Signals and Systems

Lecture # 4

Exponentials and Sinusoidal Signals Relationship

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Topics of the lecture:

- Fundamental Concepts.
- Exponential and Sinusoidal Signals Relationship.

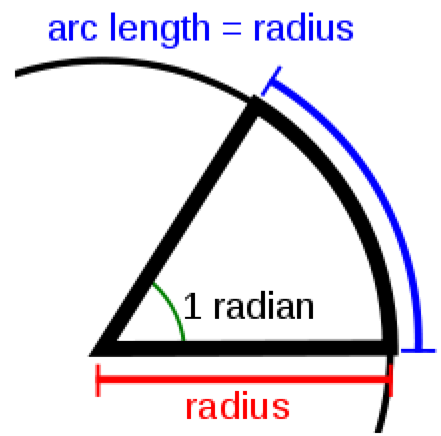
➢ Fundamental Concepts.

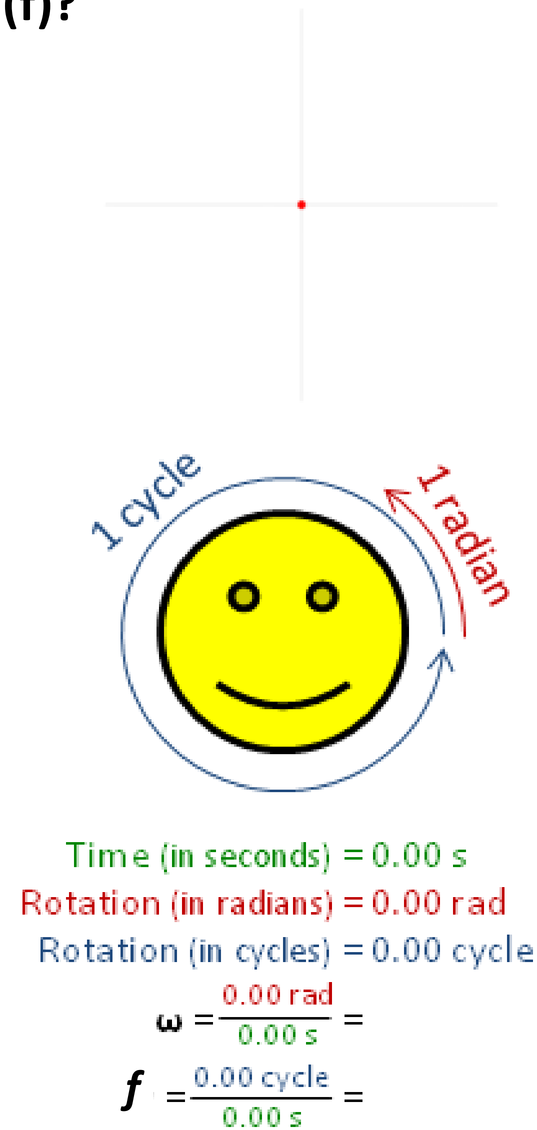
Let us first to know what is meant by the radians? And what is the difference between the angular frequency (w) and the frequency (f)?

- **Radian is the ratio between the length of an arc and its radius. The radian is the standard unit of angular measure in many areas of math.**

- **Angular Frequency (*w*) is the number of radians per second. rad/sec**

- **Frequency (*f*): is the number of cycles per second. cyc/sec**





$$f = \frac{cyc}{sec} = \frac{cyc}{rad} \times \frac{rad}{sec} = \frac{1}{2\pi} \times w = \frac{w}{2\pi} \Rightarrow w = 2\pi f$$

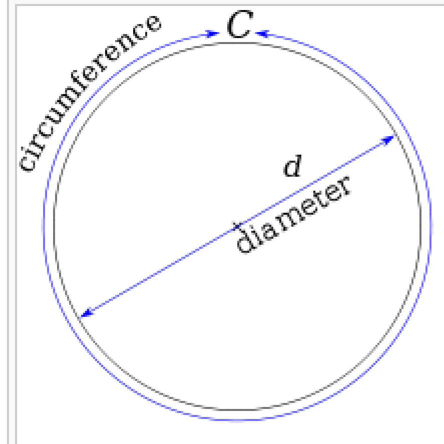
$$as\ \frac{cyc(in\,one\,second)}{rad(in\,one\,second)} = \frac{one\ cycle}{no.of\ radians\ per\,cycle} = \frac{1}{2\pi}$$

$$\frac{\theta(=1rad)}{arc(=r)} = \frac{2\pi}{circumference} \quad => \quad circumference = 2\pi\ r$$

$$as\ \ \frac{\theta 1}{arc\ of\ \theta 1} = \frac{\theta 2}{arc\ of\ \theta 2}$$

➢ Fundamental Concepts.

The constant π **(pi):** is the ratio of a circle's circumference to its diameter = 3.14159265…



The circumference of a circle is slightly more than three times as long as its diameter. The exact ratio is called π.

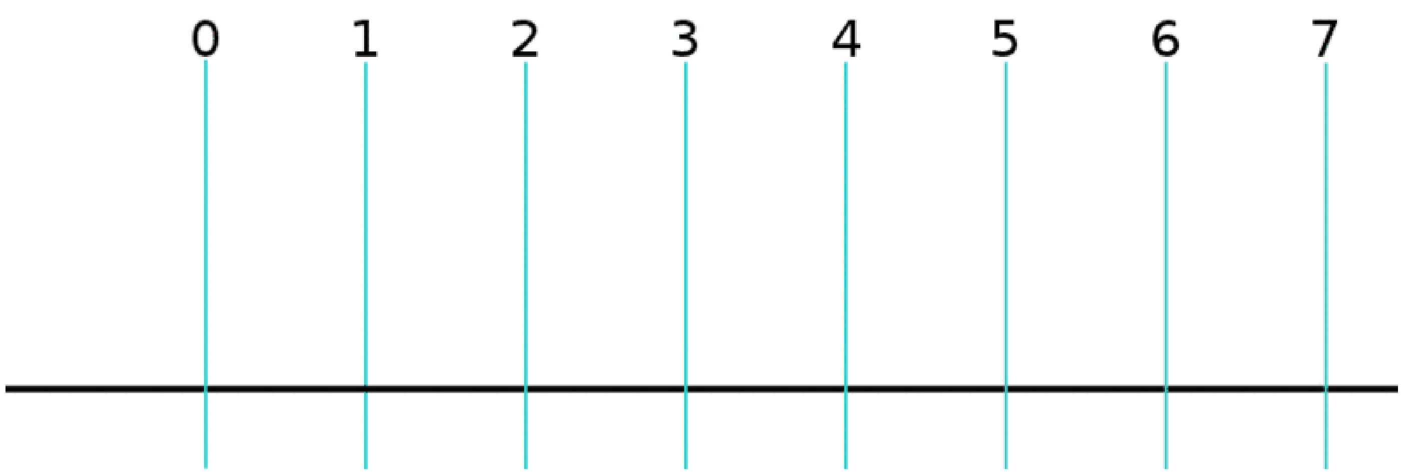
It is not 22/7 !

$$\frac{22}{7} = 3.\overline{142\,857},$$

$$\pi \approx 3.141\,592\,65\ldots$$

<u>**It is NOT rational number!**</u>

Its decimal representation never ends and never settles into a permanent repeating pattern.



