CONDITIONAL PROBABILITY

Why we have to learn the concept of conditional probability?

Conditional probability allows us for an alternation of the probability of an event in the light of additional information, it also enables us to understand better the concept of independent events.

CONDITIONAL PROBABILITY:

The **conditional probability** of event ***B*,** given event ***A***, denoted by $P(B \mid A)$, is defined by

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \qquad \text{provided that } P(A) > 0$$

It measures the probability that event ***B*** occurs when it is known that event ***A*** occurs.

Note that:

$$* \, If \;\; A \cap B = \phi \;\; \text{then} \;\; P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0}{P(A)} = 0$$

$$* \, If \;\; A \subset B \;\; \text{then} \;\; A \cap B = A \;\; \text{and then} \;\; P(B \mid A) = \frac{P(A)}{P(A)} = 1$$

Example 1:

Recall question 2.4 of throwing a fair pair of dice, in which one was green and other one was red. If we have two events, defined by

A = { the green die scores a 6 }

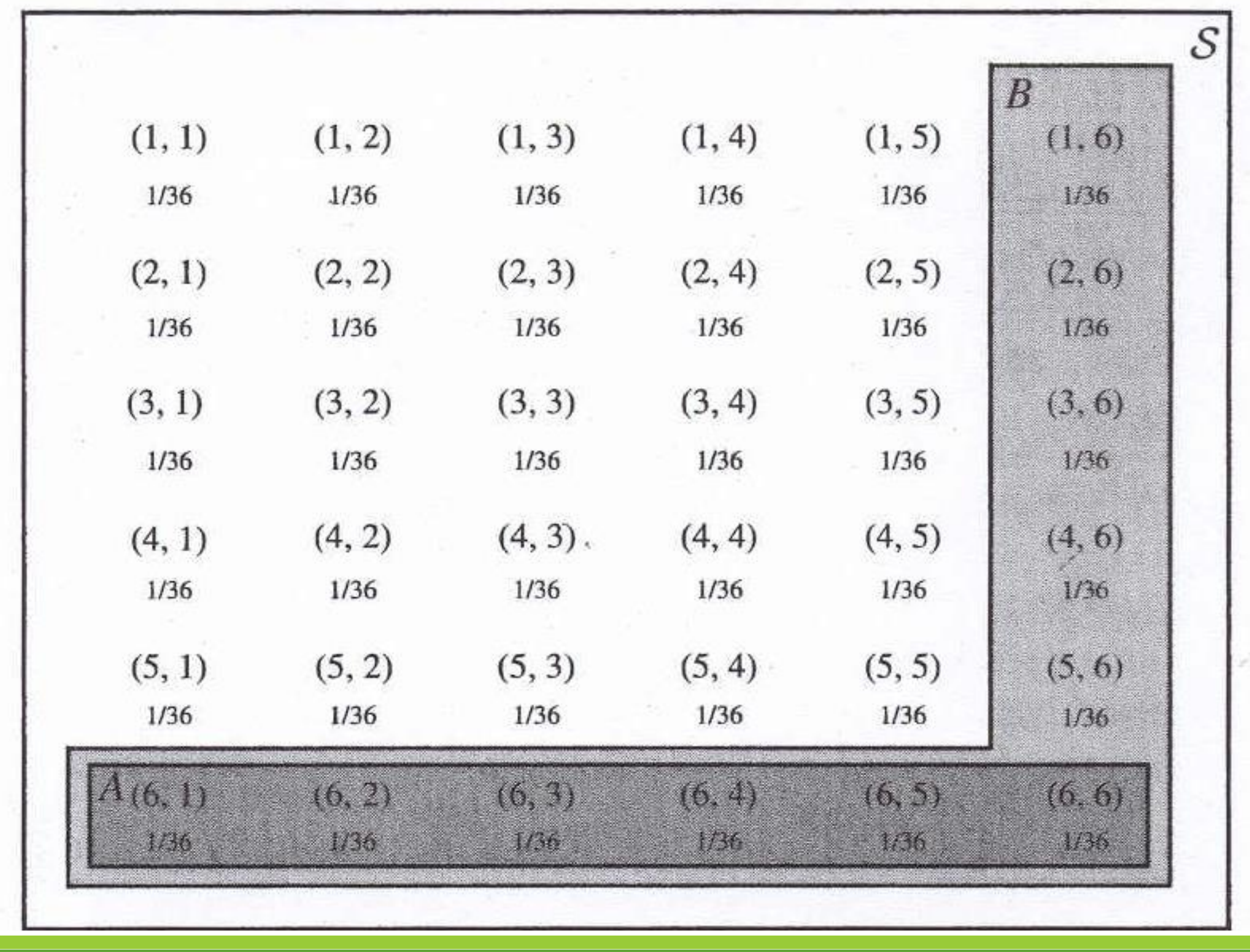
B = { at least one 6 is obtained

on the two dice }

Find the probability

(a): of A given B

(b): of A given that B has only even numbers



Solution (a):

