Compact Answers to Temporal Regular Path Queries (Supplementary Material)

ACM Reference Format:
. 2023. Compact Answers to Temporal Regular Path Queries (Supplementary Material). In *Proceedings of ACM Conference (Conference'17).* ACM, New York, NY, USA, 10 pages. https://doi.org/10.1145/nnnnnnn.nnnnnnn

1 INTRODUCTION

This document provides detailed definitions and proofs for the article *Compact answers to Temporal Regular Path Queries*, submitted at CIKM 2023.

As opposed to the structure adopted in the article, the result here are grouped by topic (inductive representation, finiteness, complexity, etc) rather than representation ($\mathcal{U}^{[t]}$, $\mathcal{U}^{[d]}$, etc.). This allows us to emphasize which proofs differ from one representation to the other.

2 NOTATION

Let $\langle o_1, o_2, \tau, \delta, b, e \rangle \in \mathcal{U}^{[t,d],b,e}$, and let $t \in \tau$.
In the article, we defined the interval δ_t for each t as

$$_{\delta}\lfloor\ b_{\delta} + \max(0, b - t)\ ,\ e_{\delta_i} - \max(0, t - e)\ \rfloor_{\delta}$$

In this supplementary material, we will use $\delta(t)$ instead of δ_t. This notation will allow us to write $\delta_1(t)$ when several tuples are involved. Note that the time points b and e in this notaton are still omitted, for conciseness, because they should be clear from the context.

Conference'17, July 2017, Washington, DC, USA

ACM ISBN 978-x-xxxx-xxxx-x/YY/MM...$15.00
https://doi.org/10.1145/nnnnnnn.nnnnnnn

3 INDUCTIVE REPRESENTATION

Let q be a TRPQ and G a TG.
Then $[\![q]\!]_G$ is the set of anwers to q over G (represented as tuples in $\mathcal{U}$).
In this section, we provide the full definition of the four inductive representations of $[\![q]\!]_G$ discussed in the article, in $\mathcal{U}^{[t]}$, $\mathcal{U}^{[d]}$, $\mathcal{U}^{[t,d]}$ and $\mathcal{U}^{[t,d],b,e}$ respectively, and prove that they are correct.
These representations are denoted as $(\!|q|\!)_G^{[t]}$, $(\!|q|\!)_G^{[d]}$, $(\!|q|\!)_G^{[t,d]}$ and $(\!|q|\!)_G^{[t,d],b,e}$ respectively.

3.1 Grammar

For convenience, we reproduce here the syntax of TRPQs (already provided in the article).
A TRPQ is an expression for the symbol path in the following grammar:

$$\begin{aligned}
\mathsf{path} &::= \mathsf{test} \mid \mathsf{axis} \mid (\mathsf{path}/\mathsf{path}) \mid (\mathsf{path} + \mathsf{path}) \mid \mathsf{path}[m, n] \mid \mathsf{path}[m, _] \\
\mathsf{test} &::= \mathit{pred} \mid (?\mathsf{path}) \mid \mathsf{test} \vee \mathsf{test} \mid \mathsf{test} \wedge \mathsf{test} \mid \neg\mathsf{test} \\
\mathsf{axis} &::= \mathrm{F} \mid \mathrm{B} \mid \mathrm{T}_\delta
\end{aligned}$$

with $\delta \in \mathsf{intv}(\mathcal{T})$, $m, n \in \mathbb{N}^+$ and $m \leq n$.

3.2 In $\mathcal{U}$

For convenience still, we reproduce the full semantics of TRPQs, i.e. the inductive definition of the evaluation $[\![q]\!]_G$ of a query q over a graph G in $\mathcal{U}$ (already provided in the article).

$$\begin{aligned}
[\![\mathit{pred}]\!]_G &= \{\langle o, o, t, 0\rangle \mid t \in \tau \text{ for some } \tau \in \mathsf{val}(o, \mathit{pred})\} \\
[\![\mathrm{T}_\delta]\!]_G &= \{\langle o, o, t, d\rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G\} \\
[\![\mathrm{F}]\!]_G &= \{\langle v, e, t, 0\rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, 0\rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\} \\
[\![\mathrm{B}]\!]_G &= \{\langle v, e, t, 0\rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, 0\rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \\
[\![?\mathsf{path}]\!]_G &= \{\langle o, o, t, 0\rangle \mid \langle o, o', t, t + d\rangle \in [\![\mathsf{path}]\!]_G \text{ for some } o' \in N \cup E, d \in \mathcal{T}\} \\
[\![\mathsf{test}_1 \vee \mathsf{test}_2]\!]_G &= [\![\mathsf{test}_1]\!]_G \cup [\![\mathsf{test}_2]\!]_G \\
[\![\mathsf{test}_1 \wedge \mathsf{test}_2]\!]_G &= [\![\mathsf{test}_1]\!]_G \cap [\![\mathsf{test}_2]\!]_G \\
[\![\neg\mathsf{test}]\!]_G &= (\{\langle o, o\rangle \mid o \in N \cup E\} \times \mathcal{T}_G \times \{0\}) \setminus [\![\mathsf{test}]\!]_G \\
[\![\mathsf{path}_1/\mathsf{path}_2]\!]_G &= \{\langle o_1, o_3, t, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, t, d_1\rangle \in [\![\mathsf{path}_1]\!]_G \wedge \langle o_2, o_3, t + d_1, d_2\rangle \in [\![\mathsf{path}_2]\!]_G\} \\
[\![\mathsf{path}_1 + \mathsf{path}_2]\!]_G &= [\![\mathsf{path}_1]\!]_G \cup [\![\mathsf{path}_2]\!]_G \\
[\![\mathsf{path}[m, n]]\!]_G &= \bigcup_{k=m}^{n} [\![\mathsf{path}^k]\!]_G \\
[\![\mathsf{path}[m, _]]\!]_G &= \bigcup_{k \geq m} [\![\mathsf{path}^k]\!]_G
\end{aligned}$$

