Signals and Systems

Lecture # 17

Fourier Transform

Prepared by:

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Topics of the lecture:

- Introduction
- Development of Continuous-Time Fourier Transform.
- Convergence of Fourier Transforms
- Examples
- THE FOURIER TRANSFORM FOR PERIODIC SIGNALS
- PROPERTIES OF THE CONTINUOUS-TIME FOURIER TRANSFORM
- Development of the Discrete-Time Fourier Transform
- Examples
- THE FOURIER TRANSFORM FOR PERIODIC SIGNALS
- PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

➢ Introduction

Claim : a rather large class of signals, including all signals with *finite energy*, can also be represented through a linear combination of complex exponentials.

Whereas for periodic signals the complex exponential building blocks are harmonically related, for aperiodic signals they are infinitesimally close in frequency, and the representation in terms of a linear combination takes the form of an *integral* rather than a sum. The resulting spectrum of coefficients in this representation is called the Fourier transform .

The idea : is that an aperiodic signal can be viewed as a periodic signal with an infinite period. More precisely, in the Fourier series representation of a periodic signal, as the period increases the fundamental frequency decreases and the harmonically related components become closer in frequency.

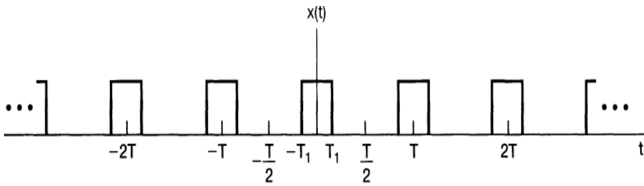
As the period becomes infinite, the frequency components form a continuum and the Fourier series sum becomes an integral.

As $w = 2\pi/T$, as $T \rightarrow \infty$ then $w \rightarrow 0$ (or $w \rightarrow dw$) $\Rightarrow \sum \rightarrow \int$

➤ Development of Continuous-Time Fourier Transform.

If x(t) is defined as :

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



The Fourier series coefficients a_k for this square wave are

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T},$$

where $\omega_0 = 2\pi/T$

bar graphs of these coefficients were shown for a fixed value of T_1 and several different values of T.

An alternative way of interpreting eq. is as samples of an envelope function, specifically,

$$T a_k = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega = k\omega_0}$$

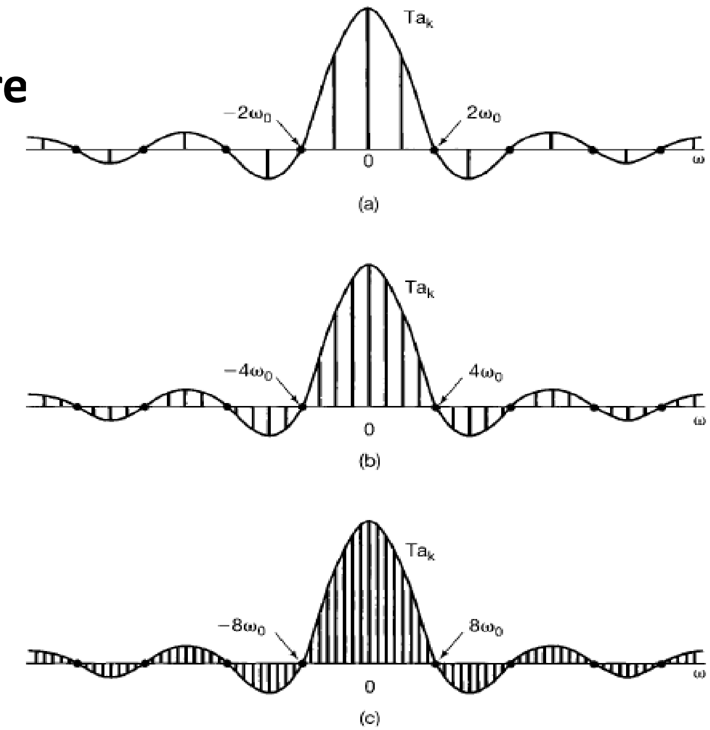


Figure The Fourier series coefficients and their envelope for the periodic square wave for several values of T (with T_1 fixed): (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$.

➢ Development of Continuous-Time Fourier Transform.

From the figure, we see that as *T increases, or equivalently, as the fundamental frequency w0 = 2π/T decreases, the* envelope is sampled with a closer and closer spacing. As *T becomes arbitrarily large,* the original periodic square wave approaches a rectangular pulse (i.e., all that remains in the time domain is an aperiodic signal corresponding to one period of the square wave). Also, the Fourier series coefficients, multiplied by *T, become more and more closely* spaced samples of the envelope, so that in some sense the set of Fourier series coefficients approaches the envelope function as *T* → ∞

This example illustrates the basic idea behind Fourier's development of a representation for aperiodic signals. Specifically, we think of an aperiodic signal as the limit of a periodic signal as the period becomes arbitrarily large, and we examine the limiting behavior of the Fourier series representation for this signal. In particular, consider a signal $x(t)$ that is of finite duration. That is, for some number T_1, $x(t) = 0$ if $|t| > T_1$, as illustrated in Figure . From this aperiodic signal, we can construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is one period, as indicated in Figure 4.3(b). As we choose the period T to be larger, $\tilde{x}(t)$ is identical to $x(t)$ over a longer interval, and as $T \to \infty$, $\tilde{x}(t)$ is equal to $x(t)$ for any finite value of t.

➢ Development of Continuous-Time Fourier Transform.

Let us now examine the effect of this on the Fourier series representation of $\tilde{x}(t)$. Rewriting eqs. here for convenience, with the integral in eq. carried out over the interval $-T/2 \leq t \leq T/2$, we have

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}dt,$$

where $\omega_0 = 2\pi/T$. Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since $x(t) = 0$ outside this interval, eq. can be rewritten as

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt.$$

Therefore, defining the envelope $X(j\omega)$ of Ta_k as $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt,$

we have, for the coefficients a_k, $a_k = \frac{1}{T}X(jk\omega_0).$

Combining eqs. , we can express $\tilde{x}(t)$ in terms of $X(j\omega)$ as $\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t},$

or equivalently, since $2\pi/T = \omega_0$, $\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0.$

As $T \rightarrow \infty$, $\tilde{x}(t)$ approaches $x(t)$, and consequently, in the limit eq. becomes a representation of $x(t)$. Furthermore, $\omega_0 \rightarrow 0$ as $T \rightarrow \infty$, and the right-hand side of eq. passes to an integral. This can be seen by considering the graphical interpretation of the equation, illustrated in Figure . Each term in the summation on the right-hand side is the area of a rectangle of height $X(jk\omega_0)e^{jk\omega_0 t}$ and width ω_0. (Here, t is regarded as fixed.) As $\omega_0 \rightarrow 0$, the summation converges to the integral of $X(j\omega)e^{j\omega t}$. Therefore, using the fact that $\tilde{x}(t) \rightarrow x(t)$ as $T \rightarrow \infty$, we see that eqs. become

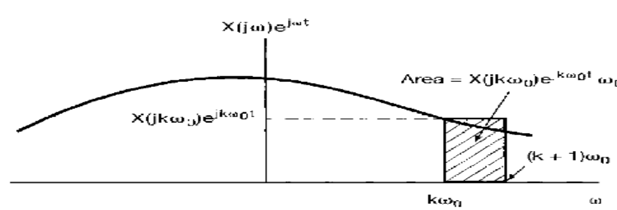
$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega}$$

Fourier Transform or *Fourier integral* of $x(t)$

and

$$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.}$$

the *inverse Fourier transform*



Convergence of Fourier Transforms

1. $x(t)$ be absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |x(t)| dt < \infty.$$

2. $x(t)$ have a finite number of maxima and minima within any finite interval.

3. $x(t)$ have a finite number of discontinuities within any finite interval. Futhermore, each of these discontinuities must be finite.

➢ Examples.