$$P(A)=\frac{6}{36}=\frac{1}{6} \text{ and } P(B)=\frac{11}{36}$$

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

$$=\frac{P(A)}{P(B)}$$

$$=\frac{1/6}{11/36}=\frac{6}{11}$$

Solution (b):

$$P(6)=\frac{1}{6}$$

$$P(6\,|\,even)=\frac{P(6\cap even)}{P(even)}=\frac{P(6)}{P(even)}$$

$$=\frac{P(6)}{P(2)+P(4)+P(6)}=\frac{1/6}{1/6+1/6+1/6}=\frac{1}{3}$$

Example 2:

As an additional illustration, suppose that our sample space S is the population of adults in a small town who have completed the requirements for a college degree. We shall categorize them according to gender and employment status. The data are given in Table 2.1.

Table 2.1: Categorization of the Adults in a Small Town

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

One of these individuals is to be selected at random for a tour throughout the country to publicize the advantages of establishing new industries in the town. We shall be concerned with the following events:

M: a man is chosen,

E: the one chosen is employed.

Using the reduced sample space E, we find that

$$P(M|E) = \frac{460}{600} = \frac{23}{30}.$$

Another approach for finding the probability is by using classical definition of probability

Let $n(A)$ denote the number of elements in any set A. Using this notation, since each adult has an equal chance of being selected, we can write

$$P(M|E) = \frac{n(E \cap M)}{n(E)} = \frac{n(E \cap M)/n(S)}{n(E)/n(S)} = \frac{P(E \cap M)}{P(E)},$$

where $P(E \cap M)$ and $P(E)$ are found from the original sample space S. To verify this result, note that

$$P(E) = \frac{600}{900} = \frac{2}{3} \quad \text{and} \quad P(E \cap M) = \frac{460}{900} = \frac{23}{45}.$$

Hence,

$$P(M|E) = \frac{23/45}{2/3} = \frac{23}{30},$$

as before.

INDEPENDENT EVENTS:

Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent.**

Theorem 1:

If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

Theorem 2:

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

Theorem 3:

If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) \\ = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

Theorem 4:

If the events $A_1, A_2, \ldots, A_k$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2) \cdots P(A_k).$$

The interpretation of two events being independent is that knowledge about one event does not affect the probability of the other event.

Example 1:

Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?
We shall let A be the event that the first fuse is defective and B the event that the second fuse is defective; then we interpret $A \cap B$ as the event that A occurs and then B occurs after A has occurred. The probability of first removing a defective fuse is $1/4$; then the probability of removing a second defective fuse from the remaining 4 is $4/19$. Hence,

$$P(A \cap B) = \left(\frac{1}{4}\right)\left(\frac{4}{19}\right) = \frac{1}{19}.$$