When a circle's radius is 1 unit, its circumference is 2π.

➢ Exponential Signals and sinusoidal Signals

The Relationship between the Complex Exponential Signals and Sinusoid Signals:

The *Complex Exponential Signals has the form*: $e^{jwt} \quad OR \quad e^{jwn}$

$$e^{jwt} = 1e^{j\angle wt} = \cos(wt) + j\sin(wt)$$

$$at \quad wt = 0, \quad e^{jwt} = 1 + 0j = 1$$

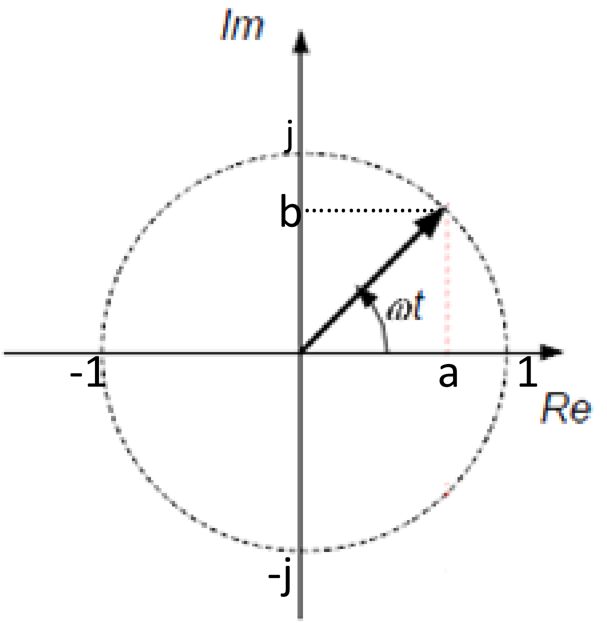
$$at \quad wt = \frac{\pi}{2}, \quad e^{jwt} = 0 + 1j = j$$

$$at \quad wt = \pi, \quad e^{jwt} = -1 + 0j = -1$$

$$at \quad wt = \frac{3\pi}{2}, \quad e^{jwt} = 0 - 1j = -j$$

$$at \quad wt = 2\pi, \quad e^{jwt} = 1 + 0j = 1$$

$$...and \ so \ on$$



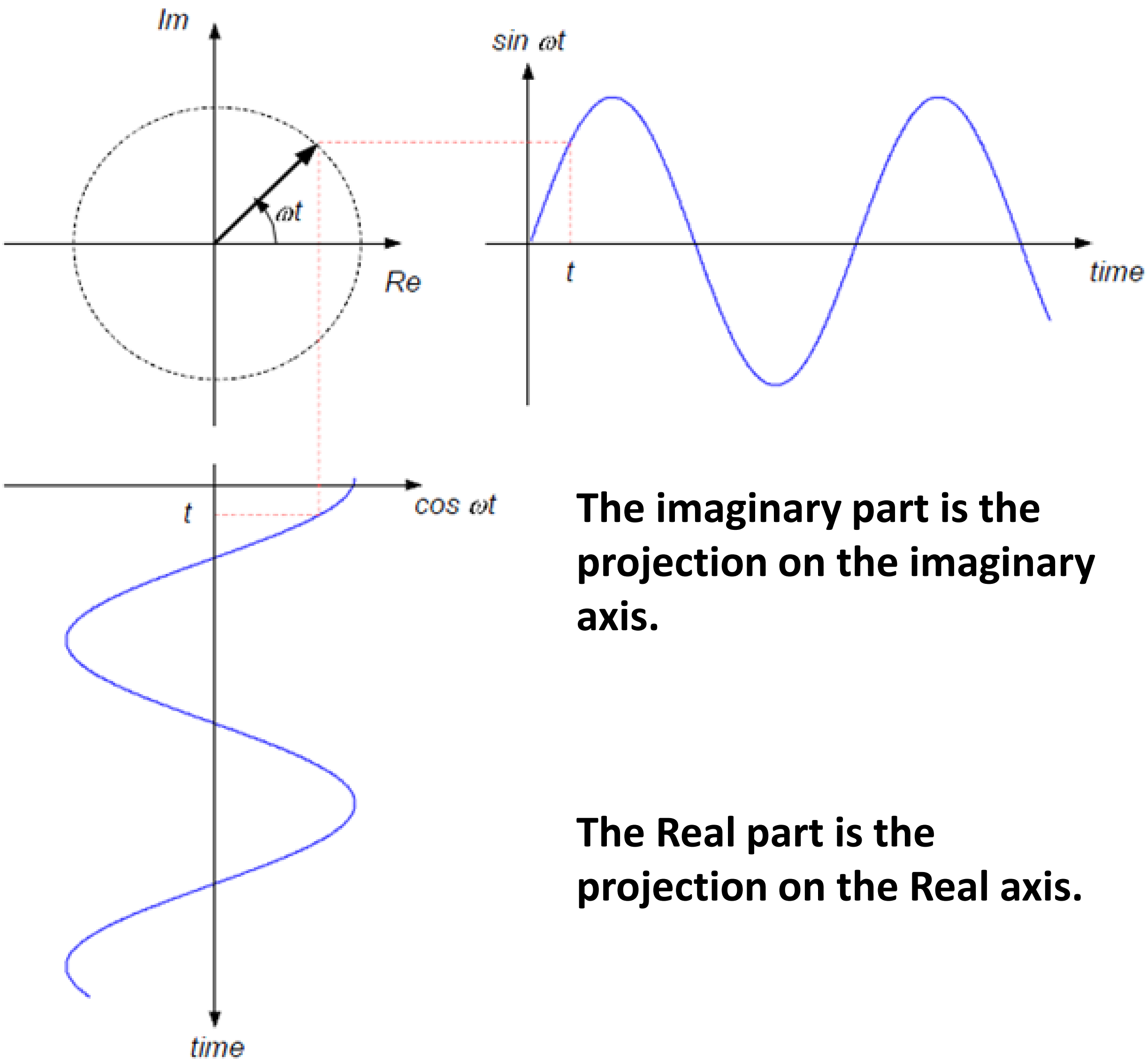
As the angle (wt) is increased, either by increasing the rotating frequency (w) or as the time (t) goes, the point representing e^{jwt} is rotating around the unit circle.

