Signals and Systems

Lectures # 10 & # 11

Continuous-time LTI Systems

(Convolution Integral)

Prepared by:

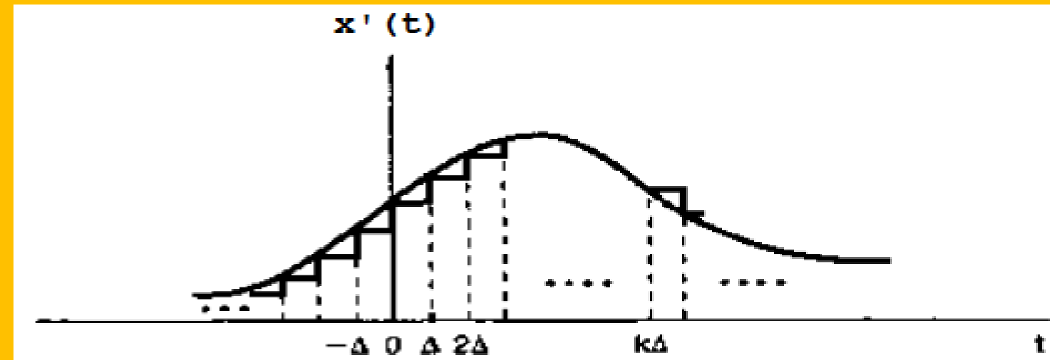
Dr. Mohammed Refaey

Topics of the lecture:

- **Convolution Integral Formula Derivation**
- **Convolution Integral Computation Algorithm**
- **Examples.**

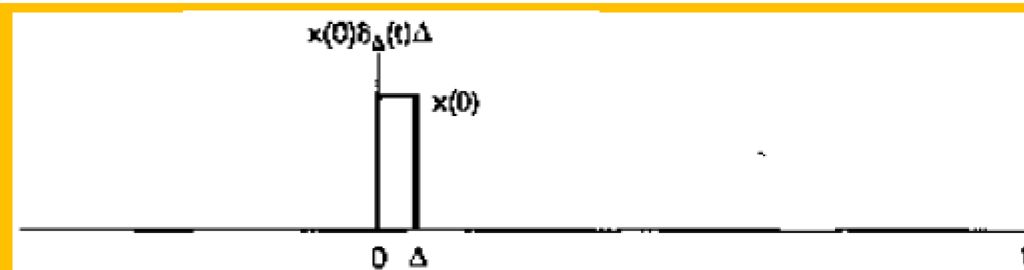
➢ Convolution Integral Formula Derivation for LTI Systems

Then the pulse approximation x'(t) of x(t) will be as in figure →



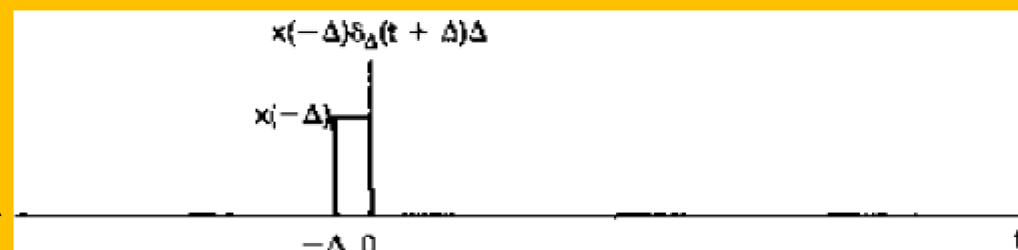
$$Recall: \quad \delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & ; \quad 0 < t < \Delta \\ 0 & ; \quad Otherwise \end{cases}$$

$$then \quad \delta_\Delta(t).\Delta = \begin{cases} 1 & ; \quad 0 < t < \Delta \\ 0 & ; \quad Otherwise \end{cases}$$



