

Compact Answers to Temporal Regular Path Queries (Supplementary Material)

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1 INTRODUCTION

This document provides detailed definitions and proofs for the article *Compact answers to Temporal Regular Path Queries*, submitted at CIKM 2023.

2 NOTATION

Let $\langle o_1, o_2, \tau, \delta, b, e \rangle \in \mathcal{U}^{[t,d],b,e}$, and let $t \in \tau$.

In the article, we defined the interval δ_t for each t as

$$\delta_t \lfloor b_\delta + \max(0, b - t), e_{\delta_t} - \max(0, t - e) \rfloor_\delta$$

In this supplementary material, we will use $\delta(t)$ instead of δ_t . This notation will allow us to write $\delta_1(t)$ when several tuples are involved. Note that the time points b and e in this notation are still omitted, for conciseness, because they should be clear from the context.

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3 INDUCTIVE REPRESENTATION

Let q be a TRPQ and G a TG.

Then $\llbracket q \rrbracket_G$ is the set of answers to q over G (represented as tuples in \mathcal{U}).

In this section, we provide the full definition of the four inductive representations of $\llbracket q \rrbracket_G$ discussed in the article, in $\mathcal{U}^{[t]}$, $\mathcal{U}^{[d]}$, $\mathcal{U}^{[t,d]}$ and $\mathcal{U}^{[t,d],b,e}$ respectively, and prove that they are correct.

These representations are denoted as $\langle\langle q \rangle\rangle_G^{[t]}$, $\langle\langle q \rangle\rangle_G^{[d]}$, $\langle\langle q \rangle\rangle_G^{[t,d]}$ and $\langle\langle q \rangle\rangle_G^{[t,d],b,e}$ respectively.

3.1 Grammar

For convenience, we reproduce here the syntax of TRPQs (already provided in the article).

Definition 3.1 (TRPQ). A TRPQ is an expression for the symbol path in the following grammar:

$$\begin{aligned} \text{path} &::= \text{test} \mid \text{axis} \mid (\text{path}/\text{path}) \mid (\text{path} + \text{path}) \mid \text{path}[m, n] \mid \text{path}[m, _] \\ \text{test} &::= \text{pred} \mid (? \text{path}) \mid \text{test} \vee \text{test} \mid \text{test} \wedge \text{test} \mid \neg \text{test} \\ \text{axis} &::= F \mid B \mid T_\delta \end{aligned}$$

with $\delta \in \text{intv}(\mathcal{T})$, $m, n \in \mathbb{N}^+$ and $m \leq n$.

3.2 In \mathcal{U}

For convenience still, we reproduce the full semantics of TRPQs, i.e. the inductive definition of the evaluation $\llbracket q \rrbracket_G$ of a query q over a graph G in \mathcal{U} (already provided in the article).

$$\begin{aligned} \llbracket \text{pred} \rrbracket_G &= \{ \langle o, o, t, 0 \rangle \mid t \in \tau \text{ for some } \tau \in \text{val}(o, \text{pred}) \} \\ \llbracket T_\delta \rrbracket_G &= \{ \langle o, o, t, d \rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G \} \\ \llbracket F \rrbracket_G &= \{ \langle v, e, t, 0 \rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G \} \cup \{ \langle e, v, t, 0 \rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G \} \\ \llbracket B \rrbracket_G &= \{ \langle v, e, t, 0 \rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G \} \cup \{ \langle e, v, t, 0 \rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G \} \\ \llbracket ? \text{path} \rrbracket_G &= \{ \langle o, o, t, 0 \rangle \mid \langle o, o', t, t + d \rangle \in \llbracket \text{path} \rrbracket_G \text{ for some } o' \in N \cup E, d \in \mathcal{T} \} \\ \llbracket \text{test}_1 \vee \text{test}_2 \rrbracket_G &= \llbracket \text{test}_1 \rrbracket_G \cup \llbracket \text{test}_2 \rrbracket_G \\ \llbracket \text{test}_1 \wedge \text{test}_2 \rrbracket_G &= \llbracket \text{test}_1 \rrbracket_G \cap \llbracket \text{test}_2 \rrbracket_G \\ \llbracket \neg \text{test} \rrbracket_G &= (\{ \langle o, o \rangle \mid o \in N \cup E \} \times \mathcal{T}_G \times \{0\}) \setminus \llbracket \text{test} \rrbracket_G \\ \llbracket \text{path}_1 / \text{path}_2 \rrbracket_G &= \{ \langle o_1, o_3, t, d_1 + d_2 \rangle \mid \exists o_2: \langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G \wedge \langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G \} \\ \llbracket \text{path}_1 + \text{path}_2 \rrbracket_G &= \llbracket \text{path}_1 \rrbracket_G \cup \llbracket \text{path}_2 \rrbracket_G \\ \llbracket \text{path}[m, n] \rrbracket_G &= \bigcup_{k=m}^n \llbracket \text{path}^k \rrbracket_G \\ \llbracket \text{path}[m, _] \rrbracket_G &= \bigcup_{k \geq m} \llbracket \text{path}^k \rrbracket_G \end{aligned}$$

3.3 In $\mathcal{U}^{[t]}$

3.3.1 Definition.

The full definition of $\langle\langle q \rangle\rangle_G^{[t]}$ is already provided in the article. We only reproduce it here for convenience.

