# COUNTING TECHNIQUES

Statisticians used the term experiment to describe any process that generates the set of data. Before start studying probability theory we have to discuss some basic terminologies, concepts and tools of probability.

**SAMPLE SPACE:**

The set of all possible outcomes of a statistical experiment is called a **Sample Space** and represented by S.

Each element in a sample space is called **element** or **sample point**. We write sample space as sets.

**EVENT:**

Any subset of a sample space is called an **Event**, represented by A, B, C etc (capital alphabets).

Keep in mind that we can have discrete or continuous sample spaces and events.

So, now first of all we recall some basic concepts of set theory and then we will discuss some counting techniques for counting the number of possible outcomes of an experiment.

# SOME DEFINITIONS FROM SET THEORY:

1. The **complement** of an event $A$, denoted by $A'$, is the set of all outcomes in $\mathcal{S}$ that are not contained in $A$.
2. The **union** of two events $A$ and $B$, denoted by $A \cup B$ and read "*A or B*," is the event consisting of all outcomes that are *either in A or in B or in both events* (so that the union includes outcomes for which both $A$ and $B$ occur as well as outcomes for which exactly one occurs)—that is, all outcomes in at least one of the events.
3. The **intersection** of two events $A$ and $B$, denoted by $A \cap B$ and read "*A and B*," is the event consisting of all outcomes that are in *both A and B*.

# TYPES OF EVENTS:

## 1. Mutually Exclusive Events:

Let $\varnothing$ denote the *null event* (the event consisting of no outcomes whatsoever). When $A \cap B = \varnothing$, $A$ and $B$ are said to be **mutually exclusive** or **disjoint** events.

Two events $A$ and $B$ are mutually exclusive, or disjoint, if $A \cap B = \phi$, that is, if $A$ and $B$ have no elements in common.

## 2. Independent Events:

Two or more events are said to be independent, in a series of an experiment if the outcome of one event does not affect the outcome of the other event and vise versa.

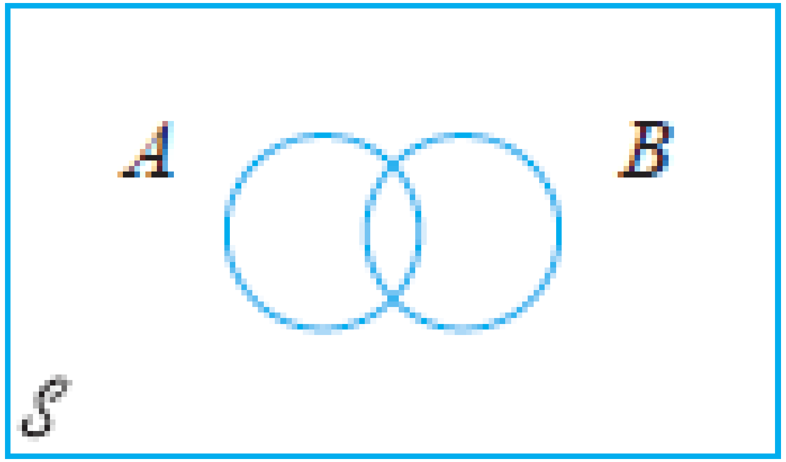
## 3. Equally likely Events:

Outcomes are said to be equally likely if all the outcomes have equal chance of occurrence.

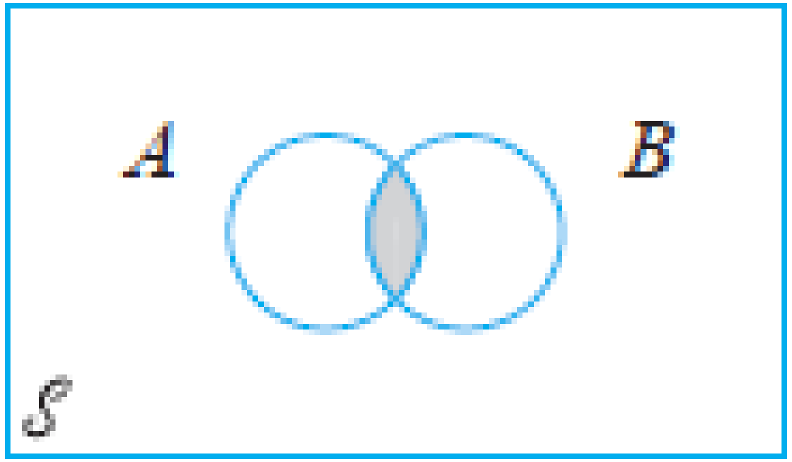
In the same manner if all the events of an experiment have equal chance of occurrence and there is no reason to expect one event to occur in presence to another then events are said to be **equally likely events**.

# GRAPHICAL REPRESENTATION:

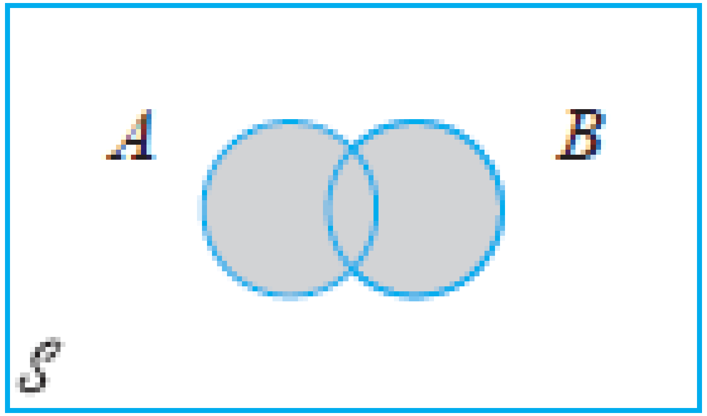
For graphical representation we use Venn diagram, to represent events we use circles and rectangles for sample space.



