Signals and Systems

Lecture # 12

LTI Systems Properties

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Topics of the lecture:

➢ **Properties of Linear Time-Invariant Systems .**

- **Commutative**
- **Distributive**
- **Associative**
- **Memoryless**
- **Invertibility**
- **Causality**
- **Stability**

Properties of Linear Time-Invariant Systems

We developed LTI System Representation as follow:

$$for\ \ discrete-time:$$

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=x[n]*h[n]$$

$$for\ \ continuous-time:$$

$$y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau=x(t)*h(t)$$

1- The commutative property:

$$let\quad r=n-k\qquad\therefore\quad k=n-r$$

$$\therefore\quad when\quad k=-\infty\quad\rightarrow\ r=n-(-\infty)=+\infty$$

$$and\quad when\quad k=+\infty\quad\rightarrow\ r=n-(+\infty)=-\infty$$

The convolution sum in terms of (r) becomes:

$$\therefore\quad y[n]=\sum_{r=+\infty}^{-\infty}x[n-r]h[r]$$

As (r) is a running dummy variable, we can rename it as (k) again and exchange the sum limits without any effect:

$$\therefore\quad y[n]=\sum_{k=-\infty}^{+\infty}h[k]x[n-k]=h[n]*x[n]$$

Similarly:

$$let\quad w=t-\tau\quad\therefore\quad\tau=t-w$$

$$\therefore\quad dw=-d\tau$$

$$\therefore\quad when\quad\tau=-\infty\quad\rightarrow\ w=t-(-\infty)=+\infty$$

$$and\quad when\quad\tau=+\infty\quad\rightarrow\ w=t-(+\infty)=-\infty$$

The convolution sum in terms of (r) becomes:

$$\therefore\quad y(t)=\int_{+\infty}^{-\infty}x(t-w)h(w)(-dw)$$

As (w) is a running dummy variable, we can rename it as (τ) again and exchange the integration limits and multiply by (-1):

$$\therefore\quad y(t)=\int_{-\infty}^{+\infty}h(\tau)x(t-\tau)d\tau=h(t)*x(t)$$

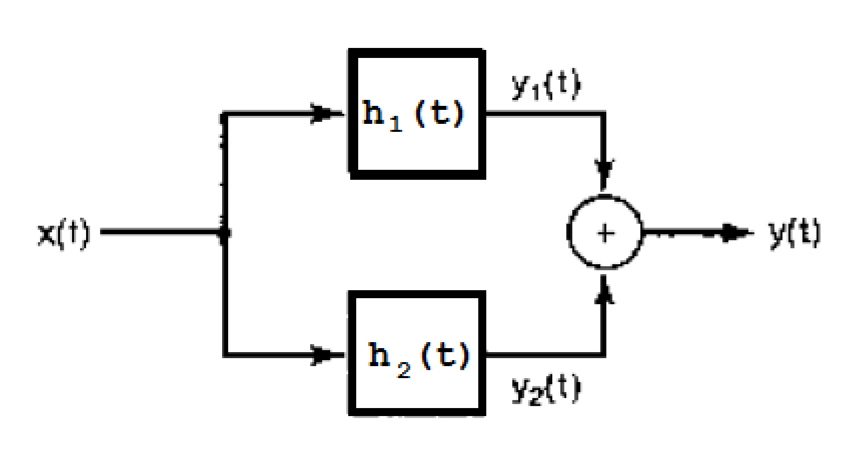
➢ Properties of Linear Time-Invariant Systems

2- The distributive property:

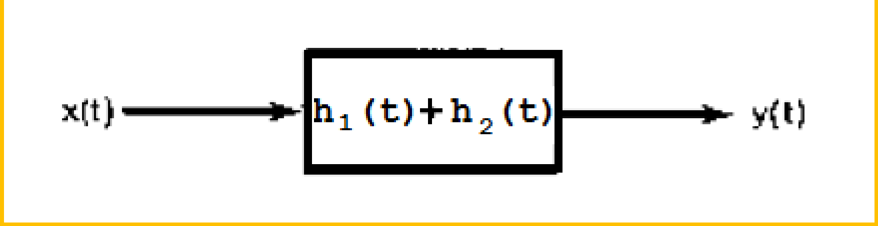
$$x(t) * \{ h_1(t) + h_2(t) \} = x(t) * h_1(t) + x(t) * h_2(t)$$

$$x[n] * \{ h_1[n] + h_2[n] \} = x[n] * h_1[n] + x[n] * h_2[n]$$

Graphical Interpretation:



If we have two systems connected in parallel then we can build an equivalent system that have an impulse response equal to the addition of the impulse responses of the original two systems.



