① Derivation of Canonical form: "CCF"

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_{n-1} z + b_n}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}$$

→ divide num & den by the highest power in den. $\div (z^n)$.

$$\therefore \frac{Y(z)}{U(z)} = \frac{b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{n-1} z^{-n+1} + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_{n-1} z^{-n+1} + a_n z^{-n}}$$

let num = $N(z)$ & den = $D(z)$.

$$\therefore \frac{Y(z)}{U(z)} = \frac{N(z)}{D(z)} \Rightarrow Y(z) = \frac{N(z)}{D(z)} U(z)$$

let $Q(z) = \frac{U(z)}{D(z)}$.

$$\therefore Y(z) = N(z)\, Q(z)$$

$$\therefore Q(z) = \frac{U(z)}{1 + a_1 z^{-1} + \cdots + a_{n-1} z^{-n+1} + a_n z^{-n}}$$

$$\therefore U(z) = Q(z) + Q(z)\left[a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n-1} z^{-n+1} + a_n z^{-n}\right]$$

$$Q(z) = U(z) - Q(z)\left[a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{n-1} z^{-n+1} + a_n z^{-n}\right] \quad ①$$

⇒ assume:

$X_1(z) = z^{-n} Q(z)$

$X_2(z) = z^{-n+1} Q(z) \Rightarrow X_2 = z X_1(z)$

$X_3(z) = z X_2(z)$

$\vdots$

$X_{n-1}(z) = z^{-2} Q(z)$

$X_n(z) = z^{-1} Q(z)$

$X_n(z) = z X_{n-1}(z)$

⇒ generally:

$X_n(z) = z X_{n-1}(z)$ applying invere z-transform

∴ in time domain

$$\boxed{X_n(k) = X_{n-1}(k+1)}$$

$\because z X_n = Q(z)$ from (1) sub about $Q(z)$
& assumptions.

$$z X_n = -a_1 X_n(z) - a_2 X_{n-1}(z) - \dots - a_n X_1(z) + U(z)$$

in time domain

$$X_n(k+1) = -a_1 X_n(k) - a_2 X_{n-1}(k) - a_3 X_{n-2}(k) - \dots - a_{n-1} X_2(k) - a_n X_1(k) + U(k)$$

For op eq.

$$Y(z) = N(z)\,Q(z)$$
$$= (b_1 z^{-1} + b_2 z^{-2} + \dots + b_n z^{-n})\,Q(z).$$

from assumption $\Rightarrow z^{-1}Q(z) = X_n(z)$ & so on.

$$= b_1 X_n(z) + b_2 X_{n-1}(z) + \dots + b_n X_1(z)$$

in time domain.

$$y(k) = b_1 X_n(k) + b_2 X_{n-1}(k) + \dots + b_n X_1(k)$$

&

$$X_n(k+1) = -a_1 X_n(k) - a_2 X_{n-1}(k) - a_3 X_{n-2}(k) - \dots - a_n X_1(k) + u(k)$$

&

$$X_n(k) = X_{n-1}(k+1)$$

SSR

$$\underline{X}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 1 \\ -a_n & -a_{n-1} & \dots & \dots & \dots & -a_1 \end{bmatrix} \underline{X}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [\,b_n \quad b_{n-1} \quad \dots \quad b_2 \quad b_1\,]\; X(k) \quad \& \; \boxed{D = 0}$$

عشان درجة البسط اقل من درجة المقام
و معناها ان مفيش علاقة مباشرة بين ال ip وال op لو درجة البسط كانت
اعلى من المقام كنت هعمل long division واعمل نفس الخطوات اللي عملتها
من شوية وساعتها ال D هيبقى ليها قيمة.

Example on CCF

$$\frac{Y(z)}{U(z)} = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

divide num & denum by z^2

$$\therefore \frac{Y(z)}{U(z)} = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{N(z)}{D(z)}$$

$$Y(z) = N(z)\,\frac{U(z)}{D(z)} \Rightarrow \text{let } Q(z) = \frac{U(z)}{D(z)}.$$

$$\therefore Y(z) = N(z)\,Q(z)$$

$$Q(z) = \frac{U(z)}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$U(z) = Q(z) + Q(z)\,[a_1 z^{-1} + a_2 z^{-2}].$$

$$Q(z) = U(z) - Q(z)\,[a_1 z^{-1} + a_2 z^{-2}] \quad \text{—} ①.$$

assume

$$X_1(z) = z^{-2} Q(z)$$

$$X_2(z) = z^{-1} Q(z) \Rightarrow X_2(z) = z X_1(z)$$

in time domain

$$\boxed{X_2(k) = X_1(k+1)} \quad (*)$$

$z X_2(z) = Q(z)$ sub from ① & assumption.

$$z X_2(z) = U(z) - a_1 X_2(z) - a_2 X_1(z).$$

in time domain.

$$X_2(k+1) = u(k) - a_1 X_2(k) - a_2 X_1(k) \quad **$$

& $X_1(k+1) = X_2(k)$ $(*)$.