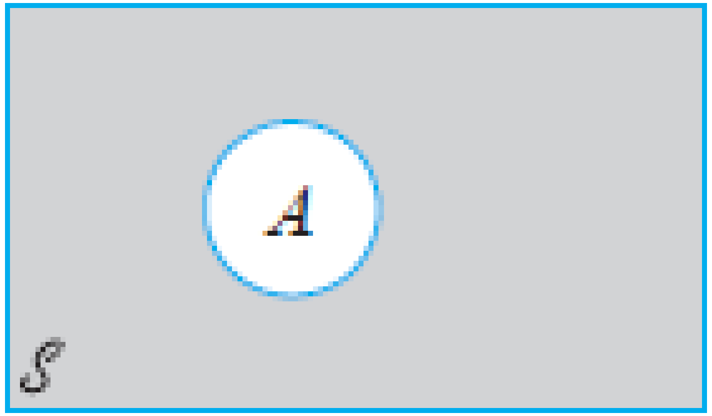

(a) Venn diagram of events $A$ and $B$



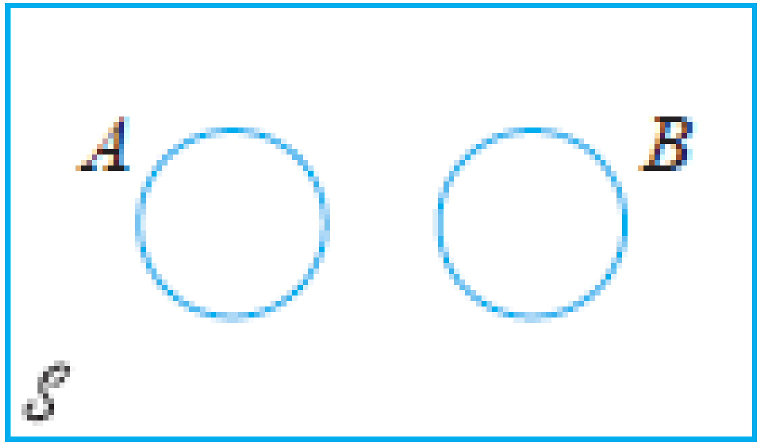

(b) Shaded region is $A \cap B$



(c) Shaded region is $A \cup B$



(d) Shaded region is $A'$



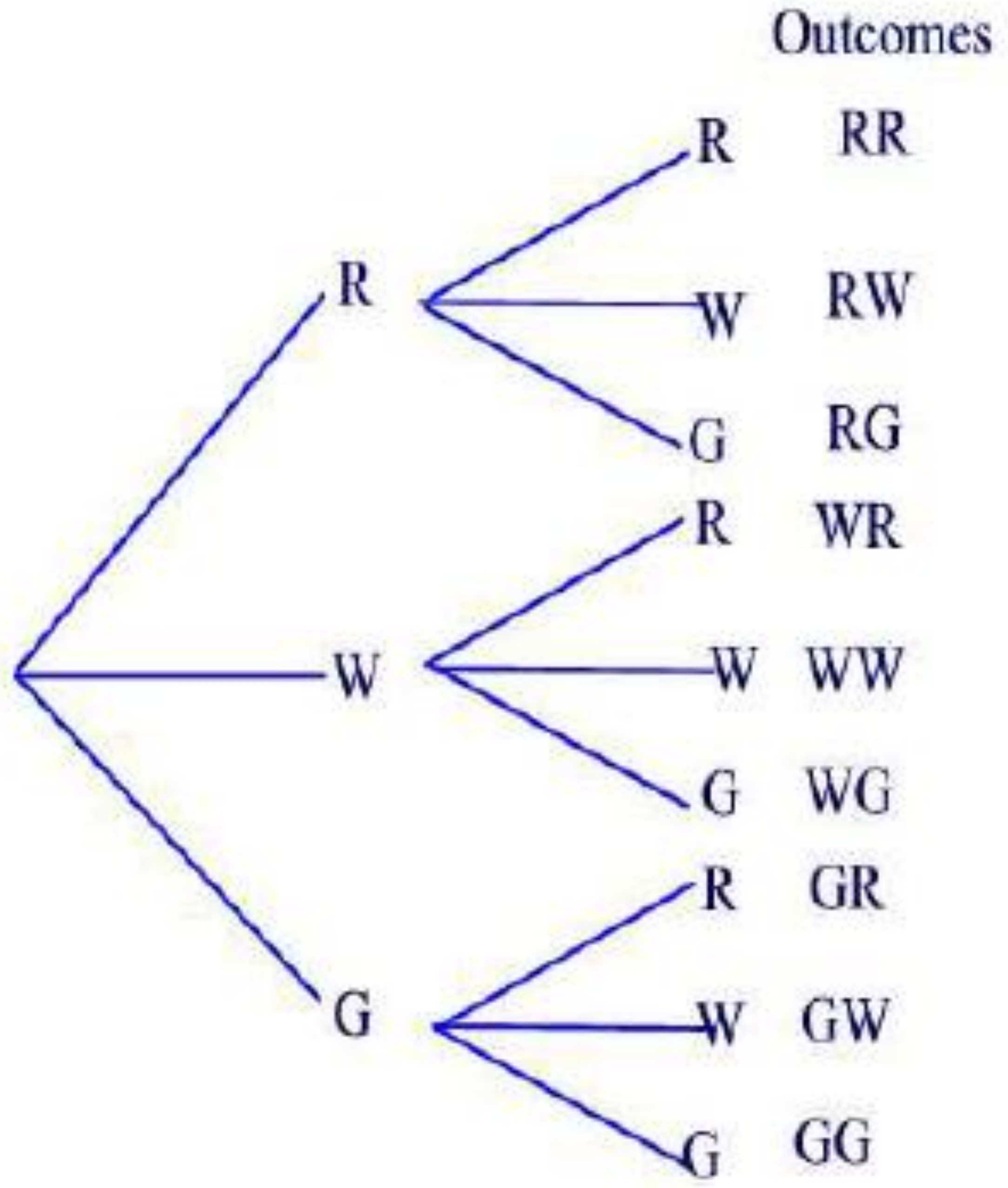
(e) Mutually exclusive events

# TREE DIAGRAMS/ POSSIBILITY TREES:

A tree diagram is a diagram used to show the total number of possible outcomes in an experiment systematically.

**Example 1**:

For example, consider the experiment of drawing two balls in succession and with replacement from a box containing one red ball (R), one white ball (W), and one green ball (G). The outcomes of this experiment, i.e. the elements of the sample space can be found in two different ways by using


Outcomes
R
RR
R
W
RW
G
RG
R
WR
W
W
WW
G
WG
R
GR
G
W
GW
G
GG

An organized table of our experiment looks like

| | R | W | G |
|---|---|---|---|
| R | RR | RW | RG |
| W | WR | WW | WG |
| G | GR | GW | GG |

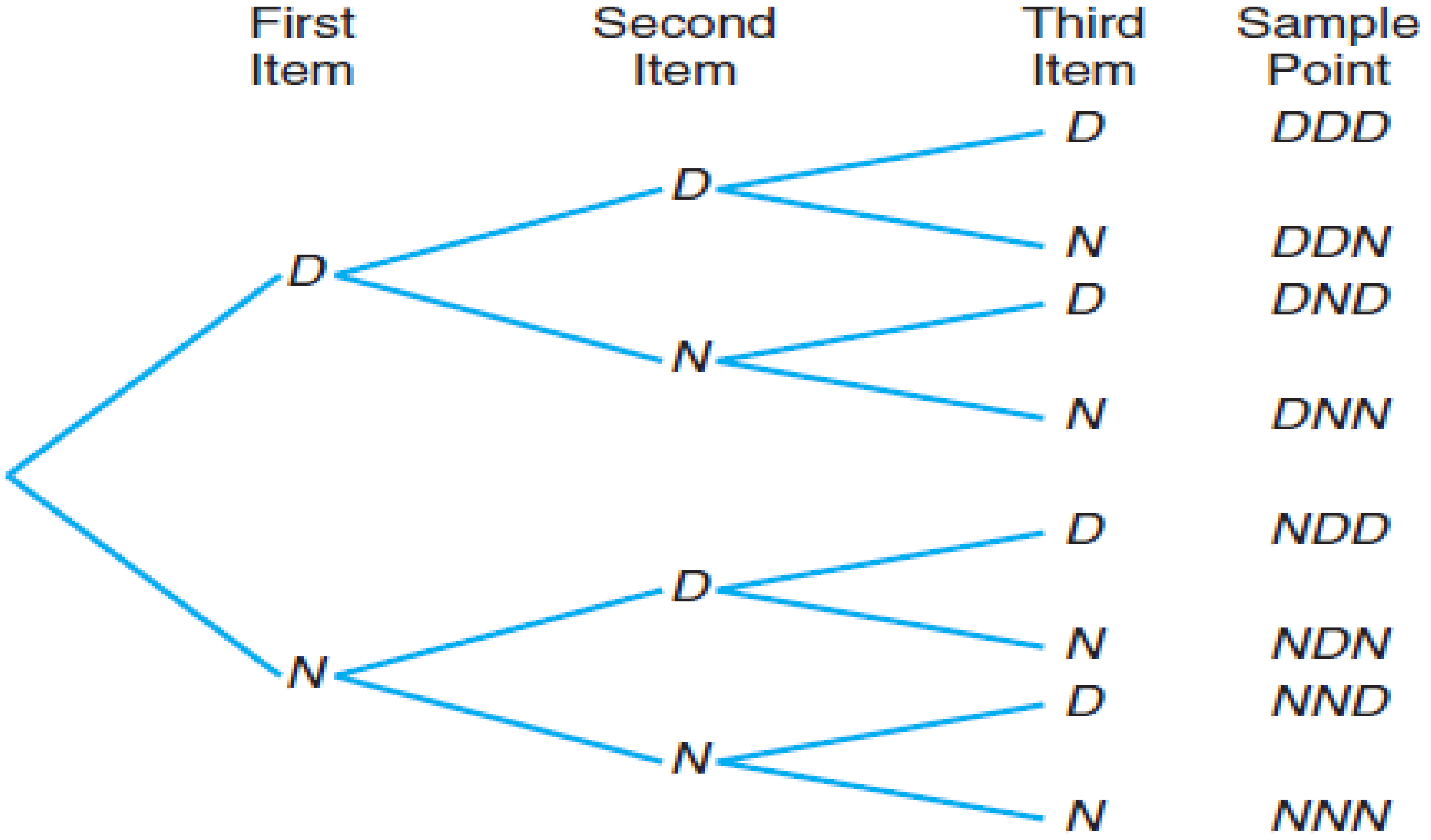
Thus, there are nine equally likely outcomes so that

