Compact Answers to Temporal Regular Path Queries (Supplementary Material)

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1 INTRODUCTION

This document provides detailed definitions and proofs for the article *Compact answers to Temporal Regular Path Queries*, submitted at CIKM 2023.

2 NOTATION

Let $\langle o_1, o_2, \tau, \delta, b, e\rangle \in \mathcal{U}^{[t,d],b,e}$, and let $t \in \tau$.
In the article, we defined the interval δ_t for each t as

$$_{\delta}\lfloor\ b_\delta + \max(0, b - t)\ ,\ e_{\delta_i} - \max(0, t - e)\ \rfloor_\delta$$

In this supplementary material, we will use $\delta(t)$ instead of δ_t. This notation will allow us to write $\delta_1(t)$ when several tuples are involved. Note that the time points b and e in this notaton are still omitted, for conciseness, because they should be clear from the context.

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3 INDUCTIVE REPRESENTATION

Let q be a TRPQ and G a TG.

Then $[\![q]\!]_G$ is the set of anwers to q over G (represented as tuples in $\mathcal{U}$).

In this section, we provide the full definition of the four inductive representations of $[\![q]\!]_G$ discussed in the article, in $\mathcal{U}^{[t]}$, $\mathcal{U}^{[d]}$, $\mathcal{U}^{[t,d]}$ and $\mathcal{U}^{[t,d],b,e}$ respectively, and prove that they are correct.

These representations are denoted as $(\!|q|\!)_G^{[t]}$, $(\!|q|\!)_G^{[d]}$, $(\!|q|\!)_G^{[t,d]}$ and $(\!|q|\!)_G^{[t,d],b,e}$ respectively.

3.1 Grammar

For convenience, we reproduce here the syntax of TRPQs (already provided in the article).

Definition 3.1 (TRPQ). A TRPQ is an expression for the symbol path in the following grammar:

$$\begin{aligned}
\mathsf{path} &::= \mathsf{test} \mid \mathsf{axis} \mid (\mathsf{path}/\mathsf{path}) \mid (\mathsf{path} + \mathsf{path}) \mid \mathsf{path}[m,n] \mid \mathsf{path}[m,_] \\
\mathsf{test} &::= \mathit{pred} \mid (?\mathsf{path}) \mid \mathsf{test} \vee \mathsf{test} \mid \mathsf{test} \wedge \mathsf{test} \mid \neg\mathsf{test} \\
\mathsf{axis} &::= \mathrm{F} \mid \mathrm{B} \mid \mathrm{T}_\delta
\end{aligned}$$

with $\delta \in \mathsf{intv}(\mathcal{T})$, $m, n \in \mathbb{N}^+$ and $m \leq n$.

3.2 In $\mathcal{U}$

For convenience still, we reproduce the full semantics of TRPQs, i.e. the inductive definition of the evaluation $[\![q]\!]_G$ of a query q over a graph G in $\mathcal{U}$ (already provided in the article).

$$\begin{aligned}
[\![\mathit{pred}]\!]_G &= \{\langle o, o, t, 0\rangle \mid t \in \tau \text{ for some } \tau \in \mathsf{val}(o, \mathit{pred})\} \\
[\![\mathrm{T}_\delta]\!]_G &= \{\langle o, o, t, d\rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G\} \\
[\![\mathrm{F}]\!]_G &= \{\langle v, e, t, 0\rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, 0\rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\} \\
[\![\mathrm{B}]\!]_G &= \{\langle v, e, t, 0\rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, 0\rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \\
[\![?\mathsf{path}]\!]_G &= \{\langle o, o, t, 0\rangle \mid \langle o, o', t, t + d\rangle \in [\![\mathsf{path}]\!]_G \text{ for some } o' \in N \cup E, d \in \mathcal{T}\} \\
[\![\mathsf{test}_1 \vee \mathsf{test}_2]\!]_G &= [\![\mathsf{test}_1]\!]_G \cup [\![\mathsf{test}_2]\!]_G \\
[\![\mathsf{test}_1 \wedge \mathsf{test}_2]\!]_G &= [\![\mathsf{test}_1]\!]_G \cap [\![\mathsf{test}_2]\!]_G \\
[\![\neg\mathsf{test}]\!]_G &= (\{\langle o, o\rangle \mid o \in N \cup E\} \times \mathcal{T}_G \times \{0\}) \setminus [\![\mathsf{test}]\!]_G \\
[\![\mathsf{path}_1/\mathsf{path}_2]\!]_G &= \{\langle o_1, o_3, t, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, t, d_1\rangle \in [\![\mathsf{path}_1]\!]_G \wedge \langle o_2, o_3, t + d_1, d_2\rangle \in [\![\mathsf{path}_2]\!]_G\} \\
[\![\mathsf{path}_1 + \mathsf{path}_2]\!]_G &= [\![\mathsf{path}_1]\!]_G \cup [\![\mathsf{path}_2]\!]_G \\
[\![\mathsf{path}[m,n]]\!]_G &= \bigcup_{k=m}^{n} [\![\mathsf{path}^k]\!]_G \\
[\![\mathsf{path}[m,_]]\!]_G &= \bigcup_{k \geq m} [\![\mathsf{path}^k]\!]_G
\end{aligned}$$

3.3 In $\mathcal{U}^{[t]}$

3.3.1 *Definition.*

The full definition of $(\!|q|\!)_G^{[t]}$ is already provided in the article. We only reproduce it here for convenience.

$$\begin{aligned}
(\!|\mathit{pred}|\!)_G^{[t]} &= \{\langle o, o, \tau, 0\rangle \mid o \in (N \cup E), \tau \in \mathsf{val}(o, \mathit{pred})\} \\
(\!|\mathrm{T}_\delta|\!)_G^{[t]} &= \{\langle o, o, [t_1, t_1], t_2 - t_1\rangle \mid o \in (N \cup E), t_1 \in \mathcal{T}_G, t_2 \in (\delta + t_1) \cap \mathcal{T}_G\} \\
(\!|\mathrm{F}|\!)_G^{[t]} &= \{\langle v, e, \mathcal{T}_G, 0\rangle \mid \mathsf{src}(e) = v\} \cup \{\langle e, v, \mathcal{T}_G, 0\rangle \mid \mathsf{tgt}(e) = v\} \\
(\!|\mathrm{B}|\!)_G^{[t]} &= \{\langle v, e, \mathcal{T}_G, 0\rangle \mid \mathsf{tgt}(e) = v\} \cup \{\langle e, v, \mathcal{T}_G, 0\rangle \mid \mathsf{src}(e) = v\} \\
(\!|(?\mathsf{path})|\!)_G^{[t]} &= \{\langle o, o, \tau, 0\rangle \mid \langle o, o', \tau, d\rangle \in (\!|\mathsf{path}|\!)_G^{[t]} \text{ for some } o' \in N \cup E, d \in \mathcal{T}\} \\
(\!|\mathsf{test}_1 \vee \mathsf{test}_2|\!)_G^{[t]} &= (\!|\mathsf{test}_1|\!)_G^{[t]} \cup (\!|\mathsf{test}_2|\!)_G^{[t]} \\
(\!|\mathsf{test}_1 \wedge \mathsf{test}_2|\!)_G^{[t]} &= \{\langle o, o, \tau_1 \cap \tau_2, 0\rangle \mid \langle o, o, \tau_1, 0\rangle \in (\!|\mathsf{test}_1|\!)_G^{[t]}, \langle o, o, \tau_2, 0\rangle \in (\!|\mathsf{test}_2|\!)_G^{[t]}, \tau_1 \cap \tau_2 \neq \emptyset\} \\
(\!|\neg\mathsf{test}|\!)_G^{[t]} &= \bigcup_{o \in N \cup E} \left\{\langle o, o, \tau, 0\rangle \mid \tau \in \mathsf{compl}\left(\{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!|\mathsf{test}|\!)_G^{[t]}\}, \mathcal{T}_G\right)\right\} \\
(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[t]} &= \left\{\langle o_1, o_3, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, \tau_1, d_1\rangle \in (\!|\mathsf{path}_1|\!)_G^{[t]} \wedge \langle o_2, o_3, \tau_2, d_2\rangle \in (\!|\mathsf{path}_2|\!)_G^{[t]} \wedge (\tau_1 + d_1) \cap \tau_2 \neq \emptyset\right\} \\
(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t]} &= (\!|\mathsf{path}_1|\!)_G^{[t]} \cup (\!|\mathsf{path}_2|\!)_G^{[t]} \\
(\!|\mathsf{path}[m,n]|\!)_G^{[t]} &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t]} \\
(\!|\mathsf{path}[m,_]|\!)_G^{[t]} &= \bigcup_{k \geq m} (\!|\mathsf{path}^k|\!)_G^{[t]}
\end{aligned}$$

We observe that when q is of the form $(\text{path}_1 + \text{path}_2)$, $(\text{path}[m, \_])$ and $(\text{path}[m, n])$, the defnition of $(\!|q|\!)^{[t]}_G$ is nearly identical to the one of $[\![q]\!]_G$. This will also be the case for the three representations below.

3.3.2 *Correctness.*

We start with a lemma:

Lemma 3.2. *Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val}\rangle$ be a TPG and let q an expression for the symbol* test *in the grammar of Definition 3.1. Then:*

- *each tuples in $[\![q]\!]_G$ is of the form $\langle o_1, o_2, t, 0\rangle$ for some o_1, o_2 and t,*
- *each tuples in $(\!|q|\!)^{[t]}_G$ is of the form $\langle o_1, o_2, \tau, 0\rangle$ for some o_1, o_2 and τ.*

Proof. Immediate from the definitions of $[\![q]\!]_G$ and $(\!|q|\!)^{[t]}_G$. □

The following result states that the representation $(\!|q|\!)^{[t]}_G$ is correct:

Proposition 3.3. *Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val}\rangle$ be a TPG and q a TRPQ. Then the unfolding of $(\!|q|\!)^{[t]}_G$ is $[\![q]\!]_G$.*

Proof.
Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val}\rangle$ be a TG, and let q be a TRPQ.
We show below that:

(I) for any $\langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$, there is a $\tau \in \text{intv}(\mathcal{T})$ such that
  (a) $\langle o_1, o_2, \tau, d\rangle \in (\!|q|\!)^{[t]}_G$, and
  (b) $t \in \tau$,
(II) for any $\langle o_1, o_2, \tau, d\rangle \in (\!|q|\!)^{[t]}_G$ for any $t \in \tau$,
  $\langle o_1, o_2, t, d\rangle$ is in $[\![q]\!]_G$.

We proceed by induction on the structure of q.
If q is of the form *pred*, F, B, (test $\vee$ test), (path + path), path$[m, n]$ or path$[m, _]$, then (I) and (II) immediately follow from the definitions of $[\![q]\!]_G$ and $(\!|q|\!)^{[t]}_G$.
So we focus below on the five remaining cases:

- $q = \text{T}_\delta$.
  From the above definitions, we have:

$$\begin{aligned}[\![q]\!]_G &= \{\langle o, o, t, d\rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G\}\\ (\!|q|\!)^{[t]}_G &= \{\langle o, o, [t_1, t_1], t_2 - t_1\rangle \mid o \in (N \cup E), t_1 \in \mathcal{T}_G, t_2 \in (\delta + t_1) \cap \mathcal{T}_G\}\end{aligned}$$

  - For (I), let $\mathbf{v} = \langle o, o, t, t + d\rangle \in [\![q]\!]_G$.
    And let $\mathbf{u} = \langle o, o, [t, t], d\rangle$ in $\mathcal{U}^{[t]}$.
    For (Ia) we show that $\mathbf{u} \in (\!|q|\!)^{[t]}_G$.
    From $\mathbf{v} \in [\![q]\!]_G$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.
    Besides, because $\mathbf{v} \in [\![q]\!]_G$ still,
$$t + d \in \mathcal{T}_G \tag{1}$$
    and
$$d \in \delta \tag{2}$$
$$t + d \in t + \delta \tag{3}$$
    So from (1) and (3)
$$t + d \in (\delta + t) \cap \mathcal{T}_G \tag{4}$$
    So there is a t_2 (namely $t + d$) such that $d = t_2 - t$ and $t_2 \in t + \delta \cap \mathcal{T}_G$.
    Together with the definition of $(\!|q|\!)^{[t]}_G$, this implies $\mathbf{u} \in (\!|q|\!)^{[t]}_G$, which concludes the proof for (Ia).
    And trivially, $t \in [t, t]$, so (Ib) is verified as well.