For o/p eq.

$Y(z) = N(z)\,Q(z) = [b_1 z^{-1} + b_2 z^{-2}]\,Q(z)$ from assumption.

$$\therefore Y(z) = b_1 X_2(z) + b_2 X_1(z)$$

in T.D

$$y(k) = b_1 X_2(k) + b_2 X_1(k) \quad ②$$

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S.S.R from eq. ** , ⊛ & ②.

$$\underline{X}(k+1) = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \underline{X}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

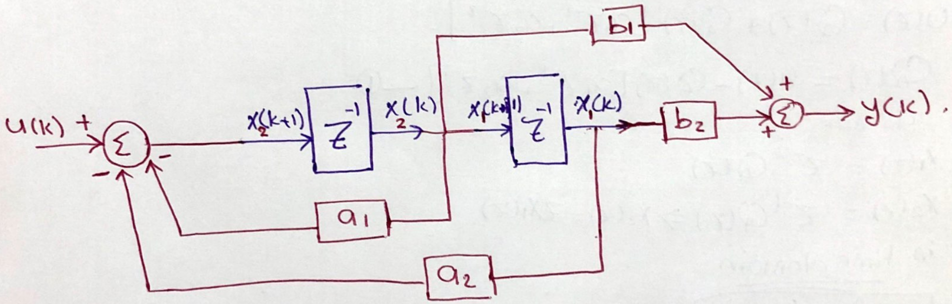
$$y(k) = [b_2 \quad b_1]\ \underline{X}(k) \quad \& D = 0.$$

for Block diagram:

بفك المعادلات تاني عشان اعرف ارسم

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = -a_2 x_1(k) - a_1 x_2(k) + u(k)$$
$$y(k) = b_2 x_1(k) + b_1 x_2(k).$$

* بحدد ال delay Blocks على حسب عندي كام state هنا عندي 2 .



Block diagram of CCF

② observable Canonical form proof.

$$\frac{Y(z)}{U(z)} = \frac{b_1 z^{-1} + b_2 z^{-2} + \dots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}}$$ "system Transfer function"

$$Y(z)\left[1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}\right] = U(z)\left[b_1 z^{-1} + b_2 z^{-2} + \dots + b_n z^{-n}\right]$$

هودي ال $Y(z)$ الناحية التانية واجمع ال coeff بتاعة z^{-1} مع بعض
وال z^{-2} مع بعض وهكذا.

$$Y(z) = U(z)\left[b_1 z^{-1} + b_2 z^{-2} + \dots + b_n z^{-n}\right] - Y(z)\left[a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}\right]$$

$$= \left(b_1 U(z) - a_1 Y(z)\right) z^{-1} + \left(b_2 U(z) - a_2 Y(z)\right) z^{-2} + \dots$$

$$+ \left(b_n U(z) - a_n Y(z)\right) z^{-n}$$

take z^{-1} Common factor

$$Y(z) = z^{-1}\left[\left(b_1 U(z) - a_1 Y(z)\right) + \left(b_2 U(z) - a_2 Y(z)\right) z^{-1} + \dots + \left(b_n U(z) - a_n Y(z)\right) z^{-n+1}\right]$$

take z^{-1} Common factor

بس من اول ال terms اللي ضمن فيها z^{-1}

$$Y(z) = z^{-1}\left[\left(b_1 U(z) - a_1 Y(z)\right) + z^{-1}\left\{\left(b_2 U(z) - a_2 Y(z)\right) + \dots + \left(b_n U(z) - a_n Y(z)\right) z^{-n+2}\right\}\right]$$

& so on

لحد لما ميبقاش في term عنده z^{-1} تاني

assume $Y(z) = X_n(z)$ —①.

$$\therefore X_n(z) = z^{-1}\left[\left(b_1 U(z) - a_1 Y(z)\right) + X_{n-1}(z)\right]$$

where $X_{n-1}(z) = z^{-1}\left[\left(b_2 U(z) - a_2 Y(z)\right) + X_{n-2}(z)\right]$

& so on till $X_1(z) = z^{-1}\left[b_n U(z) - a_n X_n(z)\right]$.