Example 1

Consider the signal

$$x(t) = e^{-at}u(t) \quad a > 0.$$

From eq. ,

$$X(j\omega) = \int_0^\infty e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^\infty.$$

That is,

$$X(j\omega) = \frac{1}{a+j\omega}, \quad a > 0.$$

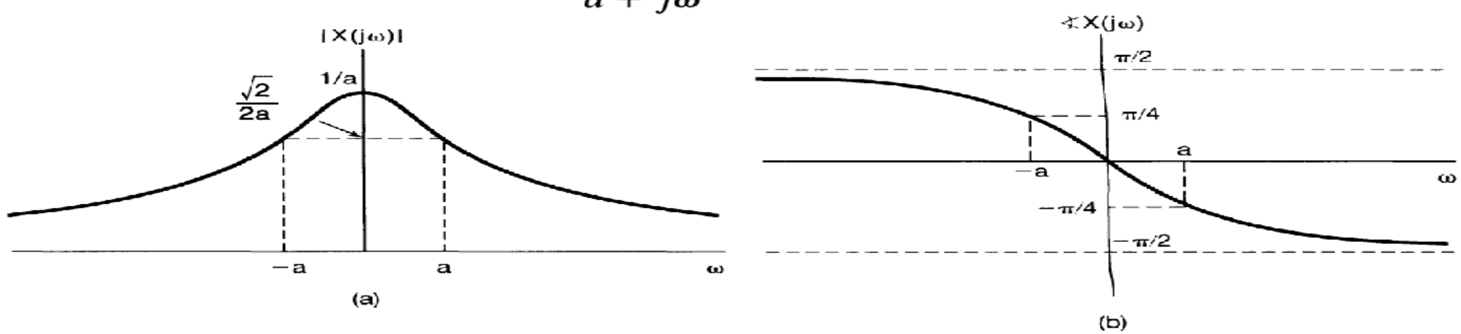
Since this Fourier transform is complex valued, to plot it as a function of ω, we express $X(j\omega)$ in terms of its magnitude and phase:

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \sphericalangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

Each of these components is sketched in Figure .

Note that if a is complex rather than real, then $x(t)$ is absolutely integrable as long as $\Re e\{a\} > 0$, and in this case the preceding calculation yields the same form for $X(j\omega)$. That is,

$$X(j\omega) = \frac{1}{a+j\omega}, \quad \Re e\{a\} > 0.$$



➢ Examples.

Example 2

Let

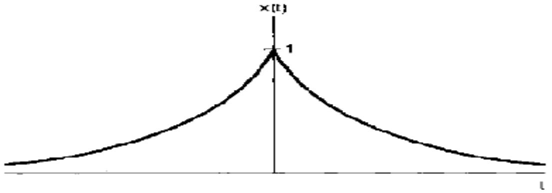
$$x(t) = e^{-a|t|}, \quad a > 0.$$

This signal is sketched in Figure . The Fourier transform of the signal is

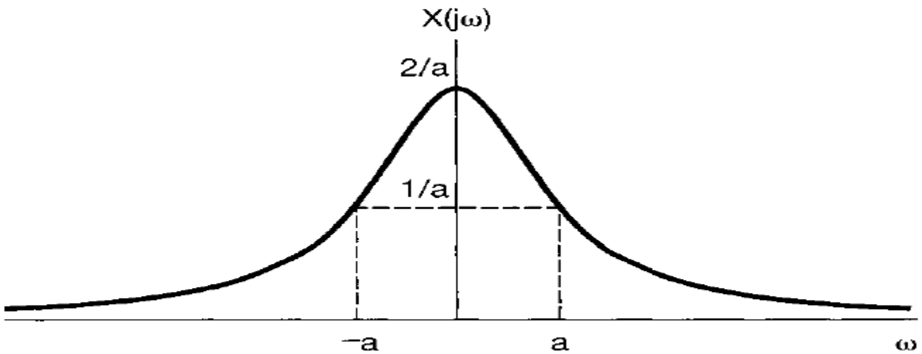
$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-a|t|}e^{-j\omega t}\,dt = \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

$$= \frac{2a}{a^2 + \omega^2}.$$



In this case $X(j\omega)$ is real, and it is illustrated in Figure .



Example 3

Now let us determine the Fourier transform of the unit impulse

$$x(t) = \delta(t).$$

Substituting into eq. (4.9) yields

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}\,dt = 1.$$

That is, the unit impulse has a Fourier transform consisting of equal contributions at *all* frequencies.

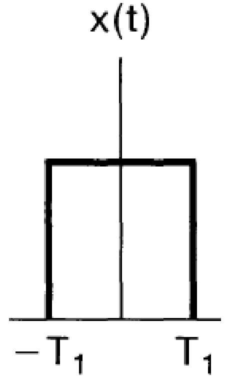
➢ Examples.

Example 4

Consider the rectangular pulse signal

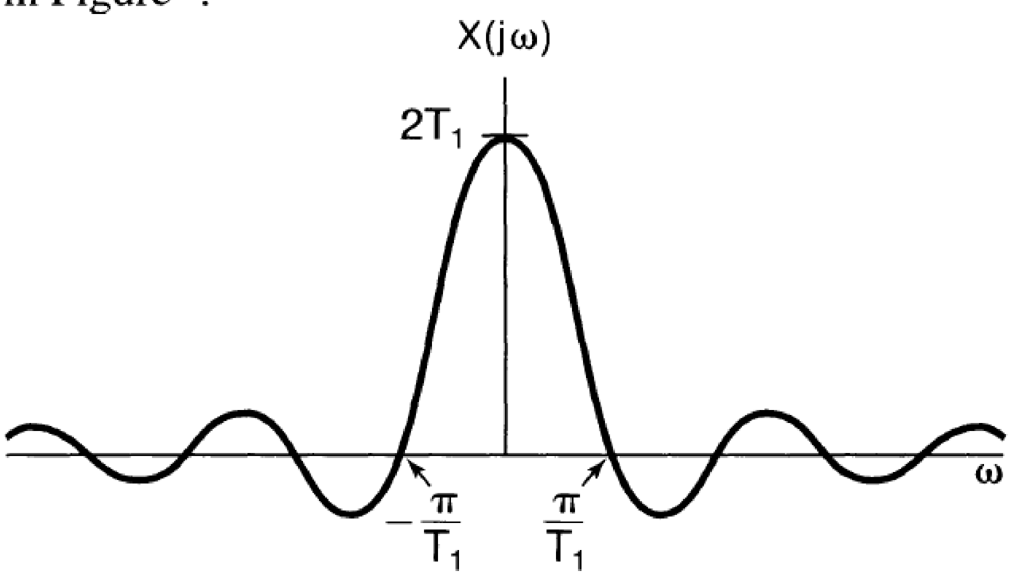
$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases},$$

as shown in Figure 4.8(a). Applying eq. , we find that the Fourier transform of this signal is



$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = 2\frac{\sin \omega T_1}{\omega},$$

as sketched in Figure .



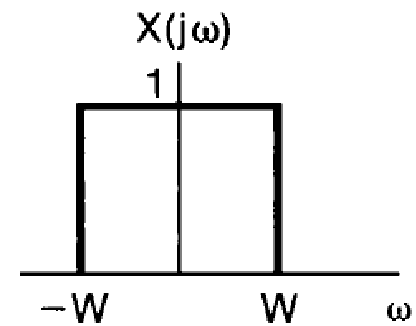
➢ Examples.