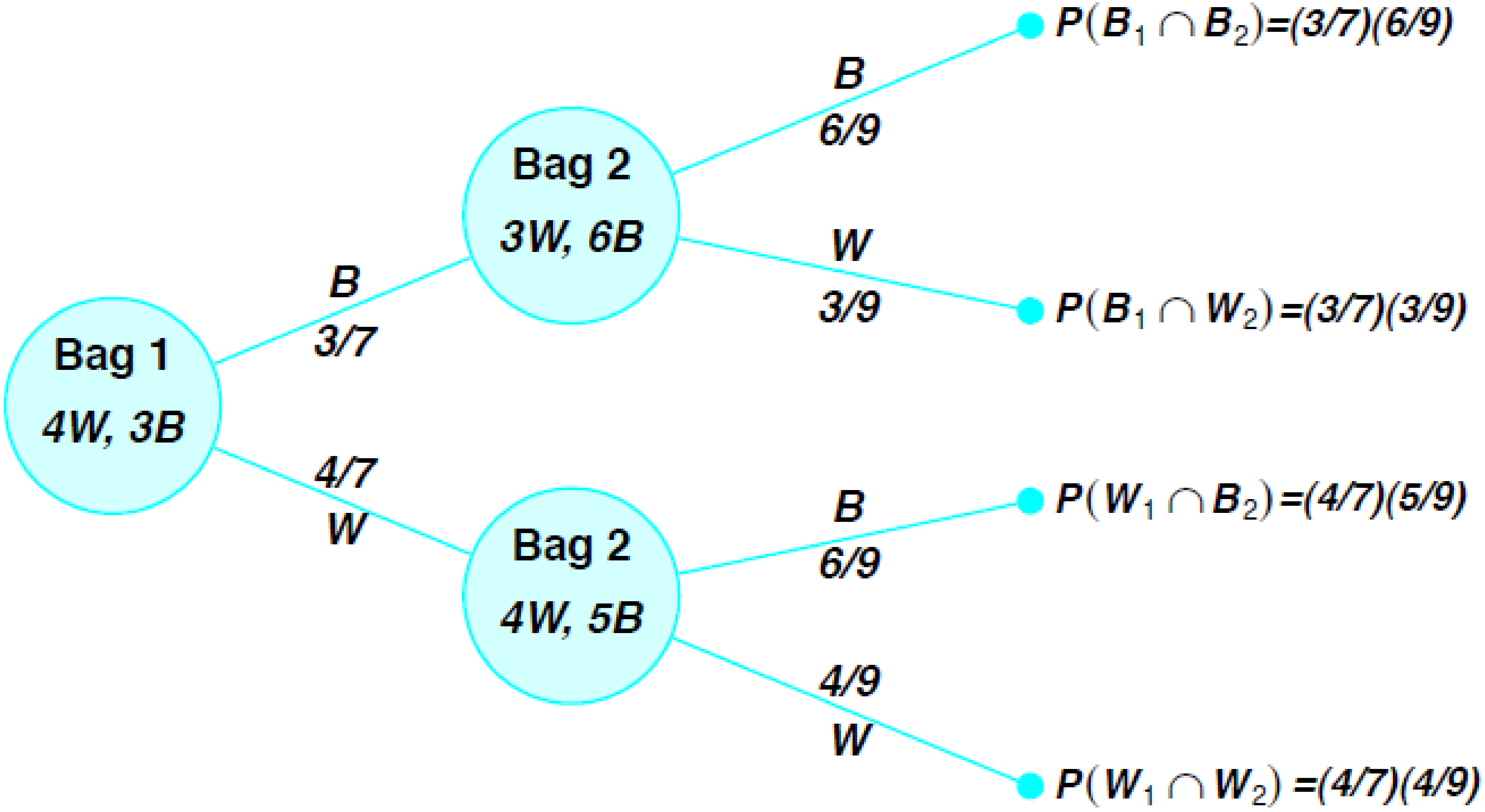
Example 2:

One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Let B_1, B_2, and W_1 represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1. We are interested in the union of the mutually exclusive events $B_1 \cap B_2$ and $W_1 \cap B_2$. The various possibilities and their probabilities are illustrated in Figure 2.8. Now

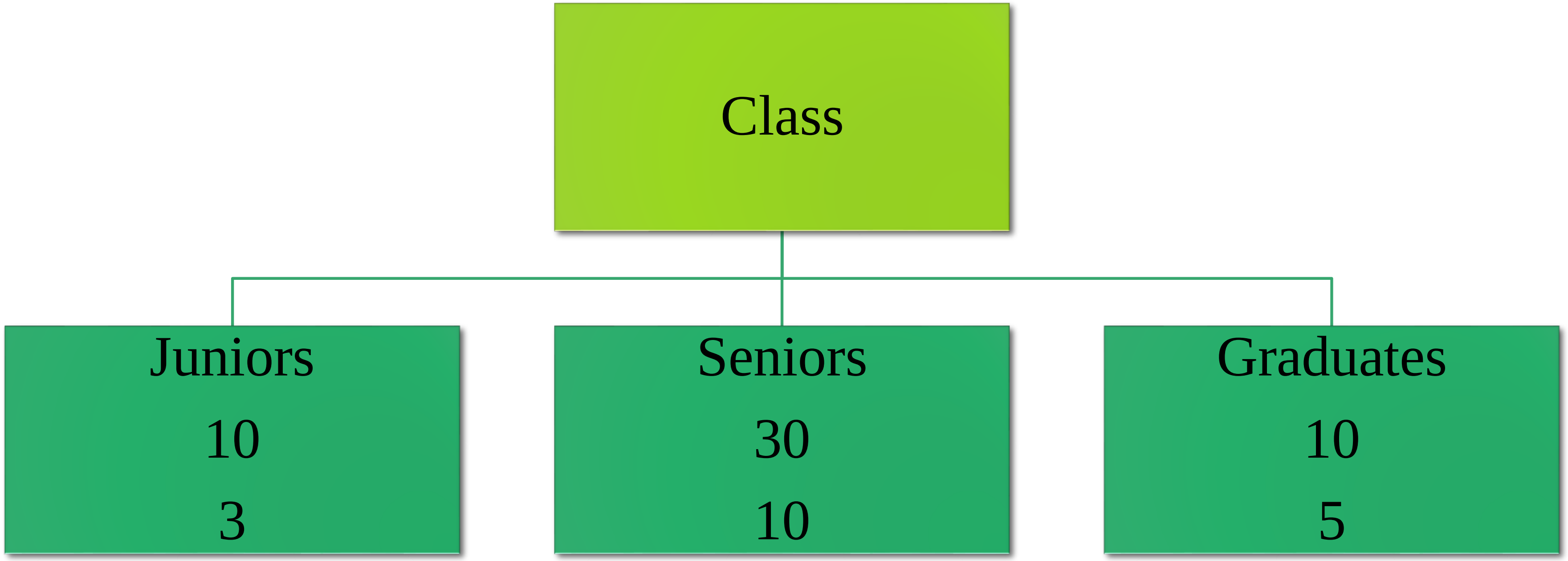
$$\begin{aligned} P[(B_1 \cap B_2) \text{ or } (W_1 \cap B_2)] &= P(B_1 \cap B_2) + P(W_1 \cap B_2) \\ &= P(B_1)P(B_2|B_1) + P(W_1)P(B_2|W_1) \\ &= \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) = \frac{38}{63}. \end{aligned}$$

