Signals and Systems

Lecture # 16

Fourier Series

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Topics of the lecture:

- Introduction
- Complex Exponentials as Eigenfunctions for LTI Systems.
- Fourier Series Representation for Periodic Signals.
- Determination of Fourier Series Representation for Continuous-Time Periodic Signals.
- Examples .
- CONVERGENCE OF THE FOURIER SERIES
- PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES
- FOURIER SERIES REPRESENTATION OF DISCRETE-TIME PERIODIC SIGNALS
- Determination of Fourier Series Representation of a Periodic Signal.
- PROPERTIES OF DISCRETE-TIME FOURIER SERIES

➢ Introduction

The basic idea behind the derivation of convolution formula for LTI systems is to represent the signal in terms of a set of basic signals that was the shifted impulses as we described in the recent few lectures.

In this topic we continue to make another representation of signals in terms of another set of basic signals that is the complex exponentials and the resulted representation called Fourier Series/Transform.

So what is the basic signal?

The basic signal is the signal that has the following two properties:

1- It can be used to represent a broad range of signals.

2- Its response when it is applied to a system is easy to be calculated.

➢ The response of LTI systems to Complex Exponentials

For a LTI system has h(t) as its impulse response, if the input is x(t) then the output of this system y(t) can be calculated from:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

If the input x(t) = e^{st}

$$\therefore \quad y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$$

Expressing $e^{s(t-\tau)}$ as $e^{st}\,e^{-s\tau}$ and noting that e^{st} can be moved outside the integral

$$\therefore \quad y(t) = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

That can be re-written as:

$$y(t) = e^{st}H(s)$$

$$where \quad H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

This means that the output of the LTI system to e^{st} is the same e^{st} but scaled by H(s).

So the signal e^{st} is called eigenfunction and H(s) is called eigenvalue.

➢ The response of LTI systems to Complex Exponentials

Is the complex exponentials are eigenfunctions too to discrete-time systems?
For a LTI system has h[n] as its impulse response, if the input is x[n] then the output of this system y[n] can be calculated from:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

If the input x[n] = z^n
Z is complex number

$$\therefore \quad y[n] = \sum_{k=-\infty}^{\infty} h[k].z^{n-k}$$

Expressing z^{n-k} as $z^n\, z^{-k}$ and noting that z^n can be moved outside the summation

$$\therefore \quad y[n] = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

That can be re-written as: $y[n] = z^n H(z)$ $\quad$ *Note : z is a complex not integer so we used ( ) instead of [ ]*

$$where \quad H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

i.e the output of Discrete-Time LTI system to z^n is the same z^n but scaled by H(z).

So the signal z^n is eigenfunction and H(z) is eigenvalue for discrete-time LTI systems.

➢ The response of LTI systems to Complex Exponentials

So, if the input x(t) is given by: $$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

From the eigenfunction property, the individual responses of each term are:

$$a_1 e^{s_1 t} \rightarrow a_1 H(s_1) e^{s_1 t}$$
$$a_2 e^{s_2 t} \rightarrow a_2 H(s_2) e^{s_2 t}$$
$$a_3 e^{s_3 t} \rightarrow a_3 H(s_3) e^{s_3 t}$$

And from the superposition property, the response to the sum is the sum of responses:

$$\therefore \quad y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

Then in general if the input can be represented as: $$x(t) = \sum_k a_k e^{s_k t}$$

Then the system output will be: $\longrightarrow$ $$y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

The same thing is applied to discrete-time case: if $$x[n] = \sum_k a_k z_k^n$$

Then the system output will be: $\longrightarrow$ $$y[n] = \sum_k a_k H(z_k) z_k^n$$

➢ The response of LTI systems to Complex Exponentials

Example I: $$y(t) = x(t-3)$$

If the input is in the form of complex exponential like $x(t) = a.e^{st} = e^{j2t}$

$$y(t) = a.H(s).e^{st}$$

$$\therefore \quad y(t) = H(j2)e^{j2t} \qquad as\ s = 2j \quad and \quad a = 1$$

The impulse response is the output when the input equal $\delta(t)$ $$\therefore \quad h(t) = \delta(t-3)$$

Then the eigenvalue is: $$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} \delta(\tau-3)e^{-s\tau}d\tau = e^{-3s}$$

Substituting in y(t): $$\therefore y(t) = e^{-6j}e^{j2t} = e^{j2(t-3)} \quad \textit{which can be derived directly from the system equation}$$

Example II: **if the input becomes** $x(t) = \cos(4t) + \cos(7t)$

$$\therefore \quad x(t) = \frac{1}{2}\left\{e^{j4t} + e^{-j4t}\right\} + \frac{1}{2}\left\{e^{j7t} + e^{-j7t}\right\}$$

$$\therefore \quad y(t) = \frac{1}{2}H(4j)e^{4jt} + \frac{1}{2}H(-4j)e^{-4jt} + \frac{1}{2}H(7j)e^{7jt} + \frac{1}{2}H(-7j)e^{-7jt}$$

$$\therefore \quad y(t) = \frac{1}{2}e^{-12j}e^{4jt} + \frac{1}{2}e^{12j}e^{-4jt} + \frac{1}{2}e^{-21j}e^{7jt} + \frac{1}{2}e^{21j}e^{-7jt}$$

$$\therefore \quad y(t) = \frac{1}{2}\left\{e^{j4(t-3)}\right\} + \frac{1}{2}\left\{e^{-4j(t-3)}\right\} + \frac{1}{2}\left\{e^{7j(t-3)}\right\} + \frac{1}{2}\left\{e^{-7j(t-3)}\right\}$$

$$\therefore \quad y(t) = \cos(4(t-3)) + \cos(7(t-3)) \quad \textit{which can be derived directly from the system equation}$$

➢ Fourier Series Representation for continuous-time periodic signals

Recall: Periodic signals have the property: $x(t) = x(t+T) \quad \text{for all } t \text{ and positive } T$

The smallest value of T that satisfy the above equation called the fundamental period T_o and $w_o=2\pi/T_o$ is the fundamental frequency.