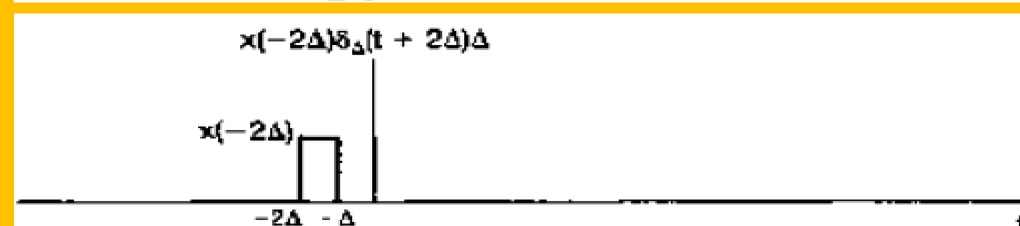
$$\Rightarrow \quad x'(t).\delta_\Delta(t).\Delta = \begin{cases} x'(0) & ; \quad 0 < t < \Delta \\ 0 & ; \quad Otherwise \end{cases}$$

$$\therefore \quad x'(t).\delta_\Delta(t).\Delta = x'(0).\delta_\Delta(t).\Delta$$



$$\Rightarrow \quad x'(t).\delta_\Delta(t+\Delta).\Delta = \begin{cases} x'(-\Delta) & ; \quad -\Delta < t < 0 \\ 0 & ; \quad Otherwise \end{cases}$$

$$\therefore \quad x'(t).\delta_\Delta(t+\Delta).\Delta = x'(-\Delta).\delta_\Delta(t+\Delta).\Delta$$



$$\Rightarrow \quad x'(t).\delta_\Delta(t+2\Delta).\Delta = \begin{cases} x'(-2\Delta) & ; \quad -2\Delta < t < -\Delta \\ 0 & ; \quad Otherwise \end{cases}$$

$$\therefore \quad x'(t).\delta_\Delta(t+2\Delta).\Delta = x'(-2\Delta).\delta_\Delta(t+2\Delta).\Delta$$

→And so on, then:

$$\therefore x'(t) = sum\ of\ all\ pulses = \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

Convolution Integral Formula Derivation for LTI Systems

$$x'(t) = sum \quad of \quad all \quad pulses$$

$$= \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\because \quad x'(t) \xrightarrow{S} y'(t)$$

$$let \quad \delta_\Delta(t) \xrightarrow{S} h'(t)$$

$$as \quad S \quad LTI \quad system:$$

$$\therefore \delta_\Delta(t-k\Delta) \xrightarrow{S} h'(t-k\Delta) \; (LTI \equiv same \;\; shift)$$

$$\therefore x'(t) = \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\xrightarrow{S} y'(t) = \sum_{k=-\infty}^{\infty} x'(k\Delta)h'(t-k\Delta)\Delta$$

$$\because \quad x(t) = \lim_{\Delta\to 0} \quad x'(t)$$

$$\therefore \quad y(t) = \lim_{\Delta\to 0} \quad y'(t)$$

$$and \quad h(t) = \lim_{\Delta\to 0} \quad h'(t)$$

$$then \quad as \quad \Delta \to 0$$

$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\xrightarrow{S} y(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x'(k\Delta)h'(t-k\Delta)\Delta$$

$as\ \Delta \to 0$ *the summation tends to be integration*

$$\sum_{-\infty}^{\infty} \quad \to \quad \int_{-\infty}^{\infty}$$

$$x'(k\Delta) \quad \to \quad x(k\Delta)$$

$$h'(t-k\Delta) \to h(t-k\Delta)$$

$$let \quad k.\Delta = \tau \qquad \therefore \Delta.dk = d\tau$$

as dk is the step of summation $\to$ *then* $dk = 1$

$$\therefore \Delta.1 = d\tau \qquad \to \qquad \Delta = d\tau$$

$$\boxed{\therefore \; y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

called Convolution Integral Formula

➢ Convolution Integral Formula Derivation for LTI Systems

$$\therefore\ y(t) = \int_{-\infty}^{\infty} x(\tau).h(t-\tau)d\tau$$

The independent variable is renamed to ***τ***

There is a time-reverse

The shift value is the value of ***t***

For each shift value ***t*** there are : multiplications + integral of multiplications