➢ Properties of Linear Time-Invariant Systems

2- The distributive property:(follow)

Example:

$$x[n] = \left\{\frac{1}{2}\right\}^n u[n] + 2^n u[-n]$$

$$h[n] = u[n]$$

Then:

An idea to get the convolution between x[n] and h[n] is as follow:

$$x[n] = \left\{\frac{1}{2}\right\}^n u[n] + 2^n u[-n]$$
$$= x_1[n] + x_2[n]$$

$$\therefore y[n] = x[n] * h[n] = h[n] * x[n]$$
$$= h[n] * \{ x_1[n] + x_2[2] \}$$
$$= h[n] * x_1[n] + h[n] * x_2[2]$$

Assignment: **Complete the example?**

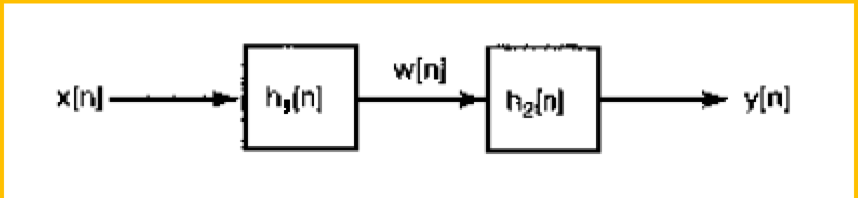
➢ Properties of Linear Time-Invariant Systems

3- The associative property:

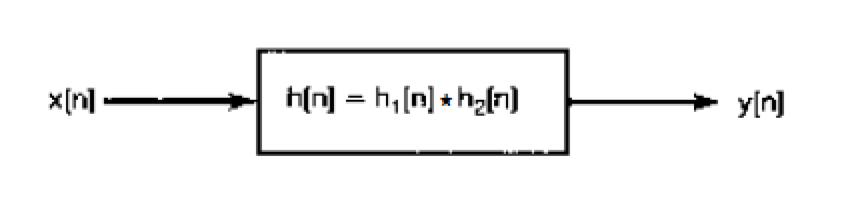
$$x(t) * \{ h_1(t) * h_2(t) \} \;=\; \{ x(t) * h_1(t) \} * h_2(t) \;=\; x(t) * h_1(t) * h_2(t)$$

$$x[n] * \{ h_1[n] * h_2[n] \} \;=\; \{ x[n] * h_1[n] \} * h_2[n] \;=\; x[n] * h_1[n] * h_2[n]$$

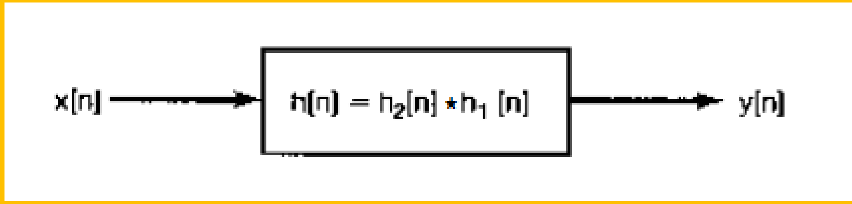
Graphical Interpretation:

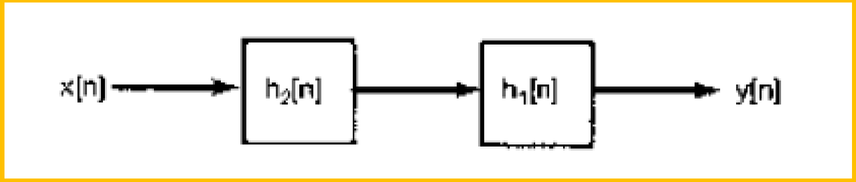


Associative Property →



Commutative Property →





➢ Properties of Linear Time-Invariant Systems

4- LTI systems with and without memory:

Remember:
The “Memoryless” system its output at any time instance depends only on the input at that time instant.

Recall :
the convolution formula :

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=x[n]*h[n]$$

Then the only way to keep the output y[n] at any time instance (n_o) to depend only on the input x[n_o] is to keep the running variable (k) at that time instance only i.e. k = n_o

Then the only condition that make that true is:

$$h[n_o-k]=0 \quad \text{for} \quad n_o \neq k$$

OR $\boxed{h[n]=0 \quad \text{for} \quad n \neq 0}$ ⟶ **The condition of any system to be memoryless**

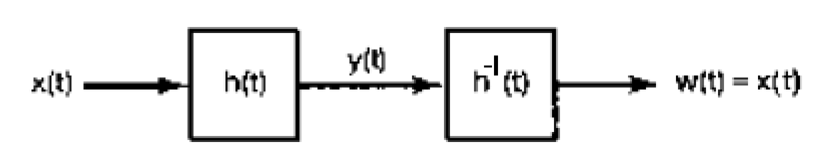
In this case $h[n] = W\,\delta[n]$ **,**
where $W = h[0]$