in T.D → from ① $y_n(k) = X_n(k)$.

$$X_n(k+1) = b_1 u(k) - a_1 X_n(k) + X_{n-1}(k)$$

$$X_{n-1}(k+1) = b_2 u(k) - a_2 X_n(k) + X_{n-2}(k)$$

$$\vdots$$

$$X_1(k+1) = b_n u(k) - a_n X_n(k)$$

SSR

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1)\end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k)\\ x_2(k)\\ \vdots \\ x_n(k)\end{bmatrix} + \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1\end{bmatrix} u(k)$$

$y(k) = [0 \ 0 \ 0 \cdots 1] \, \underline{x}(k)$ & $D = 0$.

Note that.

$G_{ocf} = G_{ccf}^T$

$C_{ocf} = H_{ccf}^T$

$H_{ocf} = C_{ccf}^T$.

Example on OCF

$$\frac{Y(z)}{U(z)} = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

$\div z^2$

$$\therefore \frac{Y(z)}{U(z)} = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$Y(z) = [b_1 z^{-1} + b_2 z^{-2}] \, U(z) - Y(z)[a_1 z^{-1} + a_2 z^{-2}].$$

$$= [b_1 U(z) - a_1 Y(z)] z^{-1} + [b_2 U(z) - a_2 Y(z)] z^{-2}$$

take z^{-1} common factor.

$$= z^{-1}\left[(b_1 U(z) - a_1 Y(z)) + (b_2 U(z) - a_2 Y(z)) z^{-1} \right]$$

let $X_2(z) = Y(z)$ ③

$\therefore X_2(z) = z^{-1} [b_1 U(z) - a_1 Y(z) + X_1(z)]$

$X_1(z) = (b_2 U(z) - a_2 Y(z)) z^{-1}$

in T.D

$x_2(k+1) = b_1 u(k) - a_1 x_2(k) + x_1(k)$ ②

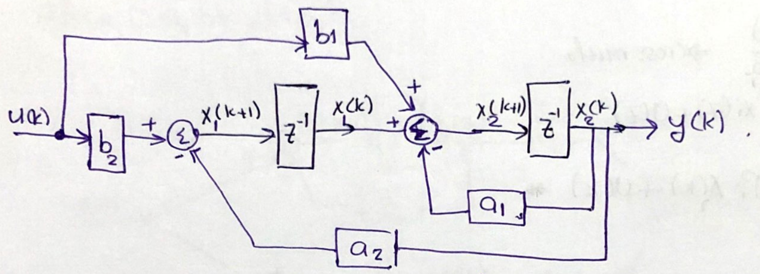
$x_1(k+1) = b_2 u(k) - a_2 x_2(k)$ ①

S.S.R

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \end{bmatrix} u(k)$$

$$y(k) = [0 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} . \qquad \& \; D = 0 .$$

Block Diagram (② , ① eq [illegible]
+③
الصفحة اللي فاتت)



Block Diagram of OCF

③ Diagonal Canonical form proof:

it's used only when the poles of the system are Real.

$$\frac{Y(z)}{U(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n}$$

degree of num = degree of denum , So ⇒ long division.

$$\frac{Y(z)}{U(z)} = b_0 + \frac{b_1' z^{n-1} + b_2' z^{n-2} + \cdots + b_n'}{(z-P_1)(z-P_2)\cdots\cdots(z-P_n)}$$

↓ partial fraction.

$$Y(z) = b_0 U(z) + \left[\frac{C_1}{z-P_1} + \frac{C_2}{z-P_2} + \cdots + \frac{C_n}{z-P_n}\right] U(z) .$$

$$Y(z) = b_0 U(z) + C_1 \frac{U(z)}{z-P_1} + C_2 \frac{U(z)}{z-P_2} + \cdots + C_n \frac{U(z)}{z-P_n}$$

where $C_i = \lim_{z \to P_i} \left(\frac{Y(z)}{U(z)} (z-P_i) \right)$

assume

$$X_1 = \frac{U(z)}{z-P_1} , \quad X_2 = \frac{U(z)}{z-P_2} \quad \cdots , \quad X_n = \frac{U(z)}{z-P_n}$$

in T.D

$$y(k) = b_0 u(k) + C_1 X_1(k) + C_2 X_2(k) + \cdots + C_n X_n(k)$$

for X

$$X_i(z) = \frac{U(z)}{z-P_i} \quad \Rightarrow \text{cross mult.}$$

$$z X_i(z) = P_i X_i(z) + U(z)$$

in T.D

$$\therefore X_i(k+1) = P_i X_i(k) + U(k)$$

S.S.R

$$\underline{X}(k+1) = \begin{bmatrix} P_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & P_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & P_3 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & \cdots & P_n \end{bmatrix} \underline{X}(k) + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [C_1 \ C_2 \ C_3 \ \cdots \ C_n] \underline{X}(k) + \underbrace{[b_0]}_{D} u(k)$$