➢ Exponential Signals and sinusoidal Signals

See the first Flash Video. (http://www.fourier-series.com/fourierseries2/flash_programs/cong/index.html)

Play the slide to see the interactive flash

➢ Exponential Signals and sinusoidal Signals

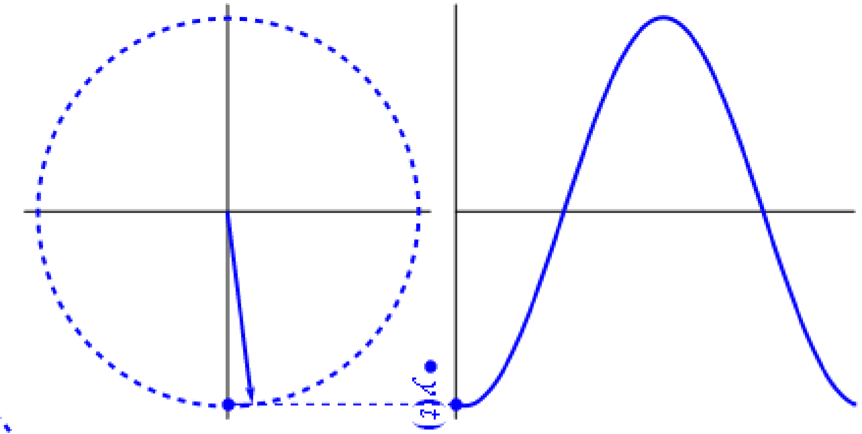


The relationship between the complex exponential and the sinusoidal signals.

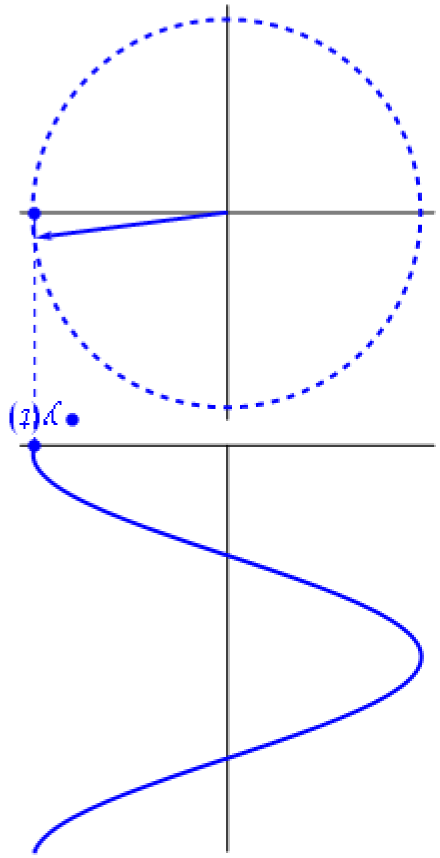
The imaginary part is the projection on the imaginary axis.

The Real part is the projection on the Real axis.

➢ Exponential Signals and sinusoidal Signals



The relationship between the complex exponential and the sinusoidal signals.



The imaginary part is the projection on the imaginary axis.

The Real part is the projection on the Real axis.

➢ Exponential Signals and sinusoidal Signals

See the second Flash Video. (**http://www.fourier-series.com/fourierseries2/flash_programs/cong_with_time/index.html**)

Play the slide to see the interactive flash

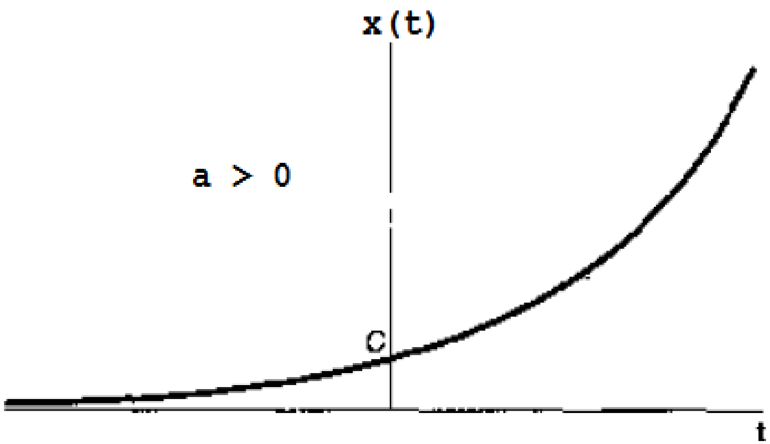
➢ Exponential Signals and sinusoidal Signals (continuous-time case)

$$The\ general\ \ form\ \Rightarrow\quad x(t) = Ce^{at}$$

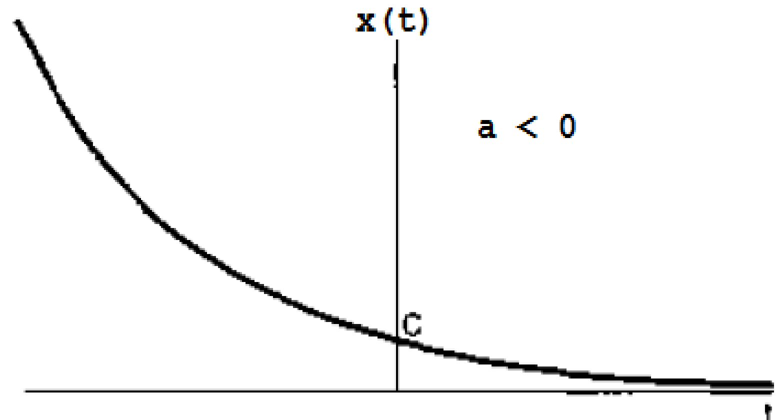
There are three important cases for *C* and *a*.