$$\begin{aligned} \langle\langle \text{pred} \rangle\rangle_G^{[t]} &= \{ \langle o, o, \tau, 0 \rangle \mid o \in (N \cup E), \tau \in \text{val}(o, \text{pred}) \} \\ \langle\langle T_\delta \rangle\rangle_G^{[t]} &= \{ \langle o, o, [t_1, t_1], t_2 - t_1 \rangle \mid o \in (N \cup E), t_1 \in \mathcal{T}_G, t_2 \in (\delta + t_1) \cap \mathcal{T}_G \} \\ \langle\langle F \rangle\rangle_G^{[t]} &= \{ \langle v, e, \mathcal{T}_G, 0 \rangle \mid \text{src}(e) = v \} \cup \{ \langle e, v, \mathcal{T}_G, 0 \rangle \mid \text{tgt}(e) = v \} \\ \langle\langle B \rangle\rangle_G^{[t]} &= \{ \langle v, e, \mathcal{T}_G, 0 \rangle \mid \text{tgt}(e) = v \} \cup \{ \langle e, v, \mathcal{T}_G, 0 \rangle \mid \text{src}(e) = v \} \\ \langle\langle (? \text{path}) \rangle\rangle_G^{[t]} &= \{ \langle o, o, \tau, 0 \rangle \mid \langle o, o', \tau, d \rangle \in \langle\langle \text{path} \rangle\rangle_G^{[t]} \text{ for some } o' \in N \cup E, d \in \mathcal{T} \} \\ \langle\langle \text{test}_1 \vee \text{test}_2 \rangle\rangle_G^{[t]} &= \langle\langle \text{test}_1 \rangle\rangle_G^{[t]} \cup \langle\langle \text{test}_2 \rangle\rangle_G^{[t]} \\ \langle\langle \text{test}_1 \wedge \text{test}_2 \rangle\rangle_G^{[t]} &= \{ \langle o, o, \tau_1 \cap \tau_2, 0 \rangle \mid \langle o, o, \tau_1, 0 \rangle \in \langle\langle \text{test}_1 \rangle\rangle_G^{[t]}, \langle o, o, \tau_2, 0 \rangle \in \langle\langle \text{test}_2 \rangle\rangle_G^{[t]}, \tau_1 \cap \tau_2 \neq \emptyset \} \\ \langle\langle \neg \text{test} \rangle\rangle_G^{[t]} &= \bigcup_{o \in N \cup E} \left\{ \langle o, o, \tau, 0 \rangle \mid \tau \in \text{compl} \left(\{ \tau' \mid \langle o, o, \tau', 0 \rangle \in \langle\langle \text{test} \rangle\rangle_G^{[t]}, \mathcal{T}_G \} \right) \right\} \\ \langle\langle \text{path}_1 / \text{path}_2 \rangle\rangle_G^{[t]} &= \left\{ \langle o_1, o_3, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2 \rangle \mid \exists o_2: \langle o_1, o_2, \tau_1, d_1 \rangle \in \langle\langle \text{path}_1 \rangle\rangle_G^{[t]} \wedge \langle o_2, o_3, \tau_2, d_2 \rangle \in \langle\langle \text{path}_2 \rangle\rangle_G^{[t]} \wedge (\tau_1 + d_1) \cap \tau_2 \neq \emptyset \right\} \\ \langle\langle \text{path}_1 + \text{path}_2 \rangle\rangle_G^{[t]} &= \langle\langle \text{path}_1 \rangle\rangle_G^{[t]} \cup \langle\langle \text{path}_2 \rangle\rangle_G^{[t]} \\ \langle\langle \text{path}[m, n] \rangle\rangle_G^{[t]} &= \bigcup_{k=m}^n \langle\langle \text{path}^k \rangle\rangle_G^{[t]} \\ \langle\langle \text{path}[m, _] \rangle\rangle_G^{[t]} &= \bigcup_{k \geq m} \langle\langle \text{path}^k \rangle\rangle_G^{[t]} \end{aligned}$$

We observe that when q is of the form $(\text{path}_1 + \text{path}_2)$, $(\text{path}[m, _])$ and $(\text{path}[m, n])$, the definition of $\llbracket q \rrbracket_G^{[t]}$ is nearly identical to the one of $\llbracket q \rrbracket_G$. This will also be the case for the three representations below.

3.3.2 Correctness.

We start with a lemma:

LEMMA 3.2. *Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$ be a TPG and let q an expression for the symbol test in the grammar of Definition 3.1. Then:*

- *each tuples in $\llbracket q \rrbracket_G$ is of the form $\langle o_1, o_2, t, 0 \rangle$ for some o_1, o_2 and t ,*
- *each tuples in $\llbracket q \rrbracket_G^{[t]}$ is of the form $\langle o_1, o_2, \tau, 0 \rangle$ for some o_1, o_2 and τ .*

PROOF. Immediate from the definitions of $\llbracket q \rrbracket_G$ and $\llbracket q \rrbracket_G^{[t]}$. □

The following result states that the representation $\llbracket q \rrbracket_G^{[t]}$ is correct:

PROPOSITION 3.3. *Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$ be a TPG and q a TRPQ. Then the unfolding of $\llbracket q \rrbracket_G^{[t]}$ is $\llbracket q \rrbracket_G$.*

PROOF.

Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$ be a TG, and let q be a TRPQ.

We show below that:

- (I) for any $\langle o_1, o_2, t, d \rangle \in \llbracket q \rrbracket_G$, there is a $\tau \in \text{intv}(\mathcal{T})$ such that
 - (a) $\langle o_1, o_2, \tau, d \rangle \in \llbracket q \rrbracket_G^{[t]}$, and
 - (b) $t \in \tau$,
- (II) for any $\langle o_1, o_2, \tau, d \rangle \in \llbracket q \rrbracket_G^{[t]}$ for any $t \in \tau$,
 - $\langle o_1, o_2, t, d \rangle$ is in $\llbracket q \rrbracket_G$.

We proceed by induction on the structure of q .

If q is of the form pred , F , B , $(\text{test} \vee \text{test})$, $(\text{path} + \text{path})$, $\text{path}[m, n]$ or $\text{path}[m, _]$, then (I) and (II) immediately follow from the definitions of $\llbracket q \rrbracket_G$ and $\llbracket q \rrbracket_G^{[t]}$.

So we focus below on the five remaining cases:

- $q = T_\delta$.

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o, o, t, d \rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G \} \\ \llbracket q \rrbracket_G^{[t]} &= \{ \langle o, o, [t_1, t_1], t_2 - t_1 \rangle \mid o \in (N \cup E), t_1 \in \mathcal{T}_G, t_2 \in (\delta + t_1) \cap \mathcal{T}_G \} \end{aligned}$$

- For (I), let $\mathbf{v} = \langle o, o, t, t + d \rangle \in \llbracket q \rrbracket_G$.

And let $\mathbf{u} = \langle o, o, [t, t], d \rangle$ in $\mathcal{U}^{[t]}$.

For (Ia) we show that $\mathbf{u} \in \llbracket q \rrbracket_G^{[t]}$.

From $\mathbf{v} \in \llbracket q \rrbracket_G$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.

Besides, because $\mathbf{v} \in \llbracket q \rrbracket_G$ still,

$$t + d \in \mathcal{T}_G \tag{1}$$

and

$$d \in \delta \tag{2}$$

$$t + d \in t + \delta \tag{3}$$

So from (1) and (3)

$$t + d \in (\delta + t) \cap \mathcal{T}_G \tag{4}$$

So there is a t_2 (namely $t + d$) such that $d = t_2 - t$ and $t_2 \in t + \delta \cap \mathcal{T}_G$.

Together with the definition of $\llbracket q \rrbracket_G^{[t]}$, this implies $\mathbf{u} \in \llbracket q \rrbracket_G^{[t]}$, which concludes the proof for (Ia).

And trivially, $t \in [t, t]$, so (Ib) is verified as well.

- For (II), let $\mathbf{u} = \langle o, o, [t, t], d \rangle \in \llbracket q \rrbracket_G^{[t]}$.

From $\mathbf{u} \in \llbracket q \rrbracket_G^{[t]}$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.

So to conclude the proof, it is sufficient to show that (i) $d \in \delta$ and (ii) $t + d \in \mathcal{T}_G$.

Because $\mathbf{u} \in \llbracket q \rrbracket_G^{[t]}$ still, we have

$$d = t_2 - t \text{ for some } t_2 \in (\delta + t) \cap \mathcal{T}_G \quad (5)$$

From (5), we get $t_2 = t + d$.

Therefore from (5) still,

$$t + d \in (\delta + t) \cap \mathcal{T}_G \quad (6)$$

which proves (ii).

And from (6), we also get

$$\begin{aligned} t + d &\in \delta + t \\ t + d - t &\in (\delta + t) - t \\ d &\in \delta \end{aligned}$$

which proves (i).

- $q = \text{test}_1 \wedge \text{test}_2$.