Example on DCF

$$\frac{Y(z)}{U(z)} = \frac{b_1 z + b_2}{(z-P_1)(z-P_2)} = \frac{C_1}{z-P_1} + \frac{C_2}{z-P_2}$$

$$Y(z) = C_1 \frac{U(z)}{z-P_1} + C_2 \frac{U(z)}{z-P_2}$$

let $X_1(z) = \frac{U(z)}{z-P_1}$ & $X_2(z) = \frac{U(z)}{z-P_2}$

$$Y(z) = C_1 X_1(z) + C_2 X_2(z)$$

for ~~general~~ X

$$\therefore z X_1(z) = P_1 X_1(z) + U(z)$$

$$z X_2(z) = P_2 X_2(z) + U(z).$$

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~~S.S.R~~ in T.D

$y(k) = C_1 X_1(k) + C_2 X_2(k)$

$X_1(k+1) = P_1 X_1(k) + u(k)$

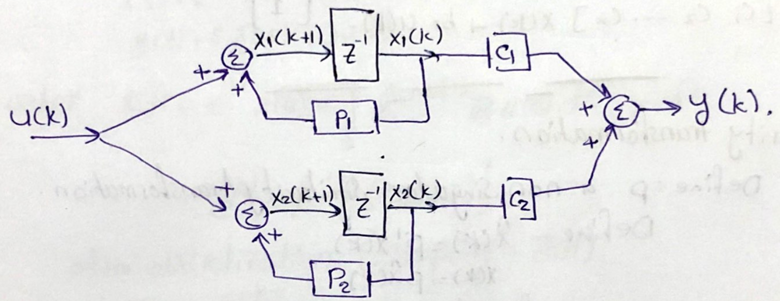
$X_2(k+1) = P_2 X_2(k) + u(k)$

S.S.R

$$X(k+1) = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \underline{X}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$y(k) = [C_1 \quad C_2]\, X(k)$ & $D = 0$

Block Diagram of DCF



for multiple poles:

$$y(z) = b_0 U(z) + C_1 \left\{ \frac{U(z)}{(z-P_1)^3} \right\} + C_2 \left[\frac{U(z)}{(z-P_1)^2} \right] + C_3 \left[\frac{U(z)}{z-P_1} \right] + C_4 \left\{ \frac{U(z)}{(z-P_4)} \right\}$$

$$\cdots + C_n \frac{U(z)}{z-P_n}$$

$X_1(z) = \dfrac{U(z)}{(z-P_1)^3}$

$X_2(z) = \dfrac{U(z)}{(z-P_1)^2}$

$X_3(z) = \dfrac{U(z)}{(z-P_1)}$

$\rightarrow X_1(z) = \dfrac{X_2(z)}{z-P_1}$

$X_2(z) = \dfrac{X_3(z)}{z-P_1}$ → due to multiple poles.

$X_4(z) = \dfrac{U(z)}{z-P_4}$

⋮

in T.D

$$x_1(k+1) = x_2(k) + P_1\, x_1(k)$$
$$x_2(k+1) = x_3(k) + P_1\, x_2(k)$$
$$x_3(k+1) = u(k) + P_1\, x_3(k)$$
$$x_4(k+1) = u(k) + P_4\, x_4(k)$$
$$\vdots$$
$$x_n(k+1) = u(k) + P_n\, x_n(k)$$

SSR.

$$x(k+1) = \begin{bmatrix} P_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & P_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & P_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & P_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & P_n \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [C_1 \; C_2 \; \cdots \; C_n]\, x(k) + b_0\, u(k).$$

---

*Similarity transformation.

Define : p a non singular similarity transformation.