Remember: $x(t) = \cos(w_o t)$ **and** $x(t) = e^{jw_o t}$

Both are periodic signals with fundamental frequency w_o and fundamental period $T_o=2\pi/w_o$

Recall the set of harmonically related complex exponentials:

$$\phi_k(t) = e^{jkw_o t} = e^{jk\frac{2\pi}{T}t} \quad ; k = 0, \pm 1, \pm 2, \ldots$$

Each one of these signals is periodic and has a fundamental frequency that is multiple of w_o and therefore has a fundamental period $T_o/|k|=2\pi/|kw_o|$ that is a fraction of $T_o=2\pi/w_o$ To

Fourier claims that any periodic signal can be represented as a linear combination of these harmonically related complex exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_o t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

In this case x(t) is also periodic with fundamental frequency w_o and fundamental period $T_o=2\pi/w_o$
The term of k=0 is constant a_k which is called the dc-component of the signal x(t).
The terms of k=1 and k=-1 are called collectively as the first harmonic (fundamental) components.
The terms of k=2 and k=-2 are called collectively as the second harmonic (fundamental) components.
and so on…The terms of k and -k are called collectively as the kth harmonic components of x(t).

➢ Fourier Series Representation for continuous-time periodic signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_o t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \quad \textit{called Fourier Seiers Representation}$$

If x(t) is real then: $x(t) = x^*(t) \quad \therefore \quad x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jkw_o t}$

Replacing k with -k : $\therefore \quad x^*(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jkw_o t}$

Then for real signals: $\therefore \quad a_{-k}^* = a_k \quad or \quad a_k^* = a_{-k}$

x(t) can be re-written as: $x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jkw_o t} + a_{-k} e^{-jkw_o t}]$

$$\therefore \quad x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jkw_o t} + a_k^* e^{-jkw_o t}] = a_0 + \sum_{k=1}^{\infty} 2Re\{a_k e^{jkw_o t}\}$$

If a_k is represented in its polar form $a_k = A_k e^{j\theta_k}$ **then:** $x(t) = a_0 + \sum_{k=1}^{\infty} 2Re\{A_k e^{j(kw_o t + \theta_k)}\}$

$$\therefore \quad x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(kw_o t + \theta_k) \quad \rightarrow \quad (I)$$

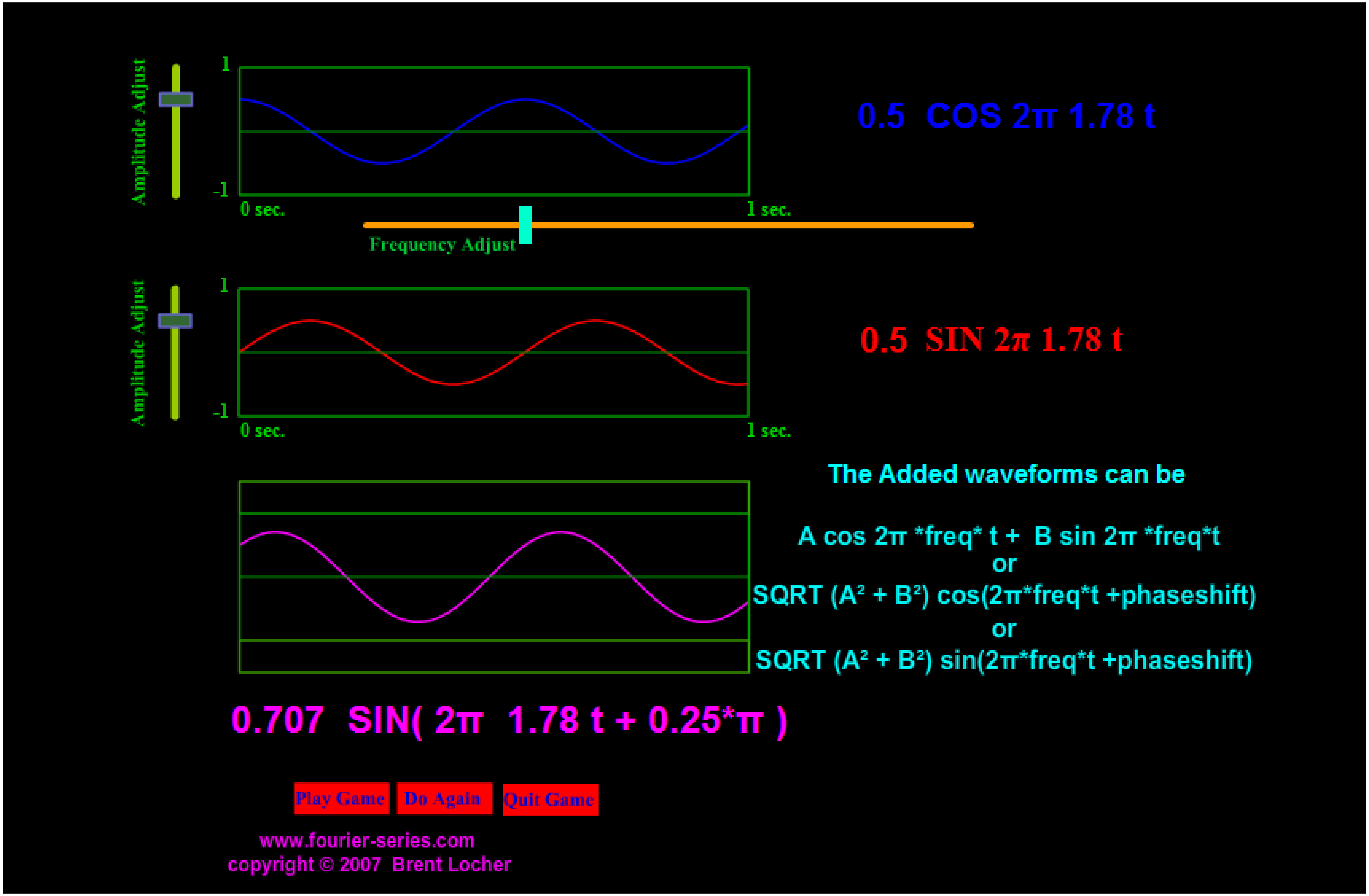
If a_k is represented in its rectangular form $a_k = B_k + jC_k$ then:

$$\therefore \quad x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k [B_k \cos(kw_o t) - C_k \sin(kw_o t)] \quad \rightarrow \quad (II)$$

Equations (I) and (II) are another Fourier Series representations for real signals in terms of sinusoidal.

