$ is $\Re e\{s\} < -1$, while the ROC for the term with pole at $s = -2$ is $\Re e\{s\} < -2$. Then,

$$-e^{-t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \Re e\{s\} < -1,$$

$$-e^{-2t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \Re e\{s\} < -2,$$

so that

$$x(t) = [-e^{-t} + e^{-2t}]u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad \Re e\{s\} < -2.$$

Finally, suppose that the ROC of $X(s)$ in eq. is $-2 < \Re e\{s\} < -1$. In this case, the ROC is to the left of the pole at $s = -1$ so that this term corresponds to the left-sided signal , while the ROC is to the right of the pole at $s = -2$ so that this term corresponds to the right-sided signal . Combining these, we find that

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad -2 < \Re e\{s\} < -1.$$

➢ Laplace Transform Properties.

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

➢ Laplace Transform Properties.

1 **Linearity**

if $x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s)$, ROC $= R_1$

and $x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s)$, ROC $= R_2$

Then $a x_1(t) \pm b x_2(t) \xleftrightarrow{\mathcal{L}} a X_1(s) \pm b X_2(s)$

with ROC containing $R_1 \cap R_2$

Proof

let $x_3(t) = a x_1(t) \pm b x_2(t) \xleftrightarrow{\mathcal{L}} \int_{-\infty}^{+\infty} x_3(t) e^{-st} dt = X_3(s)$

$\therefore X_3(s) = \int_{-\infty}^{+\infty} \{a x_1(t) \pm b x_2(t)\} e^{-st} dt$

$\therefore X_3(s) = \int_{-\infty}^{+\infty} a x_1(t) e^{-st} dt \pm \int_{-\infty}^{+\infty} b x_2(t) e^{-st} dt$

$\therefore \boxed{X_3(s) = a \; X_1(s) \pm b \; X_2(s)}$

with ROC_3 should be satisfy both terms at the same time

i.e. ROC_3 should contain $R_1 \cap R_2$

But: may be when terms be added together

Some zeros cancel some poles

$\Rightarrow ROC_3$ may be larger than $R_1 \cap R_2$

So, $\boxed{ROC_3 \text{ should contain at least } R_1 \cap R_2}$

2 **Time-shift** if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$, ROC $= R$

then $x(t - t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s)$; ROC $= R$

Proof

let $x_1(t) = x(t - t_0) \xleftrightarrow{\mathcal{L}} X_1(s) = \int_{-\infty}^{+\infty} x_1(t) e^{-st} dt$

$\therefore X_1(s) = \int_{-\infty}^{+\infty} x(t - t_0) e^{-st} dt$

let $\tau = t - t_0 \Rightarrow t = \tau + t_0 \Rightarrow dt = d\tau \Rightarrow$ $t = -\infty \Rightarrow \tau = -\infty - t_0 \therefore \tau = -\infty$; $t = +\infty \Rightarrow \tau = +\infty - t_0 \therefore \tau = +\infty$

$\therefore X_1(s) = \int_{-\infty}^{+\infty} x(\tau) e^{-s(\tau + t_0)} d\tau$

$= \int_{-\infty}^{+\infty} x(\tau) e^{-s\tau} e^{-st_0} d\tau = e^{-st_0} \cdot \int_{-\infty}^{+\infty} x(\tau) e^{-s\tau} d\tau$

$\therefore \boxed{X_1(s) = e^{-st_0} X(s)}$ with values of s to converge is the same as $\in R$

as X(s) exist $\Rightarrow \therefore e^{-s\tau}$ converge for some values of s for any τ including $\tau = t_0$ $\Rightarrow \boxed{\therefore ROC_1 = ROC = R}$

➢ Laplace Transform Properties.