- For (II), let $\mathbf{u} = \langle o, o, [t, t], d\rangle \in (\!|q|\!)_G^{[t]}$.
  From $\mathbf{u} \in (\!|q|\!)_G^{[t]} G$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.
  So to conclude the proof, it is sufficient to show that *(i)* $d \in \delta$ and *(ii)* $t + d \in \mathcal{T}_G$.
  Because $\mathbf{u} \in (\!|q|\!)_G^{[t]} G$ still, we have
  $$d = t_2 - t \text{ for some } t_2 \in (\delta + t) \cap \mathcal{T}_G \tag{5}$$
  From (5), we get $t_2 = t + d$.
  Therefore from (5) still,
  $$t + d \in (\delta + t) \cap \mathcal{T}_G \tag{6}$$
  which proves *(ii)* .
  And from (6), we also get
  $$\begin{gathered} t + d \in \delta + t \\ t + d - t \in (\delta + t) - t \\ d \in \delta \end{gathered}$$
  which proves *(i)* .

- $q = \text{test}_1 \wedge \text{test}_2$.
  From the above definitions, we have:
  $$\begin{aligned} [\![q]\!]_G =& [\![\text{test}_1]\!]_G \cap [\![\text{test}_2]\!]_G \\ (\!|q|\!)_G^{[t]} =& \{\langle o, o, \tau_1 \cap \tau_2, 0\rangle \mid \langle o, o, \tau_1, 0\rangle \in (\!|\text{test}_1|\!)_G^{[t]}, \langle o, o, \tau_2, 0\rangle \in (\!|\text{test}_2|\!)_G^{[t]}, \tau_1 \cap \tau_2 \neq \emptyset\} \end{aligned}$$

  - For (I), let $\mathbf{v} = \langle o, o, t, d\rangle \in [\![q]\!]_G$.
    From Lemma 3.2, $d = 0$.
    And from the definition of $[\![q]\!]_G$, $\mathbf{v} \in [\![\text{test}_1]\!]_G \cap [\![\text{test}_2]\!]_G$.
    So by IH, there are intervals τ_1 and τ_2 s.t. $\langle o, o, \tau_i, 0\rangle \in (\!|\text{test}_i|\!)_G^{[t]}$ for $i \in \{1, 2\}$ and $t \in \tau_1 \cap \tau_2$.
    Together with the definition of $(\!|q|\!)_G^{[t]}$, this proves (I).
  - For (II), let $\langle o, o, \tau, d\rangle \in (\!|q|\!)_G^{[t]}$.
    Then from Lemma 3.2, $d = 0$.
    And from the definition of $(\!|q|\!)_G^{[t]}$, there are two intervals τ_1 and τ_2 s.t. $\tau = \tau_1 \cap \tau_2$ and $\langle o, o, \tau_i, 0\rangle \in (\!|\text{test}_i|\!)_G^{[t]}$ for $i \in \{1, 2\}$.
    Now take any $t \in \tau$.
    Then $t \in \tau_i$ for $i \in \{1, 2\}$.
    So by IH, $\langle o, o, t, 0\rangle \in [\![\text{test}_i]\!]_G$ for each $i \in \{1, 2\}$.
    Together with the defintiion of $[\![q]\!]_G$, this proves (II).

- $q = (?\text{path})$.
  From the above definitions, we have:
  $$\begin{aligned} [\![q]\!]_G &= \{\langle o, o, t, 0\rangle \mid \langle o, o', t, t + d\rangle \in [\![\text{path}]\!]_G \text{ for some } o' \in N \cup E, d \in \mathcal{T}\} \\ (\!|q|\!)_G^{[t]} &= \{\langle o, o, \tau, 0\rangle \mid \langle o, o', \tau, d\rangle \in (\!|\text{path}|\!)_G^{[t]} \text{ for some } o' \in N \cup E, d \in \mathcal{T}\} \end{aligned}$$

  - For (I), let $\langle o, o, t, 0\rangle \in [\![q]\!]_G$.
    From the definition of $[\![q]\!]_G$, there are o' and d such that $\langle o, o', t, t + d\rangle \in [\![\text{path}]\!]_G$.
    So by IH, there is a τ s.t. $t \in \tau$ and $\langle o, o', \tau, d\rangle \in (\!|\text{path}|\!)_G^{[t]}$.
    Therefore $\langle o, o, \tau, 0\rangle \in (\!|q|\!)_G^{[t]}$, from the definition of $(\!|q|\!)_G^{[t]}$.
  - For (II), let $\langle o, o, \tau, 0\rangle \in (\!|q|\!)_G^{[t]}$.
    From the definition of $(\!|q|\!)_G^{[t]}$, there are o' and d s.t. $\langle o, o', \tau, d\rangle \in (\!|\text{path}|\!)_G^{[t]}$.
    Now take any $t \in \tau$.
    By IH, $\langle o, o', t, t + d\rangle \in [\![\text{path}]\!]_G$.
    Therefore $\langle o, o, t, 0\rangle \in [\![q]\!]_G$, from the definition of $[\![q]\!]_G$.

- $q = \neg\text{test}$.
  From the above definitions, we have:

$$\begin{aligned}
[\![q]\!]_G &= (\{\langle o, o\rangle \mid o \in N \cup E\} \times \mathcal{T}_G \times \{0\}) \setminus [\![\text{test}]\!]_G \\
(\!|q|\!)_G^{[t]} &= \bigcup_{o \in N \cup E} \left\{\langle o, o, \tau, 0\rangle \mid \tau \in \mathsf{compl}\left(\{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!|\text{test}|\!)_G^{[t]}\}, \mathcal{T}_G\right)\right\}
\end{aligned}$$

  - For (I), let $\mathbf{v} = \langle o, o, t, 0\rangle \in [\![q]\!]_G$.
    From the definition of $[\![q]\!]_G$, $\mathbf{v} \notin [\![\text{test}]\!]_G$.
    So
$$t \notin \{t' \mid \langle o, o, t', 0\rangle \in [\![\text{test}]\!]_G\} \tag{7}$$
    Now by IH, together with Lemma 3.2, we get:
$$\langle o, o, t', 0\rangle \in [\![\text{test}]\!]_G \text{ iff } t' \in \tau' \text{ for some } \tau' \text{ s.t. } \langle o, o, \tau', 0\rangle \in (\!|\text{test}|\!)_G^{[t]} \tag{8}$$
    So from (7) and (8):
$$t \notin \bigcup\{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!|\text{test}|\!)_G^{[t]}\}$$
    Therefore
$$t \in \mathcal{T}_G \setminus \bigcup\{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!|\text{test}|\!)_G^{[t]}\} \tag{9}$$
    So $t \in \tau$ for some $\tau \in \mathsf{compl}\left(\bigcup\{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!|\text{test}|\!)_G^{[t]}\}, \mathcal{T}_G\right)$.
    And $\langle o, o, \tau, 0\rangle \in (\!|q|\!)_G^{[t]}$, from the definition of $(\!|q|\!)_G^{[t]}$.
  - For (II), let $\langle o, o, \tau, 0\rangle \in (\!|q|\!)_G^{[t]}$.
    And take any $t \in \tau$.
    From the definition of $(\!|q|\!)_G^{[t]}$:
$$t \in \mathcal{T}_G \setminus \bigcup\{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!|\text{test}|\!)_G^{[t]}\}$$
    Together with (8), this implies
$$\langle o, o, t, 0\rangle \notin [\![\text{test}]\!]_G$$
    Therefore $\langle o, o, t, 0\rangle \in [\![q]\!]_G$, from the definition of $[\![q]\!]_G$.

- $q = \text{path}_1/\text{path}_2$.
  From the above definitions, we have:

$$\begin{aligned}
[\![q]\!]_G &= \{\langle o_1, o_3, t, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, t, d_1\rangle \in [\![\text{path}_1]\!]_G \wedge \langle o_2, o_3, t + d_1, d_2\rangle \in [\![\text{path}_2]\!]_G\} \\
(\!|q|\!)_G^{[t]} &= \Big\{\langle o_1, o_3, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, \tau_1, d_1\rangle \in (\!|\text{path}_1|\!)_G^{[t]} \wedge \langle o_2, o_3, \tau_2, d_2\rangle \in (\!|\text{path}_2|\!)_G^{[t]} \wedge \\
&\qquad (\tau_1 + d_1) \cap \tau_2 \neq \emptyset \Big\}
\end{aligned}$$

  - For (I), let $\mathbf{v} = \langle o_1, o_3, t, d\rangle \in [\![q]\!]_G$.
    Fom the definition of $[\![q]\!]_G$, there are o_2, d_1 and d_2 such that $\langle o_1, o_2, t, d_1\rangle \in [\![\text{path}_1]\!]_G$, $\langle o_2, o_3, t + d_1, d_2\rangle \in [\![\text{path}_2]\!]_G$ and $d = d_1 + d_2$.
    By IH, because $\langle o_1, o_2, t, d_1\rangle \in [\![\text{path}_1]\!]_G$, there is a τ_1 such that $t \in \tau_1$ and
$$\langle o_1, o_2, \tau_1, d_1\rangle \in (\!|\text{path}_1|\!)_G^{[t]} \tag{10}$$
    And similarly, because $\langle o_2, o_3, t + d_1, d_2\rangle \in [\![\text{path}_2]\!]_G$, there is a τ_2 such that $t + d_1 \in \tau_2$ and
$$\langle o_2, o_3, \tau_2, d_2\rangle \in (\!|\text{path}_2|\!)_G^{[t]} \tag{11}$$
    From $t \in \tau_1$, we get
$$t + d_1 \in \tau_1 + d_1 \tag{12}$$
    Together with the fact that $t + d_1 \in \tau_2$, this implies
$$\tau_1 + d_1 \cap \tau_2 \neq \emptyset \tag{13}$$
    So from (10), (11), (13) and the definition of $(\!|q|\!)_G^{[t]}$,
$$\langle o_1, o_2, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2\rangle \in (\!|q|\!)_G^{[t]}$$
    which proves (Ia).
    And in order to prove (Ib), we only need to show that
$$t \in ((\tau_1 + d_1) \cap \tau_2) - d_1$$

We know that $t \in \tau_1$, therefore

$$t + d_1 \in \tau_1 + d_1$$

Together with the fact that $t + d_1 \in \tau_2$, this yields

$$\begin{aligned} t + d_1 &\in (\tau_1 + d_1) \cap \tau_2 \\ t &\in ((\tau_1 + d_1) \cap \tau_2) - d_1 \end{aligned}$$