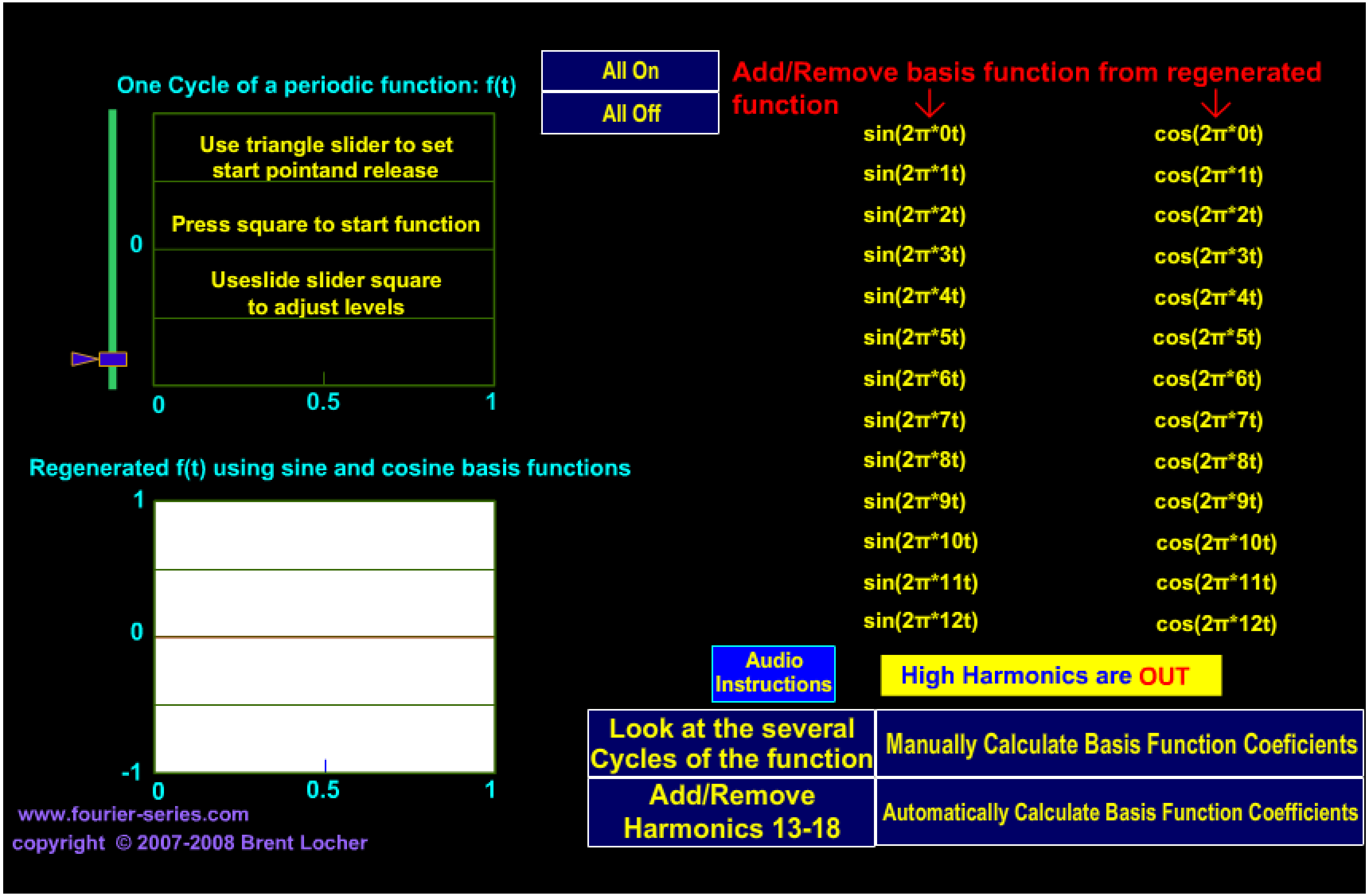
➢ Fourier Series Representation for continuous-time periodic signals

Example to show the idea of Fourier



➢ Fourier Series Representation for continuous-time periodic signals

Example to show the idea of Fourier



➢ Determination of Fourier Series Representation for Continuous-Time Periodic Signals

How can we compute a_k **in** $\longrightarrow$ $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jkw_o t}$ **?**

Multiply both sides by $e^{-jnw_o t}$ **then:** $\therefore \quad x(t)e^{-jnw_o t}=\sum_{k=-\infty}^{\infty} a_k e^{jkw_o t} e^{-jnw_o t}$

Integrating both sides from 0 to T (fundamental period of x(t)):

$$\therefore \int_0^T x(t)e^{-jnw_o t}dt=\int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jkw_o t} e^{-jnw_o t}dt$$

Interchange Integration with summation: $\therefore \int_0^T x(t)e^{-jnw_o t}dt=\sum_{k=-\infty}^{\infty} a_k \left\{\int_0^T e^{j(k-n)w_o t}dt\right\}$

$$\because \quad \int_0^T e^{j(k-n)w_o t}dt=\int_0^T \cos\big((k-n)w_o t\big)\,dt+j\int_0^T \sin\big((k-n)w_o t\big)\,dt$$

As sin(0)=0 and cos(0)=1 and as the area under the curve of one period of sin and cos = 0

$$\therefore \quad \int_0^T e^{j(k-n)w_o t}dt=\begin{cases} T & ; \quad k=n \\ 0 & ; \quad k\neq n \end{cases}$$

$$\therefore \int_0^T x(t)e^{-jnw_o t}dt=a_n T \qquad or \qquad a_n=\frac{1}{T}\int_0^T x(t)e^{-jnw_o t}dt$$

As we can choose any period (interval T)
then a_n **can written as:**

$$\therefore \quad a_k=\frac{1}{T}\int_T x(t)e^{-jkw_o t}dt \quad \textit{called Fourier Seieris Coefficients}$$

➢ Determination of Fourier Series Representation for Continuous-Time Periodic Signals

To summarize, if $x(t)$ has a Fourier series representation [i.e., if it can be expressed as a linear combination of harmonically related complex exponentials in the form of eq. , then the coefficients are given by eq. This pair of equations, then, defines the Fourier series of a periodic continuous-time signal:

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \\ a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt. \end{aligned}}$$

Here, we have written equivalent expressions for the Fourier series in terms of the fundamnetal frequency ω_0 and the fundamental period T. Equation is referred to as the *synthesis* equation and eq. as the *analysis* equation. The set of coefficients $\{a_k\}$ are often called the *Fourier series coefficients* or the *spectral coefficients* of $x(t)$. These complex coefficients measure the portion of the signal $x(t)$ that is at each harmonic of the fundamental component. The coefficient a_0 is the dc or constant component of $x(t)$ and is given by eq. with $k = 0$. That is,

$$a_0 = \frac{1}{T}\int_T x(t)\,dt,$$

which is simply the average value of $x(t)$ over one period.