$$S = \{RR, RW, RG, WR, WW, WG, GR, GW, GG\}$$

# Example 2:

Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified defective, $D$, or nondefective, $N$. To list the elements of the sample space providing the most information, we construct the tree diagram of Figure 2.2. Now, the various paths along the branches of the tree give the distinct sample points. Starting with the first path, we get the sample point $DDD$, indicating the possibility that all three items inspected are defective. As we proceed along the other paths, we see that the sample space is

$$S = \{DDD,\ DDN,\ DND,\ DNN,\ NDD,\ NDN,\ NND,\ NNN\}.$$


First Item
Second Item
Third Item
Sample Point
D
D
D
DDD
N
DDN
N
D
DND
N
DNN
N
D
D
NDD
N
NDN
N
D
NND
N
NNN

# Exercises

**2.1** List the elements of each of the following sample spaces:

(a) the set of integers between 1 and 50 divisible by 8;

(b) the set $S = \{x \mid x^2 + 4x - 5 = 0\}$;

(c) the set of outcomes when a coin is tossed until a tail or three heads appear;

(d) the set $S = \{x \mid x \text{ is a continent}\}$;

(e) the set $S = \{x \mid 2x - 4 \geq 0 \text{ and } x < 1\}$.

**2.2** Use the rule method to describe the sample space $S$ consisting of all points in the first quadrant inside a circle of radius 3 with center at the origin.

comes up 3 followed by a head and then a tail on the coin, construct a tree diagram to show the 18 elements of the sample space $S$.

**2.6** Two jurors are selected from 4 alternates to serve at a murder trial. Using the notation $A_1A_3$, for example, to denote the simple event that alternates 1 and 3 are selected, list the 6 elements of the sample space $S$.

**2.7** Four students are selected at random from a chemistry class and classified as male or female. List the elements of the sample space $S_1$, using the letter $M$ for male and $F$ for female. Define a second sample space $S_2$ where the elements represent the number of females selected.

females selected.

**2.3** Which of the following events are equal?

(a) $A = \{1, 3\}$;

(b) $B = \{x \mid x \text{ is a number on a die}\}$;

(c) $C = \{x \mid x^2 - 4x + 3 = 0\}$;

(d) $D = \{x \mid x \text{ is the number of heads when six coins are tossed}\}$.

**2.4** An experiment involves tossing a pair of dice, one green and one red, and recording the numbers that come up. If $x$ equals the outcome on the green die and $y$ the outcome on the red die, describe the sample space $S$

(a) by listing the elements $(x, y)$;

(b) by using the rule method.

**2.5** An experiment consists of tossing a die and then flipping a coin once if the number on the die is even. If the number on the die is odd, the coin is flipped twice. Using the notation $4H$, for example, to denote the outcome that the die comes up 4 and then the coin comes up heads, and $3HT$ to denote the outcome that the die

**2.8** For the sample space of Exercise 2.4,

(a) list the elements corresponding to the event $A$ that the sum is greater than 8;

(b) list the elements corresponding to the event $B$ that a 2 occurs on either die;

(c) list the elements corresponding to the event $C$ that a number greater than 4 comes up on the green die;

(d) list the elements corresponding to the event $A \cap C$;

(e) list the elements corresponding to the event $A \cap B$;

(f) list the elements corresponding to the event $B \cap C$;

(g) construct a Venn diagram to illustrate the intersections and unions of the events $A$, $B$, and $C$.

**2.9** For the sample space of Exercise 2.5,

(a) list the elements corresponding to the event $A$ that a number less than 3 occurs on the die;

(b) list the elements corresponding to the event $B$ that two tails occur;

(c) list the elements corresponding to the event $A'$;

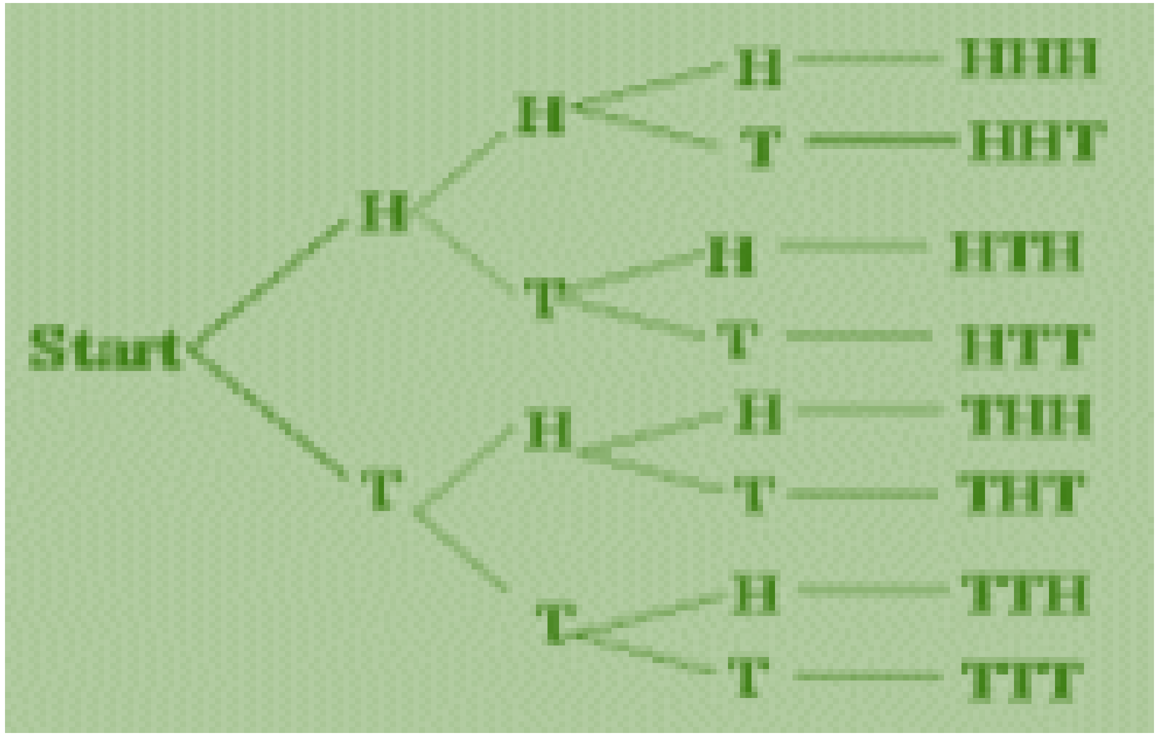
2.1 (a) $S = \{8, 16, 24, 32, 40, 48\}$.

