Compact Answers to Temporal Regular Path Queries (Supplementary Material)

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1 INTRODUCTION

This document provides detailed definitions and proofs for the article *Compact answers to Temporal Regular Path Queries*, submitted at CIKM 2023.

As opposed to the structure adopted in the article, the result here are grouped by topic (inductive representation, finiteness, complexity, etc) rather than representation ($\mathcal{U}^{[t]}$, $\mathcal{U}^{[d]}$, etc.). This allows us to emphasize which proofs differ from one representation to the other.

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2 INDUCTIVE REPRESENTATION

2.1 Grammar

For convenience, we reproduce here the syntax of TRPQs (already provided in the article).
A TRPQ is an expression for the symbol path in the following grammar:

$$\begin{aligned}
\text{path} &::= \text{test} \mid \text{axis} \mid (\text{path}/\text{path}) \mid (\text{path} + \text{path}) \mid \text{path}[m, n] \mid \text{path}[m, _] \\
\text{test} &::= \textit{pred} \mid (?\text{path}) \mid \text{test} \vee \text{test} \mid \text{test} \wedge \text{test} \mid \neg\text{test} \\
\text{axis} &::= \mathrm{F} \mid \mathrm{B} \mid \mathrm{T}_\delta
\end{aligned}$$

with $\delta \in \text{intv}(\mathcal{T})$, $m, n \in \mathbb{N}^+$ and $m \leq n$.

2.2 In $\mathcal{U}$

For convenience still, we reproduce here the full semantics of TRPQs, i.e. the inductive definition of the evaluation $[\![q]\!]_G$ of a query q over a graph G in $\mathcal{U}$ (already provided in the article).

$$\begin{aligned}
[\![\textit{pred}]\!]_G &= \{\langle o, o, t, 0\rangle \mid \langle o, t\rangle \models \textit{pred}\} \\
[\![\mathrm{T}_\delta]\!]_G &= \{\langle o, o, t, d\rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G\} \\
[\![\mathrm{F}]\!]_G &= \{\langle v, e, t, 0\rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, 0\rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G\} \\
[\![\mathrm{B}]\!]_G &= \{\langle v, e, t, 0\rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, 0\rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G\} \\
[\![?\text{path}]\!]_G &= \{\langle o, o, t, 0\rangle \mid \langle o, o', t, t + d\rangle \in [\![\text{path}]\!]_G \text{ for some } o' \in N \cup E, d \in \mathcal{T}\} \\
[\![\text{test}_1 \vee \text{test}_2]\!]_G &= [\![\text{test}_1]\!]_G \cup [\![\text{test}_2]\!]_G \\
[\![\text{test}_1 \wedge \text{test}_2]\!]_G &= [\![\text{test}_1]\!]_G \cap [\![\text{test}_2]\!]_G \\
[\![\neg\text{test}]\!]_G &= (\{\langle o, o\rangle \mid o \in N \cup E\} \times \mathcal{T}_G \times \{0\}) \setminus [\![\text{test}]\!]_G \\
[\![\text{path}_1/\text{path}_2]\!]_G &= \{\langle o_1, o_3, t, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, t, d_1\rangle \in [\![\text{path}_1]\!]_G \wedge \langle o_2, o_3, t + d_1, d_2\rangle \in [\![\text{path}_2]\!]_G\} \\
[\![\text{path}_1 + \text{path}_2]\!]_G &= [\![\text{path}_1]\!]_G \cup [\![\text{path}_2]\!]_G \\
[\![\text{path}[m, n]]\!]_G &= \bigcup_{k=m}^{n} [\![\text{path}^k]\!]_G \\
[\![\text{path}[m, _]]\!]_G &= \bigcup_{k \geq m} [\![\text{path}^k]\!]_G
\end{aligned}$$