Equations were known to both Euler and Lagrange in the middle of the 18th century. However, they discarded this line of analysis without having examined the question of how large a class of periodic signals could, in fact, be represented in such a fashion.

Example 3

Consider the signal $$x(t) = \sin \omega_0 t,$$

whose fundamental frequency is ω_0. One approach to determining the Fourier series coefficients for this signal is to apply eq. (3.39). For this simple case, however, it is easier to expand the sinusoidal signal as a linear combination of complex exponentials and identify the Fourier series coefficients by inspection. Specifically, we can express $\sin \omega_0 t$ as

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}.$$

Comparing the right-hand sides of this equation and eq. , we obtain

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$

$$a_k = 0, \qquad k \neq +1 \text{ or } -1.$$

Example 4

Let

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has fundamental frequency ω_0. As with Example 3, we can again expand $x(t)$ directly in terms of complex exponentials, so that

$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}].$$

Collecting terms, we obtain

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

Thus, the Fourier series coefficients for this example are

$$a_0 = 1,$$

$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,$$

$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j,$$

$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j),$$

$$a_k = 0, \quad |k| > 2.$$

In Figure , we show a bar graph of the magnitude and phase of a_k.

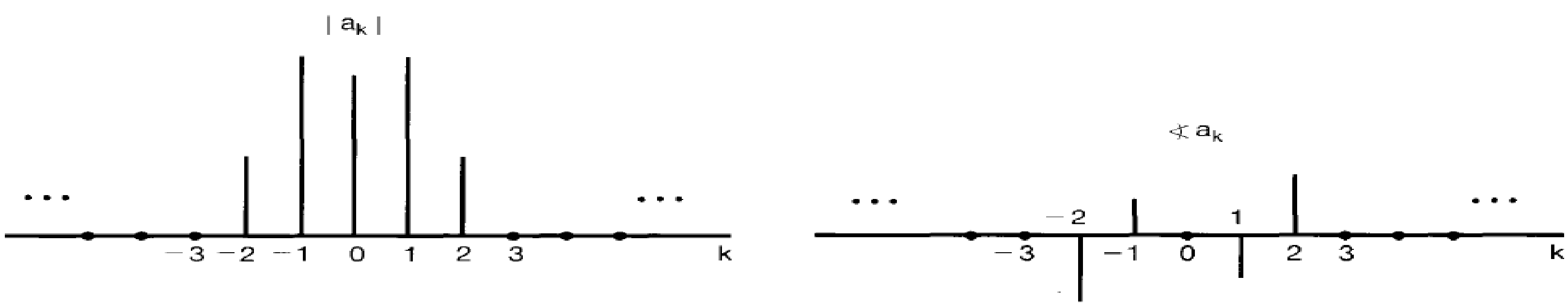


Figure Plots of the magnitude and phase of the Fourier coefficients of the signal considered in Example 4.

➢ Examples .

Example 5

The periodic square wave, sketched in Figure and defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases},$$

is a signal that we will encounter a number of times throughout this book. This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

To determine the Fourier series coefficients for $x(t)$, we use eq. Because of the symmetry of $x(t)$ about $t = 0$, it is convenient to choose $-T/2 \le t < T/2$ as the

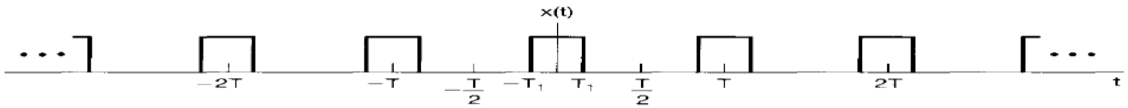


Figure Periodic square wave.

equally valid and thus will lead to the same result. Using these limits of integration and substituting from eq. we have first, for $k = 0$,

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$$

As mentioned previously, a_0 is interpreted to be the average value of $x(t)$, which in this case equals the fraction of each period during which $x(t) = 1$. For $k \neq 0$, we obtain

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1},$$

which we may rewrite as

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right].$$

Noting that the term in brackets is $\sin k\omega_0 T_1$, we can express the coefficients a_k as

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \qquad k \neq 0,$$

where we have used the fact that $\omega_0 T = 2\pi$.

Figure is a bar graph of the Fourier series coefficients for this example. In particular, the coefficients are plotted for a fixed value of T_1 and several values of T. For this specific example, the Fourier coefficients are real, and consequently, they can be depicted graphically with only a single graph. More generally, of course, the Fourier coefficients are complex, so that two graphs, corresponding to the real and imaginary parts, or magnitude and phase, of each coefficient, would be required. For $T = 4T_1$, $x(t)$ is a square wave that is unity for half the period and zero for half the period. In this case, $\omega_0 T_1 = \pi/2$, and from eq.

$$a_k = \frac{\sin(\pi k/2)}{k\pi}, \qquad k \neq 0,$$

while

$$a_0 = \frac{1}{2}.$$

From eq. $a_k = 0$ for k even and nonzero. Also, $\sin(\pi k/2)$ alternates between ± 1 for successive odd values of k. Therefore,

