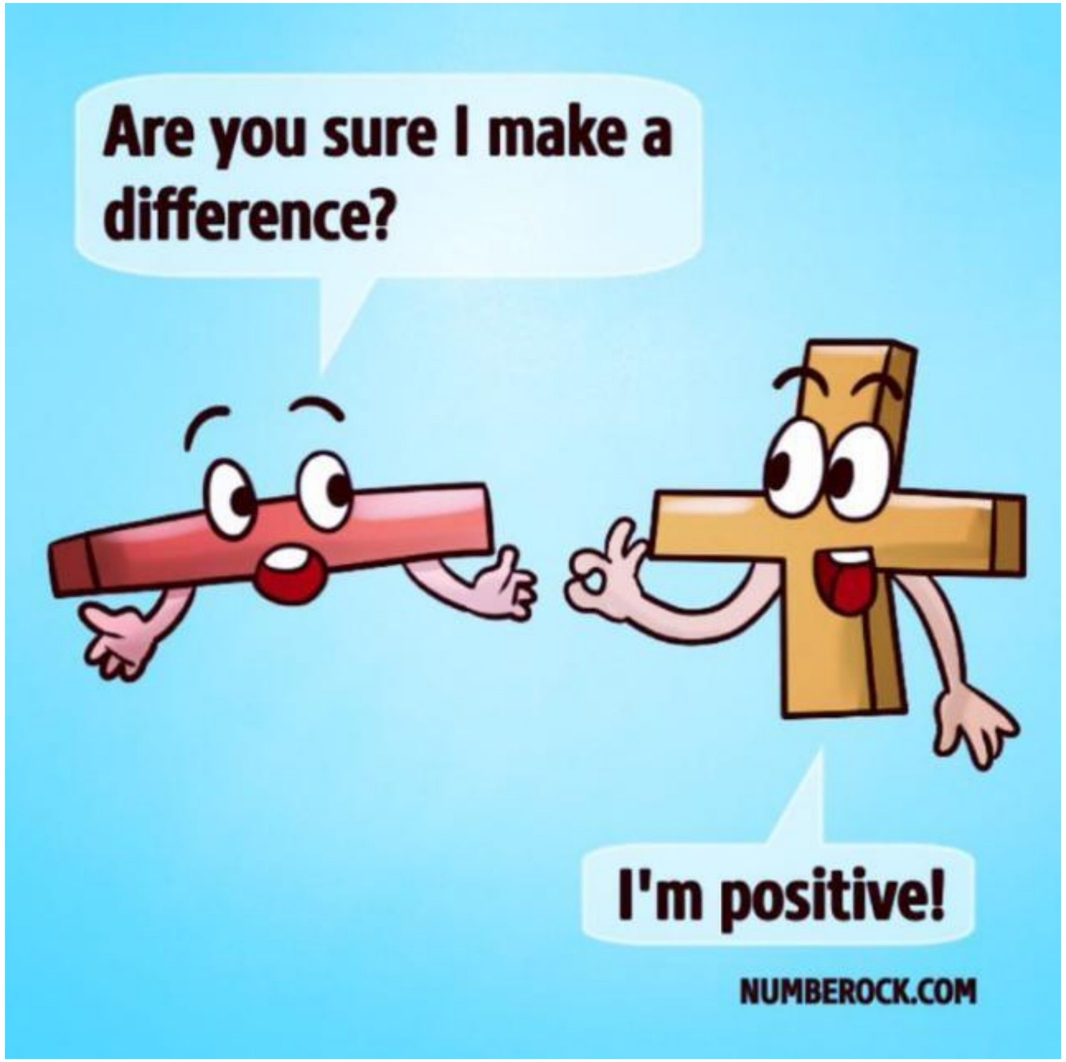
PROBABILITY THEORY

Have you noticed our WhatsApp group icon?

Have you seen the uncertainty?

Negative sign is not sure about his existence while positive sign have confidence upon him.



WHAT IS PROBABILTY:

The term probability refers to the study of randomness and uncertainty. In any situation in which one of the number of possible outcomes may occur, the discipline of probability provides methods for quantifying the chances, or likelihoods, associated with the various outcomes. The language of probability is constantly used in an informal manner in both written and spoken contexts.

For example, probably it will rain today, Pakistan will win the test match. There is an uncertain element in our statements. In statistics, we have a technique or tool which is used to measure the amount of uncertainty, called Probability.

Therefore, probability is a measure of degree of belief in a particular statement or probability is a measure of the chance that an uncertain event will occur.

There are three approaches to construct a measure of probability of occurrence of an event, they are

- Classical Approach
- Empirical or Frequency or Statistical Approach
- Axiomatic Approach

We will discuss the first and third ones.

CLASSICAL DEFINITION OF PROBABILTY:

If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A, then the probability of event A is

$$P(A) = \frac{n}{N}.$$

Or

Probability of an event A is denoted by P(A) and is defined by:

$$P(A) = \frac{Number\ of\ outcomes\ in\ event\ A}{Number\ of\ outcomes\ in\ sample\ space\ S} = \frac{n(A)}{n(S)}$$

AXIOMETIC DEFINITION OF PROBABILTY:

If A is any event in a sample space S, then

1: $0 \le P(A) \le 1$

2: $P(S) = 1$

3: $P(\phi) = 0$

4: $P(A') = 1 - P(A)$

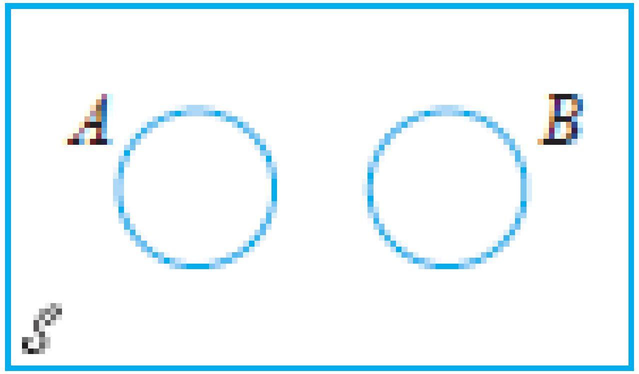
5: $If\ A_1, A_2, \ldots, A_n\ is\ a\ sequence\ of\ mutually\ exlusive\ events, then$

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

Furthermore,

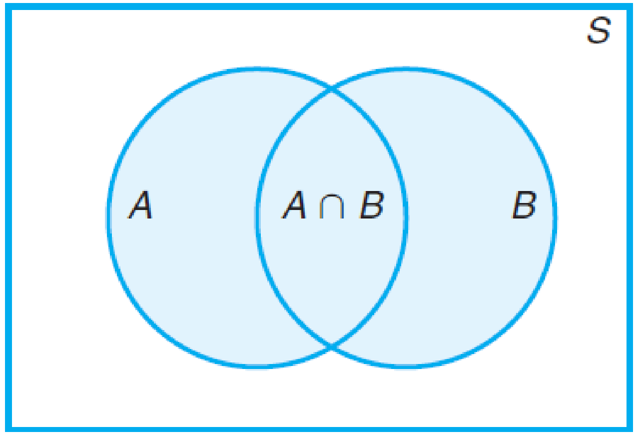
If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$