– For (II), let $\mathbf{u} = \langle o_1, o_3, \tau, d\rangle \in (\!|q|\!)_G^{[t]}$, and let $t \in \tau$.
We show that $\langle o_1, o_3, t, t + d\rangle \in [\![q]\!]_G$.
Because $\mathbf{u} \in (\!|q|\!)_G^{[t]}$, from the definition of $(\!|q|\!)_G^{[t]}$, there are τ_1, τ_2, d_1, d_2 and o_2 s.t.:
(i) $d = d_1 + d_2$
(ii) $\tau = ((\tau_1 + d_1) \cap \tau_2) - d_1$
(iii) $\langle o_1, o_2, \tau_1, d_1\rangle \in (\!|\mathsf{path}_1|\!)_G^{[t]}$
(iv) $\langle o_2, o_3, \tau_2, d_2\rangle \in (\!|\mathsf{path}_2|\!)_G^{[t]}$
Since $t \in \tau$, from (ii), we have

$$t \in ((\tau_1 + d_1 \cap \tau_2) - d_1 \tag{14}$$

$$t + d_1 \in (((\tau_1 + d_1 \cap \tau_2) - d_1) + d_1 \tag{15}$$

$$t + d_1 \in (\tau_1 + d_1) \cap \tau_2 \tag{16}$$

$$t + d_1 \in \tau_1 + d_1 \tag{17}$$

$$t \in \tau_1 \tag{18}$$

From (iii), by IH, for any $t' \in \tau_1$

$$\langle o_1, o_2, t' + d_1\rangle \in [\![q]\!]_G$$

In particular, from (18)

$$\langle o_1, o_2, t, t + d_1\rangle \in [\![q]\!]_G \tag{19}$$

And from (iv), by IH, for any $t'' \in \tau_2$

$$\langle o_2, o_3, t'', t'' + d_2\rangle \in [\![q]\!]_G$$

In particular, from (16)

$$\langle o_2, o_3, t + d_1, (t + d_1) + d_2\rangle \in [\![q]\!]_G \tag{20}$$

So from (19), (20) and the definition of $[\![q]\!]_G$

$$\langle o_1, o_3, t, t + d_1 + d_2\rangle \in [\![q]\!]_G$$

□

3.4 In $\mathcal{U}^{[d]}$

3.4.1 Definition.
We start with the case where q is an expression for the symbol test in the grammar of Definition 3.1.
The definitions of $(\!|pred|\!)_G^{[t]}$ and $(\!|\neg\mathsf{test}|\!)_G^{[t]}$ are already provided in the article, we reproduce them here for completeness:

$$\begin{aligned}
(\!|pred|\!)_G^{[d]} &= \{\langle o, o, t, [0, 0]\rangle \mid t \in \tau \text{ for some } \tau \in \mathrm{val}(o, pred)\} \\
(\!|\mathsf{F}|\!)_G^{[d]} &= \{\langle v, e, t, [0, 0]\rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, [0, 0]\rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\} \\
(\!|\mathsf{B}|\!)_G^{[d]} &= \{\langle v, e, t, [0, 0]\rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, [0, 0]\rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \\
(\!|(?\mathsf{path})|\!)_G^{[d]} &= \{\langle o, o, t, [0, 0]\rangle \mid \exists o', \delta\colon \langle o, o', t, \delta\rangle \in (\!|\mathsf{path}|\!)_G^{[d]}\} \\
(\!|\mathsf{test}_1 \vee \mathsf{test}_2|\!)_G^{[d]} &= (\!|\mathsf{test}_1|\!)_G^{[d]} \cup (\!|\mathsf{test}_2|\!)_G^{[d]} \\
(\!|\mathsf{test}_1 \wedge \mathsf{test}_2|\!)_G^{[d]} &= (\!|\mathsf{test}_1|\!)_G^{[d]} \cap (\!|\mathsf{test}_2|\!)_G^{[d]} \\
(\!|\neg\mathsf{test}|\!)_G^{[d]} &= \Big\{\langle o, o, t, [0, 0]\rangle \mid o \in N \cup E, t \in \mathcal{T}_G \setminus \{t' \mid \langle o, o, t', [0, 0]\rangle \in (\!|\mathsf{test}|\!)_G^{[d]}\}\Big\}
\end{aligned}$$

Next, we consider the operators $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m, \_])$ and $(\mathsf{path}[m, n])$.
For these cases, $(\!|q|\!)_G^{[t,d]}$ is once again defined analogously to $[\![q]\!]_G$, in terms of temporal join (a.k.a. $\mathsf{path}_1/\mathsf{path}_2$) and set union.

We only write the definitions here for the sake of completeness:

$$\begin{aligned}
(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t,d]} &= (\!|\mathsf{path}_1|\!)_G^{[t,d]} \cup (\!|\mathsf{path}_2|\!)_G^{[t,d]} \\
[\![\mathsf{path}[m,n]]\!]_G &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t,d]} \\
[\![\mathsf{path}[m,_]]\!]_G &= \bigcup_{k\geq m} (\!|\mathsf{path}^k|\!)_G^{[t,d]}
\end{aligned}$$

The only remaining operators are temporal join ($\mathsf{path}_1/\mathsf{path}_2$) and temporal navigation (T_δ), already defined in the article.
We reproduce here these two definition for convenience:

$$\begin{aligned}
(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[d]} &= \left\{\langle o_1, o_3, t_1, \delta_2 + t_2 - t_1\rangle \mid \exists o_2 \colon \langle o_1, o_2, t_1, \delta_1\rangle \in (\!|\mathsf{path}_1|\!)_G^{[d]} \land \langle o_2, o_3, t_2, \delta_2\rangle \in (\!|\mathsf{path}_2|\!)_G^{[d]} \land t_2 \in t_1 + \delta_1\right\} \\
(\!|\mathrm{T}_\delta|\!)_G^{[d]} &= \{\langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t\rangle \mid o \in N \cup E, t \in \mathcal{T}_G, (\delta + t) \cap \mathcal{T}_G \neq \emptyset\}
\end{aligned}$$

We also reproduce the alternative characterization of $(\!|\mathrm{T}_\delta|\!)_G^{[d]}$ provided in the article, as a unary operator:

$$(\!|q/\mathrm{T}_\delta|\!)_G^{[d]} = \{\langle o_1, o_2, t, (\delta' + \delta) \cap \mathcal{T}_G\rangle \mid \langle o_1, o_2, t, \delta'\rangle \in (\!|q|\!)_G^{[d]}, (t + (\delta' + \delta)) \cap \mathcal{T}_G \neq \emptyset\}$$

3.4.2 *Correctness.*

The following result states that the representation $(\!|q|\!)_G^{[d]}$ is correct:

PROPOSITION 3.4. *Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TPG and q a TRPQ. Then the unfolding of $(\!|q|\!)_G^{[d]}$ is $[\![q]\!]_G$.*

PROOF.
Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TG, and let q be a TRPQ.
We show below that:

(I) for any $\langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$, there is a $\delta \in \mathsf{intv}(\mathcal{T})$ such that
  (a) $\langle o_1, o_2, t, \delta\rangle \in (\!|q|\!)_G^{[d]}$, and
  (b) $d \in \delta$,

(II) for any $\langle o_1, o_2, t, \delta\rangle \in (\!|q|\!)_G^{[d]}$ for any $d \in \delta$,
  $\langle o_1, o_2, t, d\rangle$ is in $[\![q]\!]_G$.

We proceed once again by induction on the structure of q.
If q is of the form *pred*, F, B, (test $\lor$ test), (test $\land$ test), $\neg$test, (path + path), path$[m, n]$ or path$[m, _]$, then (I) and (II) immediately follow from the definitions of $[\![q]\!]_G$ and $(\!|q|\!)_G^{[d]}$.
If q is of the form (?path), then the proof is nearly identical to one already provided for $(\!|(?\mathsf{path})|\!)_G^{[t]}$.
So we focus below on the two remaining cases:

- $q = \mathrm{T}_\delta$.
  From the above definitions, we have:

$$\begin{aligned}
[\![q]\!]_G &= \{\langle o, o, t, d\rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G\} \\
(\!|q|\!)_G^{[d]} &= \{\langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t\rangle \mid o \in N \cup E, t \in \mathcal{T}_G, (\delta + t) \cap \mathcal{T}_G \neq \emptyset\}
\end{aligned}$$

  – For (I), let $\mathbf{v} = \langle o, o, t, d\rangle \in [\![q]\!]_G$.
    And let $\mathbf{u} = \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t\rangle$ in $\mathcal{U}^{[d]}$.
    For (Ia) we show that $\mathbf{u} \in (\!|q|\!)_G^{[d]}$.
    From $\mathbf{v} \in [\![q]\!]_G$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.
    Besides, because $\mathbf{v} \in [\![q]\!]_G$ still,

$$t + d \in \mathcal{T}_G \tag{21}$$

    and

$$d \in \delta \tag{22}$$
$$t + d \in t + \delta \tag{23}$$

    So from (21) and (23)

$$t + d \in (\delta + t) \cap \mathcal{T}_G \tag{24}$$
$$(\delta + t) \cap \mathcal{T}_G \neq \emptyset \tag{25}$$

Together with the definition of $(\!|q|\!)_G^{[d]}$, this implies $\mathbf{u} \in (\!|q|\!)_G^{[d]}$, which concludes the proof for (Ia).
Finally, from (24), we get

$$t + d - t \in ((\delta + t) \cap \mathcal{T}_G) - t \tag{26}$$
$$d \in ((\delta + t) \cap \mathcal{T}_G) - t \tag{27}$$

which proves (Ib).

– For (II), let $\mathbf{u} = \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t\rangle \in (\!|q|\!)_G^{[d]}$, and let $d \in ((\delta + t) \cap \mathcal{T}_G) - t$.
From $\mathbf{u} \in (\!|q|\!)_G^{[d]} G$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.
So to conclude the proof, it is sufficient to show that *(i)* $d \in \delta$ and *(ii)* $t + d \in \mathcal{T}_G$.
By assumption, we have

$$d \in ((\delta + t) \cap \mathcal{T}_G) - t \tag{28}$$
$$d + t \in (\delta + t) \cap \mathcal{T}_G \tag{29}$$
$$d + t \in \mathcal{T}_G \tag{30}$$

which proves *(ii)* .
And from (29), we also get

$$d + t \in \delta + t$$
$$d + t - t \in (\delta + t) - t$$
$$d \in \delta$$

which proves *(i)* .

- $q = \mathsf{path}_1/\mathsf{path}_2$.
From the above definitions, we have:

$$\begin{aligned} [\![q]\!]_G &= \{\langle o_1, o_3, t, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, t, d_1\rangle \in [\![\mathsf{path}_1]\!]_G \wedge \langle o_2, o_3, t + d_1, d_2\rangle \in [\![\mathsf{path}_2]\!]_G\} \\ (\!|q|\!)_G^{[d]} &= \left\{\langle o_1, o_3, t_1, \delta_2 + t_2 - t_1\rangle \mid \exists o_2 \colon \langle o_1, o_2, t_1, \delta_1\rangle \in (\!|\mathsf{path}_1|\!)_G^{[d]} \wedge \langle o_2, o_3, t_2, \delta_2\rangle \in (\!|\mathsf{path}_2|\!)_G^{[d]} \wedge t_2 \in t_1 + \delta_1\right\} \end{aligned}$$

– For (I), let $\mathbf{v} = \langle o_1, o_3, t, d\rangle \in [\![q]\!]_G$.
Fom the definition of $[\![q]\!]_G$, there are o_2, d_1 and d_2 such that $\langle o_1, o_2, t, d_1\rangle \in [\![\mathsf{path}_1]\!]_G$, $\langle o_2, o_3, t + d_1, d_2\rangle \in [\![\mathsf{path}_2]\!]_G$ and $d = d_1 + d_2$.
By IH, because $\langle o_1, o_2, t, d_1\rangle \in [\![\mathsf{path}_1]\!]_G$, there is a δ_1 such that $d_1 \in \delta_1$ and

$$\langle o_1, o_2, t, \delta_1\rangle \in (\!|\mathsf{path}_1|\!)_G^{[d]} \tag{31}$$

And similarly, because $\langle o_2, o_3, t + d_1, d_2\rangle \in [\![\mathsf{path}_2]\!]_G$, there is a δ_2 such that $d_2 \in \delta_2$ and

$$\langle o_2, o_3, t + d_1, \delta_2\rangle \in (\!|\mathsf{path}_2|\!)_G^{[d]} \tag{32}$$

Next, since $d \in \delta_1$

$$t + d_1 \in t + \delta_1 \tag{33}$$

So from (31), (32), (33) and the definition of $(\!|q|\!)_G^{[d]}$ (replacing t_1 with t and t_2 with $t + d_1$), we get

$$\langle o_1, o_2, t, \delta_2 + (t + d_1) - t\rangle \in (\!|q|\!)_G^{[d]}$$

which proves (Ia).
And in order to prove (Ib), we only need to show that $d \in \delta_2 + (t + d_1) - t$, or in other words that

$$d \in \delta_2 + d_1$$

We know that

$$d_2 \in \delta_2 \tag{34}$$
$$d_2 + d_1 \in \delta_2 + d_1 \tag{35}$$

Together with the fact that $d = d_1 + d_2$, this concludes the proof for (Ib).

– For (II), let $\mathbf{u} = \langle o_1, o_3, t_1, \delta\rangle \in (\!|q|\!)_G^{[d]}$, and let $d \in \delta$.