$$a_1 = a_{-1} = \frac{1}{\pi},$$

$$a_3 = a_{-3} = -\frac{1}{3\pi},$$

$$a_5 = a_{-5} = \frac{1}{5\pi},$$

$$\vdots$$

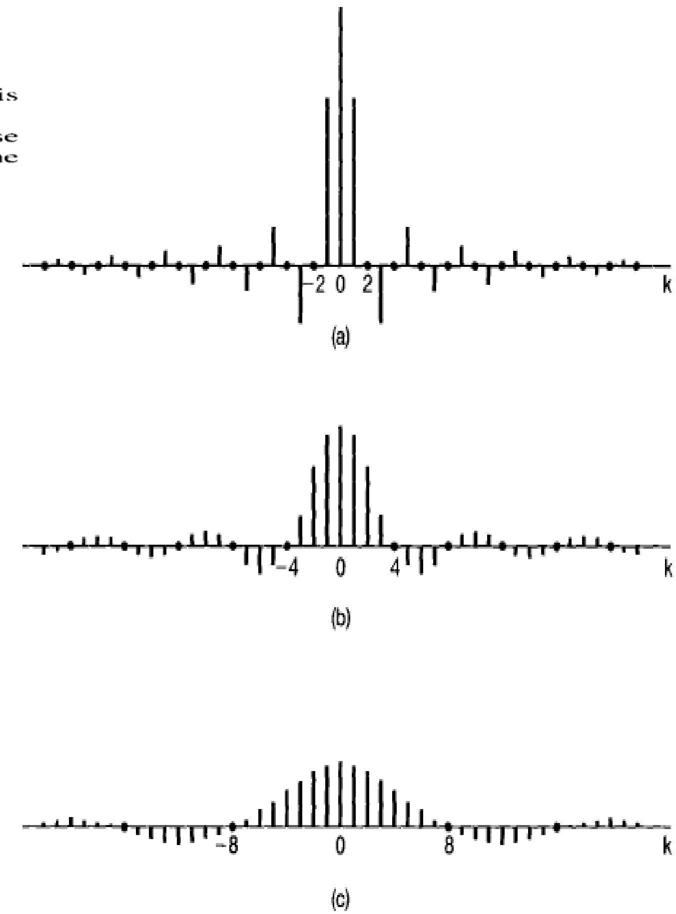


Figure Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (*a*) $T = 4T_1$; (*b*) $T = 8T_1$; (*c*) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

➢CONVERGENCE OF THE FOURIER SERIES

Condition 1. Over any period, $x(t)$ must be *absolutely integrable*; that is,

$$\int_T |x(t)|\, dt < \infty.$$

As with square integrability, this guarantees that each coefficient a_k will be finite, since

$$|a_k| \leq \frac{1}{T}\int_T |x(t)e^{-jk\omega_0 t}|\, dt = \frac{1}{T}\int_T |x(t)|\, dt.$$

So if

$$\int_T |x(t)|\, dt < \infty,$$

then

$$|a_k| < \infty.$$

A periodic signal that violates the first Dirichlet condition is

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1;$$

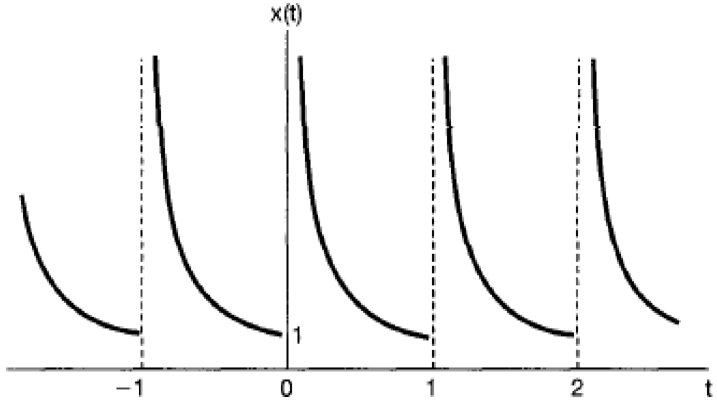
Condition 2. In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

An example of a function that meets Condition 1 but not Condition 2 is

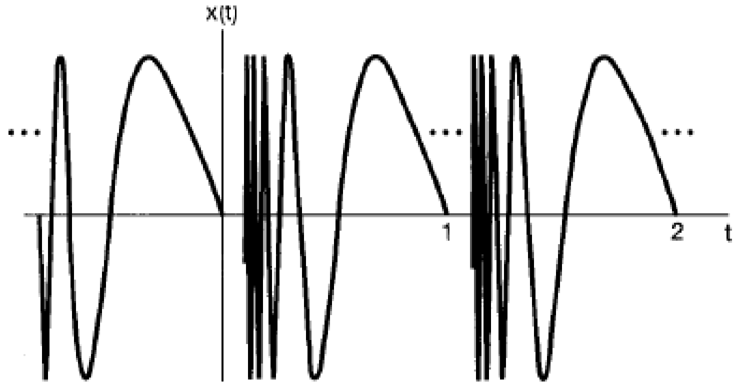
$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1,$$

Condition 3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

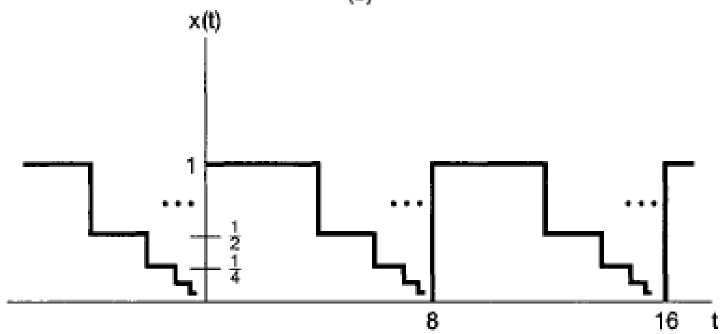
An example of a function that violates Condition 3 is illustrated in Figure (c). The



(a)