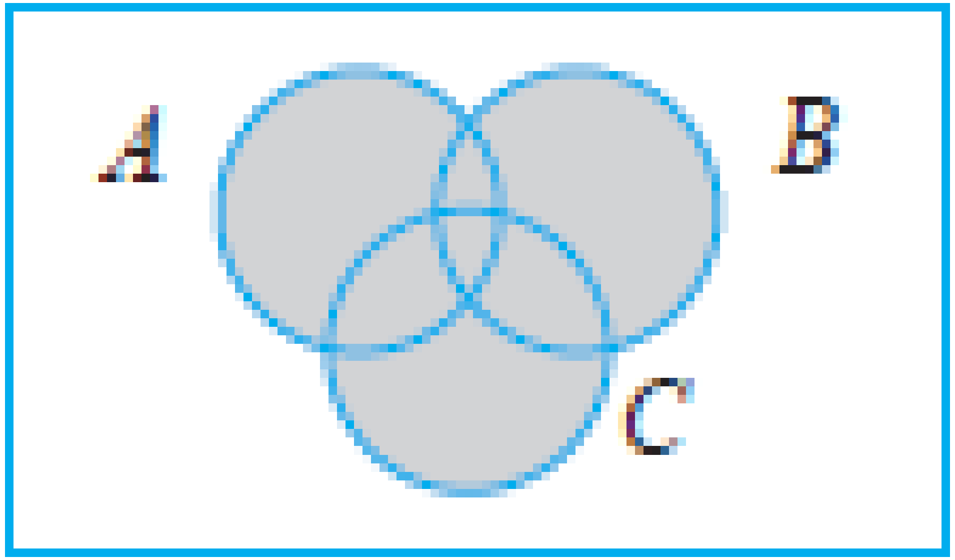
If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



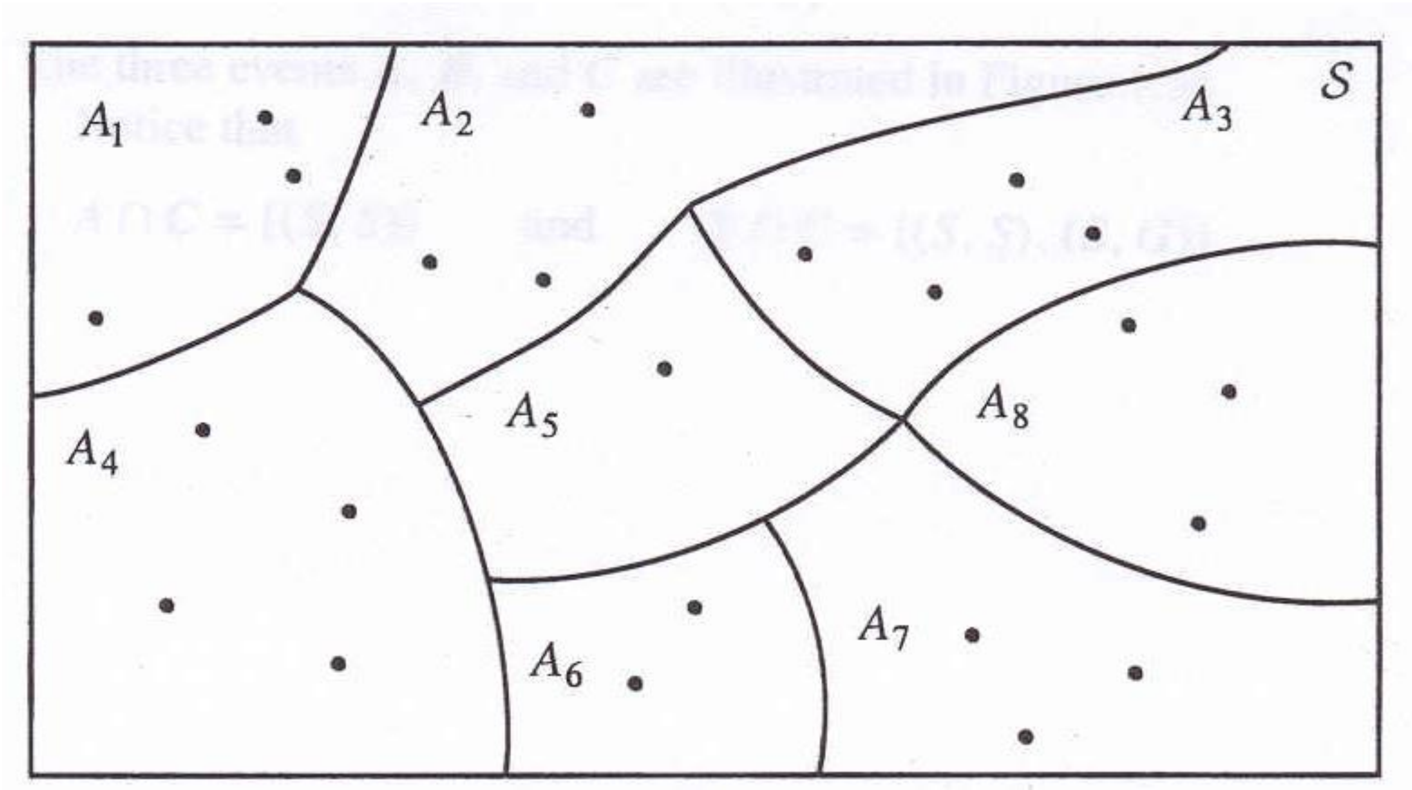
For three events A, B, and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$



If $A_1,\ A_2, \ldots, A_n$ is a partition of sample space S, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = P(S) = 1.$$



If the events $A_1, A_2, \ldots, A_k$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2)\cdots P(A_k).$$

Example 1:

A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is (a) an industrial engineering major and (b) a civil engineering or an electrical engineering major.

Denote by I, M, E, and C the students majoring in industrial, mechanical, electrical, and civil engineering, respectively. The total number of students in the class is 53, all of whom are equally likely to be selected.