Example 5

Consider the signal $x(t)$ whose Fourier transform is

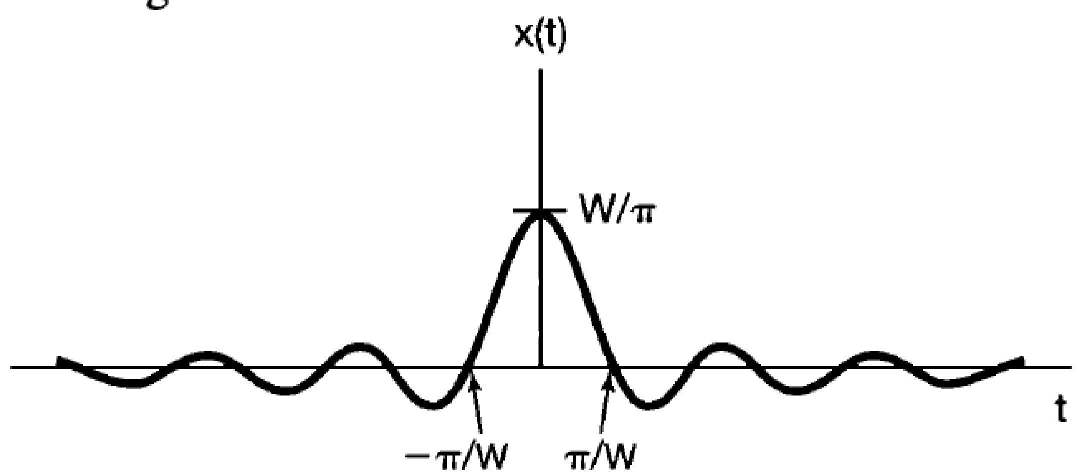
$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}.$$

This transform is illustrated in Figure . Using the synthesis equation , we can then determine



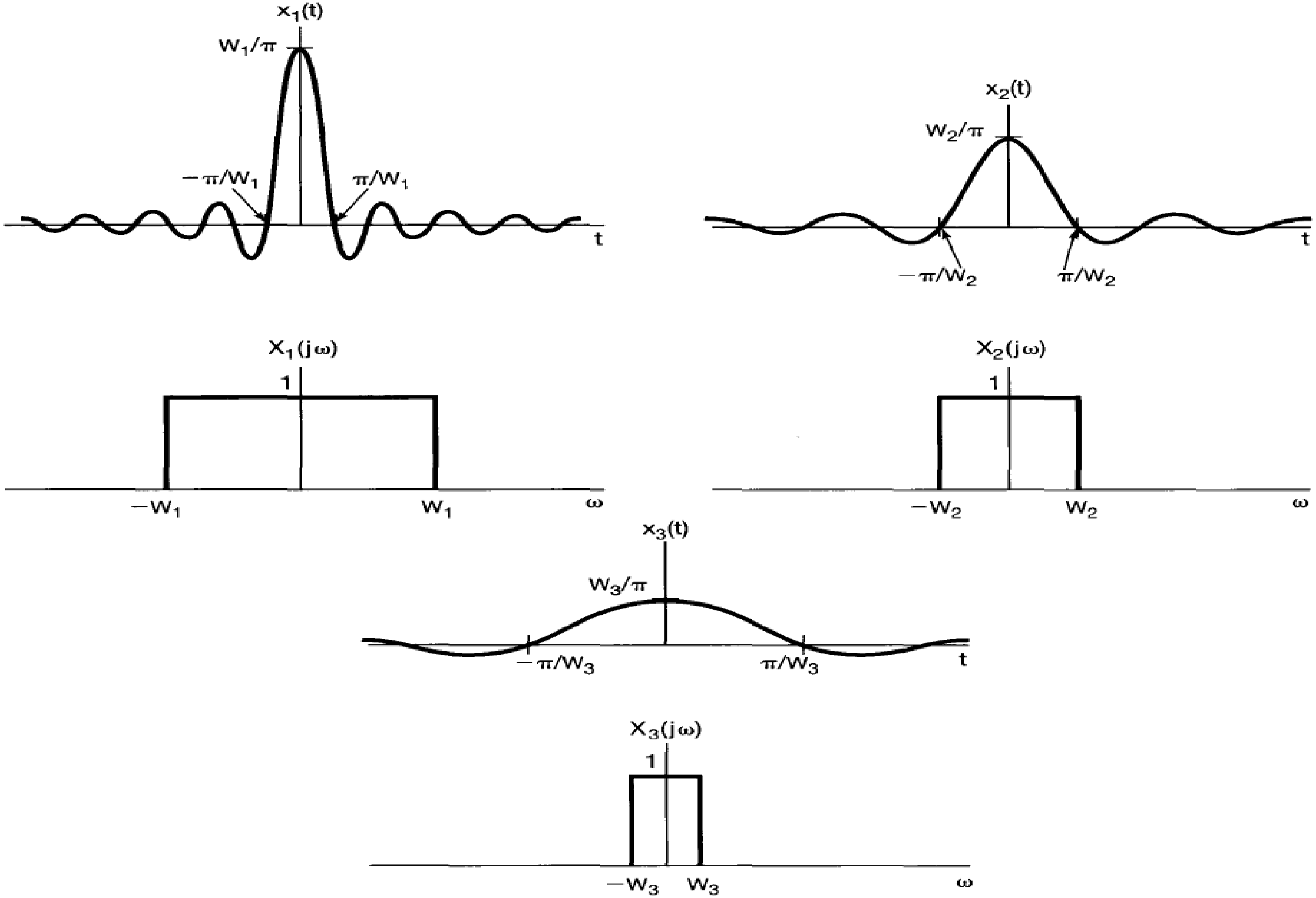
$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{\sin Wt}{\pi t},$$

which is depicted in Figure .



the *duality*

➢ Examples.



(c)

Figure Fourier transform pair for several different values of W.

From this figure, we see that as W increases, $X(j\omega)$ becomes broader

➢THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

To suggest the general result, let us consider a signal $x(t)$ with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$; that is,

$$X(j\omega) = 2\pi\delta(\omega - \omega_0).$$

To determine the signal $x(t)$ for which this is the Fourier transform, we can apply the inverse transform relation, , to obtain

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega \\ &= e^{j\omega_0 t}. \end{aligned}$$

More generally, if $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency, that is,

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0),$$

then the application of eq. yields

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}.$$

We see that eq. corresponds exactly to the Fourier *series* representation of a periodic signal, . Thus, the Fourier transform of a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the kth harmonic frequency $k\omega_0$ is 2π times the kth Fourier series coefficient a_k.

Example

A signal that we will find extremely useful in our analysis of sampling systems in Chapter 7 is the impulse train

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

which is periodic with period T, as indicated in Figure 4.14(a). The Fourier series coefficients for this signal were computed in Example 3.8 and are given by

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}.$$

That is, every Fourier coefficient of the periodic impulse train has the same value, $1/T$. Substituting this value for a_k in eq. (4.22) yields

$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right).$$

Thus, the Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2\pi/T$, as sketched in Figure . Here again, we see an illustration of the inverse relationship between the time and the frequency domains. As the spacing between the impulses in the time domain (i.e., the period) gets longer, the spacing between the impulses in the frequency domain (namely, the fundamental frequency) gets smaller.

➢PROPERTIES OF THE CONTINUOUS-TIME FOURIER TRANSFORM

TABLE PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals	$\int_{-\infty}^{+\infty} \|x(t)\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \|X(j\omega)\|^2 d\omega$	

➢PROPERTIES OF THE CONTINUOUS-TIME FOURIER TRANSFORM

TABLE BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

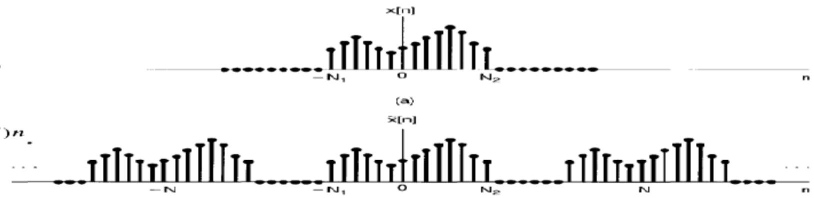
➢Development of the Discrete-Time Fourier Transform

Consider a general sequence $x[n]$ that is of finite duration. That is, for some integers N_1 and N_2, $x[n] = 0$ outside the range $-N_1 \leq n \leq N_2$. A signal of this type is illustrated in Figure . From this aperiodic signal, we can construct a periodic sequence $\tilde{x}[n]$ for which $x[n]$ is one period, as illustrated in Figure . As we choose the period N to be larger, $\tilde{x}[n]$ is identical to $x[n]$ over a longer interval, and as $N \longrightarrow \infty$, $\tilde{x}[n] = x[n]$ for any finite value of n.