Because $\mathbf{u} \in (\!|q|\!)_G^{[d]}$, from the definition of $(\!|q|\!)_G^{[d]}$, there are δ_1, δ_2, t_2 and o_2 s.t.:
(i) $\delta = \delta_2 + t_2 - t_1$
(ii) $t_2 \in t_1 + \delta_1$
(iii) $\langle o_1, o_2, t_1, \delta_1\rangle \in (\!|\mathsf{path}_1|\!)_G^{[d]}$

(iv) $\langle o_2, o_3, t_2, \delta_2\rangle \in (\!|\mathsf{path}_2|\!)^{[d]}_G$

From (i) and (ii), we get

$$\begin{aligned}\delta &= \delta_2 + (t_1 + \delta_1) - t_1\\ &= \delta_2 + \delta_1\end{aligned}$$

Together with $d \in \delta$, this implies that there are $d_1 \in \delta_1$ and $d_2 \in \delta_2$ such that $d = d_1 + d_2$.
Next, because $d_1 \in \delta_1$, from (iii), by IH

$$\langle o_1, o_2, t_1, t_1 + d_1\rangle \in [\![q]\!]_G \tag{36}$$

And similarly, because $d_2 \in \delta_2$, from (iv)

$$\langle o_2, o_3, t_2, t_2 + d_2\rangle \in [\![q]\!]_G \tag{37}$$

So from (36), (37) and the definition of $[\![q]\!]_G$

$$\langle o_1, o_3, t_1, d_1 + d_2\rangle \in [\![q]\!]_G \tag{38}$$

Together with the fact that $d = d_1 + d_2$, this concludes the proof for (II).

□

3.5 In $\mathcal{U}^{[t,d]}$

3.5.1 *Definition.*

We start with the case where q is an expression for the symbol test in the grammar of Definition 3.1.
As a consequence of Lemma 3.2, $(\!|q|\!)^{[t,d]}_G$ can be trivially defined out of $(\!|q|\!)^{[t]}_G$ by replacing the distance 0 with the interval $[0, 0]$, i.e.

$$(\!|\mathsf{test}|\!)^{[t,d]}_G = \{\langle o, o, \tau, [0,0]\rangle \mid \{\langle o, o, \tau, 0\rangle \in (\!|\mathsf{test}|\!)^{[t]}_G\}$$

Next, if q is of the form $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m, \_])$ or $(\mathsf{path}[m, n])$, then the definition of $(\!|q|\!)^{[t,d]}_G$ is once again nearly identical to the one of $[\![q]\!]_G$:

$$\begin{aligned}(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)^{[t,d]}_G &= (\!|\mathsf{path}_1|\!)^{[t,d]}_G \cup (\!|\mathsf{path}_2|\!)^{[t,d]}_G\\ [\![\mathsf{path}[m,n]]\!]_G &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)^{[t,d]}_G\\ [\![\mathsf{path}[m,\_]]\!]_G &= \bigcup_{k\geq m} (\!|\mathsf{path}^k|\!)^{[t,d]}_G\end{aligned}$$

The only remaining operators are temporal join ($\mathsf{path}_1/\mathsf{path}_2$) and temporal navigation (T_δ), already defined in the article, and reproduced here for convenience:

$$\begin{aligned}(\!|\mathsf{path}_1/\mathsf{path}_2|\!)^{[t,d]}_G &= \bigcup\{\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 \mid \mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)^{[t,d]}_G, \mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)^{[t,d]}_G\}\\ (\!|\mathrm{T}_\delta|\!)^{[t,d]}_G &= \bigcup_{o\in N\cup E} \{\langle o, o, \mathcal{T}_G, \delta\rangle \,\overline{\bowtie}\, \langle o, o, \mathcal{T}_G, [0,0]\rangle\}\end{aligned}$$

where $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2$ is defined as follows.
Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1\rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2\rangle$.
Define τ_2' as

$$\tau_2' = (\tau_1 + \delta_1) \cap \tau_2$$

If $o_2 \neq o_3$ or $\tau_2' = \emptyset$, then $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 = \emptyset$.
Otherwise, let:

$$\begin{aligned}\tau &= (\tau_2' \ominus \delta_1) \cap \tau_1\\ b &= b_{\tau_2'} - b_{\delta_1}\\ e &= e_{\tau_2'} - e_{\delta_1}\end{aligned}$$

And for every $t \in \tau$, let

$$\delta(t) = \;_{\delta_1}\!\lfloor\, b_{\delta_1} + \max(0, b - t),\; e_{\delta_1} - \max(0, t - e)\,\rfloor_{\delta_1}$$

Then

$$\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 = \{\langle o_1, o_4, [t,t], \delta(t) + \delta_2\rangle \mid t \in \tau\}$$

3.5.2 *Correctness.*

We start with a lemma:

LEMMA 3.5. *Let $\alpha, \beta \in \mathsf{intv}(\mathcal{T})$. Then*

$$\beta \ominus \alpha = \{t \mid (t + \alpha) \cap \beta \neq \emptyset\}$$

Next, if $\mathbf{u} = \langle o_1, o_2, \tau, \delta \rangle \in \mathcal{U}^{[t,d]}$, we call *temporal relation induced by* $\mathbf{u}$ the set $\{(t, t+d) \mid t \in \tau, d \in \delta\}$.
We also define the binary operator $\bowtie$: $(\mathcal{T} \times \mathcal{T}) \times (\mathcal{T} \times \mathcal{T}) \rightarrow (\mathcal{T} \times \mathcal{T})$ as in the article, i.e.

$$R_1 \bowtie R_2 = \{t_1, t_3 \mid (t_1, t_2) \in R_1 \text{ and } (t_2, t_3) \in R_2 \text{ for some } t_2\}$$

We can now formulate the following lemma:

LEMMA 3.6. *Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2 \rangle$ be two tuples in $\mathcal{U}^{[t,d]}$ such that $o_2 = o_3$. And for $i \in \{1, 2\}$, let R_i denote the temporal relation induced by $\mathbf{u}_i$. Then*

$$R_1 \bowtie R_2 = \bigcup_{\langle o_1, o_2, \tau, \delta \rangle \in \mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2} \{(t, t+d) \mid t \in \tau, d \in \delta\}$$

PROOF. $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2 \rangle$ be two tuples in $\mathcal{U}^{[t,d]}$ such that $o_2 = o_3$. And for $i \in \{1, 2\}$, let R_i denote the temporal relation induced by $\mathbf{u}_i$.

We show that:

(I) (a) If $\tau_2' = \emptyset$, then $\mathsf{dom}(R_1 \bowtie R_2) = \emptyset$,
(b) otherwise $\tau = \mathsf{dom}(R_1 \bowtie R_2)$,

(II) for each $t \in \tau$,

$$t + \delta(t) + \delta_2 = \{t' \mid (t, t') \in R_1 \bowtie R_2\}$$

We start with (I).
From the defnition of "+" (applied to two intervals):

$$\tau_1 + \delta_1 = \{t + d \mid t \in \tau_1, d_1 \in \delta_1\} \tag{39}$$

So from the definition of R_1

$$\tau_1 + \delta_1 = \mathsf{range}(R_1) \tag{40}$$

Since $\tau_2 = \mathsf{dom}(R_2)$, this implies

$$(\tau_1 + \delta_1) \cap \tau_2 = \mathsf{range}(R_1) \cap \mathsf{dom}(R_2) \tag{41}$$

$$\tau_2' = \mathsf{range}(R_1) \cap \mathsf{dom}(R_2) \tag{42}$$

If $\mathsf{range}(R_1) \cap \mathsf{dom}(R_2) = \emptyset$, then $\mathsf{dom}(R_1 \bowtie R_2) = \emptyset$, immediately from the definition of $\bowtie$, which concludes the proof of (Ia).
Otherwise, from Lemma 3.5,

$$\tau_2' \ominus \delta_1 = \{t \mid (t + \delta_1) \cap \tau_2' \neq \emptyset\} \tag{43}$$

So from (42)

$$\begin{aligned}
\tau_2' \ominus \delta_1 &= \{t \mid (t + \delta_1) \cap \mathsf{range}(R_1) \cap \mathsf{dom}(R_2) \neq \emptyset\} \\
(\tau_2' \ominus \delta_1) \cap \tau_1 &= \{t \in \tau_1 \mid (t + \delta_1) \cap \mathsf{range}(R_1) \cap \mathsf{dom}(R_2) \neq \emptyset\} \\
(\tau_2' \ominus \delta_1) \cap \tau_1 &= \mathsf{dom}(R_1 \bowtie R_2) \\
\tau &= \mathsf{dom}(R_1 \bowtie R_2)
\end{aligned}$$

which proves (Ib).

Now for (II), let $t \in \tau$.
We show below that *(i)* $t + \delta(t) = \{t' \mid (t, t') \in R_1 \text{ and } t' \in \mathsf{range}(R_1) \cap \mathsf{dom}(R_2)\}$.
Together with the definition of $\bowtie$ (and the fact that $t + \delta(t)$ is an interval), this proves (II).

We only prove the result for the case where τ, τ_2' and δ_1 are closed-closed intervals (the proof for the other 63 cases is symmetric).
First, from (Ib) and the assumption that $t \in \tau$, we have $t \in \tau_1$. So from the definition of R_1,

$$t + \delta_1 = \{t' \mid (t, t') \in R_1\} \tag{44}$$

Together with (42), this means that *(i)* is equivalent to *(ii)* $t + \delta(t) = \{(t + \delta_1) \cap \tau_2'\}$.
So in order to prove (II) (and conclude our proof), it is sufficient to prove *(ii)*.

Now since $t \in \tau$, from (Ib) and the definition of τ_2', we have $(t + \delta(t)) \cap \tau_2' \neq \emptyset$.
And since $\delta(t)$ and τ_2' are intervals, $(t + \delta(t)) \cap \tau_2'$ is an also an interval.

So in order to prove *(ii)* , it is sufficient to show that $t + b_{\delta(t)}$ (resp. $t + e_{\delta(t)}$) is the smallest (resp. greatest) value in $(t + \delta_1) \cap \tau'_2$.
We only prove the result for $t + b_{\delta(t)}$ (the proof for $t + e_{\delta(t)}$) is symmetric.
We consider two cases.

- If $b \leq t$, then

$$b_{\tau'_2} - b_{\delta_1} \leq t \quad \text{from the definition of } b \tag{45}$$

$$b_{\tau'_2} - b_{\delta_1} + b_{\delta_1} \leq t + b_{\delta_1} \tag{46}$$

$$b_{\tau'_2} \leq t + b_{\delta_1} \tag{47}$$

  And because $t \in \tau$

$$t \leq e_{\tau} \tag{48}$$

$$t \leq e_{\tau'_2} - b_{\delta_1} \quad \text{from the definition of } \tau \tag{49}$$

$$t + b_{\delta_1} \leq e_{\tau'_2} - b_{\delta_1} + b_{\delta_1} \tag{50}$$

$$t + b_{\delta_1} \leq e_{\tau'_2} \tag{51}$$

  So from (47) and (51)

$$t + b_{\delta_1} \in \tau'_2 \tag{52}$$

  Next, since $b \leq t$ (by assumption), we have

$$b - t \leq 0$$

$$\max(0, b - t) = 0$$

  So from the definition of $\delta(t)$

$$b_{\delta(t)} = b_{\delta_1} \tag{53}$$

  Therefore $t + b_{\delta(t)}$ is the smallest value in $t + \delta_1$.
  So from (52), it is also the smallest value in $t + \delta_1 \cap \tau'_2$, which concludes the proof for this case.
- If $b > t$, then

$$b - t > 0 \tag{54}$$

$$\max(0, b - t) = b - t \tag{55}$$

  So from the definition of $\delta(t)$

$$b_{\delta(t)} = b_{\delta_1} + b - t \tag{56}$$

  Besides, from (54)

$$b - t + b_{\delta_1} > b_{\delta_1} \tag{57}$$

  So from (56) and (57)

$$b_{\delta(t)} > b_{\delta_1} \tag{58}$$

  Next, since $t \in \tau$

$$b_{\tau} \leq t \tag{59}$$

  And from the definition of τ

$$b_{\tau'_2} - e_{\delta_1} \leq b_{\tau} \tag{60}$$

  So from (59) and (60)

$$b_{\tau'_2} - e_{\delta_1} \leq t \tag{61}$$

$$b_{\tau'_2} - t \leq e_{\delta_1} \tag{62}$$

$$b_{\tau'_2} - t + b_{\delta_1} - b_{\delta_1} \leq e_{\delta_1} \tag{63}$$

$$b_{\delta_1} + (b_{\tau'_2} - b_{\delta_1}) - t \leq e_{\delta_1} \tag{64}$$

$$b_{\delta_1} + b - t \leq e_{\delta_1} \quad \text{from the definition of } b \tag{65}$$

$$b_{\delta(t)} \leq e_{\delta_1} \quad \text{from (56)} \tag{66}$$

  Therefore from (58) and (66)

$$b_{\delta(t)} \in \delta_1 \tag{67}$$

$$t + b_{\delta(t)} \in t + \delta_1 \tag{68}$$

Finally, from (56) still,

$$t + b_{\delta(t)} = t + b_{\delta_1} + b - t \tag{69}$$
$$= t + b_{\delta_1} + b_{\tau_2'} - b_{\delta_1} - t \quad \text{from the definition of } b \tag{70}$$
$$= b_{\tau_2'} \tag{71}$$

So $t + b_{\delta(t)}$ is the smallest value in τ_2'.
Together with (68), this concludes the proof for this case.