2.3 In $\mathcal{U}^{[t]}$

2.4 In $\mathcal{U}^{[t,d]}$

2.5 In $\mathcal{U}^{[t,d],b,e}$

2.5.1 *Definition.* If q is an expression for the symbol test in the grammar of Section **??**, then all tuples in $[\![q]\!]_G$ must have distance 0. As a result, $(\!|q|\!)_G^{[t,d],b,e}$ can be trivially defined out of $(\!|q|\!)_G^{[t]}$, by extending each tuple $\{\langle o, o, \tau, [0, 0]\rangle$ with b_τ and e_τ,i.e.

$$(\!|q|\!)_G^{[t,d],b,e} = \{\langle o, o, \tau, [0, 0], b_\tau, e_\tau\rangle \mid \{\langle o, o, \tau, [0, 0]\rangle \in (\!|q|\!)_G^{[t]}\}$$

So for this fragment of the language, the inductive definition of $(\!|q|\!)_G^{[t,d],b,e}$ is nearly identical to the one of $(\!|q|\!)_G^{[t]}$. We only reproduce it hee for the sake of completeness.

$$\begin{aligned}
(\!|\textit{pred}|\!)_G^{[t,d],b,e} &= \{\langle o, o, \tau, [0, 0], b_\tau, e_\tau\rangle \mid \tau \in \text{val}(o, \textit{pred})\} \\
(\!|\mathrm{F}|\!)_G^{[t,d],b,e} &= \{\langle v, e, \mathcal{T}_G, 0, b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \mid \text{src}(e) = v\} \cup \{\langle e, v, \mathcal{T}_G, 0\rangle \mid \text{tgt}(e) = v\} \\
(\!|\mathrm{B}|\!)_G^{[t,d],b,e} &= \{\langle v, e, \mathcal{T}_G, 0, , b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \mid \text{tgt}(e) = v\} \cup \{\langle e, v, \mathcal{T}_G, 0\rangle \mid \text{src}(e) = v\} \\
(\!|(?\text{path})|\!)_G^{[t,d],b,e} &= \{\langle o_1, o_1, \tau, 0, b_\tau, e_\tau\rangle \mid \exists o_2, d, b, e \colon \langle o_1, o_2, \tau, d, b, e\rangle \in (\!|\text{path}|\!)_G^{[t,d],b,e}\} \\
(\!|\text{test}_1 \vee \text{test}_2|\!)_G^{[t,d],b,e} &= (\!|\text{test}_1|\!)_G^{[t,d],b,e} \cup (\!|\text{test}_2|\!)_G^{[t,d],b,e} \\
(\!|\text{test}_1 \wedge \text{test}_2|\!)_G^{[t,d],b,e} &= \{\langle o, o, \tau_1 \cap \tau_2, 0, b_{\tau_1 \cap \tau_2}, e_{\tau_1 \cap \tau_2}\rangle \mid \langle o, o, \tau_1, 0, b_{\tau_1}, e_{\tau_1}\rangle \in (\!|\text{test}_1|\!)_G^{[t,d],b,e}, \langle o, o, \tau_2, 0, b_{\tau_2}, e_{\tau_2}\rangle \in (\!|\text{test}_2|\!)_G^{[t,d],b,e}, \tau_1 \cap \tau_2 \neq \emptyset\} \\
(\!|\neg\text{test}|\!)_G^{[t,d],b,e} &= \bigcup_{o \in N \cup E} \left\{\langle o, o, \tau, 0, b_\tau, e_\tau\rangle \mid \tau \in \text{compl}\left(\{\tau' \mid \langle o, o, \tau', 0, b_{\tau'}, e_{\tau'}\rangle \in (\!|\text{test}|\!)_G^{[t]}\}, \mathcal{T}_G\right)\right\}
\end{aligned}$$