➢ Convolution Integral Computation Algorithm for LTI

To compute the convolution integral of x(t) and h(t):

1- Let the two signals as functions in the independent variable τ instead of t.
So, x(τ) instead of x(t)
and h(τ) instead of h(t) (just renaming the independent variable will not make any difference)

2- Choose one of the two signals and ***time-reverse*** it to be either x(-τ) or h(-τ).

3- Determine the ***areas of overlapping*** that have similar conditions of mathematical treatment, as the shift (t) will slide the time-reversed signal starting from (- ∞) towards (+∞). (area of overlapping means that this domain that both of the signals have non-zero values at the same time) (similar conditions of mathematical treatment means that there is one mathematical formula to compute the integration of multiplication of the two overlapping signals)

4- Compute the ***boundaries*** (*U*) and (*L*) of each overlapping area. (upper and lower limit of integration)

5- Compute the ***mathematical formula*** of each overlapping area using the formula:

$$y(t) = \int_L^U x(\tau)h(t-\tau)\,d\tau \quad (if\ \ you\ reversed\ h)$$

OR

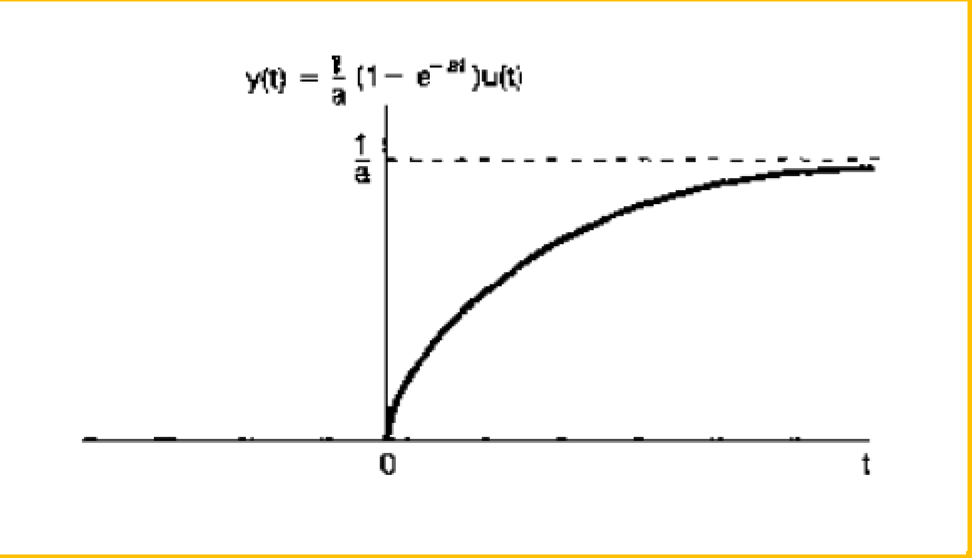
$$y(t) = \int_L^U h(\tau)x(t-\tau)\,d\tau \quad (if\ \ you\ reversed\ x)$$

6- Repeat steps 4 and 5 for each overlapping area.

➢ Convolution Integral : Examples

[1]- Compute the convolution sum of the following input x(t) and system impulse response h(t): $x(t) = e^{-at}u(t) \quad ; \quad a > 0$

$$h(t) = u(t)$$



The answer following the algorithm:

1- let the two signals as functions in (τ) instead of (t).

2- let we time-reverse either one of the two signals. Let us choose h(τ).