□

The following result states that the representation $(\!|q|\!)_G^{[t,d]}$ is correct:

Proposition 3.7. *Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TPG and q a TRPQ. Then the unfolding of $(\!|q|\!)_G^{[t,d]}$ is $[\![q]\!]_G$.*

Proof.
Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TG, and let q be a TRPQ.
We show below that:

(I) for any $\langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$, there are $\tau, \delta \in \mathsf{intv}(\mathcal{T})$ such that
  (a) $\langle o_1, o_2, \tau, \delta\rangle \in (\!|q|\!)_G^{[t,d]}$,
  (b) $t \in \tau$, and
  (c) $d \in \delta$.
(II) for any $\langle o_1, o_2, \tau, \delta\rangle \in (\!|q|\!)_G^{[t,d]}$ for any $(t, d) \in \tau \times \delta$,
  $\langle o_1, o_2, t, d\rangle$ is in $[\![q]\!]_G$.

We proceed once again by induction on the structure of q.
If q is of the form *pred*, F, B, (test ∨ test), (path + path), path$[m, n]$ or path$[m, _]$, then (I) and (II) immediately follow from the definitions of $[\![q]\!]_G$ and $(\!|q|\!)_G^{[t,d]}$.
If q is of the form test ∧ test, ¬test or (?path), then the proof is nearly identical to the one already provided for $(\!|q|\!)_G^{[t]}$.
So we focus below on the two remaining cases:

- $q = \mathsf{path}_1/\mathsf{path}_2$.
  From the above definitions, we have:

$$[\![q]\!]_G = \{\langle o_1, o_3, t, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, t, d_1\rangle \in [\![\mathsf{path}_1]\!]_G \land \langle o_2, o_3, t + d_1, d_2\rangle \in [\![\mathsf{path}_2]\!]_G\}$$
$$(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[t,d]} = \bigcup\{\mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2 \mid \mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)_G^{[t,d]}, \mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)_G^{[t,d]}\}$$

  - For (I), let $\mathbf{v} = \langle o_1, o_3, t, d\rangle \in [\![q]\!]_G$.
    Fom the definition of $[\![q]\!]_G$, there are o_2, d_1 and d_2 such that $\langle o_1, o_2, t, d_1\rangle \in [\![\mathsf{path}_1]\!]_G$, $\langle o_2, o_3, t + d_1, d_2\rangle \in [\![\mathsf{path}_2]\!]_G$ and $d = d_1 + d_2$.
    By IH, because $\langle o_1, o_2, t, d_1\rangle \in [\![\mathsf{path}_1]\!]_G$, there are τ_1 and δ_1 such that $t \in \tau_1$, $d_1 \in \delta_1$ and
    $$\langle o_1, o_2, \tau_1, \delta_1\rangle \in (\!|\mathsf{path}_1|\!)_G^{[t,d]} \tag{72}$$
    Let R_1 be the temporal relation induced by this tuple $\langle o_1, o_2, \tau_1, \delta_1\rangle$.
    Since $t \in \tau_1$ and $d_1 \in \delta_1$, we have
    $$(t, t + d_1) \in R_1 \tag{73}$$
    Similarly, because $\langle o_2, o_3, t + d_1, d_2\rangle \in [\![\mathsf{path}_2]\!]_G$, there are τ_2 and δ_2 such that $t + d_1 \in \tau_2$, $d_2 \in \delta_2$ and
    $$\langle o_2, o_3, \tau_2, \delta_2\rangle \in (\!|\mathsf{path}_2|\!)_G^{[t,d]} \tag{74}$$
    Let R_2 be the temporal relation induced by this tuple $\langle o_2, o_3, \tau_2, \delta_2\rangle$.
    Since $t + d_1 \in \tau_2$ and $d_2 \in \delta_2$, we have
    $$(t + d_1, t + d_1 + d_2) \in R_2 \tag{75}$$
    So from (73), (75) and Lemma 3.6, there are τ and δ such that $\langle o_1, o_3, \tau, \delta\rangle \in u_1 \overline{\bowtie} u_2$, $t \in \tau$ and $d_1 + d_2 = d \in \delta$, which concludes the proof for (I).
  - For (II), let $\mathbf{u} = \langle o_1, o_3, t_1, \delta\rangle \in (\!|q|\!)_G^{[t,d]}$, and let $(t, d) \in \tau \times \delta$.
    Because $\mathbf{u} \in (\!|q|\!)_G^{[t,d]}$, from the definition of $(\!|q|\!)_G^{[t,d]}$, there are $\mathbf{u}_1$ and $\mathbf{u}_2$ s.t.:
    (i) $\mathbf{u} \in \mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2$
    (ii) $\mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)_G^{[t,d]}$
    (iii) $\mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)_G^{[t,d]}$

Let R_i be the temporal relation induced by u_i for $i \in \{1, 2\}$.
From (i), and Lemma 3.6,

$$(t, t + d) \in R_1 \bowtie R_2 \tag{76}$$

Now let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1\rangle$ and $\mathbf{u}_2 = \langle o_2, o_3, \tau_2, \delta_2\rangle$ for some $o_2, \tau_1, \tau_2, \delta_1$ and δ_2.
From (76) and the definition of $\bowtie$, there must be d_1 and d_2 s.t. $d = d_1 + d_2$, $t \in \tau_1, d_1 \in \delta_1$, $t + d_1 \in \tau_2$ and $d_2 \in \delta_2$.
So from (ii), and (iii), by IH

$$\langle o_1, o_2, t, d_1\rangle \in [\![\mathsf{path}_1]\!]_G \tag{77}$$

$$\langle o_2, o_3, t + d_1, d_2\rangle \in [\![\mathsf{path}_2]\!]_G \tag{78}$$

So from (77), (78) and the definition of $[\![q]\!]_G$

$$\langle o_1, o_3, t, d_1 + d_2\rangle \in [\![q]\!]_G,$$

which concludes the proof for (II).

□

3.6 In $\mathcal{U}^{[t,d],b,e}$

3.6.1 Definition.

If q is an expression for the symbol test in the grammar of Definition **??**, then the definition of $(\!|q|\!)_G^{[t,d],b,e}$ is nearly identical to the one of $(\!|q|\!)_G^{[t,d]}$, extending each tuple $\{\langle o, o, \tau, [0, 0]\rangle$ with b_τ and e_τ, i.e.

$$(\!|\mathsf{test}|\!)_G^{[t,d],b,e} = \{\langle o, o, \tau, [0, 0], b_\tau, e_\tau\rangle \mid \{\langle o, o, \tau, [0, 0]\rangle \in (\!|\mathsf{test}|\!)_G^{[t,d]}\}$$

Next, if q is of the form $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m, \_])$ or $(\mathsf{path}[m, n])$, then the definition of $(\!|q|\!)_G^{[t,d]}$ is once again nearly identical to the one of $[\![q]\!]_G$:

$$\begin{aligned}
(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t,d],b,e} &= \quad (\!|\mathsf{path}_1|\!)_G^{[t,d],b,e} \cup (\!|\mathsf{path}_2|\!)_G^{[t,d],b,e} \\
[\![\mathsf{path}[m, n]]\!]_G &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t,d],b,e} \\
[\![\mathsf{path}[m, _]]\!]_G &= \bigcup_{k \geq m} (\!|\mathsf{path}^k|\!)_G^{[t,d],b,e}
\end{aligned}$$

So the only remaining operator are temporal join $(\mathsf{path}_1/\mathsf{path}_2)$ and temporal navigation (T_δ), already defined in the article. We reproduce here these two definition for convenience:

$$\begin{aligned}
(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[t,d],b,e} &= \quad \{\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 \mid \mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)_G^{[t,d],b,e}, \mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)_G^{[t,d],b,e}, \mathbf{u}_1 \sim \mathbf{u}_2\} \\
(\!|\mathrm{T}_\delta|\!)_G^{[t,d],b,e} &= \quad \{\langle o, o, \mathcal{T}_G, \delta, b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \,\overline{\bowtie}\, \langle o, o, \mathcal{T}_G, [0, 0], b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \mid o \in N \cup E\}
\end{aligned}$$

where $\mathbf{u}_1 \sim \mathbf{u}_2$ and $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2$ are defined as follows.
Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1, b_1, e_1\rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2, e_2, b_2\rangle$.
Define

$$\delta'_1 = \;_{\delta_1}\lfloor\, b_{\delta_1} + \max(0, b_1 - b_{\tau_1}), e_{\delta_1} - \max(0, e_{\tau_1} - e_1) \,\rfloor_{\delta_1}$$

and

$$\tau = (((\tau_1 + \delta'_1) \cap \tau_2) \ominus \delta'_1) \cap \tau_1$$

Then $\mathbf{u}_1 \sim \mathbf{u}_2$ iff $o_2 = o_3$ and $\tau \neq \emptyset$.
If $\mathbf{u}_1 \sim \mathbf{u}_2$, then $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 = \langle o_1, o_4, \tau, \delta_1 + \delta_2, b, e\rangle$, with

$$\begin{aligned}
b &= \max(b_1, b_2 - b_{\delta_1}) \\
e &= \min(e_1, e_2 - e_{\delta_1})
\end{aligned}$$

3.6.2 Correctness.

We start with two lemmas:

Lemma 3.8. *Let $\mathbf{u} = \langle o_1, o_2, \tau, \delta, b, e\rangle \in \mathcal{U}^{[t,d],b,e}$. Then for any $t_1, t_2 \in \tau$ s.t. $t_1 \leq t_2$:*

$$t_1 + b_{\delta(t_1)} \leq t_2 + b_{\delta(t_2)} \text{ and}$$
$$t_1 + e_{\delta(t_1)} \leq t_2 + e_{\delta(t_2)}$$

Lemma 3.9. *Let $\mathbf{u} = \langle o_1, o_2, \tau, \delta, b, e\rangle \in \mathcal{U}^{[t,d],b,e}$. And let τ' denote the interval $(b_\tau + b_{\delta(b_\tau)}, e_\tau + e_{\delta(e_\tau)})$. Then for any $t' \in \tau'$, there is a $t \in \tau$ s.t. $t' \in t + \delta(t)$.*

Next, similarly to what we did above for $\mathcal{U}^{[t,d]}$, if $\mathbf{u} = \langle o_1, o_2, \tau, \delta, b, e \rangle \in \mathcal{U}^{[t,d],b,e}$, we call *temporal relation induced by* $\mathbf{u}$ the set $\{(t, t+d) \mid t \in \tau, d \in \delta(t)\}$.