Given:

$$n(I)=25, \qquad n(E)=10$$

$$n(M)=10, \qquad n(C)=8$$

$$n=25+10+10+8=53$$

Solution:

(a) Since 25 of the 53 students are majoring in industrial engineering, the probability of event I, selecting an industrial engineering major at random, is

$$P(I) = \frac{25}{53}.$$

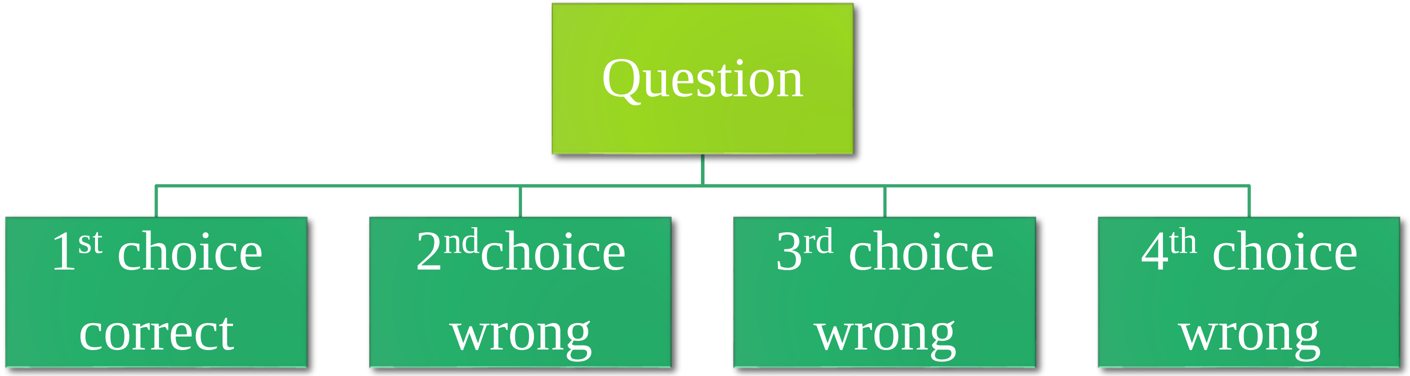
(b) Since 18 of the 53 students are civil or electrical engineering majors, it follows that

$$P(C \cup E) = \frac{18}{53}.$$

Example 2:

A quizz has 5 multiple-choice questions. Each question has 4 answer choices, of which 1 is correct answer and the other 3 are incorrect. Suppose that you guess all the answers.

(a) How many ways are there to answer the 5 questions?
(b) What is the probability of getting all 5 questions right?
(c) What is the probability of getting exactly 4 questions right and 1 wrong?
(d) What is the probability of doing well (getting at leat 4 right)?



Solution:

(a) We can look at this question as a decision consisting of five steps. There are 4 ways to do each step so that by the Fundamental Principle of Counting there are

$$(4)(4)(4)(4)(4) = 1024 \text{ ways}$$

(b) There is only one way to answer each question correctly. Using the Fundamental Principle of Counting there is $(1)(1)(1)(1)(1) = 1$ way to answer all 5 questions correctly out of 1024 possibilities. Hence,

$$\text{P(all 5 right)} = \frac{1}{1024}$$

(c) The following table lists all possible responses that involve at least 4 right answers, R stands for right and W stands for a wrong answer

Five Responses	Number of ways to fill out the test
WRRRR	$(3)(1)(1)(1)(1) = 3$
RWRRR	$(1)(3)(1)(1)(1) = 3$
RRWRR	$(1)(1)(3)(1)(1) = 3$
RRRWR	$(1)(1)(1)(3)(1) = 3$
RRRRW	$(1)(1)(1)(1)(3) = 3$

So there are 15 ways out of the 1024 possible ways that result in 4 right answers and 1 wrong answer so that

$$P(4 \text{ right,1 wrong}) = \frac{15}{1024} \approx 1.5\%$$

(d) "At least 4" means you can get either 4 right and 1 wrong or all 5 right. Thus,

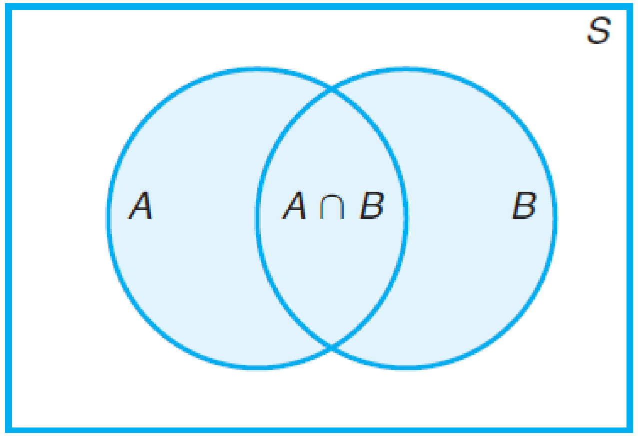
$$P(\text{at least 4 right}) = P(\text{4 right, 1 wrong}) + P(\text{5 right}) = \frac{15}{1024} + \frac{1}{1024} = \frac{16}{1024} \approx 0.016 \blacksquare$$

Example 3:

John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company A is 0.8, and his probability of getting an offer from company B is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, what is the probability that he will get at least one offer from these two companies?

Solution:

$$P(A) = 0.8, \qquad P(B) = 0.6, \qquad P(A \cap B) = 0.5$$



Using the additive rule, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.6 - 0.5 = 0.9.$$

Example 4:

If the probabilities are, respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new automobile will choose the color green, white, red, or blue, what is the probability that a given buyer will purchase a new automobile that comes in one of those colors?

Let G, W, R, and B be the events that a buyer selects, respectively, a green, white, red, or blue automobile. Since these four events are mutually exclusive, the probability is

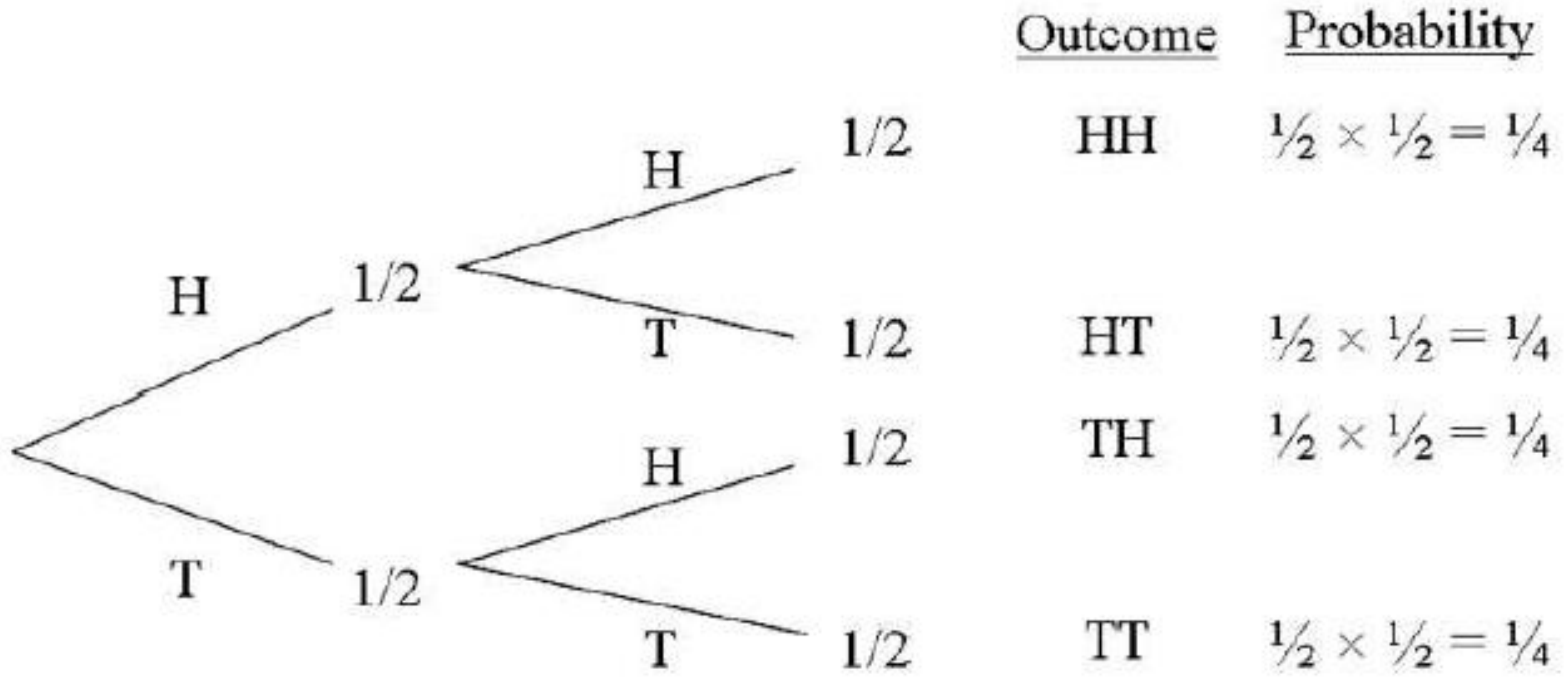
$$\begin{aligned} P(G \cup W \cup R \cup B) &= P(G) + P(W) + P(R) + P(B) \\ &= 0.09 + 0.15 + 0.21 + 0.23 = 0.68. \end{aligned}$$