Case 1: Real Exponential Continuous-Time Signals

If (*C*) and (*a*) are both real numbers then x(t) is called a real exponential signal.



e.g. Chain Reactions in atomic explosion



e.g. damped mechanical systems

If (*C*) is < 0 then x(t) will be mirrored around the horizontal t-axis

If (*a*) is = 0 then x(t) will be constant signal

➢ Exponential Signals and sinusoidal Signals (continuous-time case)

$$The\ general\ \ form\ \Rightarrow\ \ x(t) = Ce^{at}$$

Case 2: periodic complex exponentials (and sinusoidal signals)

If (*a*) is purely imaginary, i.e. *a* = jw. then x(t) = C e^{jwt} , and by ignoring the scaling factor, which not affect the periodicity property (it may only change phase and/or the magnitude), then: x(t) = e^{jwt} , which is always periodic as shown in the previous flash videos.

$$as\ the\ signal\ \ \ x(t) = e^{jwt}\ \ \ is\ \ periodic$$

$$\therefore\ \ \ \ \ \ \ \ x(t+T) = x(t)$$

$$\therefore\ \ \ e^{jw(t+T)} = e^{jwt}.e^{jwT} = e^{jwt}$$

$$\Rightarrow\ \ e^{jwT} = 1\ \ \Rightarrow\ \ as\ e^{jwT} = \cos(wT) + j\sin(wT)\ \ \Rightarrow\ \ wT = k2\pi$$

$$\Rightarrow\ \ T = k\frac{2\pi}{w},\ \ \ \ \ k\ is\ integer$$

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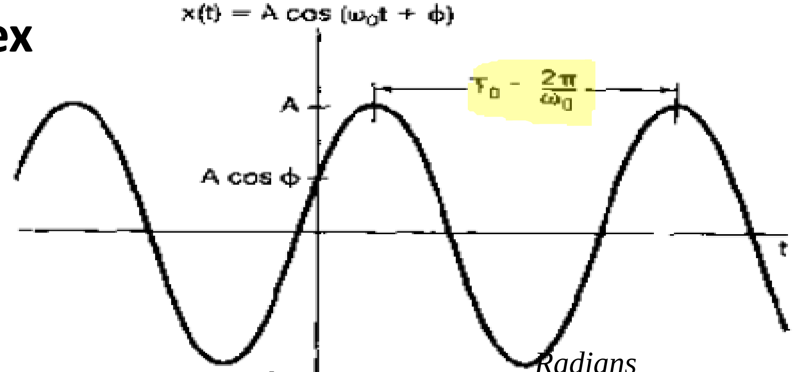
As the fundamental period is the *smallest positive* value of *T* for which $e^{jwT} = 1\ \ OR\ \ w_o T_o = 2\pi$ **:**

$$\overset{k=1}{\Rightarrow}\ \ T_o = \frac{2\pi}{|w_o|}\ \ \Rightarrow\ \ (T_o\ \ for\ e^{jwt}) = (T_o\ \ for\ e^{-jwt})\ \ \ as\ |w_o| = |-w_o|$$

➢ Exponential Signals and sinusoidal Signals (continuous-time case)

A closely related signals to the periodic complex exponential signal are the sinusoidal signals, a cosine sinusoidal general form is:

x(t)=A cos(wt + Ø)



Where (*A*) is the maximum amplitude, (*w*) is the angular frequency (rad/sec) (*w=2πf*), (*f*) the regular frequency (cyc/sec), and (*Ø*) is the phase angle (radians).

$$Angular\ Frequency = w = \frac{Radians}{Seconds}$$

if we have the smallest time of one cycle $(= T_o\ Seconds)$

then we have the smallest angle of one cycle $(= 2\pi\ Radians)$

then we have the fundamental frequency $w_o = \dfrac{2\pi}{T_o} \Rightarrow T_o = \dfrac{2\pi}{w_o}$

$$according\ to\ Euler's\ Formula: \qquad e^{jwt} = \cos(wt) + j\sin(wt)$$

$$\therefore e^{-jwt} = \cos(-wt) + j\sin(-wt) = \cos(wt) - j\sin(wt)$$

$$\therefore A\cos(wt+\phi) = \frac{A}{2}e^{j(wt+\phi)} + \frac{A}{2}e^{-j(wt+\phi)},\ and \quad A\sin(wt+\phi) = \frac{A}{2j}e^{j(wt+\phi)} - \frac{A}{2j}e^{-j(wt+\phi)}$$

$$OR: \quad A\cos(wt+\phi) = A\,\mathrm{Re}\{e^{j(wt+\phi)}\}\ ,\ and \quad A\sin(wt+\phi) = A\,\mathrm{Im}\{e^{j(wt+\phi)}\}$$

ALL these signals can be written in terms of each other and have the same fundamental

$$period \quad T_o = \frac{2\pi}{|w_o|}, \qquad where\ w_o\ is\ the\ fundamental\ frequency$$

they are also power signals?! Proof that...{applications: LC circuit, and acoustic signals}

➢ Exponential Signals and sinusoidal Signals (continuous-time case)

<u>Harmonically-Related Complex Exponentials:</u>

is a set of periodic exponentials all of which are periodic with a common period (T_o)

$$e^{jwt} \quad \textit{to be periodic} \quad \Rightarrow \quad e^{jwT} = 1$$

$$\Rightarrow \therefore \; wT = k2\pi \; ; \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\therefore w = k\frac{2\pi}{T}$$

the fundamental frequency (w_o) *is the smallest value of* (w)

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$$\overset{k=1}{\Rightarrow} w_o = \frac{2\pi}{T_o} \quad , \quad as\, T_o \textit{ is the Fundamental Period of signal having } w_o \textit{ and } w_o T_o = 2\pi$$

$$\Rightarrow w = k\, w_o \quad ; \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