We can now formulate a result analogous to Lemma 3.6:

LEMMA 3.10. *Let* $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}^{[t,d],b,e}$*, and for* $i \in \{1,2\}$*, let* R_i *denote the temporal relation induced by* $\mathbf{u}_i$*. If* $\mathbf{u}_1 \sim \mathbf{u}_2$ *and* $\mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2 = \langle o_1, o_3, \tau, \delta, b, e \rangle$*, then*

$$R_1 \bowtie R_2 = \{(t, t+d) \mid t \in \tau, d \in \delta(t)\}$$

PROOF. Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$ and $\mathbf{u}_2 = \langle o_2, o_3, \tau_2, \delta_2 \rangle$.
As explained in Section 2, for $i \in \{1,2\}$ and $t \in \tau_i$, we use $\delta_i(t)$ for the interval

$${}_{\delta_i}\lfloor\, b_{\delta_i} + \max(0, b_i - t)\ ,\ e_{\delta_i} - \max(0, t - e_i)\, \rfloor_{\delta_i}$$

We need to prove that *(i)* $\tau = \mathrm{dom}(R_1 \bowtie R_2)$ and that *(ii)* for each $t \in \tau$,

$$t + \delta(t) = \{t' \mid (t, t') \in R_1 \bowtie R_2\}$$

The proof of *(i)* is nearly identical to the one provided above for Lemma 3.6.
For *(ii)* , let $t \in \tau$.
We only provide a proof for the case where τ, δ_1 and δ_2 are closed-closed intervals (the proof for the other 63 cases is symmetric).
Since $t \in \tau$, from the defnition of τ, $t \in \tau_1$.
Therefore from the definition of R_1,

$$t + \delta_1(t) = \{t' \mid (t, t') \in R_1\} \tag{79}$$

So from *(i)* and the fact that $t \in \tau$

$$t + \delta_1(t) \cap \mathrm{dom}(R_2) \neq \emptyset \tag{80}$$

Now let a (resp. z) denote the smallest (resp. largest) value in $t + \delta_1(t) \cap \mathrm{dom}(R_2)$.
Then from (79), a (resp. z) is also the smallest value s.t. $(t, a) \in R_1$ and $a \in \mathrm{dom}(R_2)$ (resp. the largest value s.t. $(t, z) \in R_1$ and $z \in \mathrm{dom}(R_2)$).

Next, from Lemma 3.8, for any $x \in [a, z]$, we have

$$a + b_{\delta_2(a)} \leq x + b_{\delta_2(x)} \tag{81}$$

and

$$x + e_{\delta_2(x)} \leq z + e_{\delta_2(z)} \tag{82}$$

Now let a' and z' denote $a + b_{\delta_2(a)}$ and $z + e_{\delta_2(z)}$ respectively.
From (81) and the definition of R_2, a' is the smallest value s.t. $(x, a') \in R_2$ for some $x \in [a, b]$.
And similarly, from (82) and the definition of R_2, z' is the largest value s.t. $(x, z') \in R_2$ for some $x \in [a, b]$.
Together with the defnition of a (resp. of z), this implies that a' (resp. z') is also the smallest (resp. largest) value s.t. $(t, a') \in R_1 \bowtie R_2$ (resp. $(t, z') \in R_1 \bowtie R_2$).
To conclude the proof, we show that:

(1) $(t, x) \in R_1 \bowtie R_2$ for each $x \in [a', z']$, and
(2) $t + \delta(t) = [a', z']$.

We start with (1).
Consider the tuple $\mathbf{u}' = \langle o_2, o_3, [a, b], \delta_2, b_2, e_2 \rangle \in \mathcal{U}^{[t,d],b,e}$, and let R' be the temporal relation induced by $\mathbf{u}'$.
Then from he definitions of u' and $\mathbf{u}_2$:

$$R' \subseteq R_2 \tag{83}$$

Now take any $x \in [a', z']$.
From Lemma 3.9 and the definitions of a' and z', there is a $w \in [a, b]$ such that $x \in \delta_2(w)$.
Therefore

$$(w, x) \in R'$$

So from (83)

$$(w, x) \in R_2 \tag{84}$$

Finally, since $[a, b] = t + \delta_1(t)$ and $w \in [a, b]$,

$$(t, w) \in R_1 \tag{85}$$

Together with (84), this implies

$$(t, x) \in R_1 \bowtie R_2$$

which concludes the proof for (1).

For (2), we only prove that $t + b_{\delta_t} = a'$ (the proof that $t + e_{\delta_t} = z'$ is symmetric).
Following the definition of b, we consider 2 cases:

(1) $b_1 < b_2 - b_{\delta_1}$
(2) $b_1 \geq b_2 - b_{\delta_1}$

For Case (1), we get

$$\begin{aligned} b_1 <\ & b_2 - b_{\delta_1} && (86)\\ \max(b_1, b_2 - b_{\delta_1}) =\ & b_2 - b_{\delta_1} && (87)\\ b =\ & b_2 - b_{\delta_1} && \text{from the definition of } b \quad (88) \end{aligned}$$

And in Case (1) still, we get:

$$\begin{aligned} b_1 <\ & b_2 - b_{\delta_1} && (89)\\ 0 <\ & b_2 - b_{\delta_1} - b_1 && (90)\\ \max(0, b_2 - b_{\delta_1} - b_1) =\ & b_2 - b_{\delta_1} - b_1 && (91) \end{aligned}$$

Next, we consider two subcases:

(i) $t < b_2 - b_{\delta_1}$
(ii) $t \geq b_2 - b_{\delta_1}$

In Case (i), we get

$$\begin{aligned} t <\ & b_2 - b_{\delta_1} && (92)\\ 0 <\ & b_2 - b_{\delta_1} - t && (93)\\ \max(0, b_2 - b_{\delta_1} - t) =\ & b_2 - b_{\delta_1} - t && (94) \end{aligned}$$

Now from the definition of δ_t,

$$\begin{aligned} b_{\delta_t} =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && (95)\\ =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_{\delta_1} - t) && \text{from (88)} \quad (96)\\ =\ & b_{\delta_1} + b_{\delta_2} + b_2 - b_{\delta_1} - t && \text{from (94)} \quad (97)\\ =\ & b_{\delta_2} + b_2 - t && (98)\\ b_{\delta_t} + t =\ & b_{\delta_2} + b_2 - t + t && (99)\\ =\ & b_{\delta_2} + b_2 && (100) \end{aligned}$$

Next, from the definition of a'

$$\begin{aligned} a' =\ & b_{\delta_2(a)} + a && (101)\\ =\ & b_{\delta_2} + \max(0, b_2 - a) + a && (102) \end{aligned}$$

And, from the definition of a

$$\begin{aligned} a =\ & b_{\delta_1(t)} + t && (103)\\ =\ & b_{\delta_1} + \max(0, b_1 - t) + t && (104) \end{aligned}$$

Then we have two further subcases:

(I) $t \geq b_1$, or
(II) $t < b_1$

In case (I):

$$\begin{aligned} t \geq\ & b_1 && (105)\\ 0 \geq\ & b_1 - t && (106)\\ \max(0, b_1 - t) =\ & 0 && (107)\\ a =\ & b_{\delta_1} + t && \text{from (104)} \quad (108)\\ \max(0, b_2 - a) =\ & \max(0, b_2 - b_{\delta_1} - t) && (109)\\ =\ & b_2 - b_{\delta_1} - t && \text{from (94)} \quad (110)\\ =\ & b_2 - a && \text{from (108)} \quad (111) \end{aligned}$$

In case (II):

$$
\begin{aligned}
t <\ & b_1 && (112)\\
0 <\ & b_1 - t && (113)\\
\max(0, b_1 - t) =\ & b_1 - t && (114)\\
a =\ & b_{\delta_1} + b_1 - t + t && \text{from (104)} \quad (115)\\
=\ & b_{\delta_1} + b_1 && (116)\\
\max(0, b_2 - a) =\ & \max(0, b_2 - b_{\delta_1} - b_1) && (117)\\
=\ & b_2 - b_{\delta_1} - b_1 && \text{from (91)} \quad (118)\\
=\ & b_2 - a && \text{from (116)} \quad (119)\\
& && (120)
\end{aligned}
$$

So in both cases (I) and (II), we get

$$\max(0, b_2 - a) = b_2 - a$$

Thefore from (102)

$$
\begin{aligned}
a' =\ & b_{\delta_2} + b_2 - a + a && (121)\\
=\ & b_{\delta_2} + b_2 && (122)\\
=\ & t + b_{\delta_t} && \text{from (100)} \quad (123)
\end{aligned}
$$

which concludes the proof for Case (1)- (i).

We continue with Case (1)- (ii).
From Case (ii):

$$
\begin{aligned}
t \geq\ & b_2 - b_{\delta_1} && (124)\\
0 \geq\ & b_2 - b_{\delta_1} - t && (125)\\
\max(0, b_2 - b_{\delta_1} - t) =\ & 0 && (126)
\end{aligned}
$$

Now from the definition of δ_t:

$$
\begin{aligned}
b_{\delta_t} =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && (127)\\
=\ & b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_1 - t) && \text{from (88)} \quad (128)\\
=\ & b_{\delta_1} + b_{\delta_2} && \text{from (126)} \quad (129)\\
b_{\delta_t} + t =\ & b_{\delta_1} + b_{\delta_2} + t && (130)
\end{aligned}
$$

Next, from Case (1) and Case (ii), by transitivity, we get

$$
\begin{aligned}
b_1 \leq\ & t && (131)\\
\max(0, b_1 - t) =\ & 0 && (132)
\end{aligned}
$$

And from the definition of a

$$
\begin{aligned}
a =\ & b_{\delta_1(t)} + t && (133)\\
=\ & b_{\delta_1} + \max(0, b_1 - t) + t && (134)\\
=\ & b_{\delta_1} + t && \text{from (132)} \quad (135)\\
\geq\ & b_{\delta_1} + b_2 - b_{\delta_1} && \text{from Case (ii)} \quad (136)\\
\geq\ & b_2 && (137)\\
0 \geq\ & b_2 - a && (138)\\
\max(0, b_2 - a) =\ & 0 && (139)
\end{aligned}
$$

Therefore from (102) and (139)

$$
\begin{aligned}
a' =\ & b_{\delta_2} + a && (140)\\
=\ & b_{\delta_2} + b_{\delta_1} + t && \text{from (135)} \quad (141)\\
=\ & b_{\delta_t} + t && \text{from (100)} \quad (142)
\end{aligned}
$$

which concludes the proof for Case (1)- (ii).

We continute with Case (2).
In this case, we get

$$\begin{aligned} b_1 \geq & \ b_2 - b_{\delta_1} && (143)\\ \max(b_1, b_2 - b_{\delta_1}) = & \ b_1 && (144)\\ b = & \ b_1 \quad \text{from the definition of } b && (145)\end{aligned}$$

And from Case (2) still, we derive

$$\begin{aligned} b_1 \geq & \ b_2 - b_{\delta_1} && (146)\\ 0 \geq & \ b_2 - b_{\delta_1} - b_1 && (147)\\ \max(0, -b_{\delta_1} - b_1) = & \ 0 && (148)\end{aligned}$$

As well as

$$\begin{aligned} b_1 \geq & \ b_2 - b_{\delta_1} && (149)\\ b_1 + b_{\delta_1} \geq & \ b_2 && (150)\end{aligned}$$

Next, we distinguish two subcases, namely
(a) $t < b_1$ and
(b) $t \geq b_1$

We start with Case (a).
In this case,

$$\begin{aligned} t < & \ b_1 && (151)\\ 0 < & \ b_1 - t && (152)\\ \max(0, b_1 - t) = & \ b_1 - t && (153)\end{aligned}$$

And from the definition of δ_t:

$$\begin{aligned} b_{\delta_t} = & \ b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && (154)\\ = & \ b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) \quad \text{from (145)} && (155)\\ = & \ b_{\delta_1} + b_{\delta_2} + b_1 - t \quad \text{from (153)} && (156)\\ b_{\delta_t} + t = & \ b_{\delta_1} + b_{\delta_2} + b_1 - t + t && (157)\\ = & \ b_{\delta_1} + b_{\delta_2} + b_1 && (158)\end{aligned}$$

Next, from the definition of a

$$\begin{aligned} a = & \ b_{\delta_1(t)} + t && (159)\\ = & \ \max(0, b_1 - t) + b_{\delta_1} + t && (160)\\ = & \ b_1 - t + b_{\delta_1} + t \quad \text{from (153)} && (161)\\ = & \ b_1 + b_{\delta_1} && (162)\\ & \text{So from (150)} && (163)\\ a \geq & \ b_2 && (164)\\ 0 \geq & \ b_2 - a && (165)\\ \max(0, b_2 - a) = & \ 0 && (166)\\ b_{\delta_2} + \max(0, b_2 - a) = & \ b_{\delta_2} && (167)\\ b_{\delta_2(a)} = & \ b_{\delta_2} && (168)\\ b_{\delta_2(a)} + a = & \ b_{\delta_2} + a && (169)\\ a' = & \ b_{\delta_2} + a \quad \text{from the defintiion of } a' && (170)\\ a' = & \ b_{\delta_2} + b_1 + b_{\delta_1} \quad \text{from (162)} && (171)\\ a' = & \ b_{\delta_t} + t \quad \text{from (158)} && (172)\end{aligned}$$

which concludes the proof for Case (2)- (a).