(b) For $x^2 + 4x - 5 = (x + 5)(x - 1) = 0$, the only solutions are $x = -5$ and $x = 1$. $S = \{-5, 1\}$.

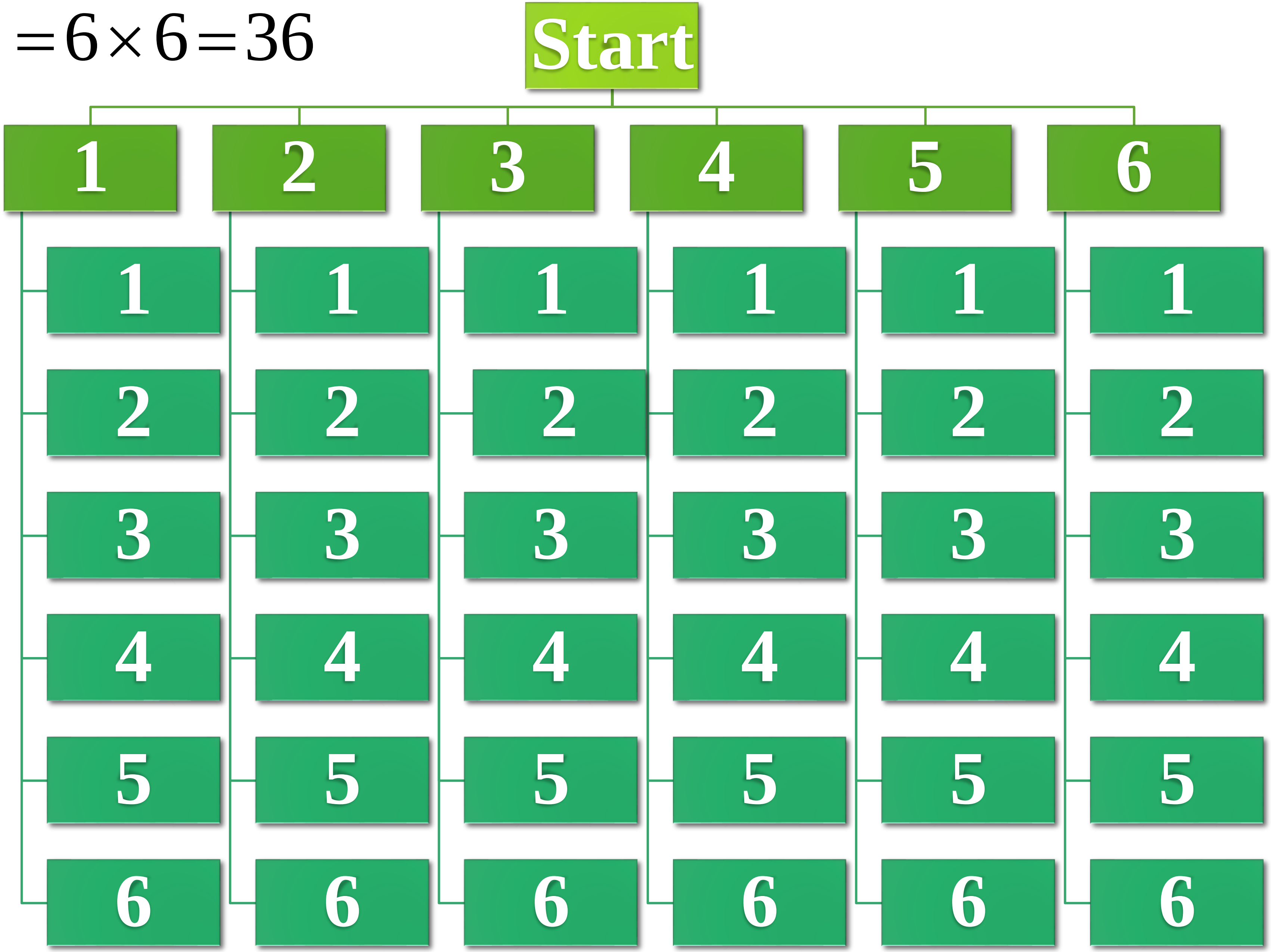
(c) $S = \{T, HT, HHT, HHH\}$.

(d) $S = \{$N. America, S. America, Europe, Asia, Africa, Australia, Antarctica$\}$.

(e) Solving $2x - 4 \geq 0$ gives $x \geq 2$. Since we must also have $x < 1$, it follows that $S = \phi$.



2.4 $n_1 \times n_2 = 6 \times 6 = 36$



2.4 (a) $S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}.$

(b) $S = \{(x,y) \mid 1 \le x, y \le 6\}.$

2.7 $S_1 = \{MMMM, MMMF, MMFM, MFMM, FMMM, MMFF, MFMF, MFFM,$
$FMFM, FFMM, FMMF, MFFF, FMFF, FFMF, FFFM, FFFF\}.$
$S_2 = \{0, 1, 2, 3, 4\}.$

$$n_1 \times n_2 = 4 \times 4 = 16$$

| | | | | |
|---|---|---|---|---|
| M | M | M | M | F |
| M | M | M | F | F |
| M | M | F | F | F |
| M | F | F | F | F |

2.8 (a) $A = \{(3,6),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6)\}.$

(b) $B = \{(1,2),(2,2),(3,2),(4,2),(5,2),(6,2),(2,1),(2,3),(2,4),$
$(2,5),(2,6)\}.$

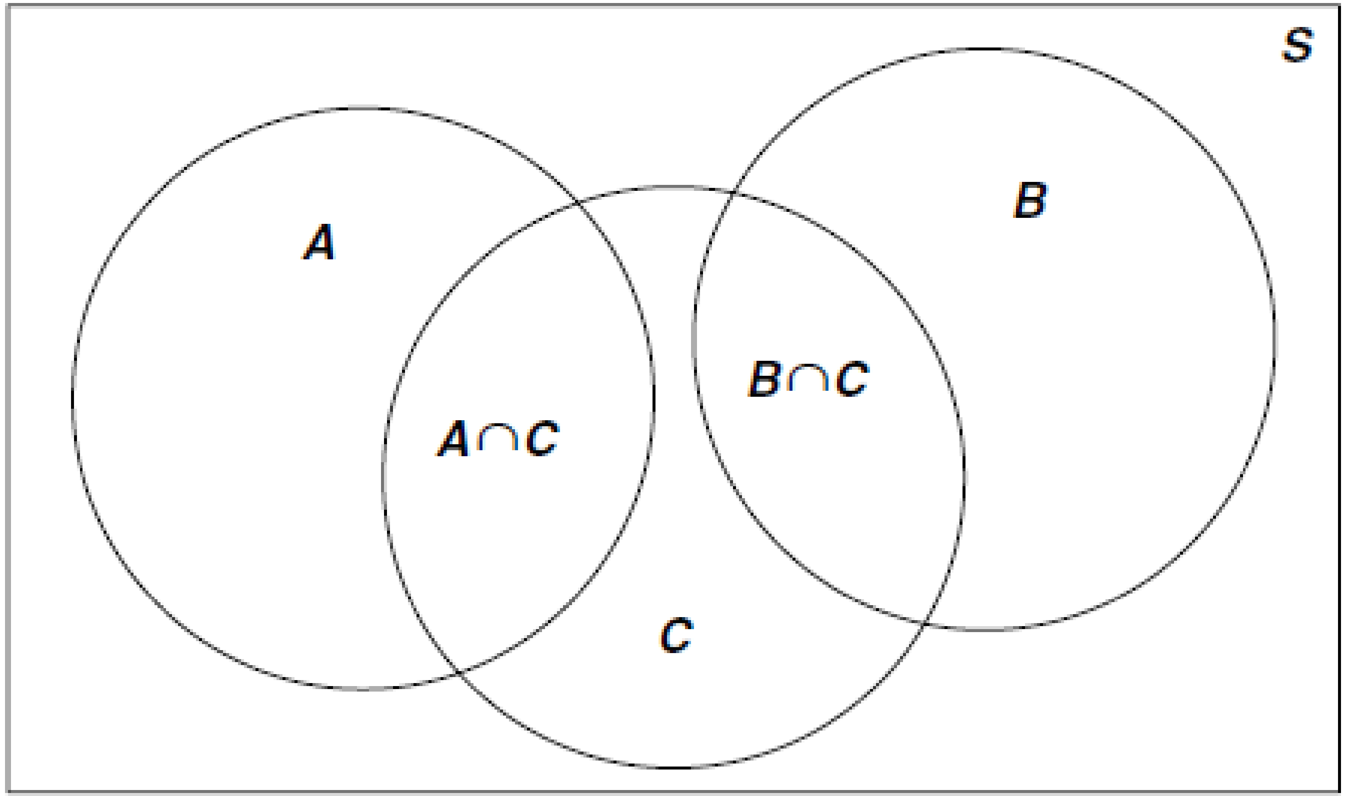
(c) $C = \{(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$.