Define $\hat{x}(k) = p^{-1} x(k)$

$x(k) = p\,\hat{x}(k)$

*prove that pulse transfer function & chlc eq. are independent on system representation (invariant).

$$\hat{T}(z) = \hat{C}(zI - \hat{G})^{-1}\hat{H} + \hat{D}$$

$$\hat{C} = Cp, \; \hat{G} = p^{-1}Gp, \; \hat{D} = D \; \& \; \hat{H} = p^{-1}H$$

$$\therefore \hat{T}(z) = Cp\,(\underset{p^{-1}p}{zI} - p^{-1}Gp)^{-1} p^{-1}H + D$$

$$= Cp\,[zp^{-1}p - p^{-1}Gp]^{-1} p^{-1}H + D$$

$$= Cp\,[p^{-1}(zp - Gp)]^{-1} p^{-1}H + D$$

$$= Cp\,(zp - Gp)^{-1}\, \underbrace{p\, p^{-1}}_{I} H + D$$

$$= Cp\,((zI - G)p)^{-1} H + D$$

$$= Cp\,p^{-1}(zI - G)^{-1}H + D = C(zI - G)^{-1}H + D = T(z)$$

$(AB)^{-1} = B^{-1}A^{-1}$

$(B^{-1})^{-1} = B$

* prove that $|zI-G| = |zI-\hat{G}|$

$$RHS = |zI-\hat{G}| = |zI-P^{-1}GP|$$

$$= |zPP^{-1}-P^{-1}GP| = |P^{-1}(zP-GP)|$$

$$= |P^{-1}(zI-G)P|$$

$$= |PP^{-1}(zI-G)| = |zI-G| = LHS$$

$$|ABC| = |CAB| = |BCA| = |ABC|$$

<u>Discretization</u>

in Cont.

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

where $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau$ — ①

$$e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}.$$

after discretization (sampling + ZOH)

$$x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)} B\,u(\tau)\,d\tau \quad \text{— ②}$$

$$x((k+1)T) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)} B\,u(\tau)\,d\tau \quad \text{— ③}$$

② * e^{AT}

$$e^{AT}x(kT) = e^{A(k+1)T}x(0) + \int_0^{kT} e^{A((k+1)T-\tau)} B\,u(\tau)\,d\tau \quad \text{— ④}$$

③ − ④

$$x((k+1)T) - e^{AT}x(kT) = \int_0^{(k+1)T} e^{A(k+1)T-\tau)} B\,u(\tau)\,d\tau - \int_0^{kT} e^{A(k+1)T-\tau)} B\,u(\tau)\,d\tau$$

$$\int_0^{T_2} f(t)\,dt - \int_0^{T_1} f(t)\,dt = \int_{T_1}^{T_2} f(t)\,dt \quad \text{where } T_1 < T_2.$$

$$\therefore X((k+1)T) - e^{AT} X(kT) = \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B\,u(\tau)\,d\tau .$$

$$X((k+1)T) = e^{AT} X(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B\,u(\tau)\,d\tau .$$

$$= e^{AT} X(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B\,u(kT)\,d\tau \qquad u(t) = u(kT)$$

$u(kT)$ is Const from kT to $(k+1)T$

$$\therefore X((k+1)T) = e^{AT} X(kT) + \left[\int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B\,d\tau\right] u(kT)$$

let: $(k+1)T - \tau = \lambda$

$d\lambda = -d\tau .$

$$\therefore X((k+1)T) = e^{AT} x(kT) - \int_{T}^{0} e^{A\lambda} B\,d\lambda \;\; u(kT)$$

$$\boxed{\therefore X((k+1)T) = e^{AT} X(kT) + \int_{0}^{T} e^{A\lambda} B\,d\lambda \;\; u(kT)}$$

$\therefore G = e^{AT}$ $\qquad C = C$

$H = \int_{0}^{T} e^{A\lambda} B\,d\lambda$ $\qquad$ & $D = D$ $\qquad$ in ~~...~~ cont.

$G^k = e^{AkT} = Z^{-1}\{(ZI - G)^{-1} Z\}$ (من المحاضرة اللي كاتب تحسب فيها الـ PTF)

$\Psi(k) = G^k \rightarrow$ state transition matrix.

*Deadbeat Response:

assume CCF system

choose $\lambda(G-HK)=0$

$\therefore X(k)=0$ for $k\leq n$.

proof

choose $\lambda_1=\lambda_2=\dots=\lambda_n=0$

$\therefore (z-\lambda_1)(z-\lambda_2)(z-\lambda_3)\dots(z-\lambda_n)=z^n=0$

$$\boxed{\therefore \alpha_i\text{'s}=0}$$

$\therefore k_i\text{'s}=\alpha_i\text{'s}-a_i$

$\boxed{\therefore k_i=-a_i}$, $k=[-a_n \quad -a_{n-1} \quad \dots \quad -a_1]$

$$G-HK=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 \dots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & \dots & \dots & 1 \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix} \rightarrow \text{Nilpotent Matrix}$$

$\therefore (G-HK)^n=0$

$\therefore x(k)=(G-HK)^k x(0) \rightarrow 0$ for $k\leq n$

So, the state will go to zero @ $k\leq n$ where k is the no. of samples & n is the order of the system.