[3] Shifting in S-Domain:- if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$; ROC = R

Then $e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s-s_0)$; ROC = R + Re{s_0}

Proof

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

$$\Rightarrow X(s-s_0) = \int_{-\infty}^{+\infty} x(t)\, e^{-(s-s_0)t}\, dt = \int_{-\infty}^{+\infty} x(t)\, e^{-st} e^{s_0 t}\, dt$$

$$= \int_{-\infty}^{+\infty} \{x(t)\, e^{s_0 t}\}\, e^{-st}\, dt$$

$$\therefore \boxed{e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s-s_0)}$$

as $X(s)$ converges with $s \in R$ $\Rightarrow$ we need converging for {s_0} Too

$\therefore$ $\boxed{\text{ROC of } X(s-s_0) \text{ should be } R + Re\{s_0\}}$

i.e. if values of s in R then values of s + Re{s_0} in ROC of $X(s-s_0)$

[4] Time-scaling :- if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$; ROC = R

Then $x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X(\frac{s}{a})$; ROC = aR

Proof

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

let $x_1(t) = x(at) \Rightarrow X_1(s) = \int_{-\infty}^{+\infty} x(at)\, e^{-st}\, dt$

let $\tau = at \Rightarrow t = \frac{\tau}{a} \Rightarrow d\tau = a\, dt$

at $t = -\infty \Rightarrow \tau = a(-\infty) = -\infty$ if $a > 0$

$= +\infty$ if $a < 0$

at $t = +\infty \Rightarrow \tau = a(+\infty) = +\infty$ if $a > 0$

$= -\infty$ if $a < 0$

$$\therefore X_1(s) = \int_{-\infty}^{+\infty} x(at)\, e^{-\frac{s}{a}\tau} \cdot a \cdot \frac{dt}{a}$$

$$= \begin{cases} \frac{1}{a} \int_{-\infty}^{+\infty} x(at)\, e^{-\frac{s}{a}\tau}\, d\tau & \text{if } a > 0 \\ \frac{1}{a} \int_{+\infty}^{-\infty} x(at)\, e^{-\frac{s}{a}\tau}\, d\tau & \text{if } a < 0 \end{cases}$$

$$= -\frac{1}{a} \int_{-\infty}^{+\infty} x(at)\, e^{-\frac{s}{a}\tau}\, d\tau \quad \text{if } a < 0 \text{ (Note limits of } \int \text{)}$$

as $at = \tau \Rightarrow \boxed{X_1(s) = \frac{1}{|a|} \int_{-\infty}^{+\infty} x(\tau)\, e^{-(\frac{s}{a})\tau}\, d\tau = \frac{1}{|a|} X(\frac{s}{a})}$

- For the values of s that $\in R \Rightarrow$ those values make the integration convergies of $X(s)$

$\therefore$ (a.s) make X_1 converge $\Rightarrow \boxed{ROC_1 = a \cdot R}$

$\Rightarrow$ scaling in S-Domain $\Rightarrow$ if pole at s_0

$\Rightarrow$ its new location as $s = s_0$ is $s = \frac{s_0}{a}$

➢ Laplace Transform Properties.

[5] Conjugation: if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$; ROC $= R$

then $x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*)$; ROC $= R$

Proof

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

$$X^*(s) = \left[\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt\right]^*$$

$$= \int_{-\infty}^{+\infty} x^*(t)\, e^{-s^*t}\, dt$$

Replace s by $s^* \Rightarrow X^*(s^*) = \int_{-\infty}^{+\infty} x^*(t)\, e^{-st}\, dt = \mathcal{L}\{x^*(t)\}$

$\therefore \boxed{x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*)}$

if $x(t)$ is real $\Rightarrow x(t) = x^*(s)$.

then $X^*(s^*) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = X(s)$

$\Rightarrow \boxed{X(s) = X^*(s^*) \text{ when } x(t) \text{ is real}}$

Then as $\{x(t)\, e^{-st}\}$ converge for some values of s which are R

Then $\{x^*(t)\, e^{-st}\}$ will converge too for the same set of values of $s = R$

$\because |x(t)| = |x^*(t)|$ $\Rightarrow$ $\boxed{\text{ROC of } X^*(s^*) = R}$

Magnitude

[6] Convolution: if $x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s)$, $ROC_1 = R_1$

and $x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s)$, $ROC_2 = R_2$

then $x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)\, X_2(s)$; ROC containing $R_1 \cap R_2$

Proof

let $x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, dt$ as $*$ means convolution