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \llbracket \text{test}_1 \rrbracket_G \cap \llbracket \text{test}_2 \rrbracket_G \\ \llbracket q \rrbracket_G^{[t]} &= \{ \langle o, o, \tau_1 \cap \tau_2, 0 \rangle \mid \langle o, o, \tau_1, 0 \rangle \in \llbracket \text{test}_1 \rrbracket_G^{[t]}, \langle o, o, \tau_2, 0 \rangle \in \llbracket \text{test}_2 \rrbracket_G^{[t]}, \tau_1 \cap \tau_2 \neq \emptyset \} \end{aligned}$$

- For (I), let $\mathbf{v} = \langle o, o, t, d \rangle \in \llbracket q \rrbracket_G$.

From Lemma 3.2, $d = 0$.

And from the definition of $\llbracket q \rrbracket_G$, $\mathbf{v} \in \llbracket \text{test}_1 \rrbracket_G \cap \llbracket \text{test}_2 \rrbracket_G$.

So by IH, there are intervals τ_1 and τ_2 s.t. $\langle o, o, \tau_i, 0 \rangle \in \llbracket \text{test}_i \rrbracket_G^{[t]}$ for $i \in \{1, 2\}$ and $t \in \tau_1 \cap \tau_2$.

Together with the definition of $\llbracket q \rrbracket_G^{[t]}$, this proves (I).

- For (II), let $\langle o, o, \tau, d \rangle \in \llbracket q \rrbracket_G^{[t]}$.

Then from Lemma 3.2, $d = 0$.

And from the definition of $\llbracket q \rrbracket_G^{[t]}$, there are two intervals τ_1 and τ_2 s.t. $\tau = \tau_1 \cap \tau_2$ and $\langle o, o, \tau_i, 0 \rangle \in \llbracket \text{test}_i \rrbracket_G^{[t]}$ for $i \in \{1, 2\}$.

Now take any $t \in \tau$.

Then $t \in \tau_i$ for $i \in \{1, 2\}$.

So by IH, $\langle o, o, t, 0 \rangle \in \llbracket \text{test}_i \rrbracket_G$ for each $i \in \{1, 2\}$.

Together with the definition of $\llbracket q \rrbracket_G$, this proves (II).

- $q = (?path)$.

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o, o, t, 0 \rangle \mid \langle o, o', t, t + d \rangle \in \llbracket path \rrbracket_G \text{ for some } o' \in N \cup E, d \in \mathcal{T} \} \\ \llbracket q \rrbracket_G^{[t]} &= \{ \langle o, o, \tau, 0 \rangle \mid \langle o, o', \tau, d \rangle \in \llbracket path \rrbracket_G^{[t]} \text{ for some } o' \in N \cup E, d \in \mathcal{T} \} \end{aligned}$$

- For (I), let $\langle o, o, t, 0 \rangle \in \llbracket q \rrbracket_G$.

From the definition of $\llbracket q \rrbracket_G$, there are o' and d such that $\langle o, o', t, t + d \rangle \in \llbracket path \rrbracket_G$.

So by IH, there is a τ s.t. $t \in \tau$ and $\langle o, o', \tau, d \rangle \in \llbracket path \rrbracket_G^{[t]}$.

Therefore $\langle o, o, \tau, 0 \rangle \in \llbracket q \rrbracket_G^{[t]}$, from the definition of $\llbracket q \rrbracket_G^{[t]}$.

- For (II), let $\langle o, o, \tau, 0 \rangle \in \llbracket q \rrbracket_G^{[t]}$.

From the definition of $\llbracket q \rrbracket_G^{[t]}$, there are o' and d s.t. $\langle o, o', \tau, d \rangle \in \llbracket path \rrbracket_G^{[t]}$.

Now take any $t \in \tau$.

By IH, $\langle o, o', t, t + d \rangle \in \llbracket path \rrbracket_G$.

Therefore $\langle o, o, t, 0 \rangle \in \llbracket q \rrbracket_G$, from the definition of $\llbracket q \rrbracket_G$.

- $q = \neg\text{test}$.

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= (\{\langle o, o \rangle \mid o \in N \cup E\} \times \mathcal{T}_G \times \{0\}) \setminus \llbracket \text{test} \rrbracket_G \\ \langle q \rangle_G^{[t]} &= \bigcup_{o \in N \cup E} \left\{ \langle o, o, \tau, 0 \rangle \mid \tau \in \text{compl} \left(\{\tau' \mid \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]} \}, \mathcal{T}_G \right) \right\} \end{aligned}$$

- For (I), let $\mathbf{v} = \langle o, o, t, 0 \rangle \in \llbracket q \rrbracket_G$.
From the definition of $\llbracket q \rrbracket_G$, $\mathbf{v} \notin \llbracket \text{test} \rrbracket_G$.
So

$$t \notin \{\tau' \mid \langle o, o, \tau', 0 \rangle \in \llbracket \text{test} \rrbracket_G\} \quad (7)$$

Now by IH, together with Lemma 3.2, we get:

$$\langle o, o, t, 0 \rangle \in \llbracket \text{test} \rrbracket_G \text{ iff } t' \in \tau' \text{ for some } \tau' \text{ s.t. } \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]} \quad (8)$$

So from (7) and (8):

$$t \notin \bigcup \{\tau' \mid \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]}\}$$

Therefore

$$t \in \mathcal{T}_G \setminus \bigcup \{\tau' \mid \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]}\} \quad (9)$$

So $t \in \tau$ for some $\tau \in \text{compl} \left(\bigcup \{\tau' \mid \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]} \}, \mathcal{T}_G \right)$.

And $\langle o, o, \tau, 0 \rangle \in \langle q \rangle_G^{[t]}$, from the definition of $\langle q \rangle_G^{[t]}$.

- For (II), let $\langle o, o, \tau, 0 \rangle \in \langle q \rangle_G^{[t]}$.
And take any $t \in \tau$.

From the definition of $\langle q \rangle_G^{[t]}$:

$$t \in \mathcal{T}_G \setminus \bigcup \{\tau' \mid \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]}\}$$

Together with (8), this implies

$$\langle o, o, t, 0 \rangle \notin \llbracket \text{test} \rrbracket_G$$

Therefore $\langle o, o, t, 0 \rangle \in \llbracket q \rrbracket_G$, from the definition of $\llbracket q \rrbracket_G$.

- $q = \text{path}_1 / \text{path}_2$.

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o_1, o_3, t, d_1 + d_2 \rangle \mid \exists o_2: \langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G \wedge \langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G \} \\ \langle q \rangle_G^{[t]} &= \left\{ \langle o_1, o_3, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2 \rangle \mid \exists o_2: \langle o_1, o_2, \tau_1, d_1 \rangle \in \langle \text{path}_1 \rangle_G^{[t]} \wedge \langle o_2, o_3, \tau_2, d_2 \rangle \in \langle \text{path}_2 \rangle_G^{[t]} \wedge \right. \\ &\quad \left. (\tau_1 + d_1) \cap \tau_2 \neq \emptyset \right\} \end{aligned}$$

- For (I), let $\mathbf{v} = \langle o_1, o_3, t, d \rangle \in \llbracket q \rrbracket_G$.
From the definition of $\llbracket q \rrbracket_G$, there are o_2, d_1 and d_2 such that $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$, $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$ and $d = d_1 + d_2$.
By IH, because $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$, there is a τ_1 such that $t \in \tau_1$ and

$$\langle o_1, o_2, \tau_1, d_1 \rangle \in \langle \text{path}_1 \rangle_G^{[t]} \quad (10)$$

And similarly, because $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$, there is a τ_2 such that $t + d_1 \in \tau_2$ and

$$\langle o_2, o_3, \tau_2, d_2 \rangle \in \langle \text{path}_2 \rangle_G^{[t]} \quad (11)$$

From $t \in \tau_1$, we get

$$t + d_1 \in \tau_1 + d_1 \quad (12)$$

Together with the fact that $t + d_1 \in \tau_2$, this implies

$$\tau_1 + d_1 \cap \tau_2 \neq \emptyset \quad (13)$$

So from (10), (11), (13) and the definition of $\langle q \rangle_G^{[t]}$,

$$\langle o_1, o_2, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2 \rangle \in \langle q \rangle_G^{[t]}$$

which proves (Ia).