Next, we consider the operators $(\text{path}_1 + \text{path}_2)$, $(\text{path}[m, \_])$ and $(\text{path}[m, n])$. For these cases, $(\!|q|\!)_G^{[t,d],b,e}$ is defined analogously as for the three previous representations, in terms of temporal join (a.k.a. $\text{path}_1/\text{path}_2$) and set union. We only write the definitions here for the sake of completeness:

$$\begin{array}{rcl} (\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t,d],b,e} & = & (\!|\mathsf{path}_1|\!)_G^{[t,d],b,e} \cup (\!|\mathsf{path}_2|\!)_G^{[t,d],b,e} \\ [\![\mathsf{path}[m,n]]\!]_G & = & \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t,d],b,e} \\ [\![\mathsf{path}[m,\_]]\!]_G & = & \bigcup_{k\geq m} (\!|\mathsf{path}^k|\!)_G^{[t,d],b,e} \end{array}$$

So the only remaining operator are temporal join ($\mathsf{path}_1/\mathsf{path}_2$) and temporal navigation (T_δ), already defined in the article. We reproduce here these two definition for convenience:

$$\begin{array}{rcl} (\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[t,d],b,e} & = & \{\mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2 \mid \mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)_G^{[t,d],b,e}, \mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)_G^{[t,d],b,e}, \mathbf{u}_1 \sim \mathbf{u}_2\} \\ (\!|\mathrm{T}_\delta|\!)_G^{[t,d],b,e} & = & \{\langle o, o, \mathcal{T}_G, \delta, b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \overline{\bowtie} \langle o, o, \mathcal{T}_G, [0,0], b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \mid o \in N \cup E\} \end{array}$$

where $\mathbf{u}_1 \sim \mathbf{u}_2$ and $\mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2$ are defined by:

$$\langle o_1, o_2, \tau_1, \delta_1, b_1, e_1\rangle \sim \langle o_3, o_4, \tau_2, \delta_2, e_2, b_2\rangle \text{ iff } o_2 = o_3$$

$$\begin{array}{rcl} \langle o_1, o_3, \tau_1, \delta_1, b_1, e_1\rangle \overline{\bowtie} & \langle o_3, o_4, \tau_2, \delta_2, e_2, b_2\rangle = & \langle o_1, o_4, \tau, \delta_1 + \delta_2, b, e\rangle \\ & \text{with} & \\ \tau = & (((\tau_1 + \delta_1) \cap \tau_2) \ominus \delta_1) \cap \tau_1 & \\ b = & \max(b_1, b_2 - b_{\delta_1}) & \\ e = & \min(e_1, e_2 - e_{\delta_1}) & \end{array}$$

2.5.2 Correctness.

Operator $\mathsf{path}_1/\mathsf{path}_2$**.**
Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1\rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2\rangle$, with $o_2 = o_3$.
Let also $\mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2 = \langle o_1, o_4, \tau_1', \delta_1 + \delta_2, b, e\rangle$, with

$$\tau = (((\tau_1 + \delta_1) \cap \tau_2) \ominus \delta_1) \cap \tau_1$$
$$b = \max(b_1, b_2 - b_{\delta_1})$$
$$e = \min(e_1, e_2 - e_{\delta_1})$$

As explained in Section XXX, for $i \in \{1, 2\}$ and $t \in \tau_i$, we use $\delta_i(t)$ for the interval

$$_{\delta_i}\lfloor\, b_{\delta_i} + \max(0, b_i - t)\ ,\ e_{\delta_i} - \max(0, t - e_i)\, \rfloor_{\delta_i}$$

And similarly to what we did for $\mathcal{U}^{[t,d]}$, we use R_i for be the binary relation over $\mathcal{T}$ specified by the time points and distances in $\mathbf{u}_i$, i.e. $R_i = \{(t, t + d) \mid t \in \tau_i, d \in \delta_i(t)\}$.
Then the intervals in the set $\mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2$ should intuitively represent this relation $R_1 \bowtie R_2$, i.e. we need to prove that *(i)* $\tau = \mathsf{dom}(R_1 \bowtie R_2)$ and that *(ii)* for each $t \in \tau$,

$$t + \delta(t) = \{t' \mid (t, t') \in R_1 \bowtie R_2\}$$

For *(ii)* , let $t \in \tau$. We use

- a for the least value s.t. $(t, a) \in \mathsf{range}(R_1) \cap \mathsf{dom}(R_2)$, and
- a' for the least value s.t. $(a, a') \in R_2$

