Part 1: Theory Questions

1. No, three-dimensional rotations are not commutative, so the order does matter. I will prove that order is important by showing you a counterexample where the same 3D vector rotation expressed in two different orders of dimensions produces different results and therefore proves that the order of application matters and that 3D axis rotations are not commutative.

   Consider the vector (3, 5, 1), first, rotate the vector 90 degrees around the x-axis and then 90 degrees around the z-axis,
   90 degrees around the x-axis:
   x = 3, y = y cos 90 – z sin 90 = -1, z = y sin 90 + z cos 90 = 5
   90 degrees around the z-axis:
   x = x cos 90 – y sin 90 =1, y = x sin 90 + y cos 90 =3, z = 5
   The resulting vector is (1, 3, 5).

   However, if you rotate the vector (3, 5, 1) along the z-axis 90 degrees first and then 90 degrees along the x-axis (the same rotations in opposite order).
   90 degrees around the z-axis:
   x = x cos 90 – y sin 90 =-5, y = x sin 90 + y cos 90 =3, z = 1
   90 degrees around the x-axis:
   x = -5, y = y cos 90 – z sin 90 = -1, z = y sin 90 + z cos 90 = 3
   The resulting vector after those rotations is (-5, -1, 3).

   Since the same 3D rotation applied in different orders created different resulting vectors (3, 5, 1) ≠ (-5, -1, 3), the order of the operations of each axis is important and therefore proves that rotations are not commutative.

2. A $= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, b $= \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$, then the least square solution is $A^T A\hat{x} = A^T b$, where $\hat{x}$ is the approximate solution.

   $A^T A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}$ and $A^T b = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

   Then $A^T A\hat{x} = A^T b \ \rightarrow \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \hat{x} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$ and $\hat{x} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

   Therefore, the least square solution for Ax=b is (3, -3).

3a. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 8 & 0 \end{pmatrix}$

3b. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3c. $\frac{1}{9}\begin{pmatrix} 6 & 6 & 11 & 9 & 9 \\ 8 & 8 & 11 & 12 & 12 \\ 7 & 8 & 12 & 12 & 11 \\ 2 & 3 & 1 & 4 & 3 \end{pmatrix}$

4a. Since K is separable Let $K=K_cK_r$, where K_r is the kernel function that returns the value of the 1 x k row matrix and K_c is the kernel function that returns the value of the k x 1 column matrix of the separable matrix K.
Then,

$$I_K(x,y) = \sum_{u=-\frac{k-1}{2}}^{\frac{k-1}{2}} K_r(u) \cdot \left(\sum_{v=-\frac{k-1}{2}}^{\frac{k-1}{2}} K_c(v) I(x-u, y-v) \right)$$

4b. **Proof:**
Since 2D convolution can be defined as $I_K = K * I$ and then further defined as

$$I_K(i,j) = \sum_{u=-\frac{k-1}{2}}^{\frac{k-1}{2}} \sum_{v=-\frac{k-1}{2}}^{\frac{k-1}{2}} K(u,v) I(i-u, j-v)$$

For all (i, j) in the image I_K.
Then since $K = gh \text{ and } K(u,v) = g(u) * g(v)$, we can substitute it in the equation as

$$= \sum_{u=-\frac{k-1}{2}}^{\frac{k-1}{2}} \sum_{v=-\frac{k-1}{2}}^{\frac{k-1}{2}} g(u) * h(v) * I(i-u,\ j-v)$$

$$= \sum_{u=-\frac{k-1}{2}}^{\frac{k-1}{2}} g(u) \sum_{v=-\frac{k-1}{2}}^{\frac{k-1}{2}} h(v) * I(i-u,\ j-v) \qquad (EQ)$$

And since the equation for 1D convolution is

$$I_X(i) = \sum_{v=-\frac{k-1}{2}}^{\frac{k-1}{2}} h(v) * I(i-u)$$

We can see that EQ is performing a 1D convolution with the h vector and input image since i-u is constant throughout that summation, and then taking the result of that 1D convolution and performing it with the g vector. Therefore, it's shown that the convolution of a 2D separable matrix can be done by doing the convolution of the input image and the K matrix's two 1D horizontal and vertical vectors g and h. Note that since convolution is commutative, the order of convolving the g vector or h vector first does not matter.