(d) $A \cap C = \{(5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6)\}$.

(e) $A \cap B = \phi$.

(f) $B \cap C = \{(5,2),(6,2)\}$.

(g) A Venn diagram is shown next.



**2.10** An engineering firm is hired to determine if certain waterways in Virginia are safe for fishing. Samples are taken from three rivers.

(a) List the elements of a sample space $S$, using the letters $F$ for safe to fish and $N$ for not safe to fish.

(b) List the elements of $S$ corresponding to event $E$ that at least two of the rivers are safe for fishing.

(c) Define an event that has as its elements the points

$$\{FFF, NFF, FFN, NFN\}.$$

(a) $S = \{FFF, FFN, FNF, NFF, FNN, NFN, NNF, NNN\}$.

(b) $E = \{FFF, FFN, FNF, NFF\}$.

(c) The second river was safe for fishing.

**2.11** The resumés of two male applicants for a college teaching position in chemistry are placed in the same file as the resumés of two female applicants. Two positions become available, and the first, at the rank of assistant professor, is filled by selecting one of the four applicants at random. The second position, at the rank of instructor, is then filled by selecting at random one of the remaining three applicants. Using the notation $M_2F_1$, for example, to denote the simple event that the first position is filled by the second male applicant and the second position is then filled by the first female applicant,

(a) list the elements of a sample space $S$;

(b) list the elements of $S$ corresponding to event $A$ that the position of assistant professor is filled by a male applicant;

(c) list the elements of $S$ corresponding to event $B$ that exactly one of the two positions is filled by a male applicant;

(d) list the elements of $S$ corresponding to event $C$ that neither position is filled by a male applicant;

(e) list the elements of $S$ corresponding to the event $A \cap B$;

(f) list the elements of $S$ corresponding to the event $A \cup C$;

(g) construct a Venn diagram to illustrate the intersections and unions of the events $A$, $B$, and $C$.

2.11 (a) $S = \{M_1M_2, M_1F_1, M_1F_2, M_2M_1, M_2F_1, M_2F_2, F_1M_1, F_1M_2, F_1F_2, F_2M_1, F_2M_2, F_2F_1\}$.

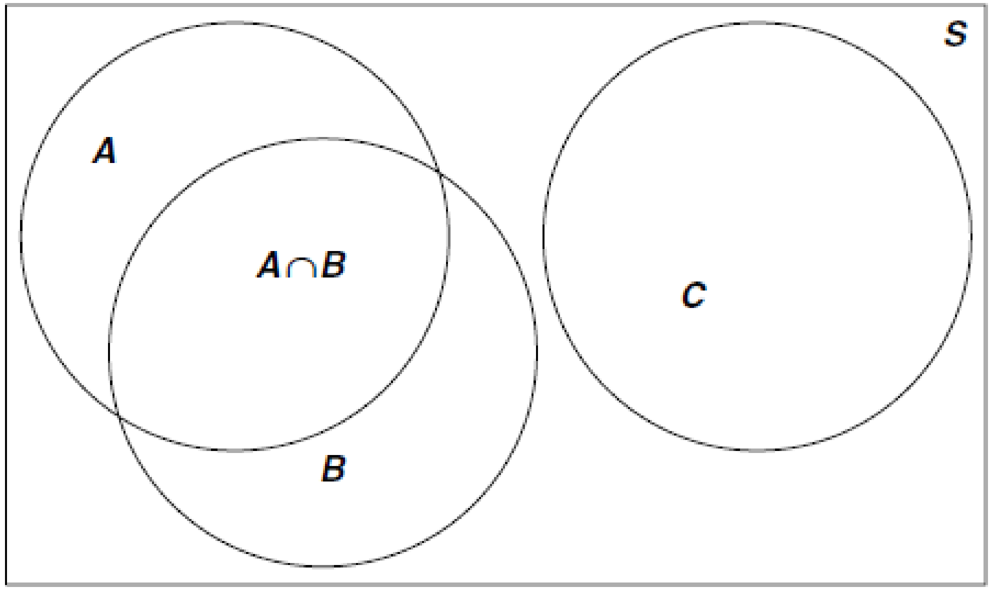
(b) $A = \{M_1M_2, M_1F_1, M_1F_2, M_2M_1, M_2F_1, M_2F_2\}$.

(c) $B = \{M_1F_1, M_1F_2, M_2F_1, M_2F_2, F_1M_1, F_1M_2, F_2M_1, F_2M_2\}$.

(d) $C = \{F_1F_2, F_2F_1\}$.

(e) $A \cap B = \{M_1F_1, M_1F_2, M_2F_1, M_2F_2\}$.

(f) $A \cup C = \{M_1M_2, M_1F_1, M_1F_2, M_2M_1, M_2F_1, M_2F_2, F_1F_2, F_2F_1\}$.



# FUNDAMENTAL THEOREM OF MULTIPLICATION:

If an operation can be performed in $n_1$ ways, and if for each of these a second operation can be performed in $n_2$ ways, and for each of the first two a third operation can be performed in $n_3$ ways, and so forth, then the sequence of $k$ operations can be performed in $n_1 n_2 \cdots n_k$ ways.

## Example 1:

Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?

Since $n_1 = 2$, $n_2 = 4$, $n_3 = 3$, and $n_4 = 5$, there are

$$n_l \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$$

different ways to order the parts.