And in order to prove (Ib), we only need to show that

$$t \in ((\tau_1 + d_1) \cap \tau_2) - d_1$$

We know that $t \in \tau_1$, therefore

$$t + d_1 \in \tau_1 + d_1$$

Together with the fact that $t + d_1 \in \tau_2$, this yields

$$\begin{aligned} t + d_1 &\in (\tau_1 + d_1) \cap \tau_2 \\ t &\in ((\tau_1 + d_1) \cap \tau_2) - d_1 \end{aligned}$$

– For (II), let $\mathbf{u} = \langle o_1, o_3, \tau, d \rangle \in \llbracket q \rrbracket_G^{[t]}$, and let $t \in \tau$.

We show that $\langle o_1, o_3, t, t + d \rangle \in \llbracket q \rrbracket_G$.

Because $\mathbf{u} \in \llbracket q \rrbracket_G^{[t]}$, from the definition of $\llbracket q \rrbracket_G^{[t]}$, there are τ_1, τ_2, d_1, d_2 and o_2 s.t.:

- (i) $d = d_1 + d_2$
- (ii) $\tau = ((\tau_1 + d_1) \cap \tau_2) - d_1$
- (iii) $\langle o_1, o_2, \tau_1, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G^{[t]}$
- (iv) $\langle o_2, o_3, \tau_2, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G^{[t]}$

Since $t \in \tau$, from (ii), we have

$$t \in ((\tau_1 + d_1) \cap \tau_2) - d_1 \quad (14)$$

$$t + d_1 \in (((\tau_1 + d_1) \cap \tau_2) - d_1) + d_1 \quad (15)$$

$$t + d_1 \in (\tau_1 + d_1) \cap \tau_2 \quad (16)$$

$$t + d_1 \in \tau_1 + d_1 \quad (17)$$

$$t \in \tau_1 \quad (18)$$

From (iii), by IH, for any $t' \in \tau_1$

$$\langle o_1, o_2, t' + d_1 \rangle \in \llbracket q \rrbracket_G$$

In particular, from (18)

$$\langle o_1, o_2, t, t + d_1 \rangle \in \llbracket q \rrbracket_G \quad (19)$$

And from (iv), by IH, for any $t'' \in \tau_2$

$$\langle o_2, o_3, t'', t'' + d_2 \rangle \in \llbracket q \rrbracket_G$$

In particular, from (16)

$$\langle o_2, o_3, t + d_1, (t + d_1) + d_2 \rangle \in \llbracket q \rrbracket_G \quad (20)$$

So from (19), (20) and the definition of $\llbracket q \rrbracket_G$

$$\langle o_1, o_3, t, t + d_1 + d_2 \rangle \in \llbracket q \rrbracket_G$$

□

3.4 In $\mathcal{U}^{[d]}$

3.4.1 Definition.

We start with the case where q is an expression for the symbol test in the grammar of Definition 3.1.

The definitions of $\llbracket \text{pred} \rrbracket_G^{[t]}$ and $\llbracket \neg \text{test} \rrbracket_G^{[t]}$ are already provided in the article, we reproduce them here for completeness:

$$\begin{aligned} \llbracket \text{pred} \rrbracket_G^{[d]} &= \{ \langle o, o, t, [0, 0] \rangle \mid t \in \tau \text{ for some } \tau \in \text{val}(o, \text{pred}) \} \\ \llbracket \text{F} \rrbracket_G^{[d]} &= \{ \langle v, e, t, [0, 0] \rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G \} \cup \{ \langle e, v, t, [0, 0] \rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G \} \\ \llbracket \text{B} \rrbracket_G^{[d]} &= \{ \langle v, e, t, [0, 0] \rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G \} \cup \{ \langle e, v, t, [0, 0] \rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G \} \\ \llbracket (? \text{path}) \rrbracket_G^{[d]} &= \{ \langle o, o, t, [0, 0] \rangle \mid \exists o', \delta: \langle o, o', t, \delta \rangle \in \llbracket \text{path} \rrbracket_G^{[d]} \} \\ \llbracket \text{test}_1 \vee \text{test}_2 \rrbracket_G^{[d]} &= \llbracket \text{test}_1 \rrbracket_G^{[d]} \cup \llbracket \text{test}_2 \rrbracket_G^{[d]} \\ \llbracket \text{test}_1 \wedge \text{test}_2 \rrbracket_G^{[d]} &= \llbracket \text{test}_1 \rrbracket_G^{[d]} \cap \llbracket \text{test}_2 \rrbracket_G^{[d]} \\ \llbracket \neg \text{test} \rrbracket_G^{[d]} &= \left\{ \langle o, o, t, [0, 0] \rangle \mid o \in N \cup E, t \in \mathcal{T}_G \setminus \{ t' \mid \langle o, o, t', [0, 0] \rangle \in \llbracket \text{test} \rrbracket_G^{[d]} \} \right\} \end{aligned}$$

Next, we consider the operators $(\text{path}_1 + \text{path}_2)$, $(\text{path}[m, _])$ and $(\text{path}[m, n])$.

For these cases, $\llbracket q \rrbracket_G^{[t, d]}$ is once again defined analogously to $\llbracket q \rrbracket_G$, in terms of temporal join (a.k.a. $\text{path}_1/\text{path}_2$) and set union.