3.3 In $\mathcal{U}^{[t]}$

3.3.1 *Definition.*

The full definition of $(\!|q|\!)_G^{[t]}$ is already provide in the article. We only reproduce it here for convenience.

$$\begin{aligned}
(\!|pred|\!)_G^{[t]} &= \{\langle o, o, \tau, 0\rangle \mid o \in (N \cup E), \tau \in \mathsf{val}(o, pred)\}\\
(\!|\mathrm{T}_\delta|\!)_G^{[t]} &= \{\langle o, o, [t_1, t_1], t_2 - t_1\rangle \mid o \in (N \cup E), t_1 \in \mathcal{T}_G, t_2 \in (\delta + t_1) \cap \mathcal{T}_G\}\\
(\!|\mathrm{F}|\!)_G^{[t]} &= \{\langle v, e, \mathcal{T}_G, 0\rangle \mid \mathsf{src}(e) = v\} \cup \{\langle e, v, \mathcal{T}_G, 0\rangle \mid \mathsf{tgt}(e) = v\}\\
(\!|\mathrm{B}|\!)_G^{[t]} &= \{\langle v, e, \mathcal{T}_G, 0\rangle \mid \mathsf{tgt}(e) = v\} \cup \{\langle e, v, \mathcal{T}_G, 0\rangle \mid \mathsf{src}(e) = v\}\\
(\!|(?\mathsf{path})|\!)_G^{[t]} &= \{\langle o_1, o_1, \tau, 0\rangle \mid \exists o_2, d \colon \langle o_1, o_2, \tau, d\rangle \in (\!|\mathsf{path}|\!)_G^{[t]}\}\\
(\!|\mathsf{test}_1 \vee \mathsf{test}_2|\!)_G^{[t]} &= (\!|\mathsf{test}_1|\!)_G^{[t]} \cup (\!|\mathsf{test}_2|\!)_G^{[t]}\\
(\!|\mathsf{test}_1 \wedge \mathsf{test}_2|\!)_G^{[t]} &= \{\langle o, o, \tau_1 \cap \tau_2, 0\rangle \mid \langle o, o, \tau_1, 0\rangle \in (\!|\mathsf{test}_1|\!)_G^{[t]}, \langle o, o, \tau_2, 0\rangle \in (\!|\mathsf{test}_2|\!)_G^{[t]}, \tau_1 \cap \tau_2 \neq \emptyset\}\\
(\!|\neg\mathsf{test}|\!)_G^{[t]} &= \bigcup_{o \in N \cup E} \left\{\langle o, o, \tau, 0\rangle \mid \tau \in \mathsf{compl}\left(\{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!|\mathsf{test}|\!)_G^{[t]}\}, \mathcal{T}_G\right)\right\}\\
(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[t]} &= \left\{\langle o_1, o_3, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, \tau_1, d_1\rangle \in (\!|\mathsf{path}_1|\!)_G^{[t]} \wedge \langle o_2, o_3, \tau_2, d_2\rangle \in (\!|\mathsf{path}_2|\!)_G^{[t]} \wedge (\tau_1 + d_1) \cap \tau_2 \neq \emptyset\right\}\\
(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t]} &= (\!|\mathsf{path}_1|\!)_G^{[t]} \cup (\!|\mathsf{path}_2|\!)_G^{[t]}\\
(\!|\mathsf{path}[m, n]|\!)_G^{[t]} &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t]}\\
(\!|\mathsf{path}[m, _]|\!)_G^{[t]} &= \bigcup_{k \geq m} (\!|\mathsf{path}^k|\!)_G^{[t]}
\end{aligned}$$

We observe that for the operators $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m, \_])$ and $(\mathsf{path}[m, n])$, the defnition of $(\!|q|\!)_G^{[t]}$ is nearly identical to the one of $[\![q]\!]_G$. It will also be the case for the three representations below.

3.3.2 Correctness. The following result states that this representation is correct:

Proposition 3.1. *Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TPG and q a TRPQ. Then the unfolding of $(\!|q|\!)_G^{[t]}$ is $[\![q]\!]_G$.*

Proof. Let $G = \langle \Omega, N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TG, and let q be a TRPQ.
We show below that:

(I) for any $\langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$, there is a $\tau \in \mathsf{intv}(\mathcal{T})$ such that
  (a) $\langle o_1, o_2, \tau, d\rangle \in (\!|q|\!)_G^{[t]}$, and
  (b) $t \in \tau$

(II) for any $\langle o_1, o_2, \tau, d\rangle \in (\!|q|\!)_G^{[t]}$ for any $t \in \tau$,
  $\langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$

By induction on the structure of q:

$q = \mathrm{F}$

From the above definitions, we have:

$$\begin{aligned}
[\![\mathrm{F}]\!]_G &= \{\langle v, e, t, 0\rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, 0\rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\}\\
(\!|\mathrm{F}|\!)_G^{[t]} &= \{\langle v, e, \mathcal{T}_G, 0\rangle \mid \mathsf{src}(e) = v\} \cup \{\langle e, v, \mathcal{T}_G, 0\rangle \mid \mathsf{tgt}(e) = v\}
\end{aligned}$$

- For (I), let $\mathbf{u} = \langle o_1, o_2, t, d\rangle \in [\![\mathrm{F}]\!]_G$.
  Then $d = 0$.
  And since $\mathbf{u} \in [\![\mathrm{F}]\!]_G$, $\mathsf{src}(o_2) = o_1$ or $\mathsf{tgt}(o_1) = o_2$ must hold.
  Therefore $\langle o_1, o_2, \mathcal{T}_G, 0\rangle \in (\!|\mathrm{F}|\!)_G^{[t]}$.
  So (Ia) is verified.
  An trivially, $t \in \mathcal{T}_G$, so (Ib) is verified as well.
- For (II), let $\mathbf{v} = \langle o_1, o_2, \mathcal{T}_G, 0\rangle \in (\!|q|\!)_G^{[t]}$, and let $t \in \mathcal{T}_G$.
  Because $\mathbf{v} \in (\!|q|\!)_G^{[t]}$, $\mathsf{src}(o_2) = o_1$ or $\mathsf{tgt}(o_1) = o_2$ must hold.
  Therefore $\langle o_1, o_2, t, t\rangle = \langle o_1, o_2, t, t + d\rangle \in [\![q]\!]_G$.

$q = \mathrm{B}$

The proof is almost identical to the case $q = \mathrm{F}$.

$q = \mathrm{T}_\delta$

From the above definitions, we have:

$$\begin{aligned}
[\![\mathrm{T}_\delta]\!]_G &= \{\langle o, o, t, d\rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G\}\\
(\!|\mathrm{T}_\delta|\!)_G^{[t]} &= \{\langle o, o, [t_1, t_1], t_2 - t_1\rangle \mid o \in (N \cup E), t_1 \in \mathcal{T}_G, t_2 \in (\delta + t_1) \cap \mathcal{T}_G\}
\end{aligned}$$

- For (I), let $\mathbf{u} = \langle o, o, t, t + d \rangle \in [\![q]\!]_G$.
  Now let $\mathbf{v} = \langle o, o, [t, t], d \rangle$ in $\mathcal{U}^{[t]}$.
  For (Ia) we show that $\mathbf{v} \in (\!|\mathrm{T}_\delta|\!)_G^{[t]}$.
  Since $\mathbf{u} \in [\![q]\!]_G$, we have $o \in N \cup E$ and $t \in \mathcal{T}_G$.
  Besides, because $\mathbf{u} \in [\![q]\!]_G$ still, $d \in \delta$ and $t + d \in \mathcal{T}_G$.
  Therefore $t + d \in t + \delta \cap \mathcal{T}_G$.
  So there is a t_2 (namely $t + d$) such that $d = t_2 - t$ and $t_2 \in t + \delta \cap \mathcal{T}_G$, which concludes the proof for (Ia).
  An trivially, $t \in [t, t]$ so (Ib) is verified as well.
- For (II), let $\mathbf{v} = \langle o, o, [t, t], d \rangle \in (\!|q|\!)_G^{[t]}$.
  Because $\mathbf{v} \in (\!|q|\!)_G^{[t]} G$, we have $o \in N \cup E$ and $t \in \mathcal{T}_G$.
  So to conclude the proof, it is sufficient to show that *(i)* $d \in \delta$ and *(ii)* $t + d \in \mathcal{T}_G$.
  Because $\mathbf{v} \in (\!|q|\!)_G^{[t]} G$ stil, we have

$$d = t_2 - t \text{ for some } t_2 \in (\delta + t) \cap \mathcal{T}_G \tag{1}$$

  From (1), $t_2 = t + d$.
  Therefore from (1) still,

$$t + d \in (\delta + t) \cap \mathcal{T}_G \tag{2}$$

  which proves *(ii)* .
  And from (2), we also get

$$\begin{aligned} d + t &\in \delta + t \\ d + t - t &\in \delta + t - t \\ d &\in \delta \end{aligned}$$

  which proves *(i)* .

□

3.4 In $\mathcal{U}^{[d]}$

3.4.1 Definition.
We start with the case where q is an expression for the symbol test in the grammar of Section 3.1.
The definitions of $(\!|pred|\!)_G^{[t]}$ and $(\!|\neg\mathsf{test}|\!)_G^{[t]}$ are already provided in the article, we reproduce them here for completeness:

$$\begin{aligned}
(\!|pred|\!)_G^{[d]} &= \{\langle o, o, t, [0,0] \rangle \mid \tau \in \mathsf{val}(o, pred) \text{ and } t \in \tau\} \\
(\!|\mathrm{F}|\!)_G^{[d]} &= \{\langle v, e, t, [0,0] \rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, [0,0] \rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\} \\
(\!|\mathrm{B}|\!)_G^{[d]} &= \{\langle v, e, t, [0,0] \rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, [0,0] \rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \\
(\!|(?\mathsf{path})|\!)_G^{[d]} &= \{\langle o_1, o_1, t, [0,0] \rangle \mid \exists o_2, \delta \colon \langle o_1, o_2, t, \delta \rangle \in (\!|\mathsf{path}|\!)_G^{[d]}\} \\
(\!|\mathsf{test}_1 \vee \mathsf{test}_2|\!)_G^{[d]} &= (\!|\mathsf{test}_1|\!)_G^{[d]} \cup (\!|\mathsf{test}_2|\!)_G^{[d]} \\
(\!|\mathsf{test}_1 \wedge \mathsf{test}_2|\!)_G^{[d]} &= (\!|\mathsf{test}_1|\!)_G^{[d]} \cap (\!|\mathsf{test}_2|\!)_G^{[d]} \\
(\!|\neg\mathsf{test}|\!)_G^{[d]} &= \left\{\langle o, o, t, [0,0] \rangle \mid o \in N \cup E, t \in \mathcal{T}_G \setminus \{t' \mid \langle o, o, t', [0,0] \rangle \in (\!|\mathsf{test}|\!)_G^{[d]}\}\right\}
\end{aligned}$$

Next, we consider the operators $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m, \_])$ and $(\mathsf{path}[m, n])$.
For these cases, $(\!|q|\!)_G^{[t,d]}$ is once again defined analogously to $[\![q]\!]_G$, in terms of temporal join (a.k.a. $\mathsf{path}_1/\mathsf{path}_2$) and set union.
We only write the definitions here for the sake of completeness:

$$\begin{aligned}
(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t,d]} &= (\!|\mathsf{path}_1|\!)_G^{[t,d]} \cup (\!|\mathsf{path}_2|\!)_G^{[t,d]} \\
[\![\mathsf{path}[m, n]]\!]_G &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t,d]} \\
[\![\mathsf{path}[m, _]]\!]_G &= \bigcup_{k \geq m} (\!|\mathsf{path}^k|\!)_G^{[t,d]}
\end{aligned}$$

The only remaining operators are temporal join $(\mathsf{path}_1/\mathsf{path}_2)$ and temporal navigation (T_δ), already defined in the article.
We reproduce here these two definition for convenience:

$$\begin{aligned}
(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[d]} &= \left\{\langle o_1, o_3, t_1, \delta_2 + t_2 - t_1 \rangle \mid \exists o_2 \colon \langle o_1, o_2, t_1, \delta_1 \rangle \in (\!|\mathsf{path}_1|\!)_G^{[d]} \wedge \langle o_2, o_3, t_2, \delta_2 \rangle \in (\!|\mathsf{path}_2|\!)_G^{[d]} \wedge t_2 \in t_1 + \delta_1\right\} \\
(\!|\mathrm{T}_\delta|\!)_G^{[d]} &= \{\langle o, o, t, (\delta + t) \cap \mathcal{T}_G \rangle \mid o \in N \cup E, t \in \mathcal{T}_G\}
\end{aligned}$$

We also reproduce the alternative characterization of $(\!|\mathrm{T}_\delta|\!)_G^{[d]}$ provided in the article, as a unary operator:

$$(\!|q/\mathrm{T}_\delta|\!)_G^{[d]} = \{\langle o_1, o_2, t, (\delta' + \delta) \cap \mathcal{T}_G\rangle \mid \langle o_1, o_2, t, \delta'\rangle \in (\!|q|\!)_G^{[d]}\}$$

3.5 In $\mathcal{U}^{[t,d]}$

3.5.1 Definition.

If q is an expression for the symbol test in the grammar of Section 3.1, then all tuples in $[\![q]\!]_G$ must have distance 0. As a result, $(\!|q|\!)_G^{[t,d]}$ can be trivially defined out of $(\!|q|\!)_G^{[t]}$ by replacing the distance 0 with the interval $[0, 0]$, i.e.

$$(\!|\mathsf{test}|\!)_G^{[t,d]} = \{\langle o, o, \tau, [0, 0]\rangle \mid \{\langle o, o, \tau, 0\rangle \in (\!|\mathsf{test}|\!)_G^{[t]}\}$$

Again, for the operators $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m, \_])$ and $(\mathsf{path}[m, n])$, the definition of $(\!|q|\!)_G^{[t,d]}$ is nearly identical to the one of $[\![q]\!]_G$:

$$\begin{aligned}
(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t,d]} &= (\!|\mathsf{path}_1|\!)_G^{[t,d]} \cup (\!|\mathsf{path}_2|\!)_G^{[t,d]}\\
[\![\mathsf{path}[m, n]]\!]_G &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t,d]}\\
[\![\mathsf{path}[m, _]]\!]_G &= \bigcup_{k\geq m} (\!|\mathsf{path}^k|\!)_G^{[t,d]}
\end{aligned}$$