Let us now examine the Fourier series representation of $\tilde{x}[n]$. Specifically, from eqs. (3.94) and (3.95), we have

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n},$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jk(2\pi/N)n}.$$



Since $x[n] = \tilde{x}[n]$ over a period that includes the interval $-N_1 \leq n \leq N_2$, it is convenient to choose the interval of summation in eq. to include this interval, so that $\tilde{x}[n]$ can be replaced by $x[n]$ in the summation. Therefore,

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-\infty}^{+\infty} x[n]e^{-jk(2\pi/N)n},$$

where in the second equality in eq. we have used the fact that $x[n]$ is zero outside the interval $-N_1 \leq n \leq N_2$. Defining the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n},$$

we see that the coefficients a_k are proportional to samples of $X(e^{j\omega})$, i.e.,

$$a_k = \frac{1}{N}X(e^{jk\omega_0}),$$

where $\omega_0 = 2\pi/N$ is the spacing of the samples in the frequency domain. Combining eqs. yields

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} \frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n}.$$

Since $\omega_0 = 2\pi/N$, or equivalently, $1/N = \omega_0/2\pi$, eq. can be rewritten as

$$\tilde{x}[n] = \frac{1}{2\pi}\sum_{k=\langle N\rangle} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0.$$

As N increases ω_0 decreases, and as $N \longrightarrow \infty$ eq. passes to an integral.

Therefore, as $N \longrightarrow \infty$, $\tilde{x}[n] = x[n]$, and eq. (5.7) becomes

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega,$$

where, since $X(e^{j\omega})e^{j\omega n}$ is periodic with period 2π, the interval of integration can be taken as *any* interval of length 2π. Thus, we have the following pair of equations:

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \\ X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \end{aligned}}$$

➢Development of the Discrete-Time Fourier Transform

Consider a general sequence $x[n]$ that is of finite duration. That is, for some integers N_1 and N_2, $x[n] = 0$ outside the range $-N_1 \leq n \leq N_2$. A signal of this type is illustrated in Figure . From this aperiodic signal, we can construct a periodic sequence $\tilde{x}[n]$ for which $x[n]$ is one period, as illustrated in Figure . As we choose the period N to be larger, $\tilde{x}[n]$ is identical to $x[n]$ over a longer interval, and as $N \rightarrow \infty$, $\tilde{x}[n] = x[n]$ for any finite value of n.

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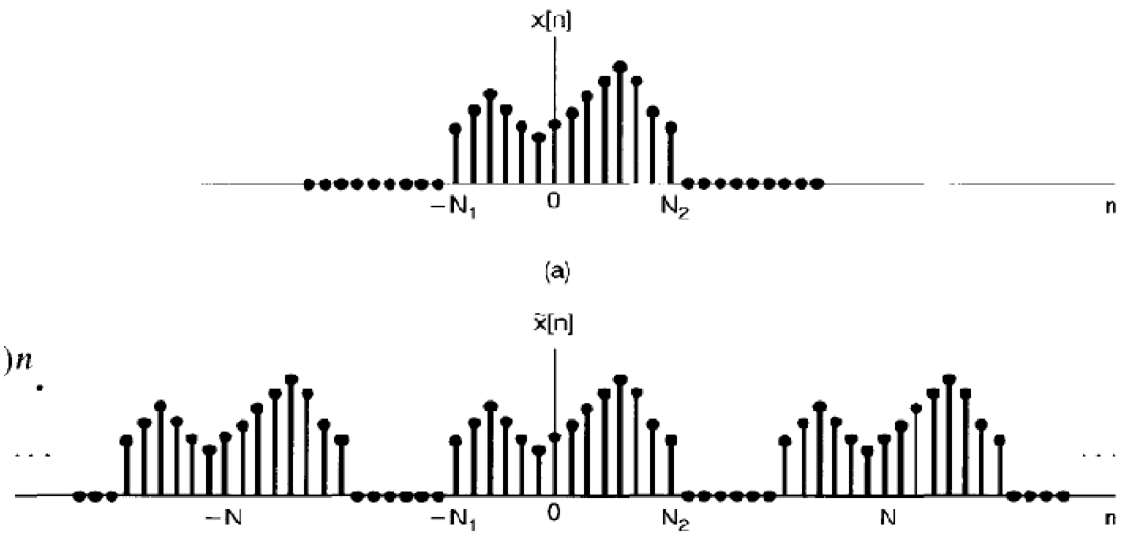

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➢Development of the Discrete-Time Fourier Transform

Therefore,

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-\infty}^{+\infty} x[n]e^{-jk(2\pi/N)n},$$

where in the second equality in eq. we have used the fact that $x[n]$ is zero outside the interval $-N_1 \le n \le N_2$. Defining the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n},$$

we see that the coefficients a_k are proportional to samples of $X(e^{j\omega})$, i.e.,

$$a_k = \frac{1}{N}X(e^{jk\omega_0}),$$

where $\omega_0 = 2\pi/N$ is the spacing of the samples in the frequency domain. Combining eqs. yields

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} \frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n}.$$

Since $\omega_0 = 2\pi/N$, or equivalently, $1/N = \omega_0/2\pi$, eq. can be rewritten as

$$\tilde{x}[n] = \frac{1}{2\pi}\sum_{k=\langle N\rangle} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0.$$

As N increases ω_0 decreases, and as $N \to \infty$ eq. passes to an integral.

Therefore, as $N \to \infty$, $\tilde{x}[n] = x[n]$, and eq. (5.7) becomes

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega,$$

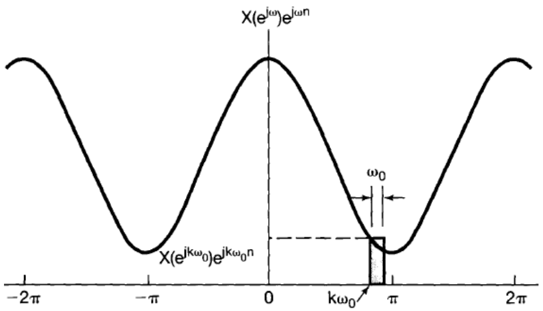


Figure Graphical interpretation

where, since $X(e^{j\omega})e^{j\omega n}$ is periodic with period 2π, the interval of integration can be taken as *any* interval of length 2π. Thus, we have the following pair of equations:

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \\ X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \end{aligned}}$$

➢Development of the Discrete-Time Fourier Transform

Equations are the discrete-time counterparts

The function $X(e^{j\omega})$ is referred to as the *discrete-time Fourier transform* and the pair of equations as the *discrete-time Fourier transform pair.* Equation is the *synthesis equation*, eq. the *analysis equation.* Our derivation of these equations indicates how an aperiodic sequence can be thought of as a linear combination of complex exponentials. In particular, the synthesis equation is in effect a representation of $x[n]$ as a linear combination of complex exponentials infinitesimally close in frequency and with amplitudes $X(e^{j\omega})(d\omega/2\pi)$. For this reason, as in continuous time, the Fourier transform $X(e^{j\omega})$ will often be referred to as the *spectrum* of $x[n]$, because it provides us with the information on how $x[n]$ is composed of complex exponentials at different frequencies.

Note also that, as in continuous time, our derivation of the discrete-time Fourier transform provides us with an important relationship between discrete-time Fourier series and transforms. In particular, the Fourier coefficients a_k of a periodic signal $\tilde{x}[n]$ can be expressed in terms of equally spaced *samples* of the Fourier transform of a finite-duration, aperiodic signal $x[n]$ that is equal to $\tilde{x}[n]$ over one period and is zero otherwise. This fact is of considerable importance in practical signal processing and Fourier analysis.