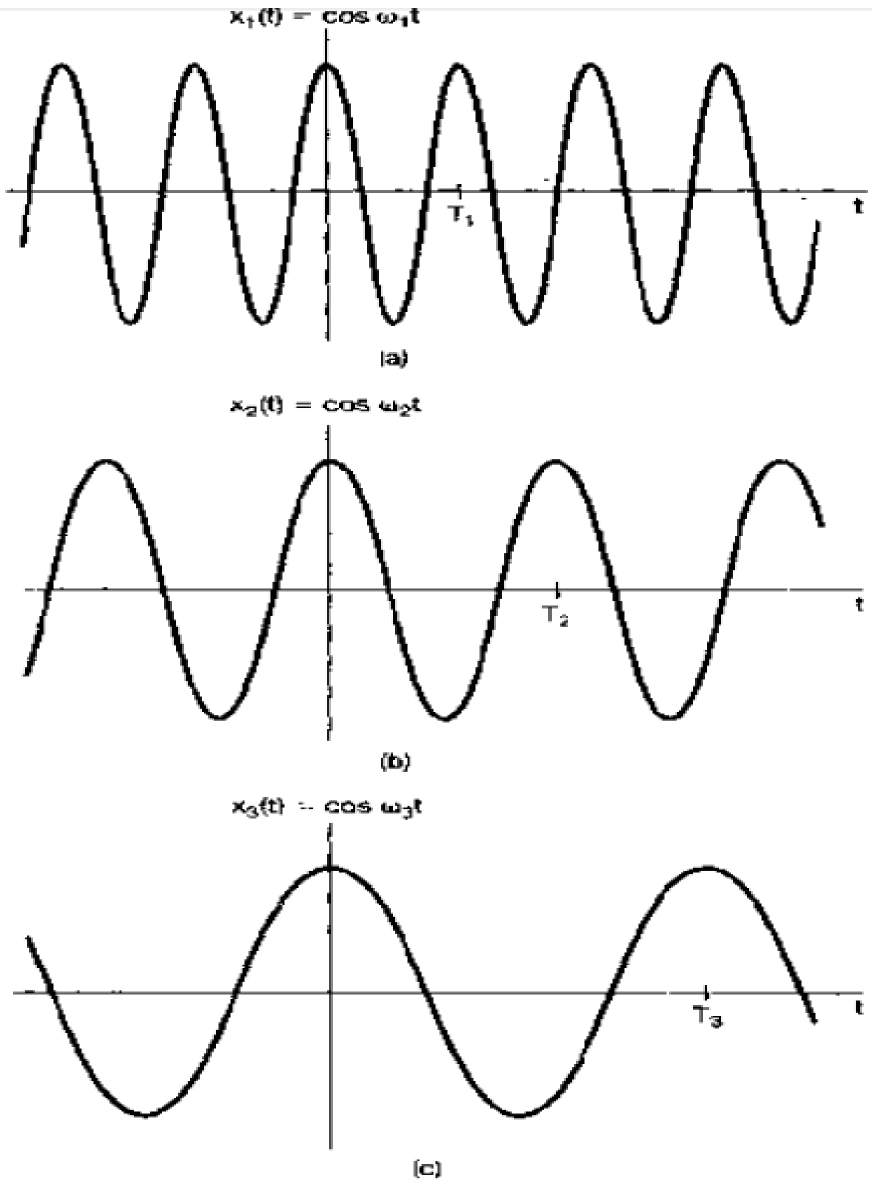
this represents a <u>set</u> of signals each one of them has a frequency (w) *that is multiple of one common frequency* (w_o)

$$\phi_k(t) = e^{jkw_o t} \quad ; \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\textit{the fundamental period of } k^{th} \textit{ harmonic is} \quad T_{o_k} = \frac{2\pi}{|kw_o|} = \frac{T_o}{|k|}$$

➢ Exponential Signals and sinusoidal Signals (continuous-time case)

You can use the previous flashes for interactive clarification of the relationship between the frequency and the fundamental period of continuous-time sinusoids



Relationship between the fundamental frequency and period for continuous-time sinusoidal signals; here, $\omega_1 > \omega_2 > \omega_3$, which implies that $T_1 < T_2 < T_3$.

➢ Exponential Signals and sinusoidal Signals (continuous-time case)

$$\text{The general form} \Rightarrow \quad x(t) = Ce^{at}$$

Case 3: General Continuous-time Complex Exponential signals:

$$\text{let } (C) \text{ and } (a) \text{ both as complex numbers}$$

$$\text{let} \quad C = |C|e^{j\theta}, \quad \text{and} \quad a = r + jw_o$$

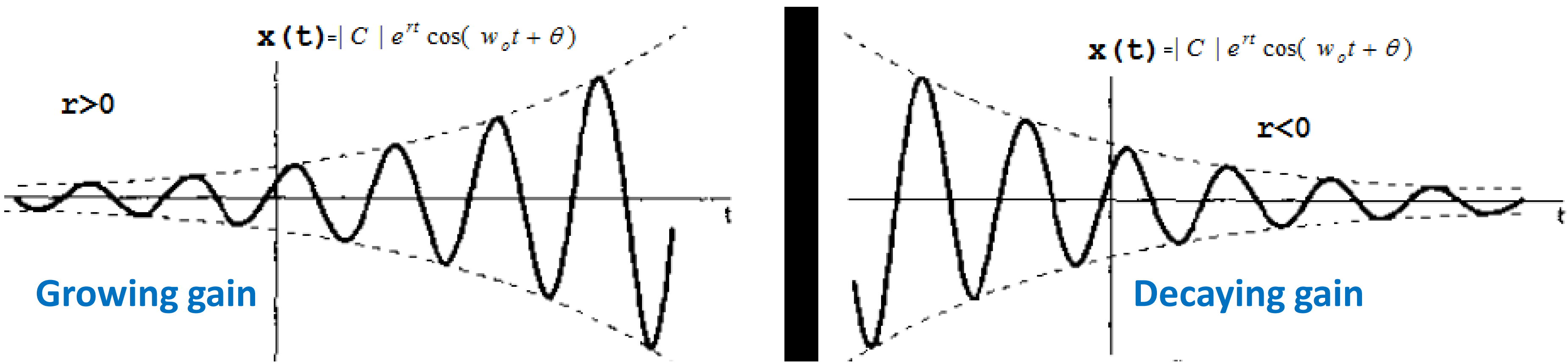
- As a is in the power so rectangular form is more suitable, while the polar form of C is more suitable to be able to add the phases together!

$$\Rightarrow \quad x(t) = Ce^{at} = |C|e^{j\theta}e^{(r+jw_o)t}$$

$$= |C|e^{j\theta}e^{rt}e^{jw_o t} = \quad |C|e^{rt}e^{j(w_o t+\theta)}$$

$$= |C|e^{rt}\left\{\cos(w_o t + \theta) + j\sin(w_o t + \theta)\right\}$$

$$= \text{variable positive gain} \times \text{periodic signal}$$



➢ Exponential Signals and sinusoidal Signals (discrete-time case)

$$The\ general\ \ form \ \Rightarrow \quad x[n] = C\alpha^{n} = Ce^{\beta n}$$

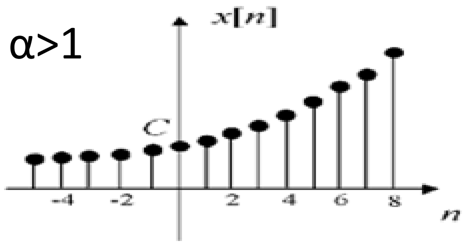
Case 1: Real Exponential Discrete-Time Signals

If (C) and (α) are both real numbers then x[n] is called a real exponential signal.