Possibility tree:



Exercises

2.74 A class in advanced physics is composed of 10 juniors, 30 seniors, and 10 graduate students. The final grades show that 3 of the juniors, 10 of the seniors, and 5 of the graduate students received an A for the course. If a student is chosen at random from this class and is found to have earned an A, what is the probability that he or she is a senior?



Solution:

$$n(J) = 10$$

$$n(E) = 30$$

$$n(G) = 10$$

$$n(S) = 10 + 30 + 10 = 50$$

$$n(A) = 3 + 10 + 5 = 18$$

$$P(E \mid A) = \frac{10}{18}$$

or one can solve this as :

$$P(E \mid A) = \frac{P(E \cap A)}{P(A)} = \frac{10/50}{18/50} = \frac{10}{18}$$

2.75 A random sample of 200 adults are classified below by sex and their level of education attained.

Education	Male	Female
Elementary	38	45
Secondary	28	50
College	22	17

Education	Male	Female
Elementary	38	45
Secondary	28	50
College	22	17

If a person is picked at random from this group, find the probability that

(a) the person is a male, given that the person has a secondary education;

(b) the person does not have a college degree, given that the person is a female.

Solution:

M: a person is a male;

S: a person has a secondary education;

C: a person has a college degree.

(a) $P(M \mid S) = 28/78 = 14/39$;

(b) $P(C' \mid M') = 95/112$.

2.76 In an experiment to study the relationship of hypertension and smoking habits, the following data are collected for 180 individuals:

	Nonsmokers	Moderate Smokers	Heavy Smokers
H	21	36	30
NH	48	26	19

where H and NH in the table stand for *Hypertension* and *Nonhypertension*, respectively. If one of these individuals is selected at random, find the probability that the person is

(a) experiencing hypertension, given that the person is a heavy smoker;

(b) a nonsmoker, given that the person is experiencing no hypertension.

Solution:

Consider the events:

A: a person is experiencing hypertension,

B: a person is a heavy smoker,

C: a person is a nonsmoker.

(a) $P(A \mid B) = 30/49$;

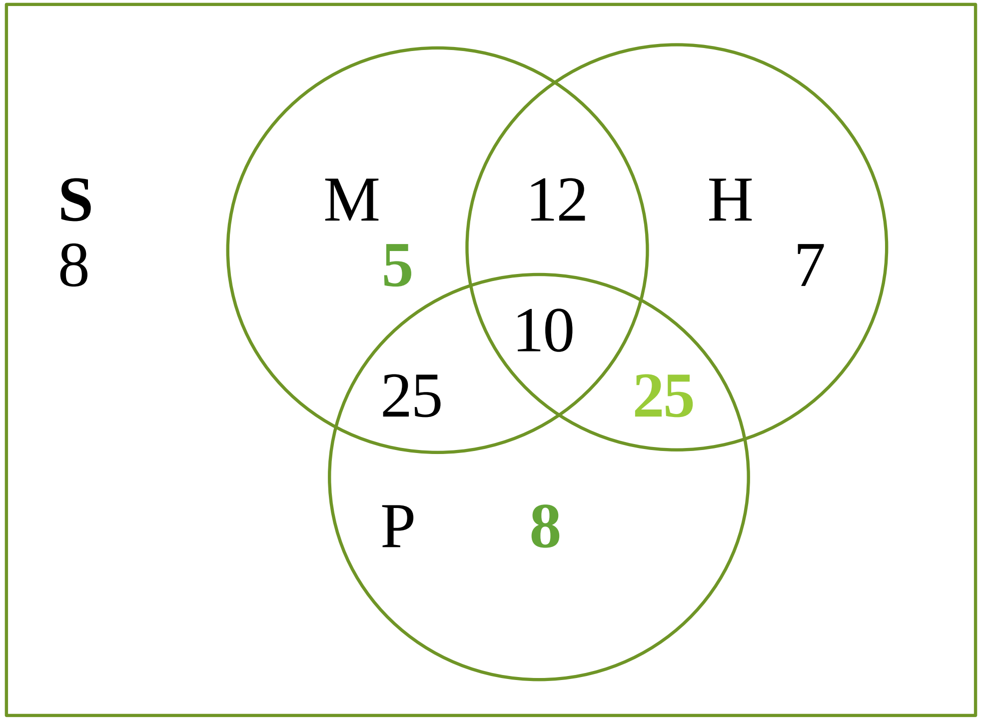
(b) $P(C \mid A') = 48/93 = 16/31$.