**Example 2: (Possibility trees and Multiplication Rule)**

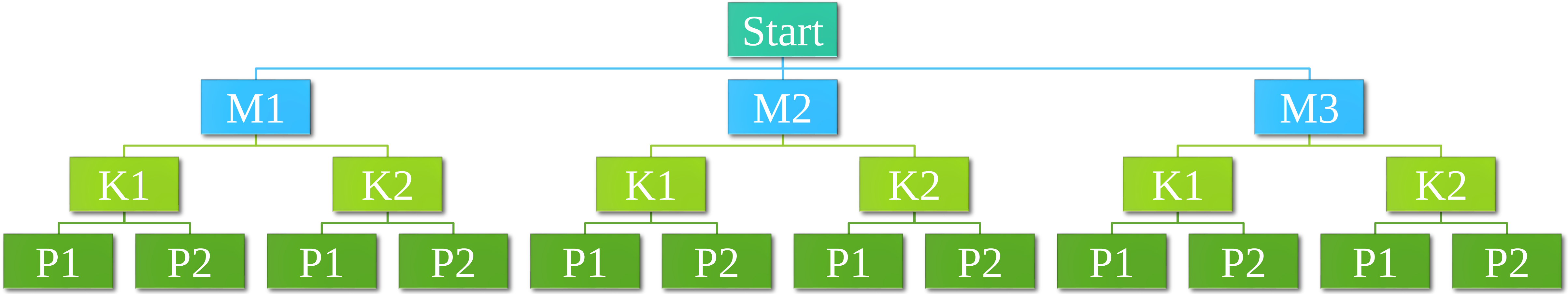
When buying a PC system, you have the choice of

- 3 models of the basic unit: M1, M2, M3 ;
- 2 models of keyboard: K1, K2 ;
- 2 models of printer: P1, P2 .

Question:

How many distinct systems can be purchased?

# The possibility tree:




The number of distinct systems is: (3)(2)(2)=12

**Example 3:**

Consider the following nested loop:

```
for i:=1 to 5
        for j:=1 to 6
            [ Statement 1 ;
              Statement 2 . ]
        next j
next i
```

Question: How many times the statements in the inner loop will be executed?

Solution: (5)(6) = 30 times (multiplication rule)

**Example 4:**

How many four digits PINs are possible from the digits 0, 1, 2, …, 9?
(i): with replacement (when repetition allowed)
(ii): without replacement (when repetition not allowed)

Solution (i):
Choosing a PIN is a 4-step operation:

- Step 1: Choose the 1st symbol (10 different ways).
- Step 2: Choose the 2nd symbol (10 different ways).
- Step 3: Choose the 3rd symbol (10 different ways).
- Step 4: Choose the 4th symbol (10 different ways).

By multiplication rule:

$$(10)(10)(10)(10) = 10{,}000 \text{ PINs are possible.}$$

| **1st symbol** | **2nd symbol** | **3rd symbol** | **4th symbol** |
|---|---|---|---|
| 10 | 10 | 10 | 10 |

Solution (ii):

Choosing a PIN is a 4-step operation:

- Step 1: Choose the 1st symbol (10 different ways).
- Step 2: Choose the 2nd symbol (9 different ways).
- Step 3: Choose the 3rd symbol (8 different ways).
- Step 4: Choose the 4th symbol (7 different ways).

By multiplication rule:

$$(10)(9)(8)(7) = 5{,}040 \text{ PINs are possible.}$$

# PERMUTATION:

A permutation is an ordered arrangement of all or a part of set of objects.

**Example 1:**

In how many ways the letters of the word “READ” can be arranged?

Solution: 4 x 3 x 2 x 1 = 12

**Example 2:** In how many ways can 5 students be lined up to get an admission form?

Solution: 5 x 4 x 3 x 2 x 1 = 12

# FACTORIAL:

For any non-negative integer $n$, $n!$, called "$n$ factorial," is defined as

$$n! = n(n-1)\cdots(2)(1),$$

with special case $0! = 1$.

# PERMUTATION OF *n* DISTINCT OBJECTS TAKEN *r* AT A TIME:

The number of permutations of $n$ distinct objects taken $r$ at a time is

$$_nP_r = \frac{n!}{(n-r)!}.$$

# Example 1:

In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

Since the awards are distinguishable, it is a permutation problem. The total number of sample points is

$$_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = (25)(24)(23) = 13,800.$$

# PERMUTATION OF *n* OBJECTS WHEN THEY ARE OF DIFFERENT KINDS:

The number of distinct permutations of $n$ things of which $n_1$ are of one kind, $n_2$ of a second kind, $\ldots$, $n_k$ of a $k$th kind is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

**Example 1**:

In how many ways can the letters of the word “STATISTICS” be arranged?

**Solution:** Here n = 10

$n_1 = 3$ ( S-three times)

$n_2 = 3$ (T-three times )

$n_3 = 1$ ( A-one time)

$n_4 = 2$ (I-two times )

$n_5 = 1$ (C-one time )

$$P = \frac{10!}{3!\,.\,3!\,.\,1!\,.\,2!\,.\,1!}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 3 \times 2 \times 1 \times 2 \times 1} = 50{,}400$$

# PERMUTATION OF *n* OBJECTS ARRANGED IN A CIRCLE:

The number of permutations of $n$ objects arranged in a circle is $(n-1)!$.

**2.43** In how many ways can 5 different trees be planted in a circle?

$$(n-1)! = (5-1)! = 4! = 24$$

**2.44** In how many ways can a caravan of 8 covered wagons from Arizona be arranged in a circle?

$$(n-1)! = (8-1)! = 7! = 5040$$

# COMBINATION:

Combinations is a technical term meaning 'selections'. We use it to refer to the number of different sets of a certain size that can be selected from a larger collection of objects where order does not matter.

The number of combinations of $n$ distinct objects taken $r$ at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

We will denote this by ${}^nC_r$ or $\left(\begin{array}{c} n \\ \\ r \end{array}\right)$ read as '$n$ choose $r$'.

The number of ways of choosing an **ordered** set of $r$ objects out of $n$ is equal to the number of ways of choosing an **unordered** set of $r$ objects out of $n$ **times** the number of ways of arranging $r$ objects in order.

That is,

$$^nP_r = {}^nC_r \times r!$$

**Example**:

How many distinct sets of 3 differently coloured scarves can be bought if the shop has scarves in 8 different colours?

Solution: The number of sets of scarves equals

$$^8C_3 = \frac{8!}{5!\,3!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56.$$

# COMBINATION OF *r* CELLS IN *n* OBJECTS :

The number of ways of partitioning a set of $n$ objects into $r$ cells with $n_1$ elements in the first cell, $n_2$ elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!},$$

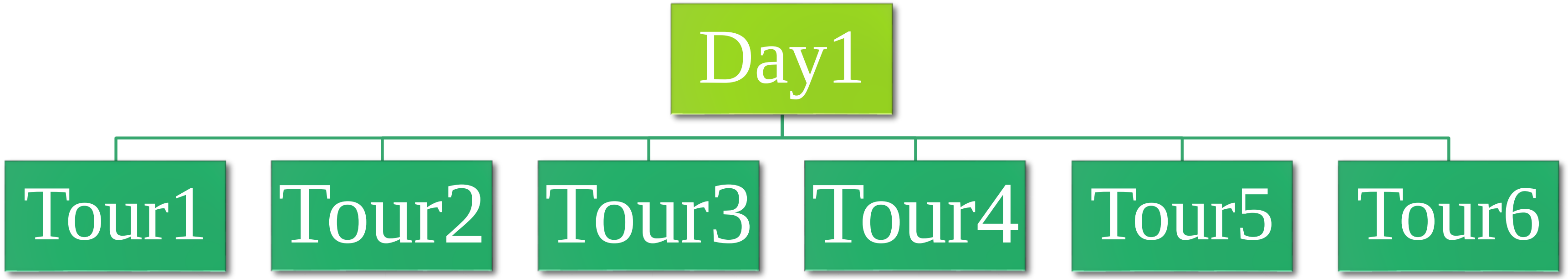
where $n_1 + n_2 + \cdots + n_r = n$.