$\therefore$ $X_3(s) = \int_{-\infty}^{+\infty} x_3(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} \left\{\int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\right\} e^{-st}\, dt$

$$= \int_{-\infty}^{+\infty} x_1(\tau) \left\{\int_{-\infty}^{+\infty} x_2(t-\tau)\, e^{-st}\, dt\right\} d\tau$$

using Time-shift property $\Rightarrow \int_{-\infty}^{+\infty} x_2(t-\tau)\, e^{-st}\, dt = e^{-s\tau} X_2(s)$

$\therefore$ $X_3(s) = \int_{-\infty}^{+\infty} x_1(\tau)\, .\, e^{-s\tau}\, .\, X_2(s)\, d\tau$

$\therefore \boxed{X_3(s) = X_2(s) \cdot \int_{-\infty}^{+\infty} x_1(\tau)\, e^{-s\tau}\, d\tau = X_2(s)\, X_1(s)} = X_1(s)\, X_2(s)$

$\therefore$ Here the ROC should contain set of values of s for which Both $X_1(s)$ and $X_2(s)$ Converge $\Rightarrow \boxed{ROC_3 \text{ containing } R_1 \cap R_2}$

as the multiplication of $X_1(s) \cdot X_2(s)$ could leads to zeros cancel poles which may lead to that ROC_3 may be bigger than $R_1 \cap R_2$

➢ Laplace Transform Properties.

[7] Differentiation in Time-Domain:- if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$, ROC $= R$

then $\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}} sX(s)$; ROC containing R

Proof

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\,e^{st}\,ds \Rightarrow \text{From inverse Laplace}$$

differentiate both sides with respect to t $\Rightarrow \frac{d}{dt}x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} s\,X(s)\,e^{st}\,ds$

Note $X(s)$ is constant

$$\therefore \frac{d\,x(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}[sX(s)]\,e^{st}\,ds \equiv \text{definition of inverse Laplace for } sX(s)$$

$$\therefore \boxed{\frac{d\,x(t)}{dt} \xleftrightarrow{\mathcal{L}} sX(s)}$$

So if $X(s)$ has a pole at $s=0$ will be cancelled

then $\boxed{\text{ROC of } sX(s) \text{ may be larger than } R \text{ but at least ROC contain } R}$

[8] Differentiation in S-Domain:- if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$; ROC $= R$

then $-t\,x(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}X(s)$; ROC $= R$

Proof

$$X(s) = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt$$

differentiate both sides with respect to s $\Rightarrow \frac{d}{ds}X(s) = \int_{-\infty}^{+\infty}(-t)\,x(t)\,e^{-st}\,dt$

Note $x(t)$ is constant

$$= \mathcal{L}\{(-t)\,x(t)\}$$

$$\Rightarrow \boxed{(-t)\,x(t) \xleftrightarrow{\mathcal{L}} \frac{d\,X(s)}{ds}}$$

Note that the same set of s required for $X(s)$ to be converge is the same set required here for $sX(s)$ to converge

$$\therefore \boxed{\text{ROC of } \frac{dX(s)}{ds} = R}$$

[9] Integration in Time:- as $e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}$; $Re\{s\} > -a$

$\Rightarrow$ as $a=0 \Rightarrow u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$; $Re\{s\} > 0$

And as $\boxed{\int_{-\infty}^{t} x(\tau)\,d\tau = x(t) * u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s)}$ with $\boxed{\text{ROC containing } R \cap \{Re\{s\}>0\}}$

with ROC may be larger if $X(s)$ has zero at $s=0$

➢ Laplace Transform Properties.