α is real is almost as if β is real

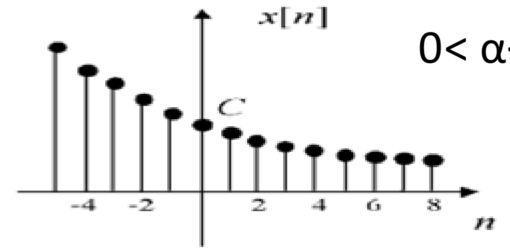
As e is real and real to the power real is real

$\alpha>1$



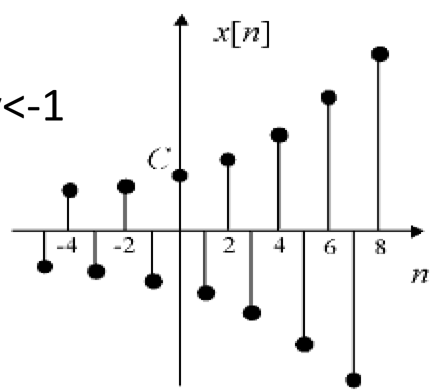
Discrete-time exponential signal growing unbounded with time.

$0<\alpha<1$



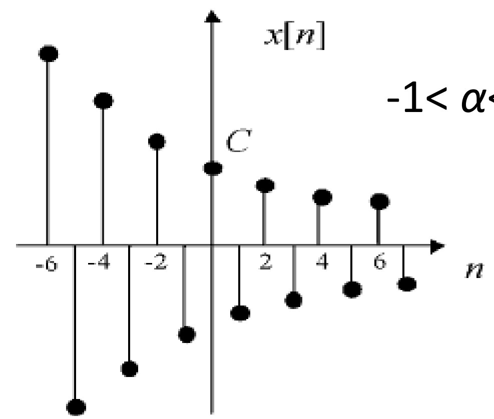
Discrete-time exponential signal tapering off to zero with time.

$\alpha<-1$



Discrete-time exponential signal alternating and growing unbounded with time.

$-1<\alpha<0$



Discrete-time exponential signal alternating and tapering off to zero with time.

➢ Exponential Signals and sinusoidal Signals (discrete-time case)

$$The\ general\ \ form\ \Rightarrow\quad x[n]=C\alpha^{n}=Ce^{\beta n}$$

Case 2: Discrete-Time Complex Exponentials:

If (β) is purely imaginary (jw) and ignoring the scaling factor (C) → $x[n]=e^{jwn}$

$$As\ before,\ according\ to\ Euler's\ Formula$$

$$e^{jwn}=\cos(wn)+j\sin(wn)$$

$$and\qquad A\cos(wn+\phi)=\frac{1}{2}\left\{e^{j(wn+\phi)}+e^{-j(wn+\phi)}\right\}$$

$$=\mathrm{Re}\left\{e^{j(wn+\phi)}\right\}$$

Again the signals $x[n]=e^{jwn}$ and A cos (wn + Ø) have same periodicity properties and parameters BUT THEY ARE NOT NECSSARILY PERIODIC (we will see it soon).

➢ Exponential Signals and sinusoidal Signals (discrete-time case)

$$\text{The general form} \Rightarrow \quad x[n] = C\alpha^n = Ce^{\beta n}$$

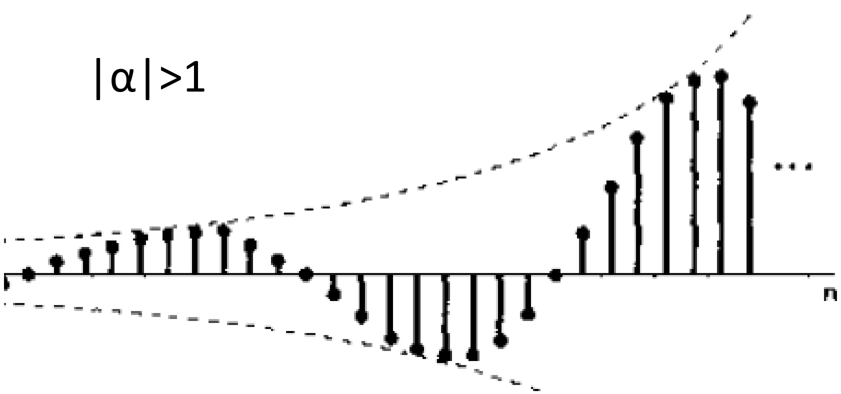
<u>Case 3: General discrete-time complex exponentials:</u>

If (C) and (α) are both complex (in polar form), i.e. let $C=|c|e^{j\theta}$ and $\alpha=|\alpha|e^{jw}$

then $x[n] = |c||\alpha|^n e^{j(wn+\theta)} = |c||\alpha|^n \cos(wn+\theta) + j|c||\alpha|^n \sin(wn+\theta)$

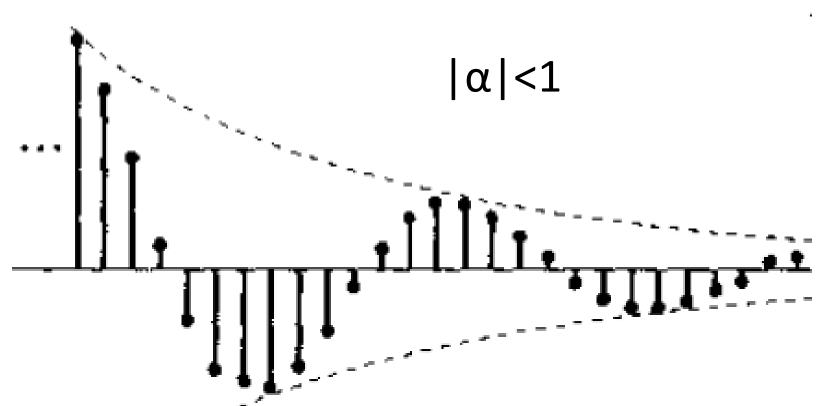
= variable positive gain x Sinusoidal signals

|α|>1



(a)

|α|<1



(b)