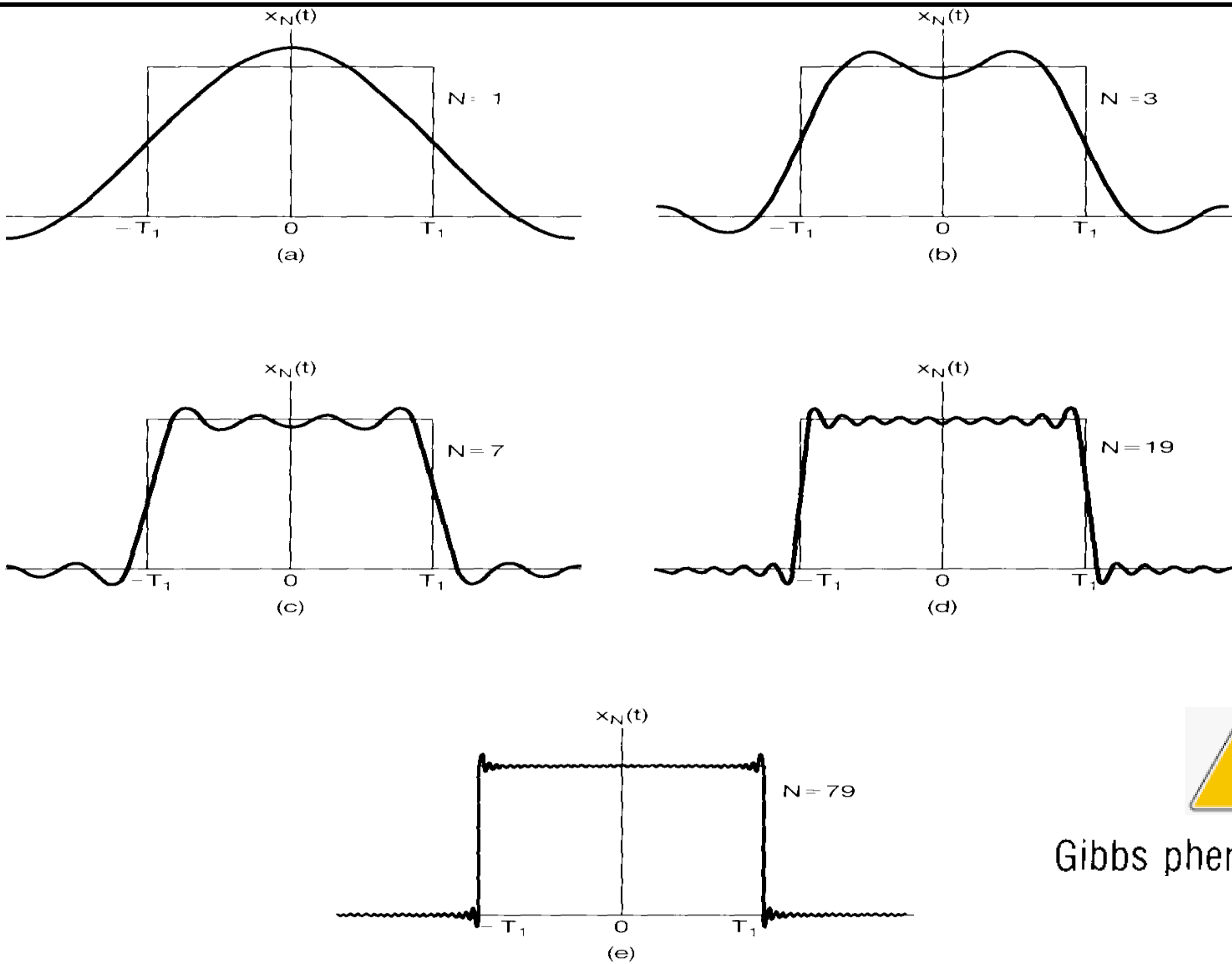


(b)



(c)

➢CONVERGENCE OF THE FOURIER SERIES



Gibbs phenomenon

Figure Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

➢PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

TABLE PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	$\begin{array}{l} a_k \\ b_k \end{array}$
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\begin{array}{l} \mathcal{Re}\{a_k\} \\ j\mathcal{Im}\{a_k\} \end{array}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

➢PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Example 6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure . We could determine the Fourier series representation of $g(t)$ directly from the analysis equation . Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2.$$

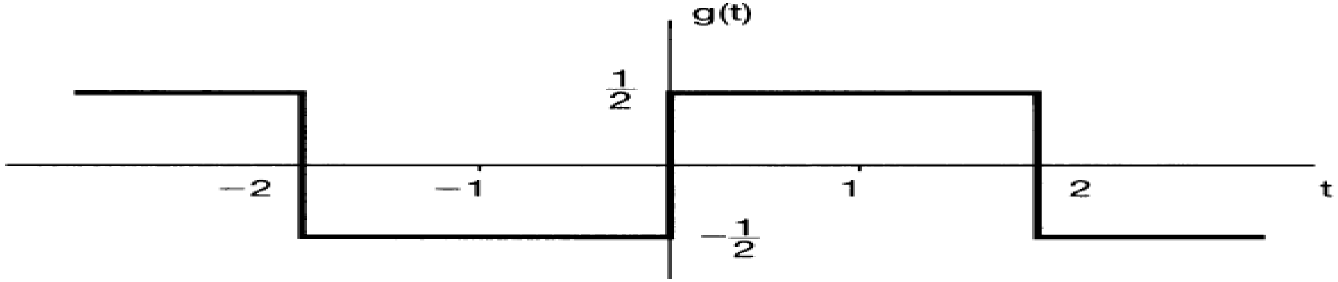


Figure Periodic signal for Example 6.

The time-shift property in Table indicates that, if the Fourier Series coefficients of $x(t)$ are denoted by a_k, the Fourier coefficients of $x(t-1)$ may be expressed as

$$b_k = a_k e^{-jk\pi/2}.$$

The Fourier coefficients of the *dc offset* in $g(t)$—i.e., the term $-1/2$ on the right-hand side of eq. —are given by

$$c_k = \begin{cases} 0, & \text{for } k \neq 0 \\ -\frac{1}{2}, & \text{for } k = 0 \end{cases}.$$

Applying the linearity property in Table , we conclude that the coefficients for $g(t)$ may be expressed as

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - \frac{1}{2}, & \text{for } k = 0 \end{cases},$$

where each a_k may now be replaced by the corresponding expression from eqs. yielding

$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}.$$

Example 7

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$ shown in Figure . The derivative of this signal is the signal $g(t)$ in Exam-

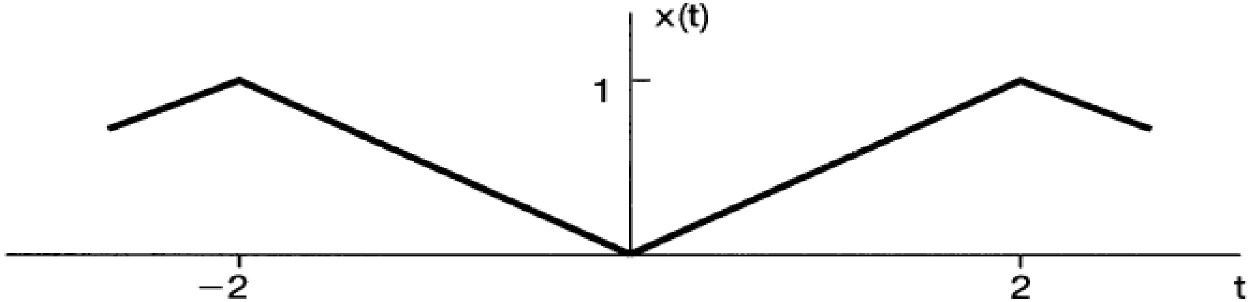


Figure Triangular wave signal in Example 7.

ple 6. Denoting the Fourier coefficients of $g(t)$ by d_k and those of $x(t)$ by e_k, we see that the differentiation property in Table indicates that

$$d_k = jk(\pi/2)e_k.$$

This equation can be used to express e_k in terms of d_k, except when $k = 0$. Specifically, from eq.

$$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(\pi k/2)}{j(k\pi)^2}e^{-jk\pi/2}, \quad k \neq 0.$$