Summary of Laplace Transform Properties and Theorems

	Property/Theorem	*Time Domain*	*Complex Frequency Domain*
1	Linearity	$c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t)$	$c_1 F_1(s) + c_2 F_2(s) + \ldots + c_n F_n(s)$
2	Time Shifting	$f(t-a)u_0(t-a)$	$e^{-as}F(s)$
3	Frequency Shifting	$e^{-as}f(t)$	$F(s+a)$
4	Time Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
5	Time Differentiation See also (2.18) through (2.20)	$\frac{d}{dt}f(t)$	$sF(s) - f(0^-)$
6	Frequency Differentiation See also (2.22)	$tf(t)$	$-\frac{d}{ds}F(s)$
7	Time Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{F(s)}{s} + \frac{f(0^-)}{s}$
8	Frequency Integration	$\frac{f(t)}{t}$	$\int_{s}^{\infty} F(s)ds$
9	Time Periodicity	$f(t+nT)$	$\frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}$
10	Initial Value Theorem	$\lim_{t \to 0} f(t)$	$\lim_{s \to \infty} sF(s) = f(0^-)$
11	Final Value Theorem	$\lim_{t \to \infty} f(t)$	$\lim_{s \to 0} sF(s) = f(\infty)$
12	Time Convolution	$f_1(t)*f_2(t)$	$F_1(s)F_2(s)$
13	Frequency Convolution	$f_1(t)f_2(t)$	$\frac{1}{2\pi j} F_1(s)*F_2(s)$

➢ Laplace Transform Properties.

Causality:

for LTI Systems

The ROC associated with the system function $H(s)$ for a causal system is a right-half plane.

impulse response is zero for $t < 0$

For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.

Stability:

for LTI Systems

An LTI system is stable if and only if the ROC of its system function $H(s)$ includes the $j\omega$-axis [i.e., $\Re e\{s\} = 0$].

$$stable\ if \int_{-\infty}^{+\infty} h(t)dt < 0$$

$$let\ s = 0\ in \int_{-\infty}^{+\infty} h(t)e^{-st}dt$$

A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane—i.e., all of the poles have negative real parts.

➢ Laplace Transform Properties.

➔ **If the degree of numerator is less than the degree of denominator**

There will be zero(s) at infinity

EX:

$$H(s) = \frac{1}{s+1}$$

If S = ∞ ➔ H(s)=0

➔ **If the degree of numerator is greater than the degree of denominator**

Then there will be pole at infinity

EX:

$$H(s) = s = \frac{s}{1}$$

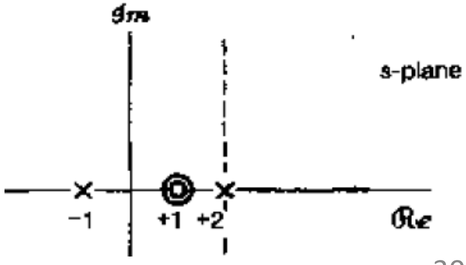
If S = ∞ ➔ H(s) = ∞

➔ **If a root (either zero or pole) has an exponent then there is a number of zeros or number of poles equal to the exponent at the location of this root**

EX:

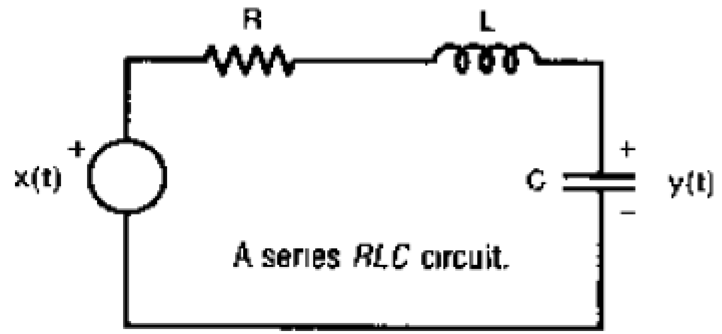
$$H(s) = \frac{(s-1)^2}{(s+1)(s+2)}$$

has two zeros at s=1 and two poles at s=-1 and at s=-2



➢ Example to show the benefits of Laplace Transform.

Consider the following circuit :



What is the system function of this system?

We have the input voltage $x(t)$ across the voltage source and the output voltage $y(t)$ across the capacitor. And we know that as **ALL** components are connected in **series** then:

The sum of individual voltages on each component = the voltage source

$$RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t) = x(t).$$

As we get the **system equation**, we can exploit the **Laplace Transform** and its **properties** to directly calculate the system function *H(s)*:

$$RC.s.Y(s) + LC.s^2.Y(s) + Y(s) = X(s)$$

$$Y(s)\left\{ sRC + s^2LC + 1 \right\} = X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{sRC + s^2LC + 1} = \frac{1/LC}{s^2 + sR/L + 1/LC}$$