The only remaining operators are temporal join $(\mathsf{path}_1/\mathsf{path}_2)$ and temporal navigation (T_δ), already defined in the article, and reproduced here for convenience:

$$\begin{aligned}
(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[t,d]} &= \bigcup\{\mathbf{u}_1 \,\bar{\bowtie}\, \mathbf{u}_2 \mid \mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)_G^{[t,d]}, \mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)_G^{[t,d]}\}\\
(\!|\mathrm{T}_\delta|\!)_G^{[t,d]} &= \bigcup_{o\in N\cup E} \{\langle o, o, \mathcal{T}_G, \delta\rangle \,\bar{\bowtie}\, \langle o, o, \mathcal{T}_G, [0, 0]\rangle\}
\end{aligned}$$

where $\mathbf{u}_1 \,\bar{\bowtie}\, \mathbf{u}_2$ is defined as follows.
Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1\rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2\rangle$.
If $o_2 \neq o_3$, then $\mathbf{u}_1 \,\bar{\bowtie}\, \mathbf{u}_2 = \emptyset$.
Otherwise, each tuple in $\mathbf{u}_1 \,\bar{\bowtie}\, \mathbf{u}_2$ is of the form $\langle o_1, o_4, \tau, \delta\rangle$ for some τ and δ.
For $i \in \{1, 2\}$, let R_i be the binary relation over $\mathcal{T}$ specified by the time points and distances in $\mathbf{u}_i$, i.e., $R_i = \{(t, t + d) \mid t \in \tau_i, d \in \delta_i\}$.
And let $R_1 \bowtie R_2$ denote $\{(t_1, t_3) \mid (t_1, t_2) \in R_1$ and $(t_2, t_3) \in R_2$ for some $t_2\}$. Then the intervals in the set $\mathbf{u}_1 \,\bar{\bowtie}\, \mathbf{u}_2$ should intuitively represent this relation $R_1 \bowtie R_2$.
For each time point $t \in \mathrm{dom}(R_1 \bowtie R_2)$, let $\delta(t)$ denote the maximal interval s.t. $(\delta(t) + t) \subseteq \tau_2$.
We define $\mathbf{u}_1 \bowtie \mathbf{u}_2$ as

$$\{\langle o_1, o_4, [t, t], \delta(t) + \delta_2\rangle \mid t \in \mathrm{dom}(R_1 \bowtie R_2)\}$$

3.6 In $\mathcal{U}^{[t,d],b,e}$

3.6.1 Definition.

If q is an expression for the symbol test in the grammar of Section 3.1, then the definition of $(\!|q|\!)_G^{[t,d],b,e}$ is nearly identical to the one of $(\!|q|\!)_G^{[t,d]}$, extending each tuple $\{\langle o, o, \tau, [0, 0]\rangle$ with b_τ and e_τ, i.e.

$$(\!|\mathsf{test}|\!)_G^{[t,d],b,e} = \{\langle o, o, \tau, [0, 0], b_\tau, e_\tau\rangle \mid \{\langle o, o, \tau, [0, 0]\rangle \in (\!|\mathsf{test}|\!)_G^{[t,d]}\}$$

Next, for the operators, $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m, \_])$ and $(\mathsf{path}[m, n])$, the definition of $(\!|q|\!)_G^{[t,d]}$ is once again nearly identical to the one of $[\![q]\!]_G$:

$$\begin{aligned}
(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t,d],b,e} &= (\!|\mathsf{path}_1|\!)_G^{[t,d],b,e} \cup (\!|\mathsf{path}_2|\!)_G^{[t,d],b,e}\\
[\![\mathsf{path}[m, n]]\!]_G &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t,d],b,e}\\
[\![\mathsf{path}[m, _]]\!]_G &= \bigcup_{k\geq m} (\!|\mathsf{path}^k|\!)_G^{[t,d],b,e}
\end{aligned}$$

So the only remaining operator are temporal join $(\mathsf{path}_1/\mathsf{path}_2)$ and temporal navigation (T_δ), already defined in the article. We reproduce here these two definition for convenience:

$$\begin{aligned}
(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[t,d],b,e} &= \{\mathbf{u}_1 \,\bar{\bowtie}\, \mathbf{u}_2 \mid \mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)_G^{[t,d],b,e}, \mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)_G^{[t,d],b,e}, \mathbf{u}_1 \sim \mathbf{u}_2\}\\
(\!|\mathrm{T}_\delta|\!)_G^{[t,d],b,e} &= \{\langle o, o, \mathcal{T}_G, \delta, b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \,\bar{\bowtie}\, \langle o, o, \mathcal{T}_G, [0, 0], b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \mid o \in N \cup E\}
\end{aligned}$$

where $\mathbf{u}_1 \sim \mathbf{u}_2$ and $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2$ are defined by:

$$\langle o_1, o_2, \tau_1, \delta_1, b_1, e_1 \rangle \sim \langle o_3, o_4, \tau_2, \delta_2, e_2, b_2 \rangle \text{ iff } o_2 = o_3$$

$$\begin{array}{rcl} \langle o_1, o_3, \tau_1, \delta_1, b_1, e_1 \rangle \,\overline{\bowtie}\, \langle o_3, o_4, \tau_2, \delta_2, e_2, b_2 \rangle & = & \langle o_1, o_4, \tau, \delta_1 + \delta_2, b, e \rangle \\ \text{with} & & \\ \tau & = & (((\tau_1 + \delta_1) \cap \tau_2) \ominus \delta_1) \cap \tau_1 \\ b & = & \max(b_1, b_2 - b_{\delta_1}) \\ e & = & \min(e_1, e_2 - e_{\delta_1}) \end{array}$$

3.6.2 *Correctness.*

Operator $\text{path}_1/\text{path}_2$**.**
Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2 \rangle$, with $o_2 = o_3$.
Let also $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 = \langle o_1, o_4, \tau_1', \delta_1 + \delta_2, b, e \rangle$, with

$$\tau = (((\tau_1 + \delta_1) \cap \tau_2) \ominus \delta_1) \cap \tau_1$$
$$b = \max(b_1, b_2 - b_{\delta_1})$$
$$e = \min(e_1, e_2 - e_{\delta_1})$$