For originel system.

$$X(n)=(G-HK)^n x(0) \Rightarrow \left(T_{ccf} G_{ccf} T_{ccf}^{-1} - T_{ccf} H_{ccf} K_{ccf} T_{ccf}^{-1}\right)^n x(0)$$

$$= T_{ccf}^n \underbrace{[G_{ccf}-H_{ccf}K_{ccf}]^n}_{=0} T_{ccf}^{-1\,n} \; x(0)$$

$\therefore X(n)\rightarrow 0$ @ $k\leq n$.

useful transformation:

[1] transformation to CCF

① construct W matrix

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & 1 & 0 & \cdots & 0 \\ a_1 & & & & \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

② Construct controllability matrix.

$$C = [\,H \quad GH \quad G^2H \cdots G^{n-1}H\,]$$

③ $T_{ccf} = CW$

④ $\boxed{G_{ccf} = T_{ccf}^{-1}\, G\, T_{ccf}}$ $\boxed{C_{ccf} = C\, T_{ccf}}$

$$H_{ccf} = T_{ccf}^{-1} H$$

[2] Transformation to OCF

① construct W matrix (same as ccf)

② construct O matrix

$$O = \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{n-1} \end{bmatrix}$$

③ $T_{ocf} = (WO)^{-1}$

④ $\boxed{G_{ocf} = T_{ocf}^{-1}\, G\, T_{ocf}}$ $\boxed{C = C\, T_{ocf}}$

$$\boxed{H = T_{ocf}^{-1} H}$$

CCF proof problem A-6-5

Consider the completely state controllable system

$$X(k+1) = GX(k) + HU(k)$$

Controllability matrix as

$$M = [H \quad GH \quad G^2H \dots G^{n-1}H]$$

show that
$$M^{-1}GM = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -a_{n-2} \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & & 1 & -a_1 \end{bmatrix}$$

حسا يزليب نثبت الشكل ده :.

Solution:
Consider 3rd order system $n=3$

$$M = [H \quad GH \quad G^2H]$$

Req. to proof that $M^{-1}GM = \begin{bmatrix} 0 & 0 & a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$

هضرب في M من الشمال

$$\therefore GM = M \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \rightarrow$$ عايزاثبت ده

$$LHS = GM = G[H \quad GH \quad G^2H] = [GH \,\vdots\, G^2H \,\vdots\, G^3H]$$

$$RHS = [H \quad GH \quad G^2H] \begin{bmatrix} 0 & 0 & a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$$

$$= [GH : G^2H : \quad -a_3H - a_2GH - a_1G^2H] \text{ — (1)}$$

from Cayly-Hamilton theorem.

$$G^n + a_1G^{n-1} + \dots + a_{n-1}G + a_nI = 0$$

for 3rd order

$$G^3 + a_1G^2 + a_2G + a_3 = 0$$

$$G^3 = -a_1G^2 - a_2G - a_3$$

from (1) $\therefore RHS = [GH : G^2H : G^3H] = LHS.$

→ the preceeding derivation can be generalized for system with order n.

Consider The completely state Controllable sys. problem A-6-6

$$x(k+1) = G\,x(k) + H\,u(k)$$

& Controllable matrix.

$$M = [H \;\vdots\; GH \;\vdots\; \cdots \;\vdots\; G^{n-1}H]$$

define.

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & & & & \\ a_1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Transformation matrix

$$T = MW$$

show that

$$T^{-1}GT = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & & \cdots & -a_1 \end{bmatrix} \rightarrow \text{عايز اثبت}$$

$$T^{-1}H = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \rightarrow \text{عايز اثبت}$$

Consider 3rd order system.

$$\because T = MW$$

$$\therefore (MW)^{-1}G(MW) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

$$LHS = W^{-1}M^{-1}GMW \qquad \text{from last Proof.}$$

$$M^{-1}GM = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$$

$$\therefore LHS = W^{-1}\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} W = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

* W from left side

$$\therefore LHS = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \underset{\text{given}}{W' = W} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

$$LHS = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}\begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$RHS = \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

$$= \begin{bmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = LHS.$$

$$\therefore T^{-1} G T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \quad 1^{st} \text{ Req.} \quad \#$$

2^{nd} Req $\quad T^{-1} H = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow$ هضرب في T
مسالشال

$\longrightarrow * T$

$$\therefore H = T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = MW \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [H \; GH \; G^2H] \underbrace{\begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{3\times 3 \;\; 3\times 1}$$

$$[H \; G \; G^2H] * \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \{ \; = H$$

$\therefore RHS = LHS.$

$$\therefore T^{-1} H = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\rightarrow \;\; \therefore T^{-1} G T = G_{ccf}$

$\rightarrow \;\; \therefore T^{-1} H = H_{ccf} \;\;$ where $T = MW$
↳ controllability matrix.