2.77 In the senior year of a high school graduating class of 100 students, 42 studied mathematics, 68 studied psychology, 54 studied history, 22 studied both mathematics and history, 25 studied both mathematics and psychology, 7 studied history but neither mathematics nor psychology, 10 studied all three subjects, and 8 did not take any of the three. Randomly select a student from the class and find the probabilities of the following events.

(a) A person enrolled in psychology takes all three subjects.

(b) A person not taking psychology is taking both history and mathematics.

Tip: draw venn diagram, write the correct numbers to the relevant event and solve



Solution:

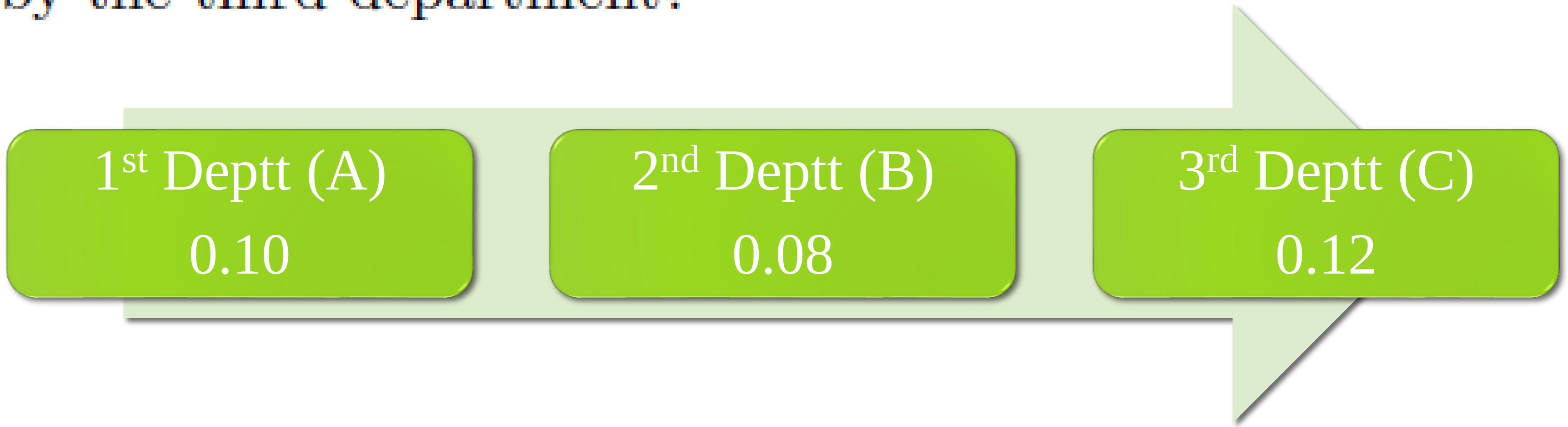
(a) $P(M \cap P \cap H) = \frac{10}{68} = \frac{5}{34}$;

(b) $P(H \cap M \mid P') = \frac{P(H \cap M \cap P')}{P(P')} = \frac{22-10}{100-68} = \frac{12}{32} = \frac{3}{8}$.

2.78 A manufacturer of a flu vaccine is concerned about the quality of its flu serum. Batches of serum are processed by three different departments having rejection rates of 0.10, 0.08, and 0.12, respectively. The inspections by the three departments are sequential and independent.

(a) What is the probability that a batch of serum survives the first departmental inspection but is rejected by the second department?

(b) What is the probability that a batch of serum is rejected by the third department?