3- if t < 0. there is no overlapping between x(τ) and h(t-τ) → y(t)=0 , t<0

4- the overlapping will start from t=0. and slightly and gradually the signal h(t-τ) will slides under x(τ) as t is increasing:
at t=0 → there is overlapping from (0) to (0)
at t=1 → there is overlapping from (0) to (1)
at t=2 → there is overlapping from (0) to (2)
And so on … then the overlapping boundaries of this area are: L=0 (as the lower limit is fixed at 0)
U=t (as the upper limit is equal to t)

5- for t ≥ 0: $y(t) = \int_{L}^{U} x(\tau)h(t-\tau)d\tau$

$$\therefore \quad y(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau$$

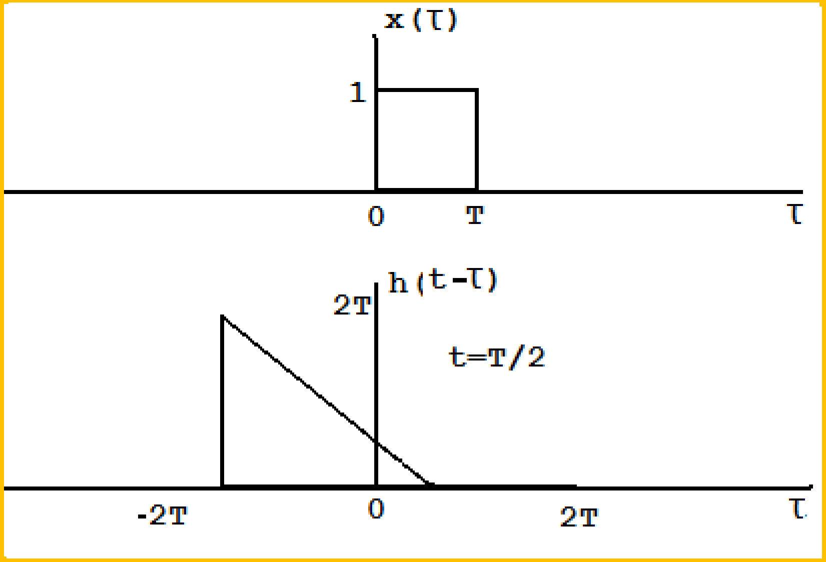
$$\therefore \quad y(t) = \int_{0}^{t} e^{-a\tau}u(\tau)u(t-\tau)d\tau$$

$$\therefore \quad y(t) = \int_{0}^{t} e^{-a\tau}d\tau = \left.\frac{e^{-a\tau}}{-a}\right|_{0}^{t} = \frac{-1}{a}\{e^{-at} - 1\}$$

$$\therefore \quad y(t) = \frac{1}{a}\{1 - e^{-at}\} \quad ; \quad t \ge 0$$

➢ Convolution Integral : Examples

[2]- Compute the convolution sum of the following input x(t) and system impulse response h(t): $x(t) = \begin{cases} 1 & ; 0<t<T \\ 0 & ; otherwise \end{cases}$; $h(t) = \begin{cases} t & ;0<t<2T \\ 0 & ;otherwise \end{cases}$



The answer following the algorithm:

1- let the two signals as functions in (τ) instead of (t).

2- let we time-reverse either one of the two signals. Let us choose h(τ).

3- if t < 0. there is no overlapping between x(τ) and h(t-τ) → y(t)=0, t<0

4- the overlapping will start from t=0. and slightly and gradually the signal h(t-τ) will slides under x(τ) as 0<t<T . (partial overlapping)

at t=0 → there is overlapping from (0) to (0)
at t=T/4 → there is overlapping from (0) to (T/4)
at t=T/2 → there is overlapping from (0) to (T/2)
And so on … then the overlapping boundaries of this area are: L=0 (as the lower limit is fixed at 0)
U=t (as the upper limit is equal to t)