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OCF. proofs

Consider completely observable system.

$$x(k+1) = G\,x(k)$$
$$y(k) = C\,x(k)$$

Define observability matrix $N = [C^* : G^*C^* : \dots : (G^*)^{n-1}C^*]$

$* \rightarrow$ transpose.

* في المحاضرة كنا واخدينها انها O وكانت الTranspose من N

$\therefore O = N^*$

Show that.

$$N^* G (N^*)^{-1} = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{vmatrix}$$

proof

for 3rd order system.

$$N^* G (N^*)^{-1} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{vmatrix}$$

هضرب في N^* عند اليمين

$$N^* G = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{vmatrix} N^*$$

$O \swarrow$

$$LHS = \begin{vmatrix} C \\ CG \\ CG^2 \end{vmatrix} G = \begin{vmatrix} CG \\ CG^2 \\ CG^3 \end{vmatrix}$$

$$RHS = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{vmatrix} \begin{vmatrix} C \\ CG \\ CG^2 \end{vmatrix} = \begin{vmatrix} CG \\ CG^2 \\ -a_3 C - a_2 CG - a_1 CG^2 \end{vmatrix}$$

from Cayly Hamilton theory.

$$G^3 + a_1 G^2 + a_2 G + a_3 I = 0$$

$$\therefore G^3 = -a_1 G^2 - a_2 G - a_3$$

$$C(-a_3 - a_2 G - a_1 G^2) = CG^3$$

$$\therefore RHS = \begin{vmatrix} CG \\ CG^2 \\ CG^3 \end{vmatrix} = LHS \quad \#$$

$$\therefore \text{generally } N^* G (N^*)^{-1} = \begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & & & & 1 \\ -a_n & -a_{n-1} & & -a_{n-2} & \cdots & -a_1 \end{vmatrix}$$

problem A-6-9

Consider the Completely state Controllable & observable System given by.

$$x(k+1) = G\, x(k) + H u(k)$$

$$y(k) = C x(k) + D u(k).$$

Define: $N = |C^* : G^* C^* : \cdots : (G^*)^{n-1} C^*|$

$$W = \begin{vmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & & & & \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 \cdots & 0 \end{vmatrix}$$

$Q = (W N^*)^{-1} \Rightarrow T_{ocf} = (WO)^{-1}$ نفس اللي في الموجة

show that $Q^{-1} G Q = \begin{vmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 1 & -a_1 \end{vmatrix}$

show that $CQ = |0 \ 0 \ \cdots \ 0 \ 1|$

$\to Q^{-1} H = H_{ocf}.$

Consider 3rd order system.

① $Q^{-1} G Q = \begin{vmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{vmatrix}$

$Q^{-1} = ((W N^*)^{-1})^{-1}$

$\cancel{= (N^*)^{-1} W^{-1}}$

$= W N^*$

$$Q^{-1}GQ = \omega \underbrace{N^* G}_{\text{from previous proof}} (N^*)^{-1} \omega^{-1} = \begin{vmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{vmatrix}$$

$$\omega \cdot \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{vmatrix} \omega^{-1} = \begin{vmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{vmatrix}$$

هضرب في ω من اليمين

$$\omega \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{vmatrix} \omega \quad , \quad \omega = \begin{vmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$LHS = \begin{vmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{vmatrix} = \begin{vmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$RHS = \begin{vmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{vmatrix} \begin{vmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$\therefore RHS = LHS$. #.

② Req. to prove $CQ = |0 \;\; 0 \;\; 1|$

$$C(\omega N^*)^{-1} = |0 \;\; 0 \;\; 1|$$

هضرب في (ωN^*) من اليمين

$\therefore C = |0 \;\; 0 \;\; 1| \omega N^*$

$$RHS = |0 \;\; 0 \;\; 1| \begin{vmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} C \\ CG \\ CG^2 \end{vmatrix} = |1 \;\; 0 \;\; 0| \begin{vmatrix} C \\ CG \\ CG^2 \end{vmatrix} = C$$

$= RHS$.