We only write the definitions here for the sake of completeness:

$$\begin{aligned} \langle \text{path}_1 + \text{path}_2 \rangle_G^{[t,d]} &= \langle \text{path}_1 \rangle_G^{[t,d]} \cup \langle \text{path}_2 \rangle_G^{[t,d]} \\ \llbracket \text{path}[m, n] \rrbracket_G &= \bigcup_{k=m}^n \langle \text{path}^k \rangle_G^{[t,d]} \\ \llbracket \text{path}[m, _] \rrbracket_G &= \bigcup_{k \geq m} \langle \text{path}^k \rangle_G^{[t,d]} \end{aligned}$$

The only remaining operators are temporal join ($\text{path}_1/\text{path}_2$) and temporal navigation (T_δ), already defined in the article. We reproduce here these two definition for convenience:

$$\begin{aligned} \langle \text{path}_1/\text{path}_2 \rangle_G^{[d]} &= \left\{ \langle o_1, o_3, t_1, \delta_2 + t_2 - t_1 \rangle \mid \exists o_2: \langle o_1, o_2, t_1, \delta_1 \rangle \in \langle \text{path}_1 \rangle_G^{[d]} \wedge \langle o_2, o_3, t_2, \delta_2 \rangle \in \langle \text{path}_2 \rangle_G^{[d]} \wedge t_2 \in t_1 + \delta_1 \right\} \\ \langle \text{T}_\delta \rangle_G^{[d]} &= \left\{ \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t \rangle \mid o \in N \cup E, t \in \mathcal{T}_G, (\delta + t) \cap \mathcal{T}_G \neq \emptyset \right\} \end{aligned}$$

We also reproduce the alternative characterization of $\langle \text{T}_\delta \rangle_G^{[d]}$ provided in the article, as a unary operator:

$$\langle q/\text{T}_\delta \rangle_G^{[d]} = \{ \langle o_1, o_2, t, (\delta' + \delta) \cap \mathcal{T}_G \rangle \mid \langle o_1, o_2, t, \delta' \rangle \in \langle q \rangle_G^{[d]}, (t + (\delta' + \delta)) \cap \mathcal{T}_G \neq \emptyset \}$$

3.4.2 Correctness.

The following result states that the representation $\langle q \rangle_G^{[d]}$ is correct:

PROPOSITION 3.4. *Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$ be a TPG and q a TRPQ. Then the unfolding of $\langle q \rangle_G^{[d]}$ is $\llbracket q \rrbracket_G$.*

PROOF.

Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$ be a TG, and let q be a TRPQ.

We show below that:

- (I) for any $\langle o_1, o_2, t, d \rangle \in \llbracket q \rrbracket_G$, there is a $\delta \in \text{intv}(\mathcal{T})$ such that
 - (a) $\langle o_1, o_2, t, \delta \rangle \in \langle q \rangle_G^{[d]}$, and
 - (b) $d \in \delta$,
- (II) for any $\langle o_1, o_2, t, \delta \rangle \in \langle q \rangle_G^{[d]}$ for any $d \in \delta$, $\langle o_1, o_2, t, d \rangle$ is in $\llbracket q \rrbracket_G$.

We proceed once again by induction on the structure of q .

If q is of the form *pred*, *F*, *B*, (*test* \vee *test*), (*test* \wedge *test*), \neg *test*, (*path* + *path*), *path*[*m*, *n*] or *path*[*m*, $_$], then (I) and (II) immediately follow from the definitions of $\llbracket q \rrbracket_G$ and $\langle q \rangle_G^{[d]}$.

If q is of the form $(? \text{path})$, then the proof is nearly identical to one already provided for $\langle (? \text{path}) \rangle_G^{[t]}$.

So we focus below on the two remaining cases:

- $q = \text{T}_\delta$.

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o, o, t, d \rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G \} \\ \langle q \rangle_G^{[d]} &= \{ \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t \rangle \mid o \in N \cup E, t \in \mathcal{T}_G, (\delta + t) \cap \mathcal{T}_G \neq \emptyset \} \end{aligned}$$

- For (I), let $\mathbf{v} = \langle o, o, t, d \rangle \in \llbracket q \rrbracket_G$.
And let $\mathbf{u} = \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t \rangle$ in $\mathcal{U}^{[d]}$.
For (Ia) we show that $\mathbf{u} \in \langle q \rangle_G^{[d]}$.
From $\mathbf{v} \in \llbracket q \rrbracket_G$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.
Besides, because $\mathbf{v} \in \llbracket q \rrbracket_G$ still,

$$t + d \in \mathcal{T}_G \tag{21}$$

and

$$d \in \delta \tag{22}$$

$$t + d \in t + \delta \tag{23}$$

So from (21) and (23)

$$t + d \in (\delta + t) \cap \mathcal{T}_G \tag{24}$$

$$(\delta + t) \cap \mathcal{T}_G \neq \emptyset \tag{25}$$

Together with the definition of $\langle q \rangle_G^{[d]}$, this implies $\mathbf{u} \in \langle q \rangle_G^{[d]}$, which concludes the proof for (Ia).
Finally, from (24), we get

$$t + d - t \in ((\delta + t) \cap \mathcal{T}_G) - t \quad (26)$$

$$d \in ((\delta + t) \cap \mathcal{T}_G) - t \quad (27)$$

which proves (Ib).

- For (II), let $\mathbf{u} = \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t \rangle \in \langle q \rangle_G^{[d]}$, and let $d \in ((\delta + t) \cap \mathcal{T}_G) - t$.

From $\mathbf{u} \in \langle q \rangle_G^{[d]}$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.

So to conclude the proof, it is sufficient to show that (i) $d \in \delta$ and (ii) $t + d \in \mathcal{T}_G$.

By assumption, we have

$$d \in ((\delta + t) \cap \mathcal{T}_G) - t \quad (28)$$

$$d + t \in (\delta + t) \cap \mathcal{T}_G \quad (29)$$

$$d + t \in \mathcal{T}_G \quad (30)$$

which proves (ii).

And from (29), we also get

$$d + t \in \delta + t$$

$$d + t - t \in (\delta + t) - t$$

$$d \in \delta$$

which proves (i).

- $q = \text{path}_1/\text{path}_2$.

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o_1, o_3, t, d_1 + d_2 \rangle \mid \exists o_2: \langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G \wedge \langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G \} \\ \langle q \rangle_G^{[d]} &= \left\{ \langle o_1, o_3, t_1, \delta_2 + t_2 - t_1 \rangle \mid \exists o_2: \langle o_1, o_2, t_1, \delta_1 \rangle \in \langle \text{path}_1 \rangle_G^{[d]} \wedge \langle o_2, o_3, t_2, \delta_2 \rangle \in \langle \text{path}_2 \rangle_G^{[d]} \wedge t_2 \in t_1 + \delta_1 \right\} \end{aligned}$$

- For (I), let $\mathbf{v} = \langle o_1, o_3, t, d \rangle \in \llbracket q \rrbracket_G$.

From the definition of $\llbracket q \rrbracket_G$, there are o_2, d_1 and d_2 such that $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$, $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$ and $d = d_1 + d_2$.

By IH, because $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$, there is a δ_1 such that $d_1 \in \delta_1$ and

$$\langle o_1, o_2, t, \delta_1 \rangle \in \langle \text{path}_1 \rangle_G^{[d]} \quad (31)$$

And similarly, because $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$, there is a δ_2 such that $d_2 \in \delta_2$ and

$$\langle o_2, o_3, t + d_1, \delta_2 \rangle \in \langle \text{path}_2 \rangle_G^{[d]} \quad (32)$$

Next, since $d \in \delta_1$

$$t + d_1 \in t + \delta_1 \quad (33)$$

So from (31), (32), (33) and the definition of $\langle q \rangle_G^{[d]}$ (replacing t_1 with t and t_2 with $t + d_1$), we get

$$\langle o_1, o_2, t, \delta_2 + (t + d_1) - t \rangle \in \langle q \rangle_G^{[d]}$$

which proves (Ia).

And in order to prove (Ib), we only need to show that $d \in \delta_2 + (t + d_1) - t$, or in other words that

$$d \in \delta_2 + d_1$$

We know that

$$d_2 \in \delta_2 \quad (34)$$

$$d_2 + d_1 \in \delta_2 + d_1 \quad (35)$$

Together with the fact that $d = d_1 + d_2$, this concludes the proof for (Ib).