➢ Exponential Signals and sinusoidal Signals (discrete-time case)

Periodicity properties of discrete-time complex exponentials:

$$consider\ the\ signal \qquad e^{jwn}$$

$$let \quad w = w_o + 2\pi \quad \Rightarrow$$

$$e^{jwn} = e^{j(w_o+2\pi)n} = e^{jw_o n} e^{j2\pi n}$$

$$2\pi n = integer \quad multiple \quad of \quad 2\pi \quad for\ all\ values\ of\ n \Rightarrow \quad e^{j2\pi n} = 1$$

$$\Rightarrow \quad e^{j(w_o+2\pi)n} = e^{jw_o n}$$

$$i.e. \quad the\ discrete-time\ complex\ exponentials\ seperated\ by\ (2\pi)$$

$$in\ frequency\ are\ identical$$

$$\Rightarrow then\ this\ means\ that\ there\ are\ only\ a\ range$$

$$of \quad 2\pi \quad for \quad w \quad of \quad e^{jwn} \quad to\ have\ distinct\ /\ different\ signals$$

$$commonly \Rightarrow \quad -\pi < w < \pi \quad OR \quad 0 < w < 2\pi$$

$$for\ continuous\ time\ exponential\ \ signals\ this\ is\ not \quad the\ case\ as$$

$$e^{j2\pi t} \neq 1 \quad for\ all\ values\ of\ \ t, \quad as\ \ t\ \ is\ not\ an\ integer$$

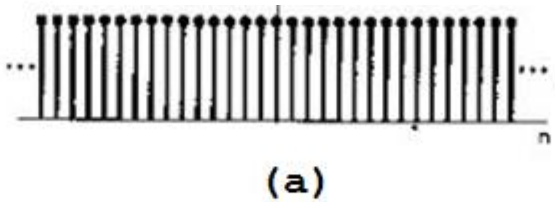
Even when t=integer it does not mean same signals but it means different signals meet at some values of t and they are totally different (one faster than another).

➢ Exponential Signals and sinusoidal Signals (discrete-time case)

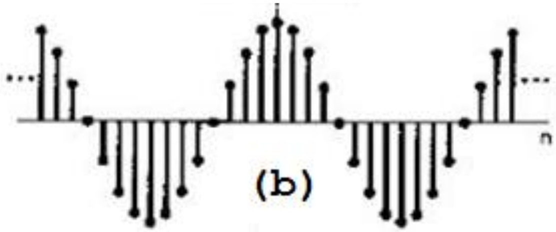
Periodicity properties of discrete-time complex exponentials:

Note that the rate of fluctuation increases then decreases with the increase in w the high frequencies are exist around π and low frequencies exist around 2 π and their multiples

X[n]=cos(0n)

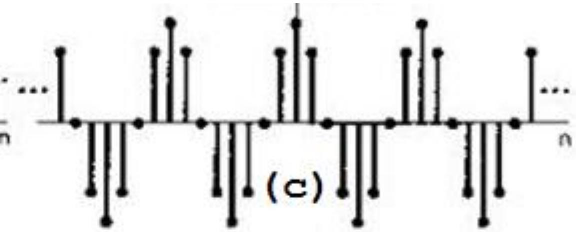


x[n]=cos(πn/8)



w*2 =>

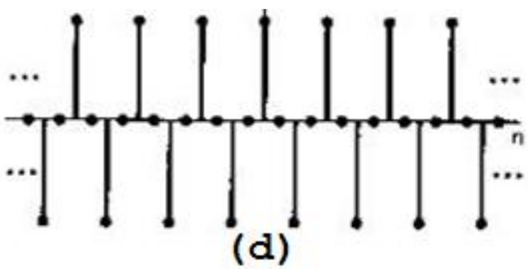
x[n]=cos(πn/4)



• Check when it is repeated (wn=k2π)
i.e. after how many samples

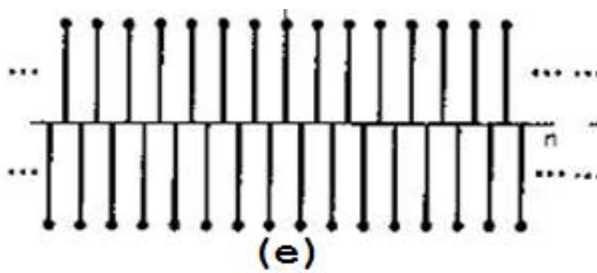
W*2 =>

X[n]=cos(π n/2)

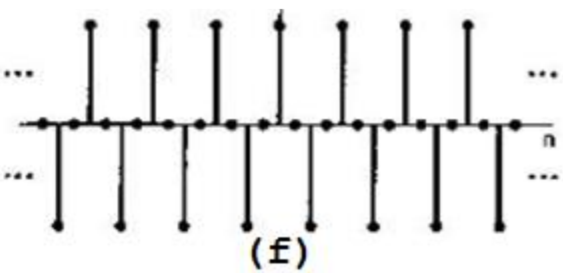


w*2 =>

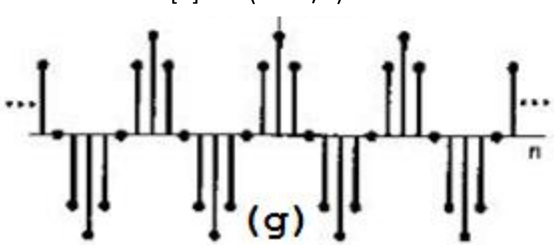
x[n]=cos(πn)



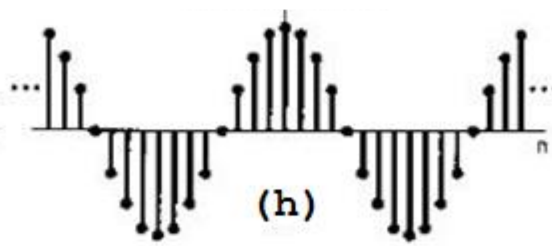
x[n]=cos(3πn/2)



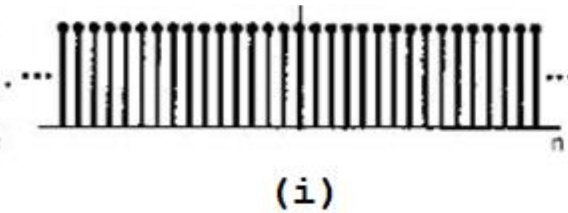
X[n]=cos(7 π n/4)



x[n]=cos(15πn/8)