Example 5:

Construct the probability tree of the experiment of flipping a fair coin twice.

Solution.

The probability tree is shown in Figure



Before start solving the questions from exercise we must understand some basic operators of probability in terms of words of English language.

- Or, either or, anyone implies union of events
- Neither nor implies compliment of union
- And, but, both implies intersection of events
- But not, not and, not both implies compliment of intersection
- Atleast “a” (any value) implies values greater than and equals to “a”
- Atmost “a” (any value) implies values less than and equals to “a”

Exercises

2.49 Find the errors in each of the following statements:

(a) The probabilities that an automobile salesperson will sell 0, 1, 2, or 3 cars on any given day in February are, respectively, 0.19, 0.38, 0.29, and 0.15.

(b) The probability that it will rain tomorrow is 0.40, and the probability that it will not rain tomorrow is 0.52.

(c) The probabilities that a printer will make 0, 1, 2, 3, or 4 or more mistakes in setting a document are, respectively, $0.19, 0.34, -0.25, 0.43$, and 0.29.

(d) On a single draw from a deck of playing cards, the probability of selecting a heart is 1/4, the probability of selecting a black card is 1/2, and the probability of selecting both a heart and a black card is 1/8.

Solution:

(a) Sum of the probabilities exceeds 1.

(b) Sum of the probabilities is less than 1.

(c) A negative probability.

(d) Probability of both a heart and a black card is zero.

2.50 Assuming that all elements of S in Exercise 2.8 on page 42 are equally likely to occur, find

(a) the probability of event A;

(b) the probability of event C;

(c) the probability of event $A \cap C$.

2.8 For the sample space of Exercise 2.4,

(a) list the elements corresponding to the event A that the sum is greater than 8;

(b) list the elements corresponding to the event B that a 2 occurs on either die;

(c) list the elements corresponding to the event C that a number greater than 4 comes up on the green die;

(d) list the elements corresponding to the event $A \cap C$;

(e) list the elements corresponding to the event $A \cap B$;

(f) list the elements corresponding to the event $B \cap C$;

(g) construct a Venn diagram to illustrate the intersections and unions of the events A, B, and C.

First we will recall the sample space of exe.2.4, considering both dice as fair ones. (equally likely outcomes)

2.4 An experiment involves tossing a pair of dice, one green and one red, and recording the numbers that come up. If x equals the outcome on the green die and y the outcome on the red die, describe the sample space S

(a) by listing the elements (x, y);

(b) by using the rule method.

$\mathcal{S}$

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Solution:

(a) $P(A) = \frac{5}{18}$;

(b) $P(C) = \frac{1}{3}$;

(c) $P(A \cap C) = \frac{7}{36}$.

$\mathcal{S}$

(1, 1) 1/36	(1, 2) 1/36	(1, 3) 1/36	(1, 4) 1/36	(1, 5) 1/36	(1, 6) 1/36
(2, 1) 1/36	(2, 2) 1/36	(2, 3) 1/36	(2, 4) 1/36	(2, 5) 1/36	(2, 6) 1/36
(3, 1) 1/36	(3, 2) 1/36	(3, 3) 1/36	(3, 4) 1/36	(3, 5) 1/36	(3, 6) 1/36
(4, 1) 1/36	(4, 2) 1/36	(4, 3) 1/36	(4, 4) 1/36	(4, 5) 1/36	(4, 6) 1/36
(5, 1) 1/36	(5, 2) 1/36	(5, 3) 1/36	(5, 4) 1/36	(5, 5) 1/36	(5, 6) 1/36
(6, 1) 1/36	(6, 2) 1/36	(6, 3) 1/36	(6, 4) 1/36	(6, 5) 1/36	(6, 6) 1/36

2.51 A box contains 500 envelopes, of which 75 contain \$100 in cash, 150 contain \$25, and 275 contain \$10. An envelope may be purchased for \$25. What is the sample space for the different amounts of money? Assign probabilities to the sample points and then find the probability that the first envelope purchased contains less than \$100.

2.52 Suppose that in a senior college class of 500 students it is found that 210 smoke, 258 drink alcoholic beverages, 216 eat between meals, 122 smoke and drink alcoholic beverages, 83 eat between meals and drink alcoholic beverages, 97 smoke and eat between meals, and 52 engage in all three of these bad health practices. If a member of this senior class is selected at random, find the probability that the student

(a) smokes but does not drink alcoholic beverages;

(b) eats between meals and drinks alcoholic beverages but does not smoke;

(c) neither smokes nor eats between meals.