(6) Req. to prove that $Q^{-1}H = H_{ocf}$

$\because \hat{x}(k+1) = Q^{-1}GQ\,\hat{x}(k) + Q^{-1}H\,u(k)$

$y(k) = CQ\,\hat{x}(k) + D\,u(k)$

assume $H_{ocf} = \begin{vmatrix} \gamma_3 \\ \gamma_2 \\ \gamma_1 \end{vmatrix}$

$$G_{ocf} = \begin{vmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{vmatrix}, \quad C_{ocf} = |0 \;\; 0 \;\; 1|$$

$$T.F = C(zI-G)^{-1}H_{ocf} + D$$

$$(zI-G)^{-1} = \begin{vmatrix} z & 0 & a_3 \\ -1 & z & a_2 \\ 0 & -1 & z+a_1 \end{vmatrix}^{-1} = \frac{1}{z^3+a_1z^2+a_2z+a_3} \begin{vmatrix} z^2+a_1z+a_2 & -a_3 & -a_3z \\ z+a_1 & z^2+a_1z & -a_2z-a_3 \\ 1 & z & z^2 \end{vmatrix}$$

$$TF = \frac{1}{z^3+a_1z^2+a_2z+a_3} \, |1 \;\; z \;\; z^2| \begin{vmatrix} \gamma_3 \\ \gamma_2 \\ \gamma_1 \end{vmatrix} + D.$$

$$= \frac{\gamma_3 + z\gamma_2 + z^2\gamma_1}{z^3+a_1z^2+a_2z+a_3} + b_0$$

$$= \frac{b_0z^3 + (\gamma_1+a_1b_0)z^2 + (\gamma_2+a_2b_0)z + (\gamma_3+a_3b_0)}{z^3+a_1z^2+a_2z+a_3}$$

$$= \frac{b_0z^3 + b_1z^2 + b_2z + b_3}{z^3+a_1z^2+a_2z+a_3}, \quad \therefore \gamma_1 = b_1 - a_1b_0$$

$$\gamma_2 = b_2 - a_2b_0$$

$$\gamma_3 = b_3 - a_3b_0$$

$$\therefore Q^{-1}H = \begin{vmatrix} b_1 - a_1b_0 \\ b_2 - a_2b_0 \\ b_3 - a_3b_0 \end{vmatrix} \quad @\; b_0 = 0$$

معلش ؟

$$\therefore Q^{-1}H = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix} = H_{ocf}$$

اشكل ال PTF في الكتاب اللي فيه الاثبات مختلف عن اللي في المحاضرة

$$PTF = \frac{b_0z^n + b_1z^{n-1} + \cdots + b_{n-1}z + b_n}{z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n}.$$

وجود ال b_0 هيأثر في ال H

ازاي ماشفنا

Pole placement.

proof of gain matrix K

$X(k+1) = G\,X(k) + H\,u(k)$ —①

$u(k) = -K\,X(k)$ sub in ①.

$X(k+1) = (G - HK)\,X(k)$ → closed loop form.

$$G_{ccf} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots \\ 0 & \cdots & & & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_1 \end{bmatrix}, \quad \& \quad H_{ccf} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$K_{ccf} = [k_n \quad k_{n-1} \quad \cdots \quad k_1]$

$$HK = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \\ k_n & k_{n-1} & \cdots & k_1 \end{bmatrix}, \quad G - HK = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & 1 \\ -a_n - k_n & -a_{n-1} - k_{n-1} & \cdots & & -a_1 - k_1 \end{bmatrix}$$

c/ch eqn. $|zI - (G - HK)|$

$$= \begin{vmatrix} z & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & z & -1 & 0 & 0 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & & & & -1 \\ a_n + k_n & a_{n-1} + k_{n-1} & \cdots & & & & z + a_1 + k_1 \end{vmatrix}$$

after simplification

$z^n + (a_1 + k_1)z^{n-1} + (a_2 + k_2)z^{n-2} + \cdots + (a_n + k_n)$ ①

Desired c/ eq.

$(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)\cdots(z - \lambda_n) = 0$

$z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \cdots + \alpha_n = 0$ ②

Compare ① & ②

DATE / / OBJECT

$\alpha_i = a_i + k_i$

$\boxed{\therefore k_i = \alpha_i - a_i}$ for $i = 1, 2, 3, \cdots n$

if the system not on CCF

$u = -kx = -k_{ccf}\,\hat{x}$

$kx = k_{ccf}\,\hat{x}$

$K\,T_{ccf}\,\hat{x} = k_{ccf}\,\hat{x}$ $\boxed{\therefore K = k_{ccf}\,T_{ccf}^{-1}}$