Then a' is also the least value s.t. $(t, a') \in R_1 \bowtie R_2$.
Analogously, we use z for the greatest value s.t. $(t, z) \in \mathsf{range}(R_1) \cap \mathsf{dom}(R_2)$, and z' for the greatest value s.t. $(z, z') \in R_2$. Then z' is also the greatest value s.t. $(t, z') \in R_1 \bowtie R_2$.

From Lemma **??**:

- $\{t\} \times (t + [a, z]) \subseteq R_1$, and
- $[a, b] \times [a', z'] \subseteq R_2$

Therefore $[a', z'] = \{c \mid (t, c) \in R_1 \bowtie R_2\}$.

To conclude the proof, we show that $t + \delta_t = [a, z]$.
We only prove that $t + b_{\delta_t} = a$ (the proof that $t + e_{\delta_t} = z$ is symmetric).
Following the definition of b, we consider 2 cases:

(1) $b_1 < b_2 - b_{\delta_1}$
(2) $b_1 \geq b_2 - b_{\delta_1}$

For Case (1), we get

$$
\begin{aligned}
b_1 <\ & b_2 - b_{\delta_1} && (1)\\
\max(b_1, b_2 - b_{\delta_1}) =\ & b_2 - b_{\delta_1} && (2)\\
b =\ & b_2 - b_{\delta_1} && \text{from the definition of } b \quad (3)
\end{aligned}
$$

And in Case (1) still, we get:

$$
\begin{aligned}
b_1 <\ & b_2 - b_{\delta_1} && (4)\\
0 <\ & b_2 - b_{\delta_1} - b_1 && (5)\\
\max(0, b_2 - b_{\delta_1} - b_1) =\ & b_2 - b_{\delta_1} - b_1 && (6)
\end{aligned}
$$

Next, we consider two subcases:

(i) $t < b_2 - b_{\delta_1}$
(ii) $t \geq b_2 - b_{\delta_1}$

In Case (i), we get

$$
\begin{aligned}
t <\ & b_2 - b_{\delta_1} && (7)\\
0 <\ & b_2 - b_{\delta_1} - t && (8)\\
\max(0, b_2 - b_{\delta_1} - t) =\ & b_2 - b_{\delta_1} - t && (9)
\end{aligned}
$$

Now from the definition of δ_t,

$$
\begin{aligned}
b_{\delta_t} =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && (10)\\
=\ & b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_{\delta_1} - t) && \text{from (3)} \quad (11)\\
=\ & b_{\delta_1} + b_{\delta_2} + b_2 - b_{\delta_1} - t && \text{from (9)} \quad (12)\\
=\ & b_{\delta_2} + b_2 - t && (13)\\
b_{\delta_t} + t =\ & b_{\delta_2} + b_2 - t + t && (14)\\
=\ & b_{\delta_2} + b_2 && (15)
\end{aligned}
$$

Next, from the definition of a'

$$
\begin{aligned}
a' =\ & b_{\delta_2(a)} + a && (16)\\
=\ & b_{\delta_2} + \max(0, b_2 - a) + a && (17)
\end{aligned}
$$

And, from the definition of a

$$
\begin{aligned}
a =\ & b_{\delta_1(t)} + t && (18)\\
=\ & b_{\delta_1} + \max(0, b_1 - t) + t && (19)
\end{aligned}
$$

Then we have two further subcases:

(I) $t \geq b_1$, or
(II) $t < b_1$

In case (I):

$$
\begin{aligned}
t \geq\ & b_1 && (20)\\
0 \geq\ & b_1 - t && (21)\\
\max(0, b_1 - t) =\ & 0 && (22)\\
a =\ & b_{\delta_1} + t && \text{from (19)} \quad (23)\\
\max(0, b_2 - a) =\ & \max(0, b_2 - b_{\delta_1} - t) && (24)\\
=\ & b_2 - b_{\delta_1} - t && \text{from (9)} \quad (25)\\
=\ & b_2 - a && \text{from (23)} \quad (26)
\end{aligned}
$$

In case (II):

$$
\begin{aligned}
t <& \;\; b_1 && && (27)\\
0 <& \;\; b_1 - t && && (28)\\
\max(0, b_1 - t) =& \;\; b_1 - t && && (29)\\
a =& \;\; b_{\delta_1} + b_1 - t + t && \text{from (19)} && (30)\\
=& \;\; b_{\delta_1} + b_1 && && (31)\\
\max(0, b_2 - a) =& \;\; \max(0, b_2 - b_{\delta_1} - b_1) && && (32)\\
=& \;\; b_2 - b_{\delta_1} - b_1 && \text{from (6)} && (33)\\
=& \;\; b_2 - a && \text{from (31)} && (34)\\
& && && (35)
\end{aligned}
$$

So in both cases (I) and (II), we get

$$\max(0, b_2 - a) = b_2 - a$$

Thefore from (17)

$$
\begin{aligned}
a' =& \;\; b_{\delta_2} + b_2 - a + a && && (36)\\
=& \;\; b_{\delta_2} + b_2 && && (37)\\
=& \;\; t + b_{\delta_t} && \text{from (15)} && (38)
\end{aligned}
$$

which concludes the proof for Case (1)- (i).

We continue with Case (1)- (ii).
From Case (ii):

$$
\begin{aligned}
t \geq& \;\; b_2 - b_{\delta_1} && (39)\\
0 \geq& \;\; b_2 - b_{\delta_1} - t && (40)\\
\max(0, b_2 - b_{\delta_1} - t) =& \;\; 0 && (41)
\end{aligned}
$$

Now from the definition of δ_t:

$$
\begin{aligned}
b_{\delta_t} =& \;\; b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (42)\\
=& \;\; b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_1 - t) && \text{from (3)} && (43)\\
=& \;\; b_{\delta_1} + b_{\delta_2} && \text{from (41)} && (44)\\
b_{\delta_t} + t =& \;\; b_{\delta_1} + b_{\delta_2} + t && && (45)
\end{aligned}
$$

Next, from Case (1) and Case (ii), by transitivity, we get

$$
\begin{aligned}
b_1 \leq& \;\; t && (46)\\
\max(0, b_1 - t) =& \;\; 0 && (47)
\end{aligned}
$$

And from the definition of a

$$
\begin{aligned}
a =& \;\; b_{\delta_1(t)} + t && && (48)\\
=& \;\; b_{\delta_1} + \max(0, b_1 - t) + t && && (49)\\
=& \;\; b_{\delta_1} + t && \text{from (47)} && (50)\\
\geq& \;\; b_{\delta_1} + b_2 - b_{\delta_1} && \text{from Case (ii)} && (51)\\
\geq& \;\; b_2 && && (52)\\
0 \geq& \;\; b_2 - a && && (53)\\
\max(0, b_2 - a) =& \;\; 0 && && (54)
\end{aligned}
$$

Therefore from (17) and (54)

$$
\begin{aligned}
a' =& \;\; b_{\delta_2} + a && && (55)\\
=& \;\; b_{\delta_2} + b_{\delta_1} + t && \text{from (50)} && (56)\\
=& \;\; b_{\delta_t} + t && \text{from (15)} && (57)
\end{aligned}
$$

which concludes the proof for Case (1)- (ii).