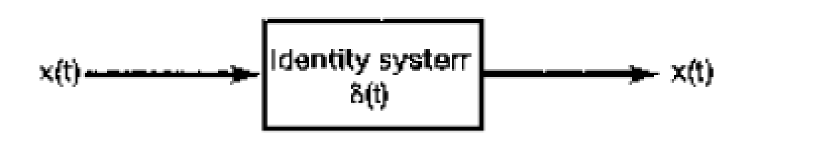
For continuous-time systems the condition to be memoryless is $\boxed{h(t)=0 \quad \text{for} \quad t \neq 0}$

➢ Properties of Linear Time-Invariant Systems

5- Invertibility of LTI systems :

The system has an inverse if →





Then:

The impulse response $h^{-1}(t)$ must satisfy the following condition: $h(t) * h^{-1}(t) = \delta(t)$

Example: $y(t) = x(t - t_o)$

If the input equal the impulse, then the output will be the impulse response → $h(t) = \delta(t - t_o)$

Then the output can be re-written as : $y(t) = x(t) * h(t)$

$$\therefore \quad x(t - t_o) = x(t) * \delta(t - t_o)$$

i.e. the convolution of a signal with a shifted impulse simply shifts the signal. Then:

As we need to shift back y(t) to return to x(t), we can do this by convolving it with a shifted impulse

$$\therefore \quad x(t) = y(t) * \delta(t + t_o)$$

$$\therefore \quad h^{-1}(t) = \delta(t + t_o)$$

$$\therefore \quad h(t) * h^{-1}(t) = \delta(t - t_o) * \delta(t + t_o) = \delta(t) \quad \#$$

Properties of Linear Time-Invariant Systems

5- Invertibility of LTI systems : (followed)

Example II: **If the impulse response →** $$h[n]=u[n]$$

Then the output can be written as : $$y[n]=x[n]*u[n]$$

$$\therefore \quad y[n]=\sum_{k=-\infty}^{\infty} x[k]*u[n-k]$$

As u[n-k] =0 for n-k <0 i.e. for k >n and u[n-k]=1 otherwise → then the output can be re-written as :

$$\therefore \quad y[n]=\sum_{k=-\infty}^{n} x[k]$$

i.e. y[n] is the running sum of x[n]. If we need to get back x[n] , we can just subtract y[n] - y[n-1]

$$\therefore \quad x[n]=y[n]-y[n-1]$$

$$\therefore \quad x[n]=y[n]*\delta[n]-y[n]*\delta[n-1]=y[n]*\{\ \delta[n]-\delta[n-1]\ \}$$

$$\therefore \quad h^{-1}[n]=\delta[n]-\delta[n-1]$$

$$\begin{aligned}\therefore \quad h[n]*h^{-1}[n] &= u[n]*\{\ \delta[n]-\delta[n-1]\ \} \\ &= u[n]*\delta[n]-u[n]*\delta[n-1] \\ &= u[n]-u[n-1] \\ &= \delta[n] \qquad \#\end{aligned}$$

➢ Properties of Linear Time-Invariant Systems

6- Causality of LTI systems :

Recall: the output of a causal system does not depend on future inputs.

As the output of LTI system given by: $$y[n] = \sum_{k=-\infty}^{\infty} x[k] * h[n-k]$$

Then at any time instance n the RHS of the previous equation should not include x[k] for k > n

to make this possible then: $$h[n-k] = 0 \quad ; \quad for\ k > n$$

$$\therefore \quad h[n-k] = 0 \quad ; \quad for\ n-k < 0$$

which can be re-written as: $$\boxed{\therefore \quad h[n] = 0 \quad ; \quad for\ n < 0}$$

The same relation applies for causal continuous-time system for which :

$$\boxed{\therefore \quad h(t) = 0 \quad ; \quad for\ t < 0}$$

If we combine this with the definition casual *signal* (x[n]=0, for n<0), then:
The LTI system is causal if its impulse response is causal.

➢ Properties of Linear Time-Invariant Systems

7- Stability of LTI systems :

Recall: the system is causal if bounded inputs lead to bounded outputs (BIBO).

If the input signal is bounded: $|x[n]| \le B \quad for\ all\ n$

If this input is applied to a LTI system with impulse response h[n] then :

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$as \quad |A+C| \le |A| + |B| \ \ then \quad \therefore |y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]x[n-k]|$$

$$as \quad |A.C| = |A|.|B| \ \ then \quad \therefore |y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$\therefore |y[n]| \le \sum_{k=-\infty}^{\infty} B|h[k]| \quad \Rightarrow \quad |y[n]| \le B \sum_{k=-\infty}^{\infty} |h[k]|$$

Then for |y[n]| to be bounded this requires: $\displaystyle\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ **OR** $\displaystyle\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

The same relationship applies for continuous-time systems too: $\displaystyle\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$