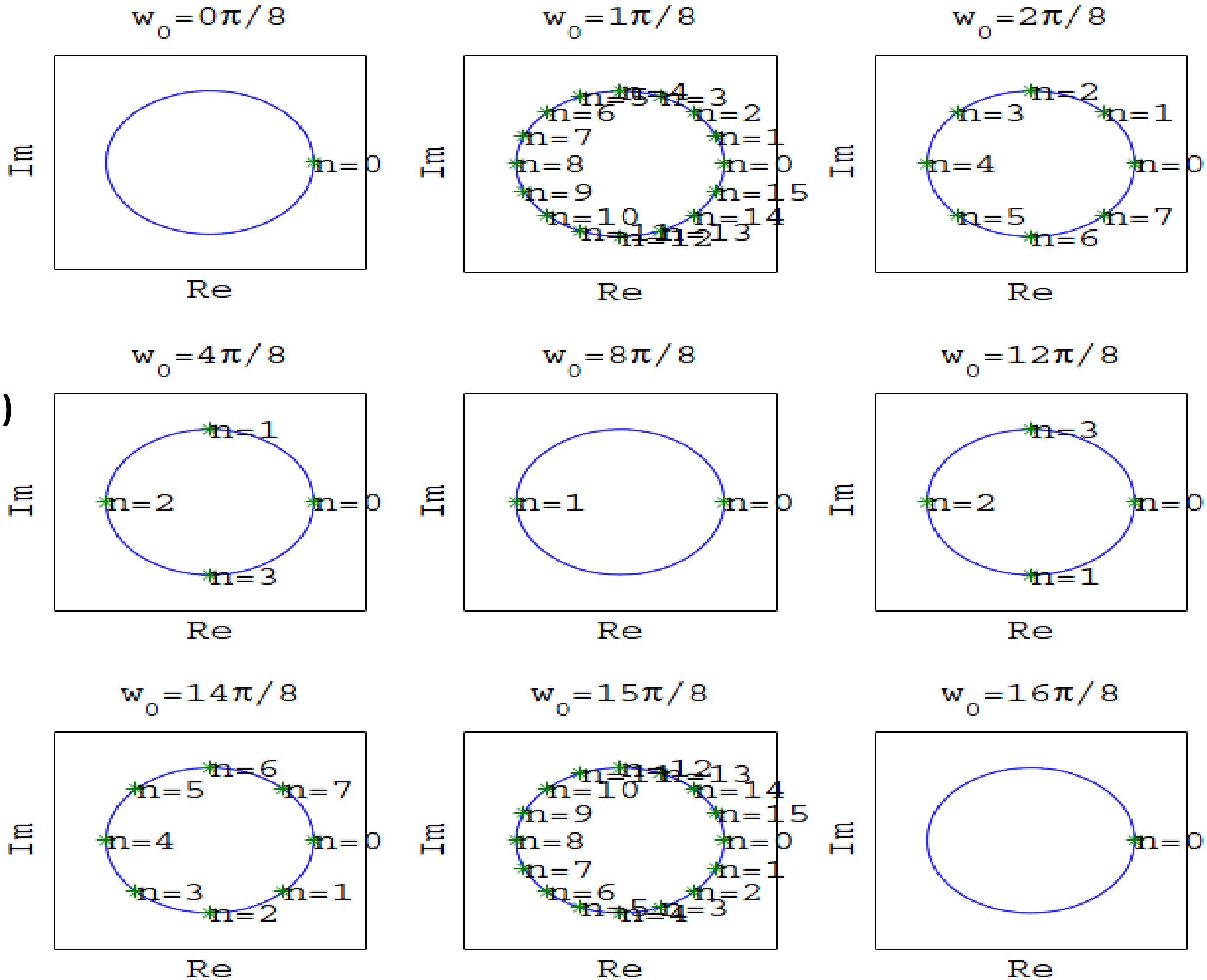
x[n]=cos(2πn)



➢ Exponential Signals and sinusoidal Signals (discrete-time case)

Periodicity properties of discrete-time complex exponentials:

Number of Samples per cycle
=
the number (n) that makes $n\, W_o = k2\pi$



➢ Exponential Signals and sinusoidal Signals (discrete-time case)

Periodicity properties of discrete-time complex exponentials:

$$The\ function \quad e^{jwn} \quad to\ be\ periodic$$

$$\therefore e^{jw_o(n+N)} = e^{jw_o n} e^{jw_o N} = e^{jw_o n}$$

$$e^{jw_o N} = 1 \quad \Rightarrow \quad w_o N = m2\pi \ ; \quad for\ \underline{\underline{m}}\ is\ an\ integer$$

$$\therefore the\ condition\ of\ e^{jw_o n}\ to\ be\ periodic\ is\ \left.\frac{w_o}{2\pi} = \frac{m}{N}\right\}\ to\ be\ simplified$$

$$rational\ number\{\ simplified \equiv no\ common\ factors\ between\ \underline{\underline{m}}\ and\ \underline{\underline{N}}\}$$

$$if\ this\ happened\ the\ signal\ e^{jw_o n}\ will\ be\ periodic\ and\ the\ fundamental$$

$$period\ will\ be\ (N),\ otherwise\ it\ will\ be\ not\ periodic$$

$$the\ same\ is\ true\ for\ discrete-time\ sinusoids$$

if you have a signal that is composed of a combination of discrete − time complex exponentials or sinusoidals then you should check every subsignal individually and if you find them ALL periodic with fundamentals {N1,N2,N3,...} then the container signal will be periodic with fundamental period N = LCM{N1,N2,N3,...},
LCM = least common multiplier.

➢ Exponential Signals and sinusoidal Signals

The difference between the continuous-time complex exponential e^{jwt} **and the discrete-time complex exponential** e^{jwn}

e^{jwt}	e^{jwn}
Distinct signals for distinct (w)	Identical signals separated in frequency (w) by 2π
Periodic for any choice of (w)	Periodic only if $\frac{w}{2\pi} = \frac{m}{N}$ is a rational number
Fundamental frequency $w = \frac{2\pi}{T_o}$	Fundamental frequency $\frac{2\pi}{N} = \frac{w}{m}$
Fundamental Period: w=0 → undefined w≠0 → $T_o = \frac{2\pi}{w}$	Fundamental Period: w=0 → N=1 w≠0 → $N = \frac{2\pi}{w} m$