# Exercises

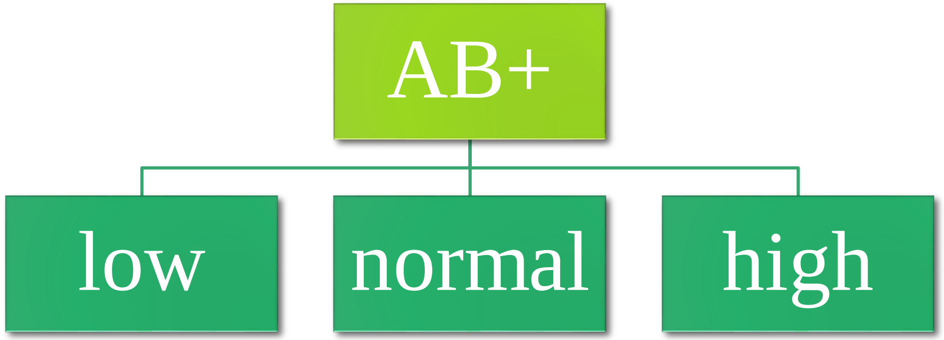
**2.21** Registrants at a large convention are offered 6 sightseeing tours on each of 3 days. In how many ways can a person arrange to go on a sightseeing tour planned by this convention?

**2.22** In a medical study, patients are classified in 8 ways according to whether they have blood type $AB^+$, $AB^-$, $A^+$, $A^-$, $B^+$, $B^-$, $O^+$, or $O^-$, and also according to whether their blood pressure is low, normal, or high. Find the number of ways in which a patient can be classified.

2.21 With $n_1 = 6$ sightseeing tours each available on $n_2 = 3$ different days, the multiplication rule gives $n_1 n_2 = (6)(3) = 18$ ways for a person to arrange a tour.




2.22 With $n_1 = 8$ blood types and $n_2 = 3$ classifications of blood pressure, the multiplication rule gives $n_1 n_2 = (8)(3) = 24$ classifications.



**2.23** If an experiment consists of throwing a die and then drawing a letter at random from the English alphabet, how many points are there in the sample space?

**2.24** Students at a private liberal arts college are classified as being freshmen, sophomores, juniors, or seniors, and also according to whether they are male or female. Find the total number of possible classifications for the students of that college.

**2.25** A certain brand of shoes comes in 5 different styles, with each style available in 4 distinct colors. If the store wishes to display pairs of these shoes showing all of its various styles and colors, how many different pairs will the store have on display?

**2.28** A drug for the relief of asthma can be purchased from 5 different manufacturers in liquid, tablet, or capsule form, all of which come in regular and extra strength. How many different ways can a doctor prescribe the drug for a patient suffering from asthma?

**2.29** In a fuel economy study, each of 3 race cars is tested using 5 different brands of gasoline at 7 test sites located in different regions of the country. If 2 drivers are used in the study, and test runs are made once under each distinct set of conditions, how many test runs are needed?

**2.30** In how many different ways can a true-false test consisting of 9 questions be answered?

2.23 Since the die can land in $n_1 = 6$ ways and a letter can be selected in $n_2 = 26$ ways, the multiplication rule gives $n_1 n_2 = (6)(26) = 156$ points in $S$.

2.24 Since a student may be classified according to $n_1 = 4$ class standing and $n_2 = 2$ gender classifications, the multiplication rule gives $n_1 n_2 = (4)(2) = 8$ possible classifications for the students.

2.30 With $n_1 = 2$ choices for the first question, $n_2 = 2$ choices for the second question, and so forth, the generalized multiplication rule yields $n_1 n_2 \cdots n_9 = 2^9 = 512$ ways to answer the test.

**2.26** A California study concluded that following 7 simple health rules can extend a man's life by 11 years on the average and a woman's life by 7 years. These 7 rules are as follows: no smoking, get regular exercise, use alcohol only in moderation, get 7 to 8 hours of sleep, maintain proper weight, eat breakfast, and do not eat between meals. In how many ways can a person adopt 5 of these rules to follow

(a) if the person presently violates all 7 rules?

(b) if the person never drinks and always eats breakfast?

2.26 Total rules = 7

(a): person violates all 7 rules so he has to adopt any 5 rules from 7, by combination rule:

There are $\binom{7}{5} = 21$ ways.

(b): the person is already following 2 rules so he has to adopt any 3, again by combination rule:

There are $\binom{5}{3} = 10$ ways.

# Task:

**2.27** A developer of a new subdivision offers a prospective home buyer a choice of 4 designs, 3 different heating systems, a garage or carport, and a patio or screened porch. How many different plans are available to this buyer?

**2.31** A witness to a hit-and-run accident told the police that the license number contained the letters RLH followed by 3 digits, the first of which was a 5. If the witness cannot recall the last 2 digits, but is certain that all 3 digits are different, find the maximum number of automobile registrations that the police may have to check.

| R | L | H | 5 | --- | --- |
|---|---|---|---|---|---|
| | | | | There are 9 possibilities here as out of 10 only digit is used | As 2 digits are used so in this position we have remaining 8 possibilities |

Since the first digit is a 5, there are $n_1 = 9$ possibilities for the second digit and then $n_2 = 8$ possibilities for the third digit. Therefore, by the multiplication rule there are $n_1 n_2 = (9)(8) = 72$ registrations to be checked.

**Example**:

In general for a number plate of any vehicle, multiplication rule is followed:

| L1 | L2 | L3 | D1 | D2 | D3 |
|---|---|---|---|---|---|
| There are 26 possibilities here (A,B,C,…,Z) | There are 25 possibilities here as out of 26 only one letter is used | There are 24 possibilities here as out of 26, two letters are used | There are 10 possibilities here (0,1,2,.., 9) | There are 9 possibilities here as out of 10 only digit is used | As 2 digits are used so in this position we have remaining 8 possibilities |

$$n_1 \times n_2 \times n_3 \times n_4 \times n_5 \times n_6 = 26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11232000$$

**2.32** (a) In how many ways can 6 people be lined up to get on a bus?

(b) If 3 specific persons, among 6, insist on following each other, how many ways are possible?

(c) If 2 specific persons, among 6, refuse to follow each other, how many ways are possible?

**2.33** If a multiple-choice test consists of 5 questions, each with 4 possible answers of which only 1 is correct,

(a) in how many different ways can a student check off one answer to each question?

(b) in how many ways can a student check off one answer to each question and get all the answers wrong?

(c) if all the men sit together to the right of all the women?

**2.39** In a regional spelling bee, the 8 finalists consist of 3 boys and 5 girls. Find the number of sample points in the sample space $S$ for the number of possible orders at the conclusion of the contest for

(a) all 8 finalists;

(b) the first 3 positions.

**2.40** In how many ways can 5 starting positions on a basketball team be filled with 8 men who can play any of the positions?

**2.41** Find the number of ways that 6 teachers can

2.32 (a) By Theorem 2.3, there are $6! = 720$ ways.

(b) A certain 3 persons can follow each other in a line of 6 people in a specified order is 4 ways or in $(4)(3!) = 24$ ways with regard to order. The other 3 persons can then be placed in line in $3! = 6$ ways. By Theorem 2.1, there are total $(24)(6) = 144$ ways to line up 6 people with a certain 3 following each other.