For $k = 0$, e_0 can be determined by finding the area under one period of $x(t)$ and dividing by the length of the period:

$$e_0 = \frac{1}{2}.$$

➢ FOURIER SERIES REPRESENTATION OF DISCRETE-TIME PERIODIC SIGNALS

In this section, we consider the Fourier series representation of discrete-time periodic signals. While the discussion closely parallels , there are some important differences. In particular, the Fourier series representation of a discrete-time periodic signal is a *finite* series, as opposed to the infinite series representation required for continuous-time periodic signals. As a consequence, there are no mathematical issues of convergence such as those discussed

As defined in Chapter 1, a discrete-time signal $x[n]$ is periodic with period N if

$$x[n] = x[n + N].$$

The fundamental period is the smallest positive integer N for which eq. holds, and $\omega_0 = 2\pi/N$ is the fundamental frequency. For example, the complex exponential $e^{j(2\pi/N)n}$ is periodic with period N. Furthermore, the set of all discrete-time complex exponential signals that are periodic with period N is given by

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \; k = 0, \pm 1, \pm 2, \ldots.$$

All of these signals have fundamental frequencies that are multiples of $2\pi/N$ and thus are harmonically related.

As mentioned there are only N distinct signals in the set given by eq.

in general, $\quad \phi_k[n] = \phi_{k+rN}[n].$

This differs from the situation in continuous time in which the signals $\phi_k(t)$ defined in eq. are all different from one another.

We now wish to consider the representation of more general periodic sequences

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n}.$$

Since the sequences $\phi_k[n]$ are distinct only over a range of N successive values of k, the summation in eq. need only include terms over this range. Thus, the summation is on k, as k varies over a range of N successive integers, beginning with any value of k. We indicate this by expressing the limits of the summation as $k = \langle N \rangle$. That is,

$$x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}.$$

➢ Determination of Fourier Series Representation of a Periodic Signal

Suppose now that we are given a sequence $x[n]$ that is periodic with fundamental period N. We would like to determine whether a representation of $x[n]$ in the form of eq. exists and, if so, what the values of the coefficients a_k are. This question can be phrased in terms of finding a solution to a set of linear equations. Specifically, if we evaluate eq. for N successive values of n corresponding to one period of $x[n]$, we obtain

$$x[0] = \sum_{k=\langle N\rangle} a_k, \quad x[1] = \sum_{k=\langle N\rangle} a_k e^{j2\pi k/N}, \quad \vdots \quad x[N-1] = \sum_{k=\langle N\rangle} a_k e^{j2\pi k(N-1)/N}$$

The basis for this result is the fact, that

$$\sum_{n=\langle N\rangle} e^{jk(2\pi/N)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}.$$

Now consider the Fourier series representation of eq. Multiplying both sides by $e^{-jr(2\pi/N)n}$ and summing over N terms, we obtain

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-r)(2\pi/N)n}.$$

Interchanging the order of summation on the right-hand side, we have

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n}.$$

From the identity in eq. , the innermost sum on n on the right-hand side of eq. is zero, unless $k-r$ is zero or an integer multiple of N. Therefore, if we choose values for r over the same range as that over which k varies in the outer summation, the innermost sum on the right-hand side of eq. equals N if $k = r$ and 0 if $k \neq r$. The right-hand side of eq. then reduces to Na_r, and we have

$$a_r = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n}.$$

This provides a closed-form expression for obtaining the Fourier series coefficients, and we have the *discrete-time Fourier series pair*:

$$\boxed{\begin{aligned} x[n] &= \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}, \\ a_k &= \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}. \end{aligned}}$$

➢ Examples .

Example 10

Consider the signal

$$x[n] = \sin \omega_0 n,$$

which is the discrete-time counterpart of the signal $x(t) = \sin \omega_0 t$ of Example 3. $x[n]$ is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers. For the case when $2\pi/\omega_0$ is an integer N, that is, when

$$\omega_0 = \frac{2\pi}{N},$$

$x[n]$ is periodic with fundamental period N, and we obtain a result that is exactly analogous to the continuous-time case. Expanding the signal as a sum of two complex exponentials, we get

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}.$$

Comparing eq. , we see by inspection that

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j},$$

and the remaining coefficients over the interval of summation are zero. As described previously, these coefficients repeat with period N; thus, a_{N+1} is also equal to $(1/2j)$ and a_{N-1} equals $(-1/2j)$. The Fourier series coefficients for this example with $N = 5$ are illustrated in Figure . The fact that they repeat periodically is indicated. However, only one period is utilized in the synthesis equation

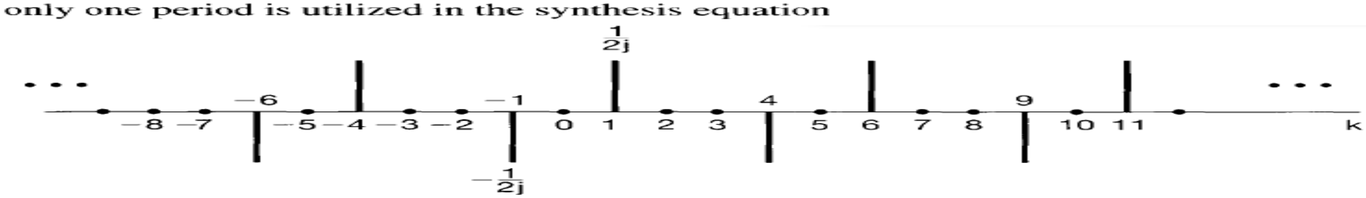


Figure 3.13 Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Consider now the case when $2\pi/\omega_0$ is a ratio of integers—that is, when

$$\omega_0 = \frac{2\pi M}{N}.$$

Assuming that M and N do not have any common factors, $x[n]$ has a fundamental period of N. Again expanding $x[n]$ as a sum of two complex exponentials, we have

$$x[n] = \frac{1}{2j} e^{jM(2\pi/N)n} - \frac{1}{2j} e^{-jM(2\pi/N)n},$$

from which we can determine by inspection that $a_M = (1/2j)$, $a_{-M} = (-1/2j)$, and the remaining coefficients over one period of length N are zero. The Fourier coefficients for this example with $M = 3$ and $N = 5$ are depicted in Figure . Again, we have indicated the periodicity of the coefficients. For example, for $N = 5$, $a_2 = a_{-3}$, which in our example equals $(-1/2j)$. Note, however, that over any period of length 5 there are only two nonzero Fourier coefficients, and therefore there are only two nonzero terms in the synthesis equation.