➢Development of the Discrete-Time Fourier Transform

As our derivation indicates, the discrete-time Fourier transform shares many similarities with the continuous-time case. The major differences between the two are the periodicity of the discrete-time transform $X(e^{j\omega})$ and the finite interval of integration in the synthesis equation. Both of these stem from a fact that we have noted several times before: Discrete-time complex exponentials that differ in frequency by a multiple of 2π are identical. we saw that, for periodic discrete-time signals, the implications of this statement are that the Fourier series coefficients are periodic and that the Fourier series representation is a finite sum. For aperiodic signals, the analogous implications are that $X(e^{j\omega})$ is periodic (with period 2π) and that the synthesis equation involves an integration only over a frequency interval that produces distinct complex exponentials (i.e., any interval of length 2π). we noted one further consequence of the periodicity of $e^{j\omega n}$ as a function of ω: $\omega = 0$ and $\omega = 2\pi$ yield the same signal. Signals at frequencies near these values or any other even multiple of π are slowly varying and therefore are all appropriately thought of as low-frequency signals. Similarly, the high frequencies in discrete time are the values of ω near odd multiples of π. Thus, the signal $x_1[n]$ shown in Figure (a) with Fourier transform depicted in Figure (b) varies more slowly than the signal $x_2[n]$ in Figure (c) whose transform is shown in Figure (d).

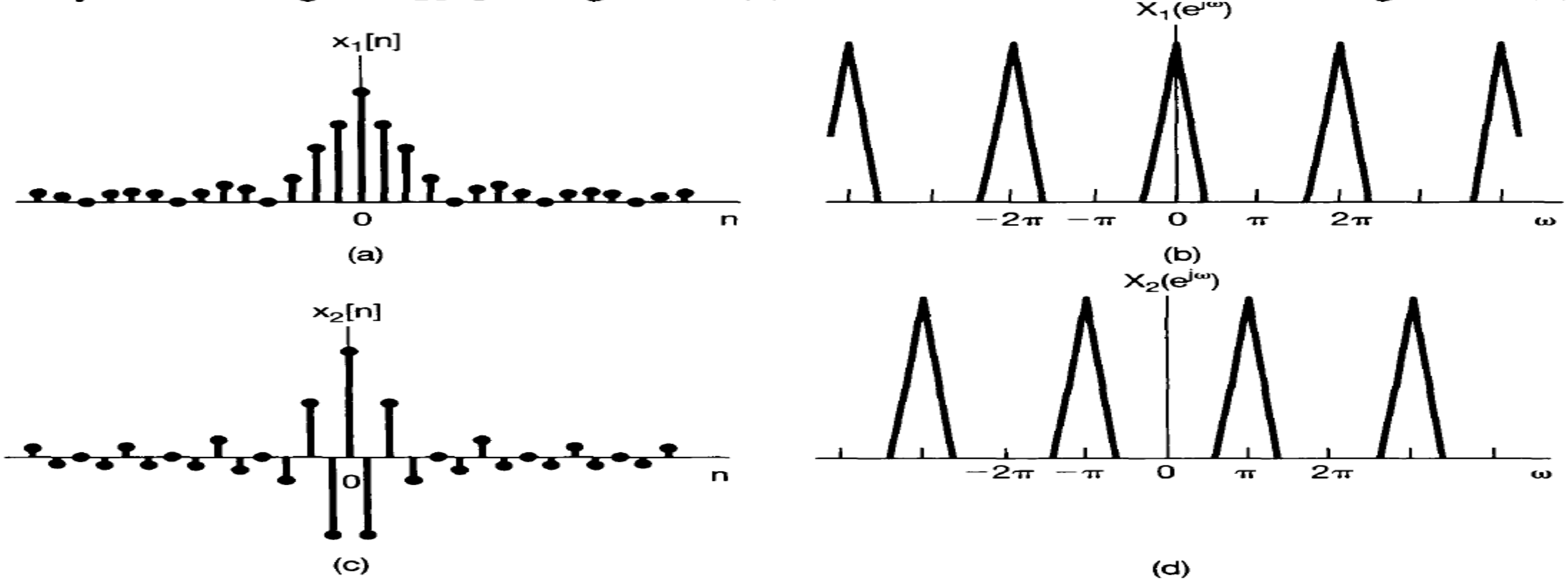


Figure (a) Discrete-time signal $x_1[n]$. (b) Fourier transform of $x_1[n]$. Note that $X_1(e^{j\omega})$ is concentrated near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$ (c) Discrete-time signal $x_2[n]$. (d) Fourier transform of $x_2[n]$. Note that $X_2(e^{j\omega})$ is concentrated near $\omega = \pm\pi, \pm 3\pi, \ldots$

➢Examples.

Example 1

Consider the signal $x[n] = a^n u[n], \qquad |a| < 1.$

In this case,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

The magnitude and phase of $X(e^{j\omega})$ are shown in Figure (a) for $a > 0$ and in Figure (b) for $a < 0$. Note that all of these functions are periodic in ω with period 2π.

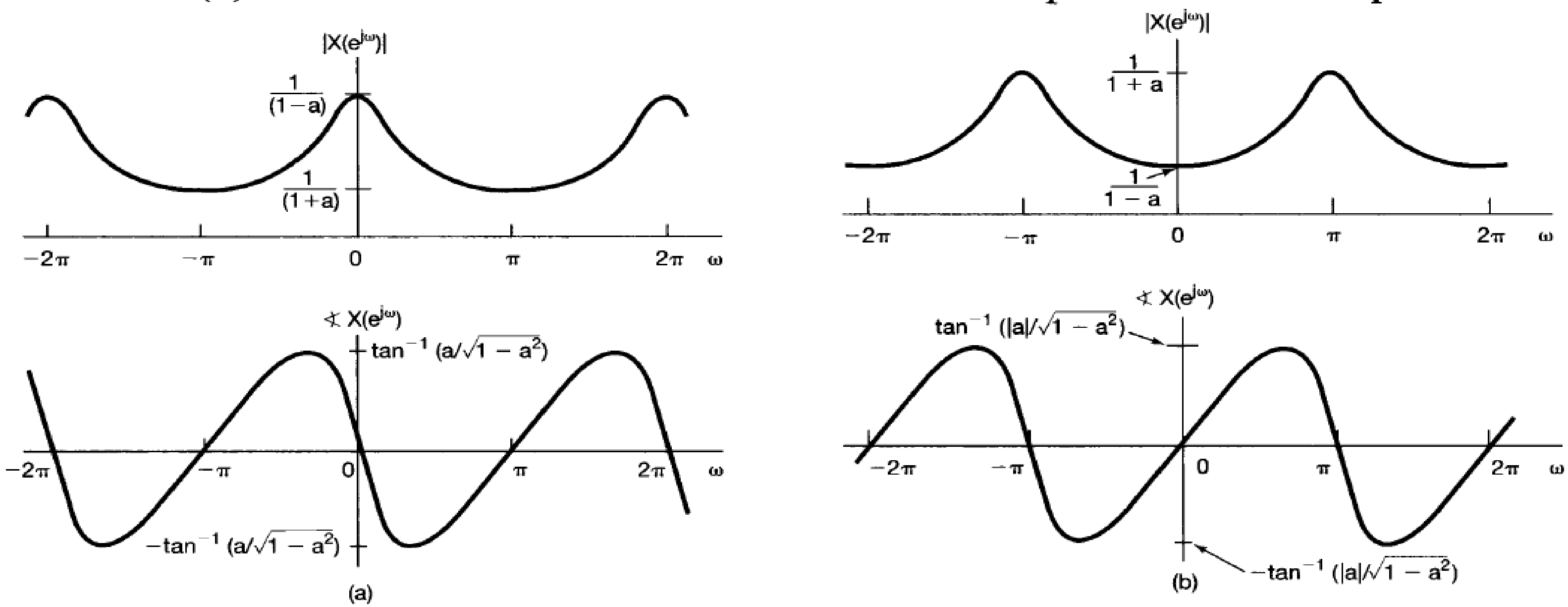


Figure Magnitude and phase of the Fourier transform of Example for (a) $a > 0$ and (b) $a < 0$.