Solution:

$$(a): \quad P(A' \cap B) = P(A')P(B)$$
$$= (1 - P(A))P(B)$$
$$= (1 - 0.10)(0.08)$$
$$= (0.9)(0.08)$$
$$= 0.072$$

$$(b): \quad P(A' \cap B' \cap C) = P(A')P(B')P(C)$$
$$= (1 - P(A))(1 - P(B))P(C)$$
$$= (1 - 0.10)(1 - 0.08)(0.12)$$
$$= (0.9)(0.92)(0.12)$$
$$= 0.099$$

2.80 The probability that an automobile being filled with gasoline also needs an oil change is 0.25; the probability that it needs a new oil filter is 0.40; and the probability that both the oil and the filter need changing is 0.14.

(a) If the oil has to be changed, what is the probability that a new oil filter is needed?

(b) If a new oil filter is needed, what is the probability that the oil has to be changed?

2.81 The probability that a married man watches a certain television show is 0.4, and the probability that a married woman watches the show is 0.5. The probability that a man watches the show, given that his wife does, is 0.7. Find the probability that

(a) a married couple watches the show;

(b) a wife watches the show, given that her husband does;

(c) at least one member of a married couple will watch the show.

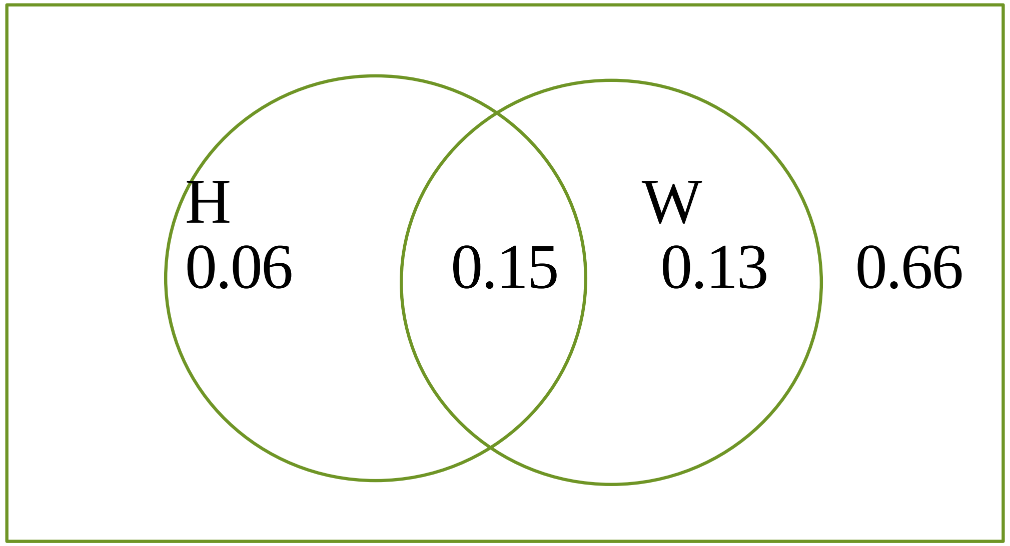
2.82 For married couples living in a certain suburb, the probability that the husband will vote on a bond referendum is 0.21, the probability that the wife will vote on the referendum is 0.28, and the probability that both the husband and the wife will vote is 0.15. What is the probability that

(a) at least one member of a married couple will vote?

(b) a wife will vote, given that her husband will vote?

(c) a husband will vote, given that his wife will not vote?

Solution 2.82:



H: the husband will vote on the bond referendum,
W: the wife will vote on the bond referendum.
Then $P(H) = 0.21$, $P(W) = 0.28$, and $P(H \cap W) = 0.15$.

(a) $P(H \cup W) = P(H) + P(W) - P(H \cap W) = 0.21 + 0.28 - 0.15 = 0.34.$

(b) $P(W \mid H) = \frac{P(H \cap W)}{P(H)} = \frac{0.15}{0.21} = \frac{5}{7}.$

(c) $P(H \mid W') = \frac{P(H \cap W')}{P(W')} = \frac{0.06}{0.72} = \frac{1}{12}.$

2.85 The probability that a doctor correctly diagnoses a particular illness is 0.7. Given that the doctor makes an incorrect diagnosis, the probability that the patient files a lawsuit is 0.9. What is the probability that the doctor makes an incorrect diagnosis and the patient sues?

Solution: Now this a bit different question.

Let D: doctor makes a correct diagnosis

P: Patient sues

Given: $P(D) = 0.7$

$$\Rightarrow P(D') = 1 - P(D) = 0.3$$

$$P(P \mid D') = 0.9$$

$$P(D' \cap P) = P(D')P(P \mid D') = (0.3)(0.9) = 0.27$$

2.89 A town has two fire engines operating independently. The probability that a specific engine is available when needed is 0.96.

(a) What is the probability that neither is available when needed?

(b) What is the probability that a fire engine is available when needed?

Solution:

Let A and B represent the availability of each fire engine.