5- for 0<t <T: $y(t) = \int_L^U x(\tau)h(t-\tau)d\tau$

6- Repeat steps 4 and 5 for T≤t<2T (total overlapping)

$$\therefore \quad y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

$$\therefore \quad y(t) = \int_0^t 1.(t-\tau)d\tau$$

$$\therefore y(t) = \int_0^t t d\tau - \int_0^t \tau d\tau \quad = t\{\tau|_0^t\} - \{\frac{\tau^2}{2}\Big|_0^t\}$$

$$= t\{t-0\} - \{\frac{t^2}{2} - 0\} \quad = t^2 - \frac{t^2}{2} \quad = \frac{t^2}{2}$$

$$\therefore y(t) = \frac{t^2}{2} \quad ; 0 \le t < T$$

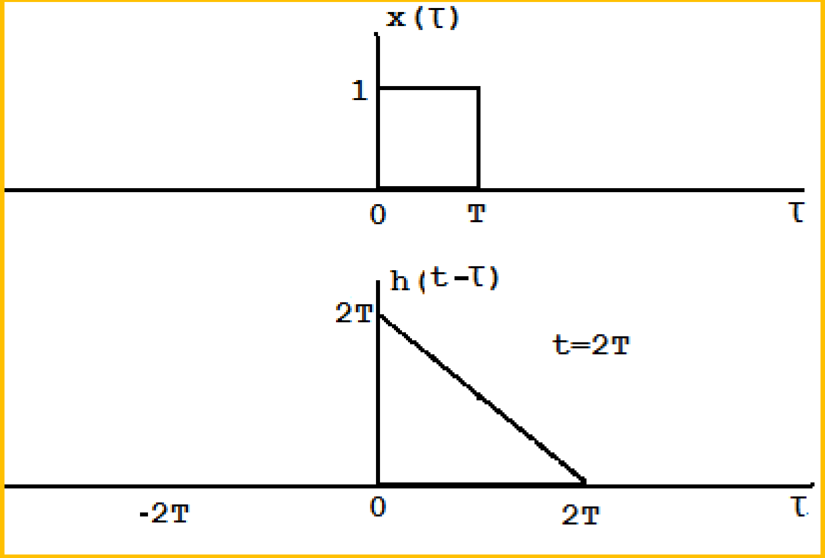
➢ Convolution Integral : Examples

[2]- Compute the convolution sum of the following input x(t) and system impulse response h(t): $x(t)=\begin{cases}1 & ;0<t<T\\ 0 & ;otherwise\end{cases}$; $h(t)=\begin{cases}t & ;0<t<2T\\ 0 & ;otherwise\end{cases}$

4'- after T till 2T there is a total overlapping :

at t=T → there is overlapping from (0) to (T)
at t=3T/2 → there is overlapping from (0) to (T)
at t=2T → there is overlapping from (0) to (T)

then the overlapping boundaries of this area are: L=0 (as the lower limit is fixed at 0)
U=T (as the upper limit is fixed to T)



5'- for T≤t <2T: $y(t)=\int_{L}^{U}x(\tau)h(t-\tau)d\tau$

$$\therefore\quad y(t)=\int_{0}^{T}x(\tau)h(t-\tau)d\tau$$

$$\therefore\quad y(t)=\int_{0}^{T}1.(t-\tau)d\tau$$

$$\therefore y(t)=\int_{0}^{T}td\tau-\int_{0}^{T}\tau d\tau$$

$$=t\{\tau|_{0}^{T}\}-\{\frac{\tau^{2}}{2}\Big|_{0}^{T}\}$$

$$=t\{T-0\}-\{\frac{T^{2}}{2}-0\}$$

$$=tT-\frac{T^{2}}{2}$$

$$\therefore y(t)=Tt-\frac{T^{2}}{2}\quad ;T\le t<2T$$

6'- Repeat steps 4 and 5 for 2T≤t<3T (partial overlapping)