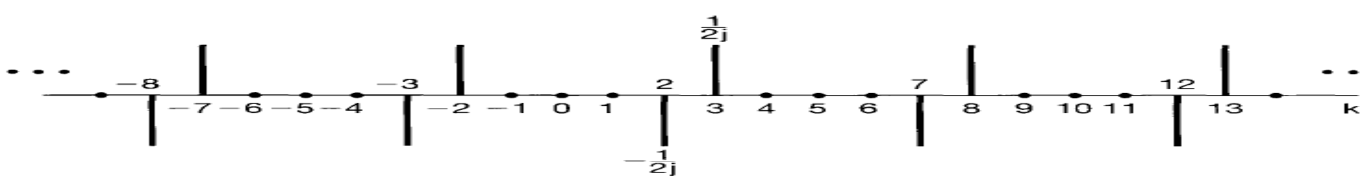


Figure Fourier coefficients for $x[n] = \sin 3(2\pi/5)n$.

➢ **Examples .**

Example 11

Consider the signal $x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right).$

This signal is periodic with period N, and, as in Example 10, we can expand $x[n]$ directly in terms of complex exponentials to obtain

$$x[n] = 1 + \frac{1}{2j}[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2}[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}] + \frac{1}{2}[e^{j(4\pi n/N+\pi/2)} + e^{-j(4\pi n/N+\pi/2)}].$$

Collecting terms, we find that

$$x[n] = 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j(2\pi/N)n} + \left(\frac{3}{2} - \frac{1}{2j}\right)e^{-j(2\pi/N)n} + \left(\frac{1}{2}e^{j\pi/2}\right)e^{j2(2\pi/N)n} + \left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2(2\pi/N)n}.$$

Thus the Fourier series coefficients for this example are

$$a_0 = 1, \qquad a_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{1}{2}j, \qquad a_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j,$$

$$a_2 = \frac{1}{2}j, \qquad a_{-2} = -\frac{1}{2}j,$$

with $a_k = 0$ for other values of k in the interval of summation in the synthesis equation . Again, the Fourier coefficients are periodic with period N, so, for example, $a_N = 1$, $a_{3N-1} = \frac{3}{2} + \frac{1}{2}j$, and $a_{2-N} = \frac{1}{2}j$. In Figure (a) we have plotted the real and imaginary parts of these coefficients for $N = 10$, while the magnitude and phase of the coefficients are depicted in Figure (b).

➢ Examples .

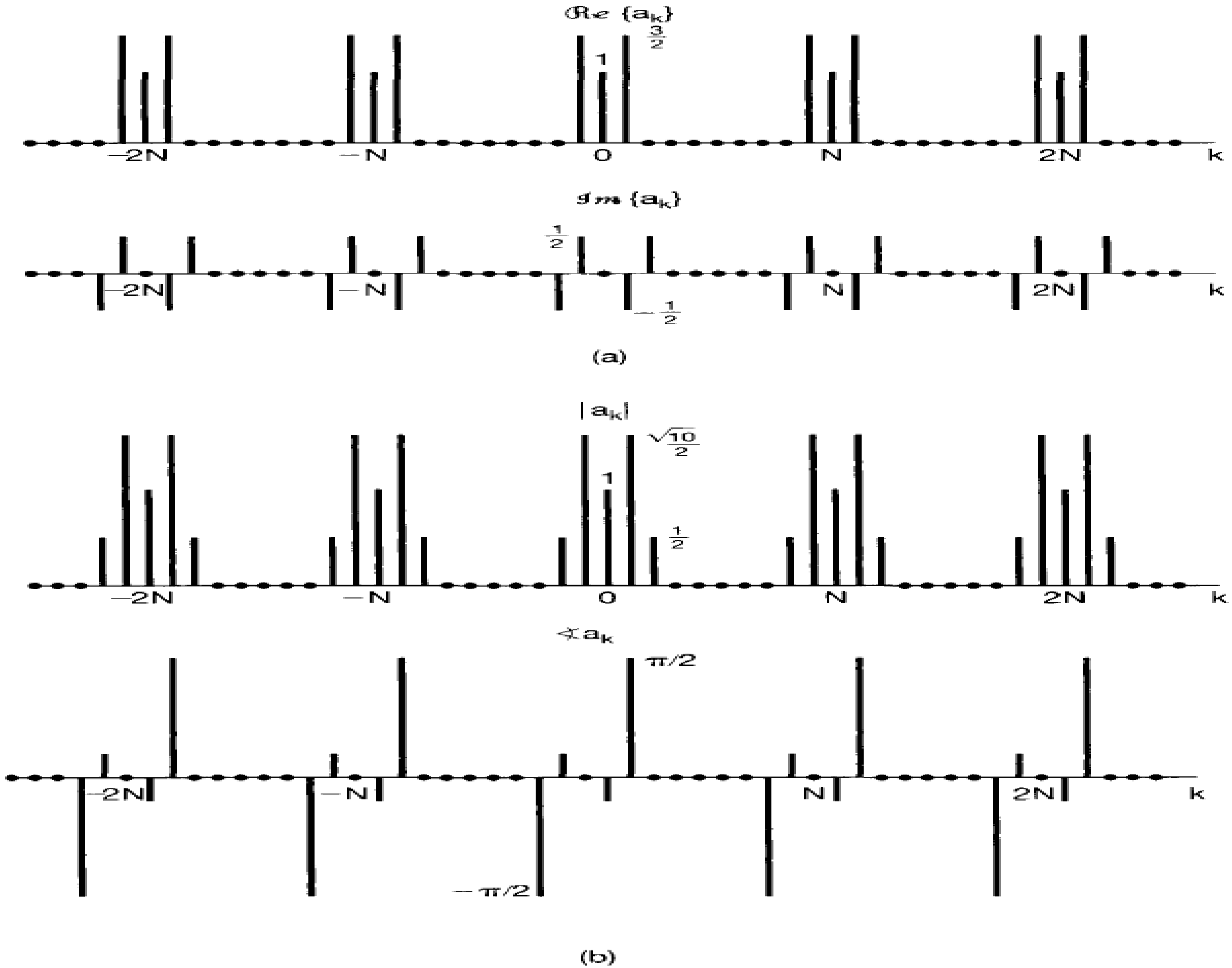


Figure (a) Real and imaginary parts of the Fourier series coefficients in Example 3.11; (b) magnitude and phase of the same coefficients.

➢PROPERTIES OF DISCRETE-TIME FOURIER SERIES

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$