As explained in Section 2, for $i \in \{1, 2\}$ and $t \in \tau_i$, we use $\delta_i(t)$ for the interval

$$_{\delta_i}\lfloor\, b_{\delta_i} + \max(0, b_i - t)\ ,\ e_{\delta_i} - \max(0, t - e_i)\, \rfloor_{\delta_i}$$

And similarly to what we did for $\mathcal{U}^{[t,d]}$, we use R_i for be the binary relation over $\mathcal{T}$ specified by the time points and distances in $\mathbf{u}_i$, i.e. $R_i = \{(t, t + d) \mid t \in \tau_i, d \in \delta_i(t)\}$.
Then the intervals in the set $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2$ should intuitively represent this relation $R_1 \bowtie R_2$, i.e. we need to prove that *(i)* $\tau = \text{dom}(R_1 \bowtie R_2)$ and that *(ii)* for each $t \in \tau$,

$$t + \delta(t) = \{t' \mid (t, t') \in R_1 \bowtie R_2\}$$

For *(ii)* , let $t \in \tau$. We use

- a for the least value s.t. $(t, a) \in \text{range}(R_1) \cap \text{dom}(R_2)$, and
- a' for the least value s.t. $(a, a') \in R_2$

Then a' is also the least value s.t. $(t, a') \in R_1 \bowtie R_2$.
Analogously, we use z for the greatest value s.t. $(t, z) \in \text{range}(R_1) \cap \text{dom}(R_2)$, and z' for the greatest value s.t. $(z, z') \in R_2$. Then z' is also the greatest value s.t. $(t, z') \in R_1 \bowtie R_2$.

From Lemma **??**:

- $\{t\} \times (t + [a, z]) \subseteq R_1$, and
- $[a, b] \times [a', z'] \subseteq R_2$

Therefore $[a', z'] = \{c \mid (t, c) \in R_1 \bowtie R_2\}$.

To conclude the proof, we show that $t + \delta_t = [a, z]$.
We only prove that $t + b_{\delta_t} = a$ (the proof that $t + e_{\delta_t} = z$ is symmetric).
Following the definition of b, we consider 2 cases:

(1) $b_1 < b_2 - b_{\delta_1}$
(2) $b_1 \geq b_2 - b_{\delta_1}$

For Case (1), we get

$$\begin{aligned} b_1 <\ & b_2 - b_{\delta_1} && (3) \\ \max(b_1, b_2 - b_{\delta_1}) =\ & b_2 - b_{\delta_1} && (4) \\ b =\ & b_2 - b_{\delta_1} && \text{from the definition of } b \quad (5) \end{aligned}$$

And in Case (1) still, we get:

$$\begin{aligned} b_1 <\ & b_2 - b_{\delta_1} && (6) \\ 0 <\ & b_2 - b_{\delta_1} - b_1 && (7) \\ \max(0, b_2 - b_{\delta_1} - b_1) =\ & b_2 - b_{\delta_1} - b_1 && (8) \end{aligned}$$

Next, we consider two subcases:

(i) $t < b_2 - b_{\delta_1}$
(ii) $t \geq b_2 - b_{\delta_1}$

In Case (i), we get

$$\begin{aligned} t <&\ b_2 - b_{\delta_1} && (9)\\ 0 <&\ b_2 - b_{\delta_1} - t && (10)\\ \max(0, b_2 - b_{\delta_1} - t) =&\ b_2 - b_{\delta_1} - t && (11) \end{aligned}$$

Now from the definition of δ_t,

$$\begin{aligned} b_{\delta_t} =&\ b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (12)\\ =&\ b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_{\delta_1} - t) && \text{from (5)} && (13)\\ =&\ b_{\delta_1} + b_{\delta_2} + b_2 - b_{\delta_1} - t && \text{from (11)} && (14)\\ =&\ b_{\delta_2} + b_2 - t && && (15)\\ b_{\delta_t} + t =&\ b_{\delta_2} + b_2 - t + t && && (16)\\ =&\ b_{\delta_2} + b_2 && && (17) \end{aligned}$$

Next, from the definition of a'

$$\begin{aligned} a' =&\ b_{\delta_2(a)} + a && (18)\\ =&\ b_{\delta_2} + \max(0, b_2 - a) + a && (19) \end{aligned}$$

And, from the definition of a

$$\begin{aligned} a =&\ b_{\delta_1(t)} + t && (20)\\ =&\ b_{\delta_1} + \max(0, b_1 - t) + t && (21) \end{aligned}$$

Then we have two further subcases:

(I) $t \geq b_1$, or
(II) $t < b_1$

In case (I):

$$\begin{aligned} t \geq&\ b_1 && && (22)\\ 0 \geq&\ b_1 - t && && (23)\\ \max(0, b_1 - t) =&\ 0 && && (24)\\ a =&\ b_{\delta_1} + t && \text{from (21)} && (25)\\ \max(0, b_2 - a) =&\ \max(0, b_2 - b_{\delta_1} - t) && && (26)\\ =&\ b_2 - b_{\delta_1} - t && \text{from (11)} && (27)\\ =&\ b_2 - a && \text{from (25)} && (28) \end{aligned}$$