We end with Case (2)- (b). In this case,

$$
\begin{aligned}
t \geq &\; b_1 && (173)\\
0 \geq &\; b_1 - t && (174)\\
\max(0, b_1 - t) = &\; 0 && (175)
\end{aligned}
$$

And from the definition of δ_t:

$$
\begin{aligned}
b_{\delta_t} = &\; b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (176)\\
= &\; b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) && \text{from (145)} && (177)\\
= &\; b_{\delta_1} + b_{\delta_2} && \text{from (175)} && (178)\\
b_{\delta_t} + t = &\; b_{\delta_1} + b_{\delta_2} + t && && (179)
\end{aligned}
$$

Next, from the definition of a

$$
\begin{aligned}
a = &\; b_{\delta_1(t)} + t && && (180)\\
= &\; \max(0, b_1 - t) + b_{\delta_1} + t && && (181)\\
= &\; b_{\delta_1} + t && \text{from (175)} && (182)
\end{aligned}
$$

Now from Case (b)

$$
\begin{aligned}
b_1 + \leq &\; t && && (183)\\
b_1 + b_{\delta_1} \leq &\; t + b_{\delta_1} && && (184)\\
b_1 + b_{\delta_1} \leq &\; a && \text{from (182)} && (185)\\
b_2 \leq &\; a && \text{from (150), by transitivity} && (186)\\
b_2 - a \leq &\; 0 && && (187)\\
\max(0, b_2 - a) = &\; 0 && && (188)\\
b_{\delta_2} + \max(0, b_2 - a) = &\; b_{\delta_2} && && (189)\\
b_{\delta_2(a)} = &\; b_{\delta_2} && && (190)\\
b_{\delta_2(a)} + a = &\; b_{\delta_2} + a && && (191)\\
a' = &\; b_{\delta_2} + a && \text{from the defintiion of } a' && (192)\\
= &\; b_{\delta_2} + b_{\delta_1} + t && \text{from (182)} && (193)\\
= &\; b_{\delta_t} + t && \text{from (179)} && (194)
\end{aligned}
$$

which concludes the proof for Case (2)- (b). □

The following result states that the representation $(\!|q|\!)_G^{[t,d],b,e}$ is correct:

Proposition 3.11. *Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TPG and q a TRPQ. Then the unfolding of $(\!|q|\!)_G^{[t,d],b,e}$ is $[\![q]\!]_G$.*

Proof. Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TG, and let q be a TRPQ.
To prove the result, it is sufficient to show that:

(I) for any $\langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$, there are $\tau, \delta \in \mathsf{intv}(\mathcal{T})$ and $b, e \in \mathcal{T}$ such that
  (a) $\langle o_1, o_2, \tau, \delta, b, e\rangle \in (\!|q|\!)_G^{[t,d],b,e}$,
  (b) $t \in \tau$, and
  (c) $d \in \delta(t)$ (where $\delta(t)$ is defined in terms of t, δ, b and e, as explained above).

(II) for any $\langle o_1, o_2, \tau, \delta, b, e\rangle \in (\!|q|\!)_G^{[t,d],b,e}$ for any $t \in \tau$ and $d \in \delta(t)$,
  $\langle o_1, o_2, t, d\rangle$ is in $[\![q]\!]_G$.

Again, the proof is by induction on the structure of q.
If q is of the form *pred*, F, B, (test $\vee$ test), (path + path), path$[m,n]$ or path$[m,_]$, then (I) and (II) immediately follow from the definitions of $[\![q]\!]_G$ and $(\!|q|\!)_G^{[t,d],b,e}$.
If q is of the form test $\wedge$ test, $\neg$test or (?path), then the proof is nearly identical to the one already provided for $(\!|q|\!)_G^{[t]}$.
And if q is of the form T_δ or path$_1$/path$_2$, then the proof is nearly identical to the one already provided for $(\!|q|\!)_G^{[t,d]}$, using Lemma 3.10 instead of 3.6. □

4 COMPLEXITY OF QUERY ANSWERING

4.1 Problem

We propose in this section a decision problem for each representation, similar to the problem COMPACT ANSWER$^{[t]}$ defined in the article.
Let x be one of $[t]$, $[d]$, $[t, d]$ or $([t, d], b, e)$.
if $\mathbf{u} \subseteq \mathcal{U}^x$, we use $\mathsf{unfold}(\mathbf{u})$ for the unfolding of $\mathbf{u}$, and we define the partial order $\sqsubseteq_x$ over $\mathcal{U}^x$ as

$$u_1 \sqsubseteq_x \mathbf{u}_2 \text{ iff } \mathsf{unfold}(u_1) \subseteq \mathsf{unfold}(\mathbf{u}_2)$$

Then we decline the notion of compact answer (defined in the article) in four flavors, as follows:

Definition 4.1 (Compact answer). Let G be a TG, let q be a TRPQ and let $\mathbf{u} \in \mathcal{U}^x$.
$\mathbf{u}$ is a *compact answer* to q over G (in $\mathcal{U}^x$) if $\mathbf{u} \in \max_{\sqsubseteq_x}\{\mathbf{u}' \in \mathcal{U}^x \mid \mathsf{unfold}(\mathbf{u}') = [\![q]\!]_G\}$

And we decline the associated decision problem analogously:

COMPACT ANSWER^x	
Input:	TG G, TRPQ q, tuple $\mathbf{u} \in \mathcal{U}^x$
Decide:	$\mathbf{u}$ is a compact answer to q over G (in $\mathcal{U}^x$)

However, this definition of compact answers does not capture the distinction made in the article between possibly redundant and non-redundant *sets* of answers in $\mathcal{U}^{[t,d]}$.
For this reason, we propose an additional problem, called COMPACT ANSWER$_{nr}^{[t,d]}$, defined as follows.
Let U be a subset of $\mathcal{U}^{[t,d]}$.
We call U *non-redundant* if all tuples in U have disjoint unfoldings.
We also say that U is an *answer set* to q over G if U has unfolding $[\![q]\!]_G$, and a *compact answer set* to q over G if it is a (finite) cardinality-minimal answer set to q over G.

We can now define our problem:

COMPACT ANSWER$_{nr}^{[t,d]}$	
Input:	TG G, TRPQ q, tuple $\mathbf{u} \in \mathcal{U}^{[t,d]}$
Decide:	$\mathbf{u} \in U$ for some compact non-redundant answer set U to q over G

4.2 Results

Our results simply leverage the ones already provided in [1] for answering TRPQS in $\mathcal{U}$.
We reproduce here the corresponding decision problem, for the sake of completeness.

COMPACT ANSWER	
Input:	TG G over discrete time, TRPQ q, tuple $\mathbf{u} \in \mathcal{U}$
Decide:	$\mathbf{u} \in [\![q]\!]_G$

4.2.1 Membership.

If $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ is a *TG* and q a TRPQ, we use $\mathsf{boundaries}(G, q)$ for the set of all interval boundaries that appear in G and q, i.e.

$$\mathsf{boundaries}(G, q) = \bigcup\{\{b_\delta, e_\delta\} \mid \mathrm{T}_\delta \text{ appears in } q\} \cup \{b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\} \bigcup\{\{b_\tau, e_\tau\} \mid \tau \in \mathsf{val}(o, p) \text{ for some } o \in N \cup E \text{ and } p \in \mathit{Pred}\}$$

Note that $\mathsf{boundaries}(G, q)$ is finite.

Next, if $Q \subseteq \mathbb{Q}$, we use Q^{+-} to denote the smallest superset of Q that is closed under addition and subtraction.
We can now formulate the two following lemmas:

LEMMA 4.2. *Let G be a TG, let q be a TRPQ, let $\mathbf{u} = \langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$, let $Q = \mathsf{boundaries}(G, q) \cup \{d\}$, and let τ be the largest interval s.t. $t \in \tau$ and $\langle o_1, o_2, t', d\rangle \in [\![q]\!]_G$ for all $t' \in \tau$. Then*

$$b_\tau \in Q^{+-} \text{ and } e_\tau \in Q^{+-}$$

LEMMA 4.3. *Let G be a TG, let q be a TRPQ, let $\mathbf{u} = \langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$, let $Q = \mathsf{boundaries}(G, q) \cup \{t\}$ and let δ be the largest interval s.t. $d \in \delta$ and $\langle o_1, o_2, t, d'\rangle \in [\![q]\!]_G$ for all $d' \in \delta$. Then*

$$b_\delta \in Q^{+-} \text{ and } e_\delta \in Q^{+-}$$

Next, let $\sqsubseteq$ denote set inclusion lifted to pairs of intervals, i.e.

$$(\tau_1, \delta_1) \sqsubseteq (\tau_2, \delta_2) \text{ iff } \tau_1 \subseteq \tau_2 \text{ and } \delta_1 \subseteq \delta_2$$

The following is an immediate consequence of Lemmas 4.2 and 4.3

Corollary 4.4. *Let G be a TG, let q be a TRPQ, let $\mathbf{u} = \langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$, let $Q = \text{boundaries}(G, q) \cup \{t, d\}$, let $P = \{(t, d) \mid \langle o_1, o_2, t, d\rangle \in [\![q]\!]_G\}$, and let $(\tau, \delta) \in \max_{\sqsubseteq}\{(\tau', \delta') \in \text{intv}(\mathcal{T}) \times \text{intv}(\mathcal{T}) \mid t \in \tau \text{ and } d \in \delta\}$. Then*

$$\{b_\tau, e_\tau, b_\delta, e_\delta\} \subseteq Q^{+-}$$

We can now prove our membership results.

Proposition 4.5. *Compact Answer$^{[t]}$ is in PSpace*

Proof.
Let G be a TG, let q be a TRPQ, and let $\mathbf{u} = \langle o_1, o_2, \tau, d\rangle \in \mathcal{U}^{[t]}$.
We use Q for boundaries$(G, q) \cup \{d\}$, and T for the set defined by

$$T = \{t \mid \langle o_1, o_2, t, d\rangle \in [\![q]\!]_G\}$$

We also use k to denote the product of the denominators of all numbers in Q, i.e.

$$k = \Pi\{j \mid \frac{i}{j} \in Q \text{ for some i } \in \mathbb{Z}\}$$

Note that k (encoded in binary) can be computed in time polynomial (therefore using space polynomial) in the cumulated sizes of G, q and $\mathbf{u}$.
We also use $\frac{1}{k}\mathbb{Z}$ (resp. $\frac{1}{2k}\mathbb{Z}$) for the set of all multiples of $\frac{1}{k}$ (resp. $\frac{1}{2k}$), i.e

$$\frac{1}{k}\mathbb{Z} = \{\frac{i}{k} \mid i \in \mathbb{Z}\}$$

and

$$\frac{1}{2k}\mathbb{Z} = \{\frac{i}{2k} \mid i \in \mathbb{Z}\}$$

Note that

$$Q^{+-} \subset \frac{1}{k}\mathbb{Z} \subset \frac{1}{2k}\mathbb{Z}$$

Now let τ' be the largest interval such that $\tau \subseteq \tau' \subseteq T$.
Recall that by assumption, $\tau \neq \emptyset$.
Under this assumption, $\langle G, q, u\rangle$ is an instance of Compact Answer$^{[t]}$ iff $\tau = \tau'$.
We show that $\tau = \tau'$ can be decided using space polynomial in the cumulated size of (the encodings of) G, q and u.

First, fom Lemma 4.2, we observe that $b_\tau \notin \frac{1}{k}\mathbb{Z}$ or $e_\tau \notin \frac{1}{k}\mathbb{Z}$ implies $\tau \neq \tau'$.
And $b_\tau \in \frac{1}{k}\mathbb{Z}$ (resp. $e_\tau \in \frac{1}{k}\mathbb{Z}$) can be decided in time polynomial in the encoding of b_τ (resp. e_τ).
So we can focus on the case where $b_\tau \in \frac{1}{k}\mathbb{Z}$ and $e_\tau \in \frac{1}{k}\mathbb{Z}$.
We use b_{inf} for the largest element of $(\frac{1}{2k}\mathbb{Z}) \setminus \tau$ that satisfies $b_{\text{inf}} \leq b_\tau$.
And similarly we use e_{sup} for the smallest element of $(\frac{1}{2k}\mathbb{Z}) \setminus \tau$ that satisfies $e_\tau \leq e_{\text{inf}}$.
Observe that b_{inf} and e_{sup} can be computed using space polynomial in (the encoding of) τ.