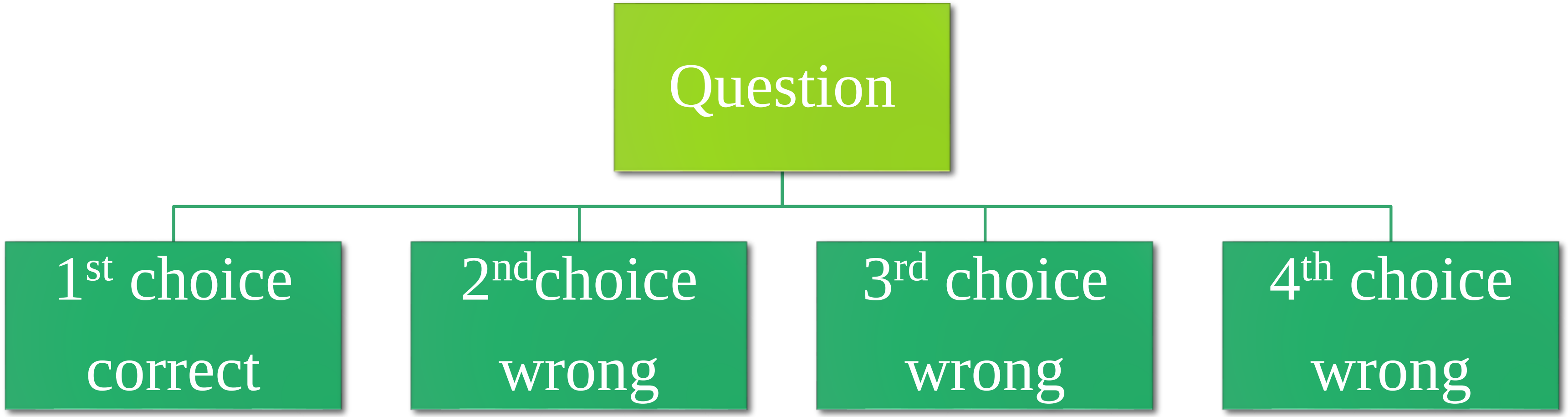
(c) Similar as in (b), the number of ways that a specified 2 persons can follow each other in a line of 6 people is $(5)(2!)(4!) = 240$ ways. Therefore, there are $720 - 240 = 480$ ways if a certain 2 persons refuse to follow each other.

**Explanation for part (b)**:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 1 | 2 | 3 | 5 | 6 |
| 4 | 5 | 1 | 2 | 3 | 6 |
| 4 | 5 | 6 | 1 | 2 | 3 |

**Explanation for part (c)**: As its difficult to find which 2 persons will refuse to follow each other while, its easy to find the compliment 2 persons will follow each other.

2.33 (a) With $n_1 = 4$ possible answers for the first question, $n_2 = 4$ possible answers for the second question, and so forth, the generalized multiplication rule yields $4^5 = 1024$ ways to answer the test.




(b) With $n_1 = 3$ wrong answers for the first question, $n_2 = 3$ wrong answers for the second question, and so forth, the generalized multiplication rule yields

$$n_1 n_2 n_3 n_4 n_5 = (3)(3)(3)(3)(3) = 3^5 = 243$$

wrong?

**2.34** (a) How many distinct permutations can be made from the letters of the word *COLUMNS*?

(b) How many of these permutations start with the letter $M$?

**2.35** A contractor wishes to build 9 houses, each different in design. In how many ways can he place these houses on a street if 6 lots are on one side of the street and 3 lots are on the opposite side?

**2.36** (a) How many three-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, and 6 if each digit can be used only once?

(b) How many of these are odd numbers?

(c) How many are greater than 330?

**2.37** In how many ways can 4 boys and 5 girls sit in a row if the boys and girls must alternate?

**2.38** Four married couples have bought 8 seats in the same row for a concert. In how many different ways can they be seated

(a) with no restrictions?

(b) if each couple is to sit together?

**2.41** Find the number of ways that 6 teachers can be assigned to 4 sections of an introductory psychology course if no teacher is assigned to more than one section.

**2.42** Three lottery tickets for first, second, and third prizes are drawn from a group of 40 tickets. Find the number of sample points in $S$ for awarding the 3 prizes if each contestant holds only 1 ticket.

**2.43** In how many ways can 5 different trees be planted in a circle?

**2.44** In how many ways can a caravan of 8 covered wagons from Arizona be arranged in a circle?

**2.45** How many distinct permutations can be made from the letters of the word *INFINITY*?

**2.46** In how many ways can 3 oaks, 4 pines, and 2 maples be arranged along a property line if one does not distinguish among trees of the same kind?

**2.47** How many ways are there to select 3 candidates from 8 equally qualified recent graduates for openings in an accounting firm?

**2.48** How many ways are there that no two students will have the same birth date in a class of size 60?

2.36 (a) Any of the 6 nonzero digits can be chosen for the hundreds position, and of the remaining 6 digits for the tens position, leaving 5 digits for the units position. So, there are $(6)(6)(5) = 180$ three digit numbers.

| Hundreds | tens | unit |
|---|---|---|
| Remember we cannot put zero in the hundred position as it will make our digit 2 digit number not 3.<br>So, for this place we have only 6 choices (1,2,3,4,5,6) | As here we can put zero and one digit we have already use in hundred place so here we have 6 choices again (0 plus remaining 5 digits) | As we have used 2 digits out of 7 therefore we have 5 choices here |

(b) The units position can be filled using any of the 3 odd digits. Any of the remaining 5 nonzero digits can be chosen for the hundreds position, leaving a choice of 5 digits for the tens position. By Theorem 2.2, there are $(3)(5)(5) = 75$ three digit odd numbers.

| Hundreds | tens | unit |
|---|---|---|
| Remember we cannot put zero in the hundred position as it will make our digit 2 digit number not 3. So, for this place we have only 5 choices (any one from 2 odd number as one we placed at unit position, remaining 1 odd numbers, 2, 4, 6) | As here we can put zero and one digit we have already use in hundred place so here we have 6 choices again (0 plus remaining 5 digits) | For odd numbers we have decide unit place first. We can only put 1, 3, 5 here. So, our choice is 3 digits for unit position |

(c) If a 4, 5, or 6 is used in the hundreds position there remain 6 and 5 choices, respectively, for the tens and units positions. This gives $(3)(6)(5) = 90$ three digit numbers beginning with a 4, 5, or 6. If a 3 is used in the hundreds position, then a 4, 5, or 6 must be used in the tens position leaving 5 choices for the units position. In this case, there are $(1)(3)(5) = 15$ three digit number begin with a 3. So, the total number of three digit numbers that are greater than 330 is $90 + 15 = 105$.

Greater than 330

When 4,5 or 6 is used in hundred place

(3)(6)(5)=90

When 3 is used in hundred place

(1)(3)(5)=15

2.38 (a) $8! = 40320$.

(b) There are 4! ways to seat 4 couples and then each member of a couple can be interchanged resulting in $2^4(4!) = 384$ ways.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| C1w | C1h | C2w | C2h | C3w | C3h | C4w | C4h |

**(c) if all the men sit together to the right of all the women?**

(c) By Theorem 2.3, the members of each gender can be seated in 4! ways. Then using Theorem 2.1, both men and women can be seated in $(4!)(4!) = 576$ ways.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| C1h | C2h | C3h | C4h | C1w | C2w | C3w | C4w |

2.44 $n=40, \quad r=3$

$$^{40}P_3 = \frac{40!}{(40-3)!} = 59280 \text{ ways.}$$

2.46 $n_1 = 3$

$n_2 = 4$

$n_3 = 2$

$n = n_1 + n_2 + n_3 = 3 + 4 + 2 = 9$

$$\text{there are } \frac{9!}{3!\,4!\,2!} = 1260 \text{ ways.}$$