In case (II):

$$\begin{aligned} t <&\ b_1 && && (29)\\ 0 <&\ b_1 - t && && (30)\\ \max(0, b_1 - t) =&\ b_1 - t && && (31)\\ a =&\ b_{\delta_1} + b_1 - t + t && \text{from (21)} && (32)\\ =&\ b_{\delta_1} + b_1 && && (33)\\ \max(0, b_2 - a) =&\ \max(0, b_2 - b_{\delta_1} - b_1) && && (34)\\ =&\ b_2 - b_{\delta_1} - b_1 && \text{from (8)} && (35)\\ =&\ b_2 - a && \text{from (33)} && (36)\\ & && && (37) \end{aligned}$$

So in both cases (I) and (II), we get

$$\max(0, b_2 - a) = b_2 - a$$

Thefore from (19)

$$a' = b_{\delta_2} + b_2 - a + a \tag{38}$$
$$= b_{\delta_2} + b_2 \tag{39}$$
$$= t + b_{\delta_t} \quad \text{from (17)} \tag{40}$$

which concludes the proof for Case (1)- (i).

We continue with Case (1)- (ii).
From Case (ii):

$$t \geq b_2 - b_{\delta_1} \tag{41}$$
$$0 \geq b_2 - b_{\delta_1} - t \tag{42}$$
$$\max(0, b_2 - b_{\delta_1} - t) = 0 \tag{43}$$

Now from the definition of δ_t:

$$b_{\delta_t} = b_{\delta_1} + b_{\delta_2} + \max(0, b - t) \tag{44}$$
$$= b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_1 - t) \quad \text{from (5)} \tag{45}$$
$$= b_{\delta_1} + b_{\delta_2} \quad \text{from (43)} \tag{46}$$
$$b_{\delta_t} + t = b_{\delta_1} + b_{\delta_2} + t \tag{47}$$

Next, from Case (1) and Case (ii), by transitivity, we get

$$b_1 \leq t \tag{48}$$
$$\max(0, b_1 - t) = 0 \tag{49}$$

And from the definition of a

$$a = b_{\delta_1(t)} + t \tag{50}$$
$$= b_{\delta_1} + \max(0, b_1 - t) + t \tag{51}$$
$$= b_{\delta_1} + t \quad \text{from (49)} \tag{52}$$
$$\geq b_{\delta_1} + b_2 - b_{\delta_1} \quad \text{from Case (ii)} \tag{53}$$
$$\geq b_2 \tag{54}$$
$$0 \geq b_2 - a \tag{55}$$
$$\max(0, b_2 - a) = 0 \tag{56}$$

Therefore from (19) and (56)

$$a' = b_{\delta_2} + a \tag{57}$$
$$= b_{\delta_2} + b_{\delta_1} + t \quad \text{from (52)} \tag{58}$$
$$= b_{\delta_t} + t \quad \text{from (17)} \tag{59}$$

which concludes the proof for Case (1)- (ii).

We continute with Case (2).
In this case, we get

$$b_1 \geq b_2 - b_{\delta_1} \tag{60}$$
$$\max(b_1, b_2 - b_{\delta_1}) = b_1 \tag{61}$$
$$b = b_1 \quad \text{from the definition of } b \tag{62}$$

And from Case (2) still, we derive

$$b_1 \geq b_2 - b_{\delta_1} \tag{63}$$
$$0 \geq b_2 - b_{\delta_1} - b_1 \tag{64}$$
$$\max(0, -b_{\delta_1} - b_1) = 0 \tag{65}$$

As well as

$$b_1 \geq b_2 - b_{\delta_1} \tag{66}$$
$$b_1 + b_{\delta_1} \geq b_2 \tag{67}$$

Next, we distinguish two subcases, namely

(a) $t < b_1$ and
(b) $t \geq b_1$

We start with Case (a).
In this case,

$$\begin{aligned}
t <\ & b_1 && (68)\\
0 <\ & b_1 - t && (69)\\
\max(0, b_1 - t) =\ & b_1 - t && (70)
\end{aligned}$$

And from the definition of δ_t:

$$\begin{aligned}
b_{\delta_t} =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (71)\\
=\ & b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) && \text{from (62)} && (72)\\
=\ & b_{\delta_1} + b_{\delta_2} + b_1 - t && \text{from (70)} && (73)\\
b_{\delta_t} + t =\ & b_{\delta_1} + b_{\delta_2} + b_1 - t + t && && (74)\\
=\ & b_{\delta_1} + b_{\delta_2} + b_1 && && (75)
\end{aligned}$$

Next, from the definition of a

$$\begin{aligned}
a =\ & b_{\delta_1(t)} + t && && (76)\\
=\ & \max(0, b_1 - t) + b_{\delta_1} + t && && (77)\\
=\ & b_1 - t + b_{\delta_1} + t && \text{from (70)} && (78)\\
=\ & b_1 + b_{\delta_1} && && (79)
\end{aligned}$$

So from (67) (80)

$$\begin{aligned}
a \geq\ & b_2 && && (81)\\
0 \geq\ & b_2 - a && && (82)\\
\max(0, b_2 - a) =\ & 0 && && (83)\\
b_{\delta_2} + \max(0, b_2 - a) =\ & b_{\delta_2} && && (84)\\
b_{\delta_2(a)} =\ & b_{\delta_2} && && (85)\\
b_{\delta_2(a)} + a =\ & b_{\delta_2} + a && && (86)\\
a' =\ & b_{\delta_2} + a && \text{from the defintiion of } a' && (87)\\
a' =\ & b_{\delta_2} + b_1 + b_{\delta_1} && \text{from (79)} && (88)\\
a' =\ & b_{\delta_t} + t && \text{from (75)} && (89)
\end{aligned}$$

which concludes the proof for Case (2)- (a).