➢Examples.

Example 2

Let

$$x[n] = a^{|n|}, \qquad |a| < 1.$$

This signal is sketched for $0 < a < 1$ in Figure 5.5(a). Its Fourier transform is obtained from eq. (5.9):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}.$$

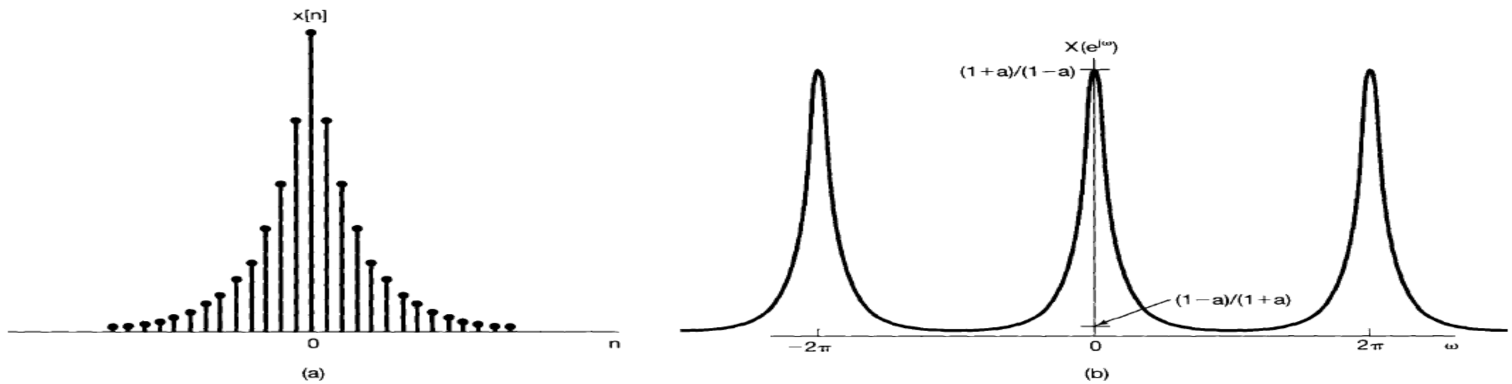


Figure (a) Signal $x[n] = a^{|n|}$ of Example 5.2 and (b) its Fourier transform ($0 < a < 1$).

Making the substitution of variables $m = -n$ in the second summation, we obtain

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{m=1}^{\infty} (ae^{j\omega})^m.$$

Both of these summations are infinite geometric series that we can evaluate in closed form, yielding

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$$= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}.$$

In this case, $X(e^{j\omega})$ is real and is illustrated in Figure (b), again for $0 < a < 1$.

➢Examples.

Example 3

Consider the rectangular pulse $x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$,

which is illustrated in Figure 5.6(a) for $N_1 = 2$. In this case,

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}.$$

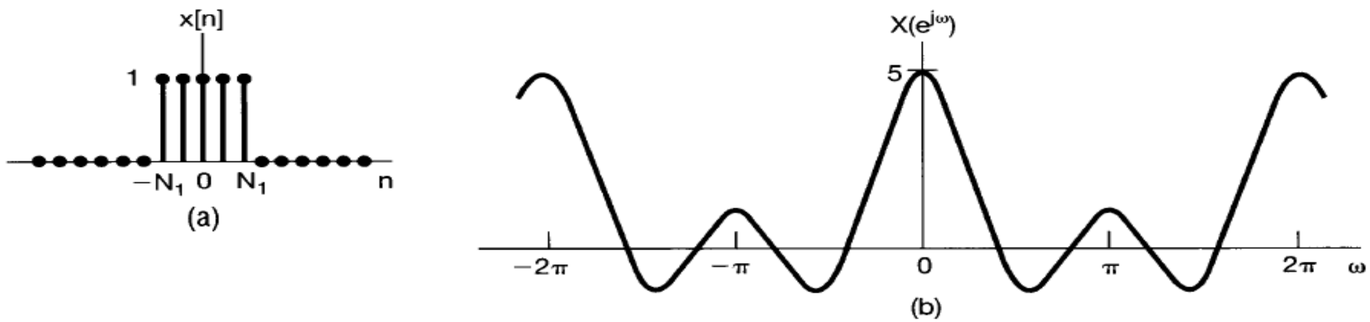


Figure (a) Rectangular pulse signal of Example 5.3 for $N_1 = 2$ and (b) its Fourier transform.

Using calculations similar to those employed
we can write

$$X(e^{j\omega}) = \frac{\sin\omega\left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}.$$

This Fourier transform is sketched in Figure (b) for $N_1 = 2$. The function in eq.
is the discrete-time counterpart of the sinc function, which appears in the Fourier transform of the continuous-time rectangular pulse . An important difference between these two functions is that the function in eq. is periodic with period 2π, whereas the sinc function is aperiodic.

➢Examples.

Example 4

Let $x[n]$ be the unit impulse; that is,

$$x[n] = \delta[n].$$

In this case the analysis equation is easily evaluated, yielding

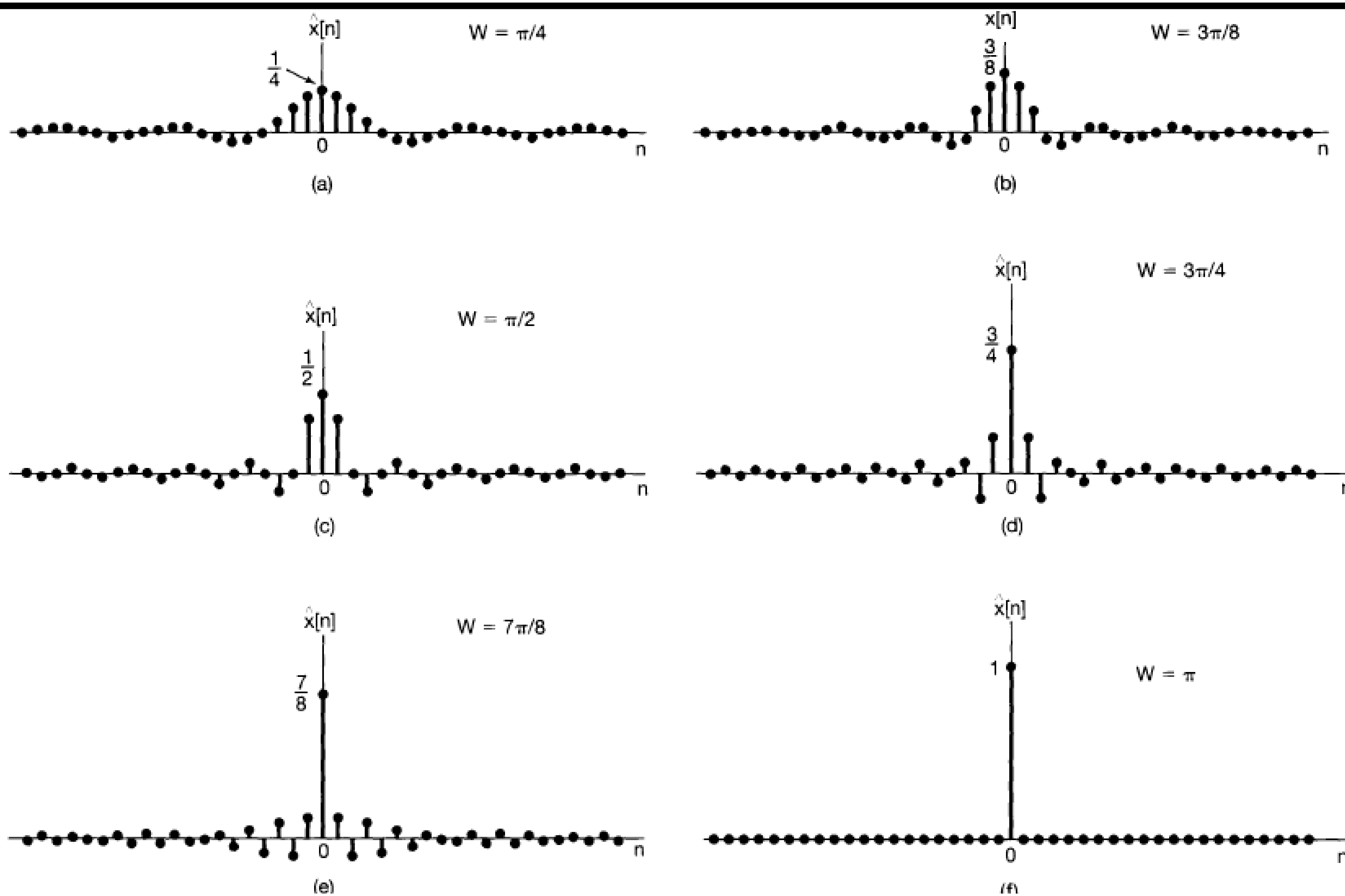
$$X(e^{j\omega}) = 1.$$

In other words, just as in continuous time, the unit impulse has a Fourier transform representation consisting of equal contributions at all frequencies. If we then apply eq. to this example, we obtain