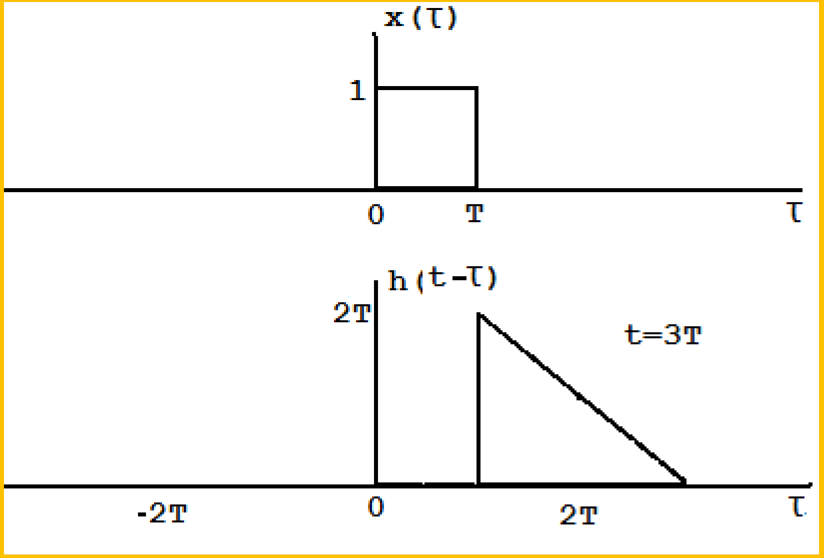
➢ Convolution Integral : Examples

[2]- Compute the convolution sum of the following input x(t) and system impulse response h(t): $x(t)=\begin{cases}1 & ;0<t<T\\ 0 & ;otherwise\end{cases}$; $h(t)=\begin{cases}t & ;0<t<2T\\ 0 & ;otherwise\end{cases}$

4''- after 2T till 3T there is a partial overlapping :

at t=2T → there is overlapping from (0) to (T)
at t=5T/2 → there is overlapping from (T/2) to (T)
at t=3T → there is overlapping from (T) to (T)

then the overlapping boundaries of this area are: L=t-2T (as the lower limit is fixed at 0)
U=T (as the upper limit is fixed to T)



5''- for 2T≤t <3T: $y(t)=\int_{L}^{U}x(\tau)h(t-\tau)d\tau$

$$\therefore\quad y(t)=\int_{t-2T}^{T}x(\tau)h(t-\tau)d\tau$$

$$\therefore\quad y(t)=\int_{t-2T}^{T}1.(t-\tau)d\tau$$

$$\therefore y(t)=\int_{t-2T}^{T}t d\tau-\int_{t-2T}^{T}\tau\, d\tau$$

$$=t\{\tau|_{t-2T}^{T}\}-\{\frac{\tau^2}{2}\Big|_{t-2T}^{T}\}$$

$$=t\{T-(t-2T)\}-\{\frac{T^2}{2}-\frac{(t-2T)^2}{2}\}$$

$$=t\{-t+3T\}-\{\frac{T^2-(t^2-4tT+4T^2)}{2}\}$$

$$=-t^2+3tT+\frac{3T^2}{2}+\frac{t^2}{2}-2tT$$

$$\therefore\quad y(t)=-\frac{t^2}{2}+Tt+\frac{3T^2}{2}\quad;\quad 2T\le t<3T$$

➢ Convolution Integral : Examples

[2]- Compute the convolution sum of the following input x(t) and system impulse response h(t): $x(t) = \begin{cases} 1 & ; 0<t<T \\ 0 & ; otherwise \end{cases}$; $h(t) = \begin{cases} t & ;0<t<2T \\ 0 & ;otherwise \end{cases}$

4’’’- after 3T there is NO overlapping

→ y(t)=0 , t>3T

$$\therefore y(t) = \begin{cases} 0 & ; \quad t<0 \\ \dfrac{t^2}{2} & ; \quad 0 \le t < T \\ Tt - \dfrac{T^2}{2} & ; \quad T \le t \le 2T \\ -\dfrac{t^2}{2} + Tt + \dfrac{3T^2}{2} & ; \quad 2T < t \le 3T \\ 0 & ; \quad t > 3T \end{cases}$$