(a) $P(A' \cap B') = P(A')P(B') = (0.04)(0.04) = 0.0016.$

(b) $P(A \cup B) = 1 - P(A' \cap B') = 1 - 0.0016 = 0.9984.$

2.92 Suppose the diagram of an electrical system is as given in Figure 2.10. What is the probability that the system works? Assume the components fail independently.

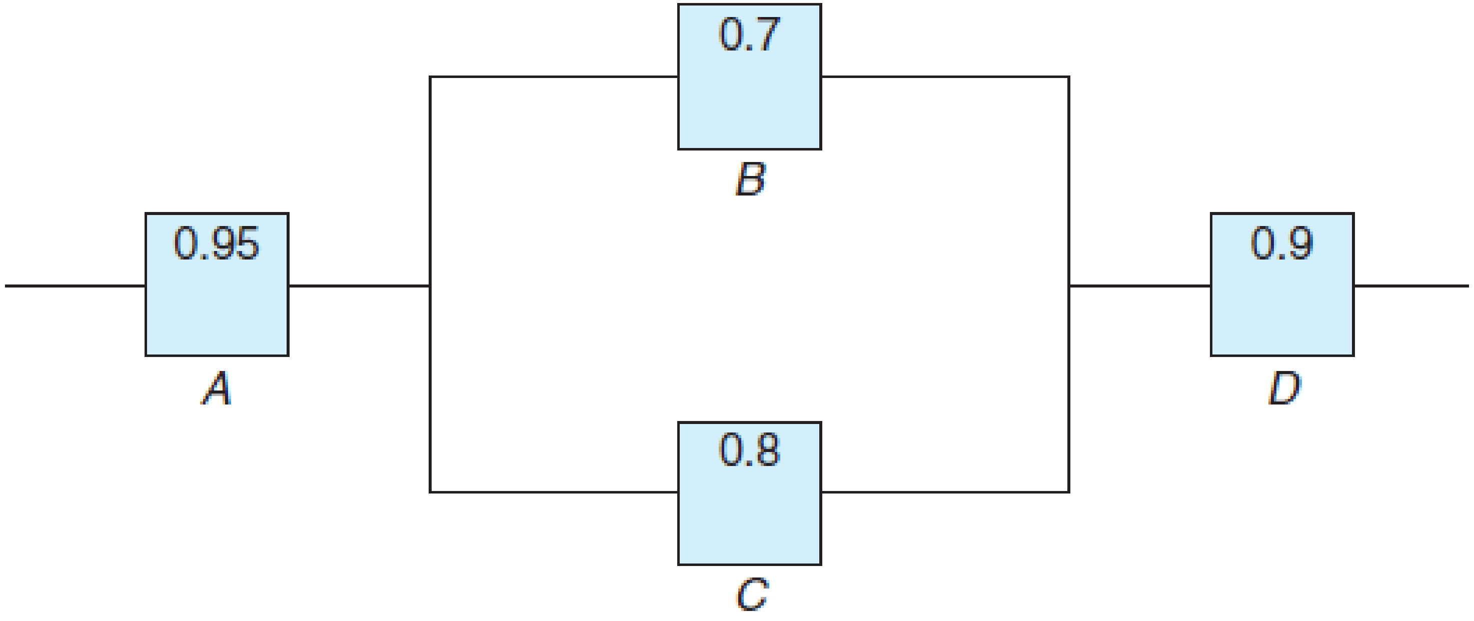


Figure 2.10: Diagram for Exercise 2.92.

Note that this is a basic circuit diagram in which A,B,D and A,C,D are in series while BC are parallel. System works either B or C, or both are working.

Therefore, system will work when A, B and D are working or when A, C and D are working or both. Hence our probability equation will be

$$P(A\cap(B\cup C)\cap D)=P(A)P(B\cup C))P(D)$$

$$=P(A))(1-P(B\cup C)')P(D) \qquad \because P(A)=1-P(A')$$

$$=P(A))(1-P(B'\cap C'))P(D) \qquad \because (A\cup B)'=A'\cap B'$$

$$(De\,Morgan's\,law)$$

$$=P(A))(1-P(B')P(C'))P(D)$$

$$=P(A))(1-(1-P(B))(1-P(C)))P(D)$$

$$=(0.95)(1-(1-0.7)(1-0.8))(0.9)$$

$$=(0.95)(1-(0.3)(0.2))(0.9)$$

$$=(0.95)(1-(0.94))(0.9)$$

$$=0.8037$$

2.93 A circuit system is given in Figure 2.11. Assume the components fail independently.

(a) What is the probability that the entire system works?

(b) Given that the system works, what is the probability that the component A is not working?