- For (II), let $\mathbf{u} = \langle o_1, o_3, t_1, \delta \rangle \in \langle q \rangle_G^{[d]}$, and let $d \in \delta$.

Because $\mathbf{u} \in \langle q \rangle_G^{[d]}$, from the definition of $\langle q \rangle_G^{[d]}$, there are δ_1, δ_2, t_2 and o_2 s.t.:

$$(i) \quad \delta = \delta_2 + t_2 - t_1$$

$$(ii) \quad t_2 \in t_1 + \delta_1$$

$$(iii) \quad \langle o_1, o_2, t_1, \delta_1 \rangle \in \langle \text{path}_1 \rangle_G^{[d]}$$

(iv) $\langle o_2, o_3, t_2, \delta_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G^{[d]}$
 From (i) and (ii), we get

$$\begin{aligned}\delta &= \delta_2 + (t_1 + \delta_1) - t_1 \\ &= \delta_2 + \delta_1\end{aligned}$$

Together with $d \in \delta$, this implies that there are $d_1 \in \delta_1$ and $d_2 \in \delta_2$ such that $d = d_1 + d_2$.

Next, because $d_1 \in \delta_1$, from (iii), by IH

$$\langle o_1, o_2, t_1, t_1 + d_1 \rangle \in \llbracket q \rrbracket_G \quad (36)$$

And similarly, because $d_2 \in \delta_2$, from (iv)

$$\langle o_2, o_3, t_2, t_2 + d_2 \rangle \in \llbracket q \rrbracket_G \quad (37)$$

So from (36), (37) and the definition of $\llbracket q \rrbracket_G$

$$\langle o_1, o_3, t_1, d_1 + d_2 \rangle \in \llbracket q \rrbracket_G \quad (38)$$

Together with the fact that $d = d_1 + d_2$, this concludes the proof for (II). \square

3.5 In $\mathcal{U}^{[t,d]}$

3.5.1 Definition.

We start with the case where q is an expression for the symbol test in the grammar of Definition 3.1.

As a consequence of Lemma 3.2, $\llbracket q \rrbracket_G^{[t,d]}$ can be trivially defined out of $\llbracket q \rrbracket_G^{[t]}$ by replacing the distance 0 with the interval $[0, 0]$, i.e.

$$\llbracket \text{test} \rrbracket_G^{[t,d]} = \{ \langle o, o, \tau, [0, 0] \rangle \mid \{ \langle o, o, \tau, 0 \rangle \in \llbracket \text{test} \rrbracket_G^{[t]} \}$$

Next, if q is of the form $(\text{path}_1 + \text{path}_2)$, $(\text{path}[m, _])$ or $(\text{path}[m, n])$, then the definition of $\llbracket q \rrbracket_G^{[t,d]}$ is once again nearly identical to the one of $\llbracket q \rrbracket_G$:

$$\begin{aligned}\llbracket \text{path}_1 + \text{path}_2 \rrbracket_G^{[t,d]} &= \llbracket \text{path}_1 \rrbracket_G^{[t,d]} \cup \llbracket \text{path}_2 \rrbracket_G^{[t,d]} \\ \llbracket \text{path}[m, n] \rrbracket_G &= \bigcup_{k=m}^n \llbracket \text{path}^k \rrbracket_G^{[t,d]} \\ \llbracket \text{path}[m, _] \rrbracket_G &= \bigcup_{k \geq m} \llbracket \text{path}^k \rrbracket_G^{[t,d]}\end{aligned}$$

The only remaining operators are temporal join $(\text{path}_1/\text{path}_2)$ and temporal navigation (T_δ) , already defined in the article, and reproduced here for convenience:

$$\begin{aligned}\llbracket \text{path}_1/\text{path}_2 \rrbracket_G^{[t,d]} &= \bigcup \{ \mathbf{u}_1 \bowtie \mathbf{u}_2 \mid \mathbf{u}_1 \in \llbracket \text{path}_1 \rrbracket_G^{[t,d]}, \mathbf{u}_2 \in \llbracket \text{path}_2 \rrbracket_G^{[t,d]} \} \\ \llbracket T_\delta \rrbracket_G^{[t,d]} &= \bigcup_{o \in N \cup E} \{ \langle o, o, \mathcal{T}_G, \delta \rangle \bowtie \langle o, o, \mathcal{T}_G, [0, 0] \rangle \}\end{aligned}$$

where $\mathbf{u}_1 \bowtie \mathbf{u}_2$ is defined as follows.

Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2 \rangle$.

Define τ'_2 as

$$\tau'_2 = (\tau_1 + \delta_1) \cap \tau_2$$

If $o_2 \neq o_3$ or $\tau \tau'_2 = \emptyset$, then $\mathbf{u}_1 \bowtie \mathbf{u}_2 = \emptyset$.

Otherwise, let:

$$\begin{aligned}\tau &= (\tau'_2 \ominus \delta_1) \cap \tau_1 \\ b &= b_{\tau'_2} - b_{\tau_1} \\ e &= e_{\tau'_2} - e_{\tau_1}\end{aligned}$$

And for every $t \in \tau$, let

$$\delta(t) = \delta_1 \lfloor b_{\delta_1} + \max(0, b - t), e_{\delta_1} - \max(0, t - e) \rfloor_{\delta_1}$$

Then

$$\mathbf{u}_1 \bowtie \mathbf{u}_2 = \{ \langle o_1, o_4, [t, t], \delta(t) + \delta_2 \rangle \mid t \in \tau \}$$

3.5.2 Correctness.

We start with a lemma

LEMMA 3.5. *Let $\alpha, \beta \in \text{intv}(\mathcal{T})$. Then*

$$\beta \ominus \alpha = \{t \mid (t + \alpha) \cap \beta \neq \emptyset\}$$

Next, if $\mathbf{u} = \langle o_1, o_2, \tau, \delta \rangle \in \mathcal{U}^{[t,d]}$, we call *temporal relation induced by \mathbf{u}* the set $\{(t, t + d) \mid t \in \tau, d \in \delta\}$.

We also define the binary operator \bowtie : $(\mathcal{T} \times \mathcal{T}) \times (\mathcal{T} \times \mathcal{T}) \rightarrow (\mathcal{T} \times \mathcal{T})$ as in the article, i.e.

$$R_1 \bowtie R_2 = \{t_1, t_3 \mid (t_1, t_2) \in R_1 \text{ and } (t_2, t_3) \in R_2 \text{ for some } t_2\}$$

We can now formulate the following lemma:

LEMMA 3.6. *Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2 \rangle$ be two tuples in $\mathcal{U}^{[t,d]}$ such that $o_2 = o_3$. And for $i \in \{1, 2\}$, let R_i denote the temporal relation induced by \mathbf{u}_i . Then*

$$R_1 \bowtie R_2 = \bigcup_{\langle o_1, o_2, \tau, \delta \rangle \in \mathbf{u}_1 \bowtie \mathbf{u}_2} \{(t, t + d) \mid t \in \tau, d \in \delta\}$$

The following result states that the representation $\langle q \rangle_G^{[t,d]}$ is correct:

PROPOSITION 3.7. *Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$ be a TPG and q a TRPQ. Then the unfolding of $\langle q \rangle_G^{[t,d]}$ is $\llbracket q \rrbracket_G$.*

PROOF.

Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$ be a TG, and let q be a TRPQ.

We show below that:

- (I) for any $\langle o_1, o_2, t, d \rangle \in \llbracket q \rrbracket_G$, there are $\tau, \delta \in \text{intv}(\mathcal{T})$ such that
 - (a) $\langle o_1, o_2, \tau, \delta \rangle \in \langle q \rangle_G^{[t,d]}$,
 - (b) $t \in \tau$, and
 - (c) $d \in \delta$.
- (II) for any $\langle o_1, o_2, \tau, \delta \rangle \in \langle q \rangle_G^{[t,d]}$ for any $(t, d) \in \tau \times \delta$, $\langle o_1, o_2, t, d \rangle$ is in $\llbracket q \rrbracket_G$.

We proceed once again by induction on the structure of q .

If q is of the form *pred*, *F*, *B*, (*test* \vee *test*), (*path* $+$ *path*), *path*[m, n] or *path*[$m, _$], then (I) and (II) immediately follow from the definitions of $\llbracket q \rrbracket_G$ and $\langle q \rangle_G^{[t,d]}$.

If q is of the form *test* \wedge *test*, \neg *test* or (*?path*), then the proof is nearly identical to the one already provided for $\langle q \rangle_G^{[t]}$.

So we focus below on the two remaining cases:

- $q = \text{path}_1 / \text{path}_2$.

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o_1, o_3, t, d_1 + d_2 \rangle \mid \exists o_2: \langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G \wedge \langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G \} \\ \langle \text{path}_1 / \text{path}_2 \rangle_G^{[t,d]} &= \bigcup \{ \mathbf{u}_1 \bowtie \mathbf{u}_2 \mid \mathbf{u}_1 \in \langle \text{path}_1 \rangle_G^{[t,d]}, \mathbf{u}_2 \in \langle \text{path}_2 \rangle_G^{[t,d]} \} \end{aligned}$$

- For (I), let $\mathbf{v} = \langle o_1, o_3, t, d \rangle \in \llbracket q \rrbracket_G$.

From the definition of $\llbracket q \rrbracket_G$, there are o_2, d_1 and d_2 such that $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$, $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$ and $d = d_1 + d_2$.

By IH, because $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$, there are τ_1 and δ_1 such that $t \in \tau_1$, $d_1 \in \delta_1$ and

$$\langle o_1, o_2, \tau_1, \delta_1 \rangle \in \langle \text{path}_1 \rangle_G^{[t,d]} \quad (39)$$

Let R_1 be the temporal relation induced by this tuple $\langle o_1, o_2, \tau_1, \delta_1 \rangle$.

Since $t \in \tau_1$ and $d_1 \in \delta_1$, we have

$$(t, t + d_1) \in R_1 \quad (40)$$

Similarly, because $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$, there are τ_2 and δ_2 such that $t + d_1 \in \tau_2$, $d_2 \in \delta_2$ and

$$\langle o_2, o_3, \tau_2, \delta_2 \rangle \in \langle \text{path}_2 \rangle_G^{[t,d]} \quad (41)$$

Let R_2 be the temporal relation induced by this tuple $\langle o_2, o_3, \tau_2, \delta_2 \rangle$.

Since $t + d_1 \in \tau_2$ and $d_2 \in \delta_2$, we have

$$(t + d_1, t + d_1 + d_2) \in R_2 \quad (42)$$

So from (40), (42) and Lemma 3.6, there are τ and δ such that $\langle o_1, o_3, \tau, \delta \rangle \in \mathbf{u}_1 \bowtie \mathbf{u}_2$, $t \in \tau$ and $d_1 + d_2 = d \in \delta$, which concludes the proof for (I).

- For (II), let $\mathbf{u} = \langle o_1, o_3, t_1, \delta \rangle \in \llbracket q \rrbracket_G^{[t,d]}$, and let $(t, d) \in \tau \times \delta$.
Because $\mathbf{u} \in \llbracket q \rrbracket_G^{[t,d]}$, from the definition of $\llbracket q \rrbracket_G^{[t,d]}$, there are \mathbf{u}_1 and \mathbf{u}_2 s.t.:

- (i) $\mathbf{u} \in \mathbf{u}_1 \bowtie \mathbf{u}_2$
- (ii) $\mathbf{u}_1 \in \llbracket \text{path}_1 \rrbracket_G^{[t,d]}$
- (iii) $\mathbf{u}_2 \in \llbracket \text{path}_2 \rrbracket_G^{[t,d]}$

Let R_i be the temporal relation induced by \mathbf{u}_i for $i \in \{1, 2\}$.

From (i), and Lemma 3.6,

$$(t, t + d) \in R_1 \bowtie R_2 \quad (43)$$

Now let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$ and $\mathbf{u}_2 = \langle o_2, o_3, \tau_2, \delta_2 \rangle$ for some $o_2, \tau_1, \tau_2, \delta_1$ and δ_2 .

From (43) and the definition of \bowtie , there must be d_1 and d_2 s.t. $d = d_1 + d_2$, $t \in \tau_1$, $d_1 \in \delta_1$, $t + d_1 \in \tau_2$ and $d_2 \in \delta_2$.

So from (ii), and (iii), by IH

$$\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G \quad (44)$$

$$\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G \quad (45)$$

So from (44), (45) and the definition of $\llbracket q \rrbracket_G$

$$\langle o_1, o_3, t, d_1 + d_2 \rangle \in \llbracket q \rrbracket_G,$$

which concludes the proof for (II). \square

3.6 In $\mathcal{U}^{[t,d],b,e}$

3.6.1 Definition.

If q is an expression for the symbol test in the grammar of Definition ??, then the definition of $\llbracket q \rrbracket_G^{[t,d],b,e}$ is nearly identical to the one of $\llbracket q \rrbracket_G^{[t,d]}$, extending each tuple $\{\langle o, o, \tau, [0, 0] \rangle\}$ with b_τ and e_τ , i.e.

$$(\text{test})_G^{[t,d],b,e} = \{ \langle o, o, \tau, [0, 0], b_\tau, e_\tau \rangle \mid \langle o, o, \tau, [0, 0] \rangle \in (\text{test})_G^{[t,d]} \}$$

Next, if q is of the form $(\text{path}_1 + \text{path}_2)$, $(\text{path}[m, _])$ or $(\text{path}[m, n])$, then the definition of $\llbracket q \rrbracket_G^{[t,d]}$ is once again nearly identical to the one of $\llbracket q \rrbracket_G$:

$$\begin{aligned} \llbracket \text{path}_1 + \text{path}_2 \rrbracket_G^{[t,d],b,e} &= \llbracket \text{path}_1 \rrbracket_G^{[t,d],b,e} \cup \llbracket \text{path}_2 \rrbracket_G^{[t,d],b,e} \\ \llbracket \text{path}[m, n] \rrbracket_G &= \bigcup_{k=m}^n \llbracket \text{path}^k \rrbracket_G^{[t,d],b,e} \\ \llbracket \text{path}[m, _] \rrbracket_G &= \bigcup_{k \geq m} \llbracket \text{path}^k \rrbracket_G^{[t,d],b,e} \end{aligned}$$