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\,d\omega = \frac{\sin Wn}{\pi n}.$$

This is plotted in Figure for several values of W. As can be seen, the frequency of the oscillations in the approximation increases as W is increased, which is similar to what we observed in the continuous-time case. On the other hand, in contrast to the continuous-time case, the amplitude of these oscillations decreases relative to the magnitude of $\hat{x}[0]$ as W is increased, and the oscillations disappear entirely for $W = \pi$.

➢Examples.



we would expect not to see any behavior like the Gibbs phenomenon in evaluating the discrete-time Fourier transform

Figure Approximation to the unit sample obtained as in eq. using complex exponentials with frequencies $|\omega| \leq W$: (a) $W = \pi/4$; (b) $W = 3\pi/8$; (c) $W = \pi/2$; (d) $W = 3\pi/4$; (e) $W = 7\pi/8$; (f) $W = \pi$. Note that for $W = \pi$, $\hat{x}[n] = \delta[n]$.

➢THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

As in the continuous-time case, discrete-time periodic signals can be incorporated within the framework of the discrete-time Fourier transform by interpreting the transform of a periodic signal as an impulse train in the frequency domain. To derive the form of this representation, consider the signal

$$x[n] = e^{j\omega_0 n}.$$

In continuous time, we saw that the Fourier transform of $e^{j\omega_0 t}$ can be interpreted as an impulse at $\omega = \omega_0$. Therefore, we might expect the same type of transform to result for the discrete-time signal of eq. . However, the discrete-time Fourier transform must be periodic in ω with period 2π. This then suggests that the Fourier transform of $x[n]$ in eq. should have impulses at ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on. In fact, the Fourier transform of $x[n]$ is the impulse train

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi l),$$

which is illustrated in Figure . In order to check the validity of this expression, we must evaluate its inverse transform. Substituting eq. into the synthesis equation , we find that

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi l)e^{j\omega n}\,d\omega.$$

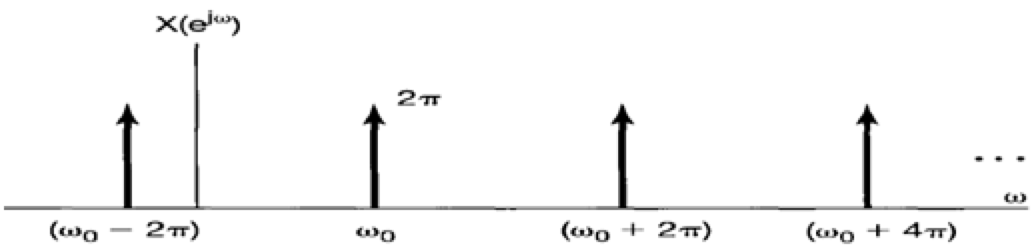


Figure Fourier transform of $x[n] = e^{j\omega_0 n}$.

Note that any interval of length 2π includes exactly one impulse in the summation given in eq. (5.18). Therefore, if the interval of integration chosen includes the impulse located at $\omega_0 + 2\pi r$, then

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = e^{j(\omega_0 + 2\pi r)n} = e^{j\omega_0 n}.$$

➢THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

Now consider a periodic sequence $x[n]$ with period N and with the Fourier series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.$$

In this case, the Fourier transform is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \,\delta\left(\omega - \frac{2\pi k}{N}\right),$$

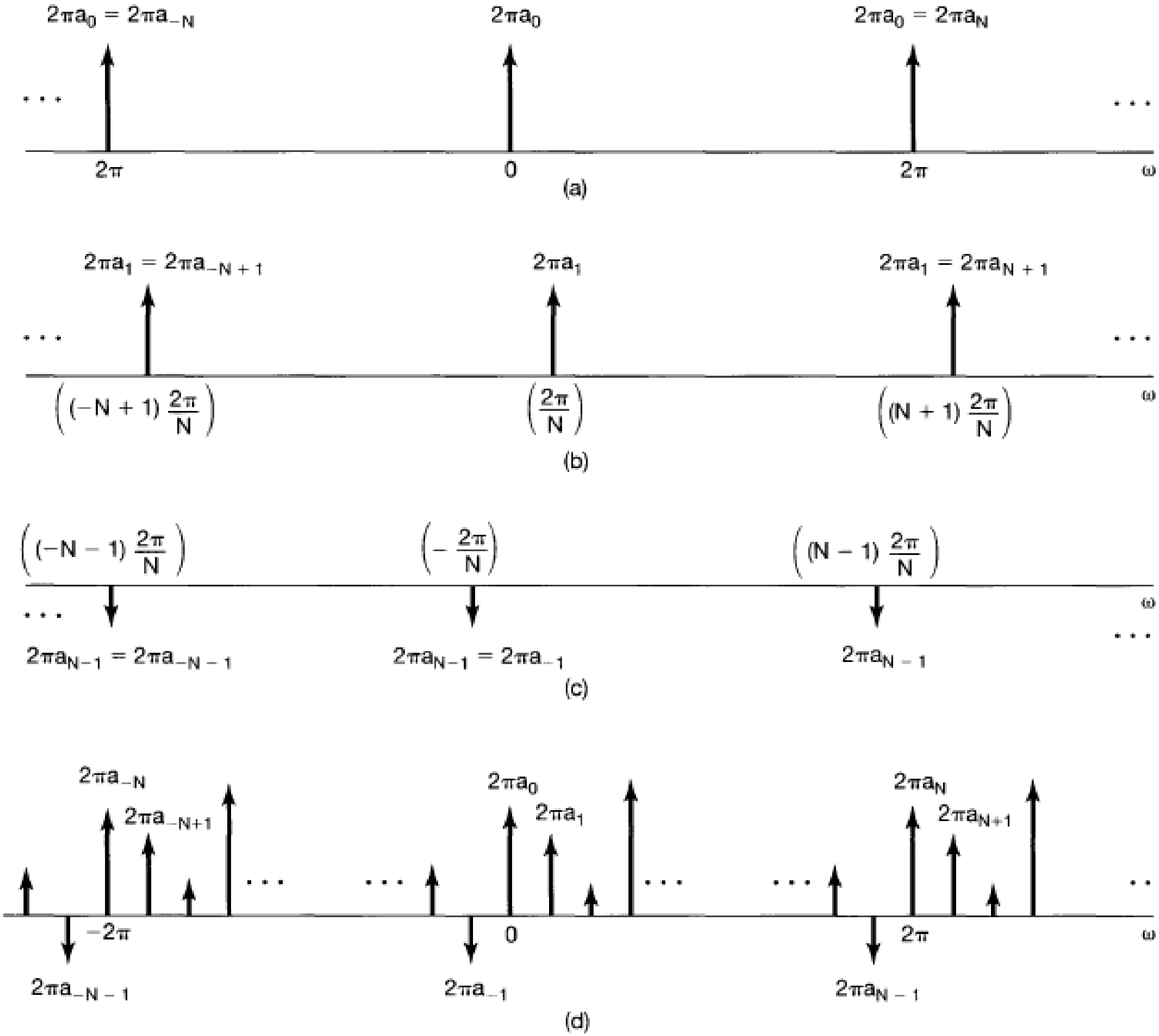
so that the Fourier transform of a periodic signal can be directly constructed from its Fourier coefficients.