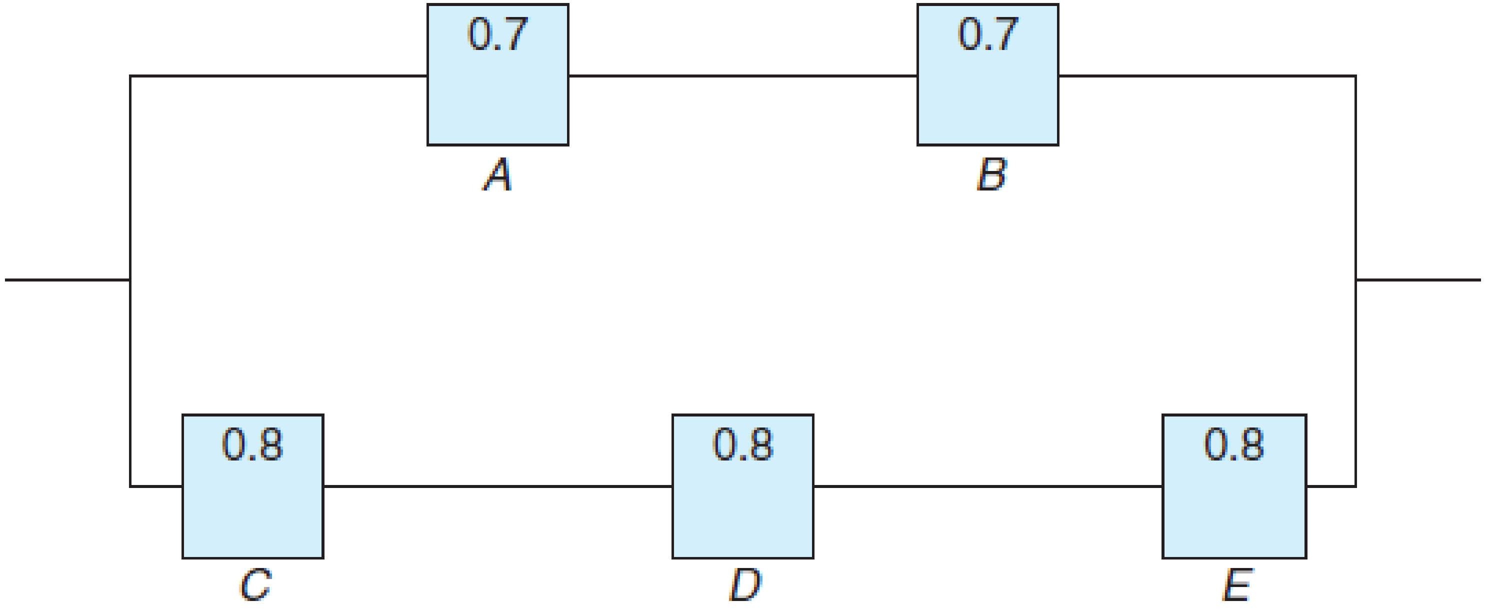


Figure 2.11: Diagram for Exercise 2.93.

Note that in this diagram A,B and C,D,E are in series while A,B and C,D,E are parallel. System is a parallel system of two series subsystems. Therefore, P(System works) will be

$$P((A\cap B)\cup(C\cap D\cap E))=1-P[\{(A\cap B)\cup(C\cap D\cap E)\}']$$

$$=1-\{P((A\cap B)'\cap(C\cap D\cap E)')\} \quad \because P(A)=1-P(A')$$

$$=1-\{P((A\cap B)')P((C\cap D\cap E)')\} \quad \because (A\cup B)'=A'\cap B'$$

$$=1-\{1-P(A\cap B)\}\{1-P(C\cap D\cap E)\}$$

$$=1-\{1-P(A)P(B)\}\{1-P(C)P(D)(E)\}$$

$$=1-\{1-(0.7)(0.7)\}\{1-(0.8)(0.8)(0.8)\}$$

$$=1-\{1-0.49\}\{1-0.512\}$$

$$=1-\{0.51\}\{0.488\}$$

$$=1-0.24888=0.75112$$

$$(b): P(\text{component } A \text{ is not working given that system is working}) = \frac{P(A' \cap C \cap D \cap E)}{P(\text{system works})}$$

$$= \frac{P(A')P(C)P(D)P(E)}{0.75112}$$

$$= \frac{\{1 - P(A)\}P(C)P(D)P(E)}{0.75112}$$

$$= \frac{(0.3)(0.8)(0.8)(0.8)}{0.75112}$$

$$= \frac{0.1536}{0.75112}$$

$$= 0.2044946214$$

$$\cong 0.2045$$

2.94 In the situation of Exercise 2.93, it is known that the system does not work. What is the probability that the component A also does not work?

Now you can solve this by yourself.