We continute with Case (2).
In this case, we get

$$\begin{aligned} b_1 \geq & \ b_2 - b_{\delta_1} && (58)\\ \max(b_1, b_2 - b_{\delta_1}) = & \ b_1 && (59)\\ b = & \ b_1 \qquad \text{from the definition of } b && (60) \end{aligned}$$

And from Case (2) still, we derive

$$\begin{aligned} b_1 \geq & \ b_2 - b_{\delta_1} && (61)\\ 0 \geq & \ b_2 - b_{\delta_1} - b_1 && (62)\\ \max(0, -b_{\delta_1} - b_1) = & \ 0 && (63) \end{aligned}$$

As well as

$$\begin{aligned} b_1 \geq & \ b_2 - b_{\delta_1} && (64)\\ b_1 + b_{\delta_1} \geq & \ b_2 && (65) \end{aligned}$$

Next, we distinguish two subcases, namely

(a) $t < b_1$ and
(b) $t \geq b_1$

We start with Case (a).
In this case,

$$\begin{aligned} t < & \ b_1 && (66)\\ 0 < & \ b_1 - t && (67)\\ \max(0, b_1 - t) = & \ b_1 - t && (68) \end{aligned}$$

And from the definition of δ_t:

$$\begin{aligned} b_{\delta_t} = & \ b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && (69)\\ = & \ b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) \qquad \text{from (60)} && (70)\\ = & \ b_{\delta_1} + b_{\delta_2} + b_1 - t \qquad \text{from (68)} && (71)\\ b_{\delta_t} + t = & \ b_{\delta_1} + b_{\delta_2} + b_1 - t + t && (72)\\ = & \ b_{\delta_1} + b_{\delta_2} + b_1 && (73) \end{aligned}$$

Next, from the definition of a

$$\begin{aligned} a = & \ b_{\delta_1(t)} + t && (74)\\ = & \ \max(0, b_1 - t) + b_{\delta_1} + t && (75)\\ = & \ b_1 - t + b_{\delta_1} + t \qquad \text{from (68)} && (76)\\ = & \ b_1 + b_{\delta_1} && (77)\\ & \text{So from (65)} && (78)\\ a \geq & \ b_2 && (79)\\ 0 \geq & \ b_2 - a && (80)\\ \max(0, b_2 - a) = & \ 0 && (81)\\ b_{\delta_2} + \max(0, b_2 - a) = & \ b_{\delta_2} && (82)\\ b_{\delta_2(a)} = & \ b_{\delta_2} && (83)\\ b_{\delta_2(a)} + a = & \ b_{\delta_2} + a && (84)\\ a' = & \ b_{\delta_2} + a \qquad \text{from the defintiion of } a' && (85)\\ a' = & \ b_{\delta_2} + b_1 + b_{\delta_1} \qquad \text{from (77)} && (86)\\ a' = & \ b_{\delta_t} + t \qquad \text{from (73)} && (87) \end{aligned}$$

which concludes the proof for Case (2)- (a).

We end with Case (2)- (b). In this case,

$$t \geq \; b_1 \tag{88}$$
$$0 \geq \; b_1 - t \tag{89}$$
$$\max(0, b_1 - t) = \; 0 \tag{90}$$

And from the definition of δ_t:

$$\begin{aligned}
b_{\delta_t} &= b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && (91)\\
&= b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) && \text{from (60)} \quad (92)\\
&= b_{\delta_1} + b_{\delta_2} && \text{from (90)} \quad (93)\\
b_{\delta_t} + t &= b_{\delta_1} + b_{\delta_2} + t && (94)
\end{aligned}$$

Next, from the definition of a

$$\begin{aligned}
a &= b_{\delta_1(t)} + t && (95)\\
&= \max(0, b_1 - t) + b_{\delta_1} + t && (96)\\
&= b_{\delta_1} + t && \text{from (90)} \quad (97)
\end{aligned}$$