We show below that for any (nonempty) interval α with boundaries in $\frac{1}{k}\mathbb{Z}$,

$$\alpha \subseteq T \text{ iff } \alpha \cap \frac{1}{2k}\mathbb{Z} \subseteq T \tag{195}$$

Therefore in order to decide whether $\tau = \tau'$, it is sufficient to decide whether

(I) $\tau \cap \frac{1}{2k}\mathbb{Z} \subseteq T$, and
(II) $\{b_{\text{inf}}, e_{\text{sup}}\} \cap T = \emptyset$

Now observe that:

- (I) can be decided with a finite number of independent calls to a procedure for Compact Answer, and
- (II) can be decided with two calls to such a procedure.

And it was shown in[1] that Compact Answer is in PSpace.

To complete the proof, we show that (195) holds.
The right direction ($\alpha \subseteq T$ implies $\alpha \cap \frac{1}{2k}\mathbb{Z} \subseteq T$) is trivial.
For the left direction, assume by contradiction that $\alpha \cap \frac{1}{2k}\mathbb{Z} \subseteq T$ but $\alpha \nsubseteq T$.
Take any $t \in \alpha \setminus T$.
Since $\alpha \cap \frac{1}{2k}\mathbb{Z} \subseteq T$ and $t \notin T$, we have

$$t \notin \frac{1}{2k}\mathbb{Z} \tag{196}$$

Next, since α has boundaries in $\frac{1}{k}\mathbb{Z}$,

$$\alpha \cap \frac{1}{2k}\mathbb{Z} \neq \emptyset \tag{197}$$

(for instance, $b_\alpha + \frac{1}{2k} \in \alpha \cap \frac{1}{2k}\mathbb{Z}$).
Together with (196), this implies that there is a t' in $\alpha \cap \frac{1}{2k}\mathbb{Z}$ s.t. either $t' < t$ or $t < t'$.
Let us assume w.l.o.g. that the former holds (the proof for the latter case is symmetric).
And let t_{inf} be the largest value that satisfies $t_{\mathsf{inf}} \in \alpha \cap \frac{1}{2k}\mathbb{Z}$ and $t_{\mathsf{inf}} < t$.
Then

$$t - t_{\mathsf{inf}} < \frac{1}{2k} \tag{198}$$

Now recall that by assumption, $\alpha \cap \frac{1}{2k}\mathbb{Z} \subseteq T$.
Therefore $t_{\mathsf{inf}} \in T$.
So from Lemma 4.2, there is a β with boundaries in $\frac{1}{k}\mathbb{Z}$ s.t. $\beta \subseteq T$ and $t_{\mathsf{inf}} \in \beta$.
Then we consider two cases: either $e_\beta \neq t_{\mathsf{inf}}$ or $e_\beta = t_{\mathsf{inf}}$.

- ($e_\beta \neq t_{\mathsf{inf}}$).
  In this case, since $e_\beta \in \frac{1}{k}\mathbb{Z}$, and $t_{\mathsf{inf}} \in \frac{1}{2k}\mathbb{Z}$,

  $$\frac{1}{2k} \leq e_\beta - t_{\mathsf{inf}}$$

  Together with (198), this yields (by transitivity)

  $$t - t_{\mathsf{inf}} < e_\beta - t_{\mathsf{inf}} \tag{199}$$

  $$t < e_\beta \tag{200}$$

  Now since $t_{\mathsf{inf}} \in \beta$,

  $$b_\beta \leq t_{\mathsf{inf}} \tag{201}$$

  Together with $t_{\mathsf{inf}} < t$, this implies

  $$b_\beta < t \tag{202}$$

  Together with (200), this yields

  $$t \in \beta$$

  Since $\beta \subseteq T$, this implies $t \in T$, which contradicts the definition of t.

- ($e_\beta = t_{\mathsf{inf}}$).
  In this case, since β has boundaries in $\frac{1}{k}\mathbb{Z}$,

  $$t_{\mathsf{inf}} \in \frac{1}{k}\mathbb{Z} \tag{203}$$

  And because $t \in \alpha$ and $t_{\mathsf{inf}} < t$

  $$t_{\mathsf{inf}} < t \leq e_\alpha \tag{204}$$

  $$t_{\mathsf{inf}} < e_\alpha \tag{205}$$

  Together with (203) and $e_\alpha \in \frac{1}{k}\mathbb{Z}$, this implies

  $$\frac{1}{k} \leq e_\alpha - t_{\mathsf{inf}} \tag{206}$$

  Now let $t_{\mathsf{sup}} = t_{\mathsf{inf}} + \frac{1}{2k}$.
  From (206), we get

  $$t_{\mathsf{sup}} < e_\alpha \tag{207}$$

Next, since $t_{\mathsf{inf}} \in \alpha$ and $t_{\mathsf{inf}} < t_{\mathsf{sup}}$

$$b_\alpha < t_{\mathsf{sup}} \tag{208}$$

So from (207) and (208)

$$t_{\mathsf{sup}} \in \alpha$$

So from Lemma 4.2, there is a β' with boundaries in $\frac{1}{k}\mathbb{Z}$ s.t. $\beta' \subseteq T$ and $t_{\mathsf{sup}} \in \beta'$.
Next, from (198) and the definition of t_{sup}

$$t_{\mathsf{sup}} - t < \frac{1}{2k} \tag{209}$$

So with an argument symmetric to the one used above to show $t \in \beta$, we get $t \in \beta'$, which once again contradicts $t \notin T$.

□

Proposition 4.6. *Compact Answer*$^{[d]}$ *is in PSpace*

Proof.
The proof is symmetric to the one provided above for Proposition 4.5, using Lemma 4.3 instead of Lemma 4.2. □

Proposition 4.7. *Compact Answer*$^{[t,d]}$ *is in PSpace*

Proof.
The proof is analogous to the one provided above for Proposition 4.5, using Corollary 4.4 instead of Lemma 4.2.
More precisely, let G be a TG, let q be a TRPQ, and let $\mathbf{u} = \langle o_1, o_2, \tau, \delta \rangle \in \mathcal{U}^{[t,d]}$.
We use Q for boundaries$(G, q) \cup \{t, d\}$, and P for the set defined by

$$P = \{(t, d) \mid \langle o_1, o_2, t, d \rangle \in [\![q]\!]_G\}$$

We also define k, $\frac{1}{k}\mathbb{Z}$ and $\frac{1}{2k}\mathbb{Z}$ identically as in the proof of Proposition 4.5.
Then analogously to what we showed in this proof, for any pair of intervals (α_1, α_2) with boundaries in $\frac{1}{k}\mathbb{Z}$,

$$\alpha_1 \times \alpha_2 \subseteq P \text{ iff } (\alpha_1 \cap \frac{1}{2k}\mathbb{Z}) \times (\alpha_2 \cap \frac{1}{2k}\mathbb{Z}) \subseteq P$$

So with a similar argument, deciding whether $\langle o_1, o_2, \tau, \delta \rangle$ is a compact answer to q over G can be reduced to deciding

- $\{b_\tau, e_\tau, b_\delta, e_\delta\} \subseteq \frac{1}{k}\mathbb{Z}$,
- $(\tau \cap \frac{1}{2k}\mathbb{Z}) \times (\delta \cap \frac{1}{2k}\mathbb{Z}) \subseteq P$,
- $\left(\{b^\tau_{\mathsf{inf}}, e^\tau_{\mathsf{sup}}\} \times (\delta \cap \frac{1}{2k}\mathbb{Z})\right) \cap P = \emptyset$ and
- $\left((\tau \cap \frac{1}{2k}\mathbb{Z}) \times \{b^\delta_{\mathsf{inf}}, e^\delta_{\mathsf{sup}}\}\right) \cap P = \emptyset$

where b^τ_{inf} is the largest element in $(\frac{1}{2k}\mathbb{Z}) \setminus \tau$ that satisfies $b^\tau_{\mathsf{inf}} \leq b_\tau$, e^τ_{sup} is the smallest element in $(\frac{1}{2k}\mathbb{Z}) \setminus \tau$ that satisfies $e_\tau \leq e^\tau_{\mathsf{sup}}$, and b^δ_{inf} and e^δ_{sup} are defined analogously.

□

4.3 Hardness

Proposition 4.8. *Compact Answer*$^{[t]}$ *is PSpace-hard*

Proof. The proof is a straightforward reduction from Compact Answer.
Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val} \rangle$ be a TG, let q be a TRPQ and let $\mathbf{u} = \{o_1, o_2, t, d\} \in \mathcal{U}$.
W.l.o.g., let us assume that $\{o_1, o_2\} \subseteq N$ (the proof for the 3 other cases is symmetric).

Now let $P \subseteq \mathit{Pred}$ be the set of predicates defined by $p \in \mathit{Pred}$ iff there is an o s.t. $\mathsf{val}(o, p) \neq \emptyset$.
Take two fresh predicates $p_1, p_2 \in \mathit{Pred} \setminus P$, two fresh nodes $n_1, n_2 \notin N$ and two fresh edges $e_1, e_2 \notin E$.
And let $G' = \langle N \cup \{n_1, n_2\}, E \cup \{e_1, e_2\}, \mathsf{conn}', \mathcal{T}_G, \mathsf{val}' \rangle$ be the TG identical to G, except for the functions conn' and val', defined by

- $\mathsf{conn}'(e) = \mathsf{conn}(e)$ for all $e \in E$,
- $\mathsf{conn}'(e_1) = (n_1, o_1)$,
- $\mathsf{conn}'(e_2) = (o_2, n_2)$,
- $\mathsf{val}'(o, p) = \mathsf{val}(o, p)$ for all $(o, p) \in (N \cup E) \times (\mathit{Pred} \setminus \{p_1, p_2\})$,
- $\mathsf{val}'(n_1, p_1) = \{[t, t]\}$, and
- $\mathsf{val}'(n_2, p_2) = \{[t + d, t + d]\}$

Now let q' be the TRPQ defined by

$$q' = p_1/\mathrm{F}/q/\mathrm{F}/p_2$$

Then immediately from the semantics of TRPQs:

$$\mathbf{u} \in [\![q]\!]_G \text{ iff } [\![q']\!]_{G'} = \{\langle n_1, n_2, t, d\rangle\} \tag{210}$$

Now consider the tuple $\mathbf{u'} = \{n_1, n_2, [t, t], d\} \in \mathcal{U}^{[t]}$.
Then from (210), $\mathbf{u} \in [\![q]\!]_G$ iff $\mathbf{u'}$ is a compact answer to q over G in $\mathcal{U}^{[t]}$.

Clearly, the input $\langle G', q, \mathbf{u'}\rangle$ to Compact Answer$^{[t]}$ can be computed in time polynomial in the size of (the encodings of) G, q and $\mathbf{u}$.
And it was shown in [1] that Compact Answer is PSpace-complete.

□

Proposition 4.9. *Compact Answer$^{[d]}$, Compact Answer$^{[t,d]}$, Compact Answer$_{nr}^{[t,d]}$ and Compact Answer$^{[t,d],b,e}$ are PSpace-hard.*

Proof. The proofs are nearly identical to the one provided above for Compact Answer.
The graph G' is defined identically in all cases, so that the reductions only differ w.r.t. to the tuple $\mathbf{u'}$.
This tuple is defined as follows:

- $\{n_1, n_2, t, [d, d]\}$ for Compact Answer$^{[d]}$,
- $\{n_1, n_2, [t, t], [d, d]\}$ for Compact Answer$^{[t,d]}$ and Compact Answer$_{nr}^{[t,d]}$,
- $\{n_1, n_2, [t, t], [d, d], t, t\}$ for Compact Answer$^{[t,d],b,e}$.

Note in particular that for Compact Answer$_{nr}^{[t,d]}$, the only compact answer *set* to q over G' is $\{\mathbf{u'}\}$ if $\mathbf{u} \in [\![q]\!]_G$, and the empty set otherwise. □

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