To verify that eq. is in fact correct, note that $x[n]$ in eq. is a linear combination of signals of the form in eq. , and thus the Fourier transform of $x[n]$ must be a linear combination of transforms of the form of eq. . In particular, suppose that we choose the interval of summation in eq. as $k = 0, 1, \ldots, N-1$, so that

$$\begin{aligned} x[n] = {} & a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} \\ & + \cdots + a_{N-1} e^{j(N-1)(2\pi/N)n} \end{aligned}$$

Thus, $x[n]$ is a linear combination of signals, as in eq. , with $\omega_0 = 0, 2\pi/N, 4\pi/N, \ldots, (N-1)2\pi/N$. The resulting Fourier transform is illustrated in Figure . In Figure (a), we have depicted the Fourier transform of the first term on the right-hand side of eq. : The Fourier transform of the constant signal $a_0 = a_0 e^{j0\cdot n}$ is a periodic impulse train, as in eq. , with $\omega_0 = 0$ and a scaling of $2\pi a_0$ on each of the impulses. Moreover, we know that the Fourier series coefficients a_k are periodic with period N, so that $2\pi a_0 = 2\pi a_N = 2\pi a_{-N}$. In Figure we have illustrated the Fourier transform of the second term in eq. , where we have again used eq. , in this case for $a_1 e^{j(2\pi/N)n}$, and the fact that $2\pi a_1 = 2\pi a_{N+1} = 2\pi a_{-N+1}$. Similarly, Figure (c) depicts the final term. Finally, Figure (d) depicts the entire expression for $X(e^{j\omega})$. Note that because of the periodicity of the a_k, $X(e^{j\omega})$ can be interpreted as a train of impulses occurring at multiples of the fundamental frequency $2\pi/N$, with the area of the impulse located at $\omega = 2\pi k/N$ being $2\pi a_k$, which is exactly what is stated in eq.

➢THE FOURIER TRANSFORM FOR PERIODIC SIGNALS



➢THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

Example 5

Consider the periodic signal

$$x[n] = \cos \omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \qquad \text{with} \quad \omega_0 = \frac{2\pi}{5}.$$

From eq. , we can immediately write

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right).$$

That is,

$$X(e^{j\omega}) = \pi\,\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\,\delta\left(\omega + \frac{2\pi}{5}\right), \qquad -\pi \le \omega < \pi,$$

and $X(e^{j\omega})$ repeats periodically with a period of 2π, as illustrated in Figure .

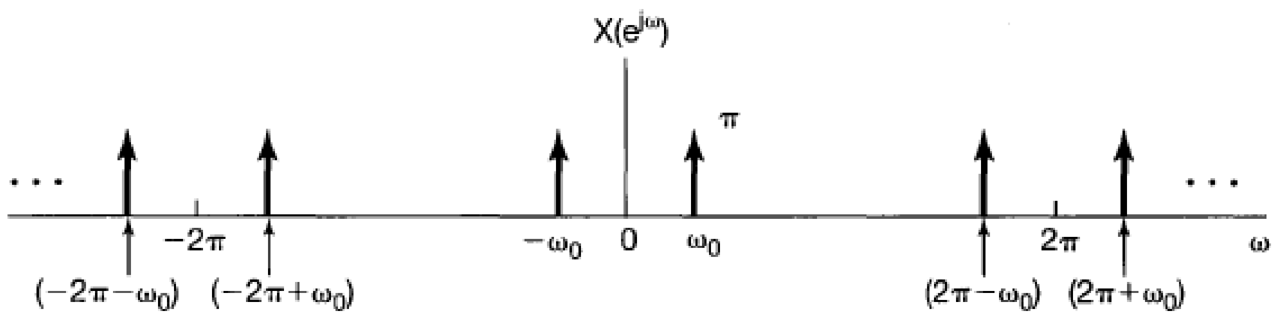


Figure Discrete-time Fourier transform of $x[n] = \cos \omega_0 n$.

➢THE FOURIER TRANSFORM FOR PERIODIC SIGNALS

Example 6

The discrete-time counterpart of the periodic impulse train is the sequence

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN],$$

as sketched in Figure 5.11(a). The Fourier series coefficients for this signal can be calculated directly from eq. (3.95):

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk(2\pi/N)n}.$$

Choosing the interval of summation as $0 \leq n \leq N-1$, we have

$$a_k = \frac{1}{N}.$$

Using eqs. , we can then represent the Fourier transform of the signal as

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right),$$

which is illustrated in Figure 5.11(b).

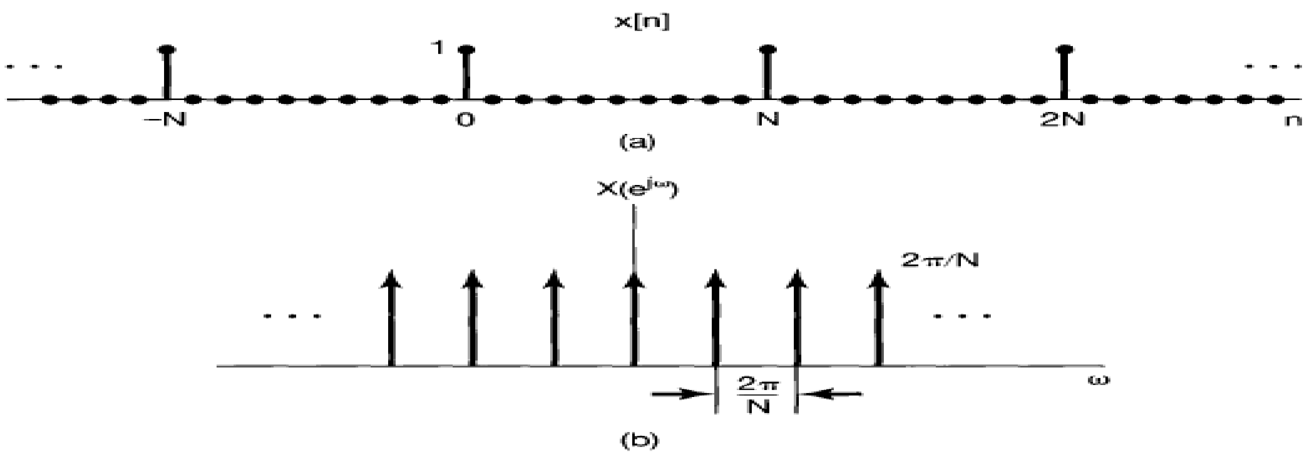


Figure (a) Discrete-time periodic impulse train; (b) its Fourier transform.

➢PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{array}{l}X(e^{j\omega})\\Y(e^{j\omega})\end{array}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n]=\begin{cases}x[n/k], & \text{if } n=\text{multiple of } k\\ 0, & \text{if } n\neq\text{multiple of } k\end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases}X(e^{j\omega})=X^*(e^{-j\omega})\\ \mathcal{R}e\{X(e^{j\omega})\}=\mathcal{R}e\{X(e^{-j\omega})\}\\ \mathcal{I}m\{X(e^{j\omega})\}=-\mathcal{I}m\{X(e^{-j\omega})\}\\ \lvert X(e^{j\omega})\rvert=\lvert X(e^{-j\omega})\rvert\\ \sphericalangle X(e^{j\omega})=-\sphericalangle X(e^{-j\omega})\end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n]=\mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n]=\mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty}|x[n]|^2=\frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2d\omega$$

➢PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

TABLE BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}$, $k \ne 0, \pm N, \pm 2N, \dots$ $a_k = \frac{2N_1 + 1}{N}$, $k = 0, \pm N, \pm 2N, \dots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r-1)!} a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—