2.52 Given:

$$n = 500$$

$$n(M) = 210,$$

$$n(D) = 258,$$

$$n(E) = 216$$

$$n(M \cap D) = 122,$$

$$n(E \cap D) = 83,$$

$$n(M \cap E) = 97,$$

$$n(M \cap D \cap E) = 52$$

$$n = 500$$

$$n(M) = 210,$$

$$n(D) = 258,$$

$$M \cap D = 122 - 52 = 70$$

$$E \cap D = 83 - 52 = 31$$

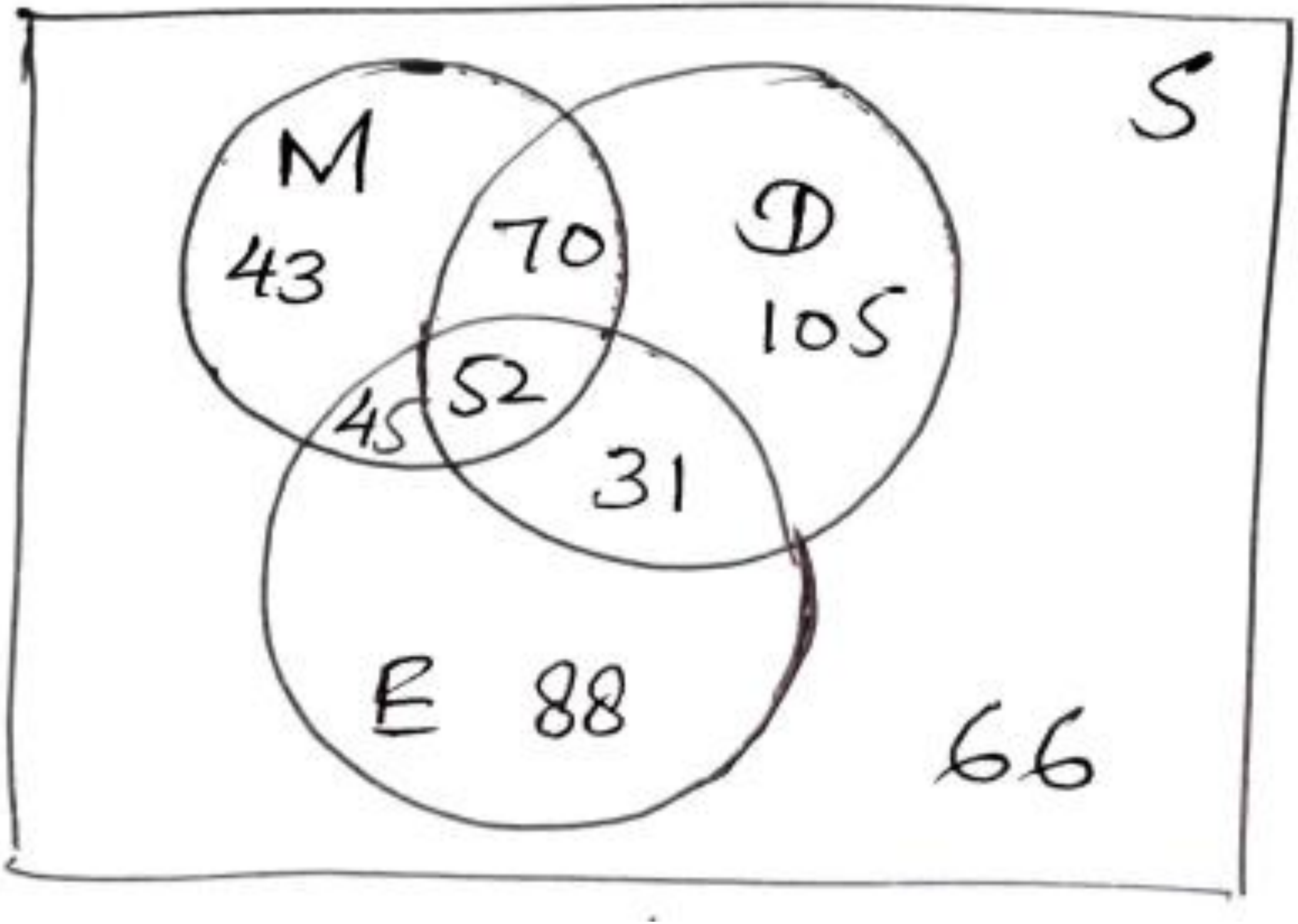
$$M \cap E = 97 - 52 = 45$$

$$M = 210 - (52 + 45 + 70) = 43$$

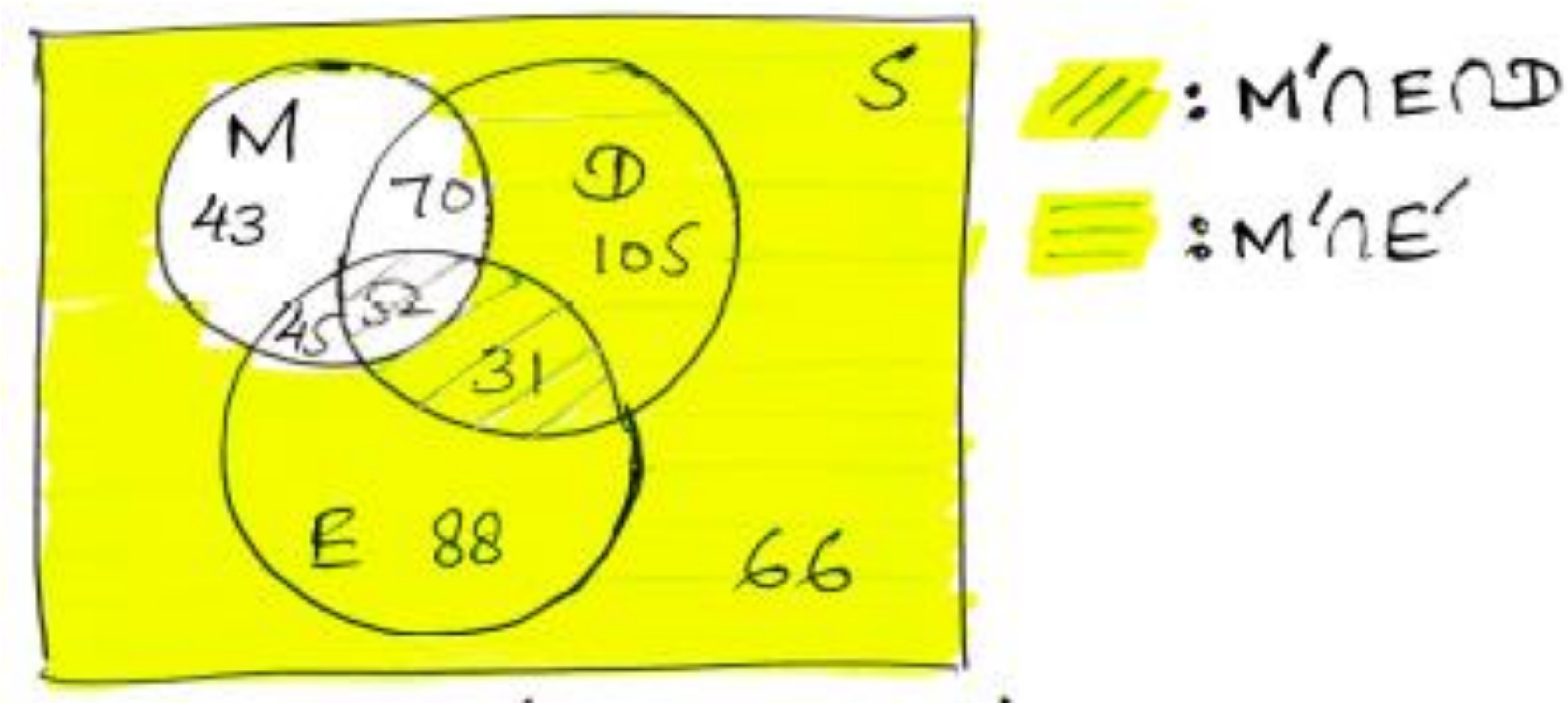
$$D = 258 - (31 + 52 + 70) = 105$$

$$E = 216 - (31 + 52 + 45) = 88$$

$$S = 66$$


S
M
D
E
43
70
105
52
45
31
88
66

Solution:



$$(a): P(M \cap D') = \frac{88}{500}$$

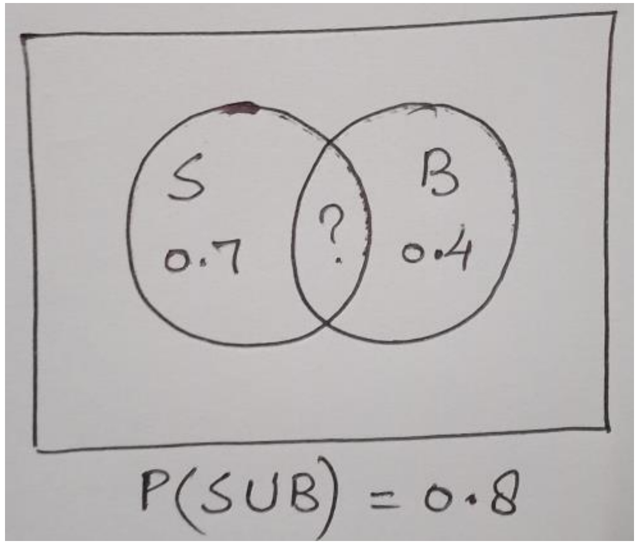
$$(b): P(E \cap D \cap M') = \frac{31}{500}$$

$$(c): P(M' \cap E') = \frac{105+66}{500} = \frac{171}{500}$$

2.53 The probability that an American industry will locate in Shanghai, China, is 0.7, the probability that it will locate in Beijing, China, is 0.4, and the probability that it will locate in either Shanghai or Beijing or both is 0.8. What is the probability that the industry will locate

(a) in both cities?

(b) in neither city?



Solution:

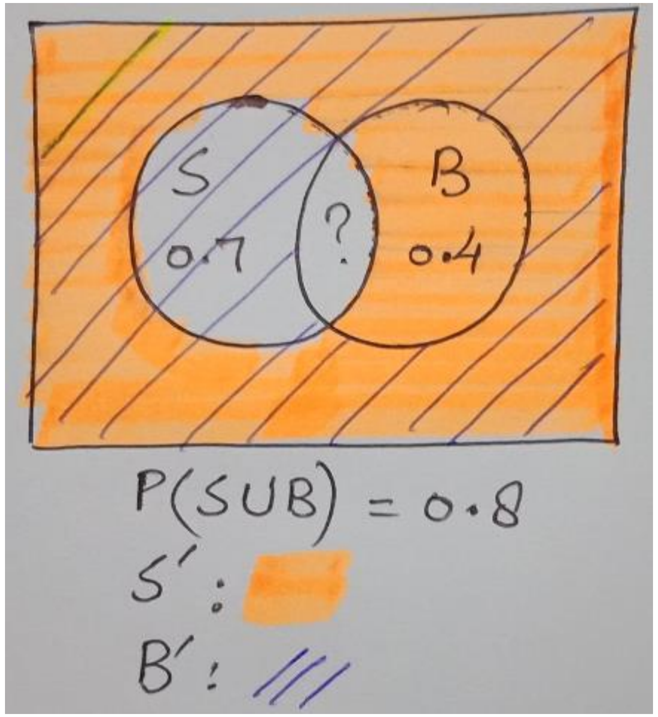
Consider the events

S: industry will locate in Shanghai,

B: industry will locate in Beijing.

(a) $P(S \cap B) = P(S) + P(B) - P(S \cup B) = 0.7 + 0.4 - 0.8 = 0.3.$

(b) $P(S' \cap B') = 1 - P(S \cup B) = 1 - 0.8 = 0.2.$



2.54 From past experience, a stockbroker believes that under present economic conditions a customer will invest in tax-free bonds with a probability of 0.6, will invest in mutual funds with a probability of 0.3, and will invest in both tax-free bonds and mutual funds with a probability of 0.15. At this time, find the probability that a customer will invest

(a) in either tax-free bonds or mutual funds;

(b) in neither tax-free bonds nor mutual funds.

2.55 If each coded item in a catalog begins with 3 distinct letters followed by 4 distinct nonzero digits, find the probability of randomly selecting one of these coded items with the first letter a vowel and the last digit even.

Solution 2.55: (recall the example of number plate)

L1	**L2**	**L3**	**D1**	**D2**	**D3**	**D4**
26	25	24	9	8	7	6
5	25	24	8	7	6	**4**

By multiplication rule:

$$N = (26)(25)(24)(9)(8)(7)(6) = 47174400$$

$$n = (5)(25)(24)(8)(7)(6)(4) = 4032000$$

$$P(given\,event) = \frac{4032000}{47174400} = \frac{10}{117}$$

2.56 An automobile manufacturer is concerned about a possible recall of its best-selling four-door sedan. If there were a recall, there is a probability of 0.25 of a defect in the brake system, 0.18 of a defect in the transmission, 0.17 of a defect in the fuel system, and 0.40 of a defect in some other area.

(a) What is the probability that the defect is the brakes or the fueling system if the probability of defects in both systems simultaneously is 0.15?

(b) What is the probability that there are no defects in either the brakes or the fueling system?

Solve by yourself

2.57 If a letter is chosen at random from the English alphabet, find the probability that the letter

(a) is a vowel exclusive of y;

(b) is listed somewhere ahead of the letter j;

(c) is listed somewhere after the letter g.

y is not a vowel, focus on given information

Ahead remember traffic signs on roads like U-turn ahead

2.58 A pair of fair dice is tossed. Find the probability of getting

(a) a total of 8;

(b) at most a total of 5.

Solution 2.57:

(a) Since 5 of the 26 letters are vowels, we get a probability of $5/26$.

(b) Since 9 of the 26 letters precede j, we get a probability of $9/26$.

(c) Since 19 of the 26 letters follow g, we get a probability of $19/26$.