We end with Case (2)- (b). In this case,

$$\begin{aligned}
t \geq\ & b_1 && (90)\\
0 \geq\ & b_1 - t && (91)\\
\max(0, b_1 - t) =\ & 0 && (92)
\end{aligned}$$

And from the definition of δ_t:

$$\begin{aligned}
b_{\delta_t} =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (93)\\
=\ & b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) && \text{from (62)} && (94)\\
=\ & b_{\delta_1} + b_{\delta_2} && \text{from (92)} && (95)\\
b_{\delta_t} + t =\ & b_{\delta_1} + b_{\delta_2} + t && && (96)
\end{aligned}$$

Next, from the definition of a

$$\begin{aligned}
a =\ & b_{\delta_1(t)} + t && && (97)\\
=\ & \max(0, b_1 - t) + b_{\delta_1} + t && && (98)\\
=\ & b_{\delta_1} + t && \text{from (92)} && (99)
\end{aligned}$$

Now from Case (b)

$$
\begin{aligned}
b_1 + \leq\ & t && (100)\\
b_1 + b_{\delta_1} \leq\ & t + b_{\delta_1} && (101)\\
b_1 + b_{\delta_1} \leq\ & a && \text{from (99)} \quad (102)\\
b_2 \leq\ & a && \text{from (67), by transitivity} \quad (103)\\
b_2 - a \leq\ & 0 && (104)\\
\max(0, b_2 - a) =\ & 0 && (105)\\
b_{\delta_2} + \max(0, b_2 - a) =\ & b_{\delta_2} && (106)\\
b_{\delta_2(a)} =\ & b_{\delta_2} && (107)\\
b_{\delta_2(a)} + a =\ & b_{\delta_2} + a && (108)\\
a' =\ & b_{\delta_2} + a && \text{from the defintiion of } a' \quad (109)\\
=\ & b_{\delta_2} + b_{\delta_1} + t && \text{from (99)} \quad (110)\\
=\ & b_{\delta_t} + t && \text{from (96)} \quad (111)
\end{aligned}
$$

which concludes the proof for Case (2)- (b). □

4 FINITENESS

5 COMPLEXITY OF QUERY ANSWERING

5.1 Problem

We define in this section a decision problem for each representation, similar to the problem COMPACT ANSWER$^{[t]}$ defined in the article.
First, we define a (possibly partial) order over tuples of each representation, in the expected way, i.e.:

- over $\mathcal{U}^{[t]}$: $\langle o_1, o_2, \tau_1, d_1\rangle \sqsubseteq_{[t]} \langle o_3, o_4, \tau_2, d_2\rangle$ iff $\langle o_1, o_2, d_1\rangle = \langle o_3, o_4, d_2\rangle$ and $\tau_1 \subseteq \tau_2$,
- over $\mathcal{U}^{[d]}$: $\langle o_1, o_2, t_1, \delta_1\rangle \sqsubseteq_{[d]} \langle o_3, o_4, t_2, \delta_1\rangle$ iff $\langle o_1, o_2, t_1\rangle = \langle o_3, o_4, t_2\rangle$ and $\delta_1 \subseteq \delta_2$,
- over $\mathcal{U}^{[t,d]}$: $\langle o_1, o_2, \tau_1, \delta_1\rangle \sqsubseteq_{[t,d]} \langle o_3, o_4, \tau_2, \delta_2\rangle$ iff $\langle o_1, o_2\rangle = \langle o_3, o_4\rangle$, $\tau_1 \subseteq \tau_2$ and $\delta_1 \subseteq \delta_2$,
- over $\mathcal{U}^{[t,d],b,e}$: $\langle o_1, o_2, \tau_1, \delta_1, b_1, b_2\rangle \sqsubseteq_{[t,d],b,e} \langle o_3, o_4, \tau_2, \delta_2, b_2, e_2\rangle$ iff $\langle o_1, o_2\rangle = \langle o_3, o_4\rangle$, $\tau_1 \subseteq \tau_2$ and $\delta_1(t) \subseteq \delta_2(t)$ for all $t \in \tau_1 \cap \tau_2$ (the notation of $\delta_i(t)$ is explained above, in Section 2).

Now let x be one of $[t]$, $[d]$, $[t,d]$ or $([t,d], b, e)$.
We decline the notion of compact answer defined in Section XXX in four flavors, as follows:

Definition 5.1. Let G be a TG, let q be a TRPQ and let $\mathbf{u} \in \mathcal{U}^x$.
We say that $\mathbf{u}$ is a *compact answer* to q over G (in $\mathcal{U}^x$) if $\mathbf{u} \in \max_{\sqsubseteq_x}\{\mathbf{u}' \in \mathcal{U}^x \mid \mathsf{unfold}(\mathbf{u}') = [\![q]\!]_G\}$

And similary, we get four decision problems:

COMPACT ANSWER^x	
Input:	TG G, TRPQ q, tuple $\mathbf{u} \in \mathcal{U}^x$
Decide:	$\mathbf{u}$ is a compact answer to q over G (in $\mathcal{U}^x$)

5.2 Hardness

5.3 Membership

6 MINIMIZATION

7 SIZE OF COMPACT ANSWERS