Now from Case (b)

$$\begin{aligned}
b_1 + \leq &\; t && (98)\\
b_1 + b_{\delta_1} \leq &\; t + b_{\delta_1} && (99)\\
b_1 + b_{\delta_1} \leq &\; a && \text{from (97)} \quad (100)\\
b_2 \leq &\; a && \text{from (65), by transitivity} \quad (101)\\
b_2 - a \leq &\; 0 && (102)\\
\max(0, b_2 - a) = &\; 0 && (103)\\
b_{\delta_2} + \max(0, b_2 - a) = &\; b_{\delta_2} && (104)\\
b_{\delta_2(a)} = &\; b_{\delta_2} && (105)\\
b_{\delta_2(a)} + a = &\; b_{\delta_2} + a && (106)\\
a' = &\; b_{\delta_2} + a && \text{from the defintiion of } a' \quad (107)\\
= &\; b_{\delta_2} + b_{\delta_1} + t && \text{from (97)} \quad (108)\\
= &\; b_{\delta_t} + t && \text{from (94)} \quad (109)
\end{aligned}$$

which concludes the proof for Case (2)- (b). □

3 FINITENESS

4 COMPLEXITY OF QUERY ANSWERING

4.1 Problem

We define in this section a decision problem for each representation, similar to the problem COMPACT ANSWER$^{[t]}$ defined in Section XXX. First, we define a (possibly partial) order over tuples of each representation, in the expected way, i.e.:

- over $\mathcal{U}^{[t]}$: $\langle o_1, o_2, \tau_1, d_1 \rangle \sqsubseteq_{[t]} \langle o_3, o_4, \tau_2, d_2 \rangle$ iff $\langle o_1, o_2, d_1 \rangle = \langle o_3, o_4, d_2 \rangle$ and $\tau_1 \subseteq \tau_2$,
- over $\mathcal{U}^{[d]}$: $\langle o_1, o_2, t_1, \delta_1 \rangle \sqsubseteq_{[d]} \langle o_3, o_4, t_2, \delta_1 \rangle$ iff $\langle o_1, o_2, t_1 \rangle = \langle o_3, o_4, t_2 \rangle$ and $\delta_1 \subseteq \delta_2$,
- over $\mathcal{U}^{[t,d]}$: $\langle o_1, o_2, \tau_1, \delta_1 \rangle \sqsubseteq_{[t,d]} \langle o_3, o_4, \tau_2, \delta_2 \rangle$ iff $\langle o_1, o_2 \rangle = \langle o_3, o_4 \rangle$, $\tau_1 \subseteq \tau_2$ and $\delta_1 \subseteq \delta_2$,
- over $\mathcal{U}^{[t,d],b,e}$: $\langle o_1, o_2, \tau_1, \delta_1, b_1, b_2 \rangle \sqsubseteq_{[t,d],b,e} \langle o_3, o_4, \tau_2, \delta_2, b_2, e_2 \rangle$ iff $\langle o_1, o_2 \rangle = \langle o_3, o_4 \rangle$, $\tau_1 \subseteq \tau_2$ and $\delta_1(t) \subseteq \delta_2(t)$ for all $t \in \tau_1 \cap \tau_2$ (the notation of $\delta_i(t)$ is explained above, in Section XXX).

Now let x be one of $[t]$, $[d]$, $[t, d]$ or $([t, d], b, e)$.
We decline the notion of compact answer defined in Section XXX in four flavors, as follows:

Definition 4.1. Let G be a TG, let q be a TRPQ and let $\mathbf{u} \in \mathcal{U}^x$.
We say that $\mathbf{u}$ is a *compact answer* to q over G (in $\mathcal{U}^x$) if $\mathbf{u} \in \max_{\sqsubseteq_x}\{\mathbf{u}' \in \mathcal{U}^x \mid \mathsf{unfold}(\mathbf{u}') = [\![q]\!]_G\}$

And similary, we get four decision problems:

COMPACT ANSWER^x	
Input:	TG G, TRPQ q, tuple $\mathbf{u} \in \mathcal{U}^x$
Decide:	$\mathbf{u}$ is a compact answer to q over G (in $\mathcal{U}^x$)

4.2 Hardness

4.3 Membership

5 MINIMIZATION

6 SIZE OF COMPACT ANSWERS