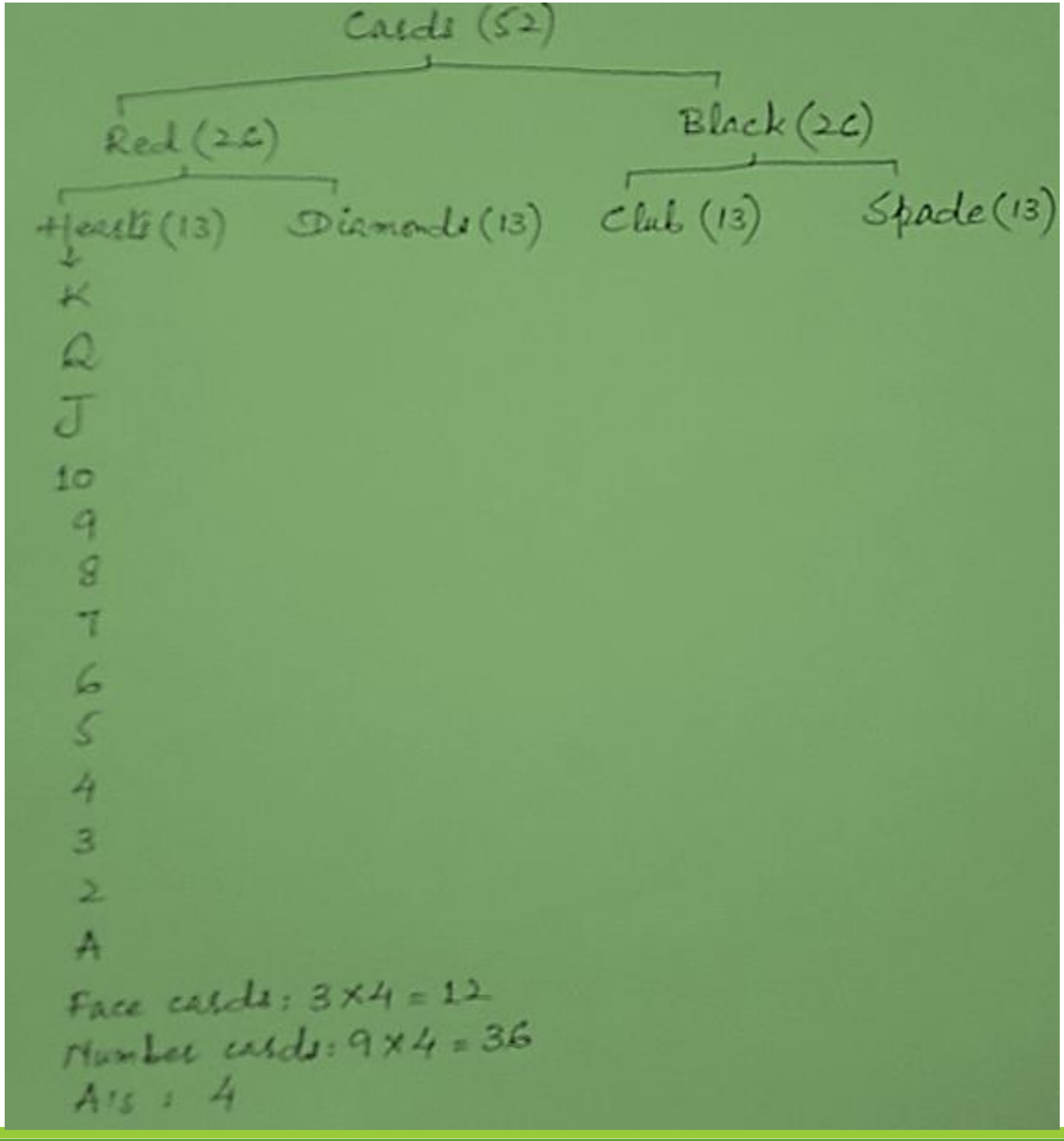
2.59 In a poker hand consisting of 5 cards, find the probability of holding

(a) 3 aces;

(b) 4 hearts and 1 club.

Before finding the solution of this question lets recall how many cards in a deck and what are their types.

There are 52 cards in total in which 26 are of red color and 26 are of black, they are further divided in 4 groups named, diamonds, hearts, spades and clubs. For your convince a detailed tree draw on next slide


Cards (52)
Red (26)
Black (26)
Hearts (13)
Diamonds (13)
Club (13)
Spade (13)
K
Q
J
10
9
8
7
6
5
4
3
2
A
Face cards: 3×4 = 12
Number cards: 9×4 = 36
A's : 4

Solution 2.59:

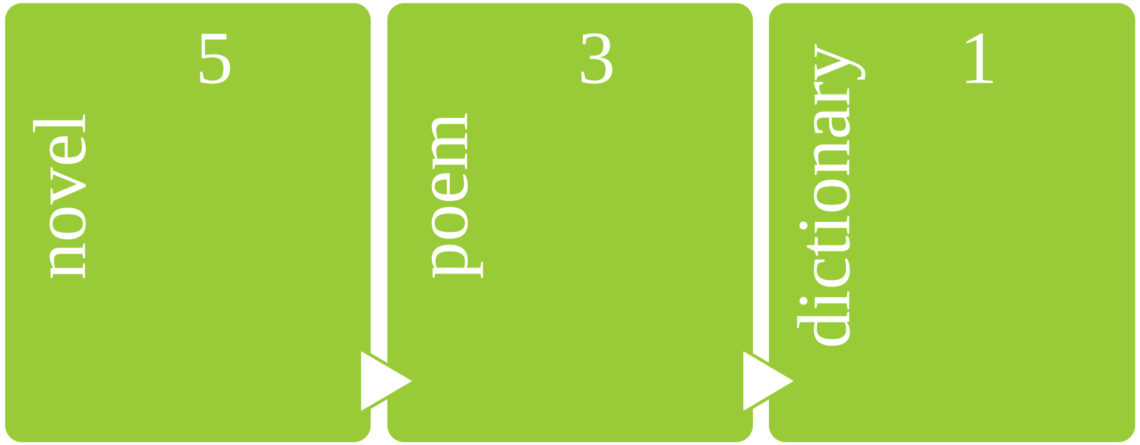
$$(a): P(3\,\text{Aces}) = \frac{{}^{4}C_{3}\,{}^{48}C_{2}}{{}^{52}C_{5}} = \frac{94}{54145}$$

$$(b): P(4\,\text{hearts}\,\text{and}\,1\,\text{club}) = \frac{{}^{13}C_{4}\,{}^{13}C_{1}}{{}^{52}C_{5}} = \frac{143}{39984}$$

2.60 If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary, what is the probability that

(a) the dictionary is selected?

(b) 2 novels and 1 book of poems are selected?



Solution 2.60:

$$(a): \frac{\binom{5}{2}\binom{3}{0}\binom{1}{1}}{\binom{9}{3}} + \frac{\binom{5}{1}\binom{3}{1}\binom{1}{1}}{\binom{9}{3}} + \frac{\binom{5}{0}\binom{3}{2}\binom{1}{1}}{\binom{9}{3}} = \frac{1}{3}$$

$$(b): \frac{\binom{5}{2}\binom{3}{1}}{\binom{9}{3}} = \frac{5}{14}$$

One can also solve this as:

$$(a): \frac{\binom{8}{2}\binom{1}{1}}{\binom{9}{3}} = \frac{1}{3}, \qquad (b): \frac{\binom{5}{2}\binom{3}{1}\binom{1}{0}}{\binom{9}{3}} = \frac{5}{14}$$

2.61 In a high school graduating class of 100 students, 54 studied mathematics, 69 studied history, and 35 studied both mathematics and history. If one of these students is selected at random, find the probability that

(a) the student took mathematics or history;

(b) the student did not take either of these subjects;

(c) the student took history but not mathematics.

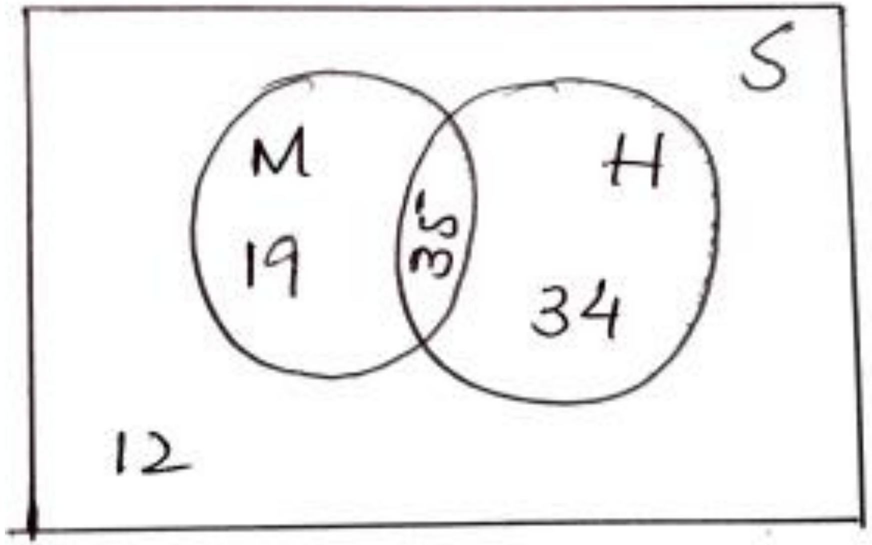
Given:

$$n(S) = 100$$

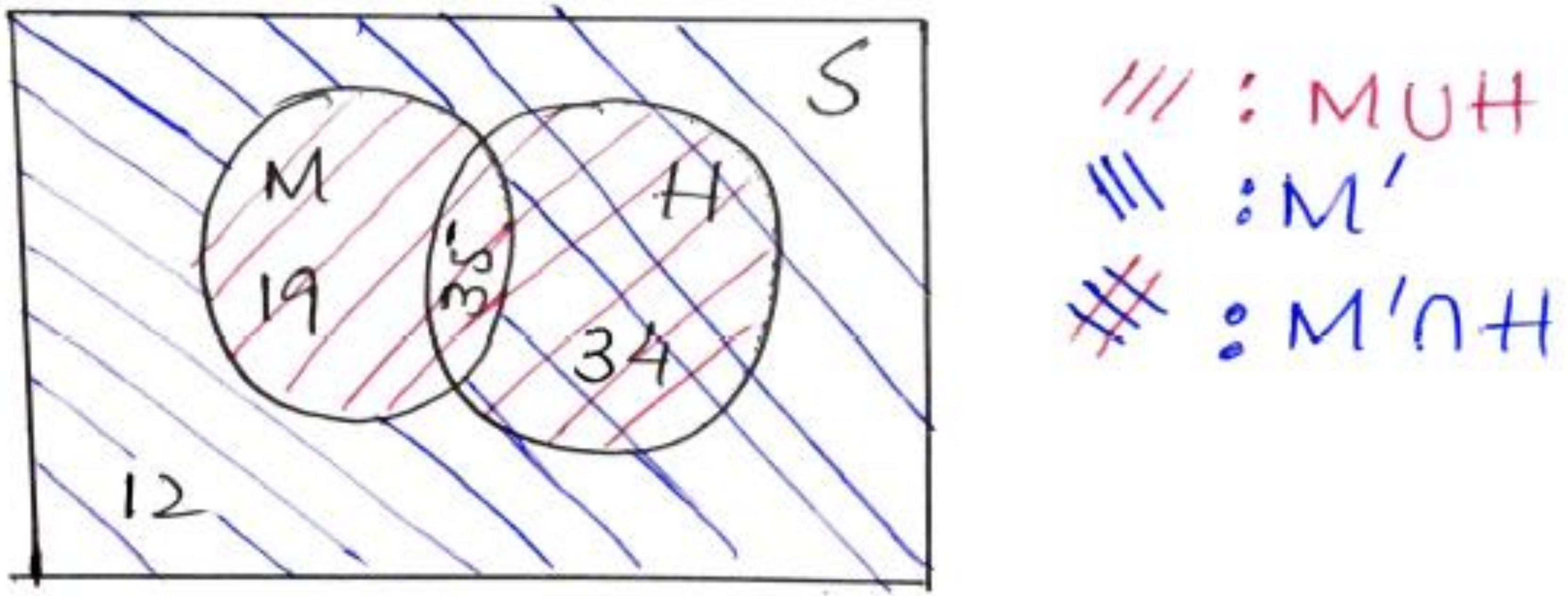
$$n(M) = 54$$

$$n(H) = 69$$

$$n(M \cap H) = 35$$



Solution 2.61:



$$(a): P(M \cup H) = \frac{88}{100}$$

$$(b): P(M' \cap H') = P((M \cup H)') = 1 - P(M \cup H) = \frac{12}{100}$$

$$(c): P(M' \cap H) = \frac{34}{100}$$

2.62 Dom's Pizza Company uses taste testing and statistical analysis of the data prior to marketing any new product. Consider a study involving three types of crusts (thin, thin with garlic and oregano, and thin with bits of cheese). Dom's is also studying three sauces (standard, a new sauce with more garlic, and a new sauce with fresh basil).

(a) How many combinations of crust and sauce are involved?

(b) What is the probability that a judge will get a plain thin crust with a standard sauce for his first taste test?

2.63 According to *Consumer Digest* (July/August 1996), the probable location of personal computers (PC) in the home is as follows:

Adult bedroom:	0.03
Child bedroom:	0.15
Other bedroom:	0.14
Office or den:	0.40
Other rooms:	0.28

(a) What is the probability that a PC is in a bedroom?

(b) What is the probability that it is not in a bedroom?

(c) Suppose a household is selected at random from households with a PC; in what room would you expect to find a PC?

2.66 Factory workers are constantly encouraged to practice zero tolerance when it comes to accidents in factories. Accidents can occur because the working environment or conditions themselves are unsafe. On the other hand, accidents can occur due to carelessness or so-called human error. In addition, the worker's shift, 7:00 A.M.–3:00 P.M. (day shift), 3:00 P.M.–11:00 P.M. (evening shift), or 11:00 P.M.–7:00 A.M. (graveyard shift), may be a factor. During the last year, 300 accidents have occurred. The percentages of the accidents for the condition combinations are as follows:

Shift	Unsafe Conditions	Human Error
Day	5%	32%
Evening	6%	25%
Graveyard	2%	30%

If an accident report is selected randomly from the 300 reports,

(a) what is the probability that the accident occurred on the graveyard shift?

(b) what is the probability that the accident occurred due to human error?

(c) what is the probability that the accident occurred due to unsafe conditions?

(d) what is the probability that the accident occurred on either the evening or the graveyard shift?