So the only remaining operator are temporal join $(\text{path}_1/\text{path}_2)$ and temporal navigation (T_δ) , already defined in the article. We reproduce here these two definition for convenience:

$$\begin{aligned} \llbracket \text{path}_1/\text{path}_2 \rrbracket_G^{[t,d],b,e} &= \{ \mathbf{u}_1 \bowtie \mathbf{u}_2 \mid \mathbf{u}_1 \in \llbracket \text{path}_1 \rrbracket_G^{[t,d],b,e}, \mathbf{u}_2 \in \llbracket \text{path}_2 \rrbracket_G^{[t,d],b,e}, \mathbf{u}_1 \sim \mathbf{u}_2 \} \\ \llbracket \text{T}_\delta \rrbracket_G^{[t,d],b,e} &= \{ \langle o, o, \mathcal{T}_G, \delta, b_{\mathcal{T}_G}, e_{\mathcal{T}_G} \rangle \bowtie \langle o, o, \mathcal{T}_G, [0, 0], b_{\mathcal{T}_G}, e_{\mathcal{T}_G} \rangle \mid o \in N \cup E \} \end{aligned}$$

where $\mathbf{u}_1 \sim \mathbf{u}_2$ and $\mathbf{u}_1 \bowtie \mathbf{u}_2$ are defined by:

$$\begin{aligned} \langle o_1, o_2, \tau_1, \delta_1, b_1, e_1 \rangle &\sim \langle o_3, o_4, \tau_2, \delta_2, e_2, b_2 \rangle \text{ iff } o_2 = o_3 \\ \langle o_1, o_3, \tau_1, \delta_1, b_1, e_1 \rangle &\bowtie \langle o_3, o_4, \tau_2, \delta_2, e_2, b_2 \rangle = \langle o_1, o_4, \tau, \delta_1 + \delta_2, b, e \rangle \\ &\text{with} \\ \tau &= ((\tau_1 + \delta_1) \cap \tau_2) \ominus \delta_1 \cap \tau_1 \\ b &= \max(b_1, b_2 - b_{\delta_1}) \\ e &= \min(e_1, e_2 - e_{\delta_1}) \end{aligned}$$

3.6.2 *Correctness.* Similarly to what we did above for $\mathcal{U}^{[t,d]}$, if $\mathbf{u} = \langle o_1, o_2, \tau, \delta, b, e \rangle \in \mathcal{U}^{[t,d],b,e}$, we call *temporal relation induced by \mathbf{u}* the set $\{(t, t + d) \mid t \in \tau, d \in \delta(t)\}$.

We can now formulate a lemma analogous to Lemma 3.6:

LEMMA 3.8. Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}^{[t,d],b,e}$, and for $i \in \{1, 2\}$, let R_i denote the temporal relation induced by \mathbf{u}_i . If $\mathbf{u}_1 \sim \mathbf{u}_2$ and $\mathbf{u}_1 \bowtie \mathbf{u}_2 = \langle o_1, o_3, \tau, \delta, b, e \rangle$, then

$$R_1 \bowtie R_2 = \{(t, t + d) \mid t \in \tau, d \in \delta(t)\}$$

PROOF. Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$ and $\mathbf{u}_2 = \langle o_2, o_3, \tau_2, \delta_2 \rangle$. As explained in Section 2, for $i \in \{1, 2\}$ and $t \in \tau_i$, we use $\delta_i(t)$ for the interval

$$\delta_i \downarrow b_{\delta_i} + \max(0, b_i - t), e_{\delta_i} - \max(0, t - e_i) \downarrow \delta_i$$

We need to prove that (i) $\tau = \text{dom}(R_1 \bowtie R_2)$ and that (ii) for each $t \in \tau$,

$$t + \delta(t) = \{t' \mid (t, t') \in R_1 \bowtie R_2\}$$

The proof of (i) is nearly identical to the one provided above for Lemma 3.6.

For (ii), let $t \in \tau$. We use

- a for the least value s.t. $(t, a) \in \text{range}(R_1) \cap \text{dom}(R_2)$, and
- a' for the least value s.t. $(a, a') \in R_2$

Then a' is also the least value s.t. $(t, a') \in R_1 \bowtie R_2$.

Analogously, we use z for the greatest value s.t. $(t, z) \in \text{range}(R_1) \cap \text{dom}(R_2)$, and z' for the greatest value s.t. $(z, z') \in R_2$.

Then z' is also the greatest value s.t. $(t, z') \in R_1 \bowtie R_2$.

From Lemma ??:

- $\{t\} \times (t + [a, z]) \subseteq R_1$, and
- $[a, b] \times [a', z'] \subseteq R_2$

Therefore $[a', z'] = \{c \mid (t, c) \in R_1 \bowtie R_2\}$.

To conclude the proof, we show that $t + \delta_t = [a, z]$.

We only prove that $t + b_{\delta_t} = a$ (the proof that $t + e_{\delta_t} = z$ is symmetric).

Following the definition of b , we consider 2 cases:

- (1) $b_1 < b_2 - b_{\delta_1}$
- (2) $b_1 \geq b_2 - b_{\delta_1}$

For Case (1), we get

$$b_1 < b_2 - b_{\delta_1} \tag{46}$$

$$\max(b_1, b_2 - b_{\delta_1}) = b_2 - b_{\delta_1} \tag{47}$$

$$b = b_2 - b_{\delta_1} \quad \text{from the definition of } b \tag{48}$$

And in Case (1) still, we get:

$$b_1 < b_2 - b_{\delta_1} \tag{49}$$

$$0 < b_2 - b_{\delta_1} - b_1 \tag{50}$$

$$\max(0, b_2 - b_{\delta_1} - b_1) = b_2 - b_{\delta_1} - b_1 \tag{51}$$

Next, we consider two subcases:

- (i) $t < b_2 - b_{\delta_1}$
- (ii) $t \geq b_2 - b_{\delta_1}$

In Case (i), we get

$$t < b_2 - b_{\delta_1} \tag{52}$$

$$0 < b_2 - b_{\delta_1} - t \tag{53}$$

$$\max(0, b_2 - b_{\delta_1} - t) = b_2 - b_{\delta_1} - t \tag{54}$$

Now from the definition of δ_t ,

$$b_{\delta_t} = b_{\delta_1} + b_{\delta_2} + \max(0, b - t) \tag{55}$$

$$= b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_{\delta_1} - t) \quad \text{from (48)} \tag{56}$$

$$= b_{\delta_1} + b_{\delta_2} + b_2 - b_{\delta_1} - t \quad \text{from (54)} \tag{57}$$

$$= b_{\delta_2} + b_2 - t \tag{58}$$

$$b_{\delta_t} + t = b_{\delta_2} + b_2 - t + t \tag{59}$$

$$= b_{\delta_2} + b_2 \tag{60}$$

Next, from the definition of a'

$$a' = b_{\delta_2(a)} + a \quad (61)$$

$$= b_{\delta_2} + \max(0, b_2 - a) + a \quad (62)$$

And, from the definition of a

$$a = b_{\delta_1(t)} + t \quad (63)$$

$$= b_{\delta_1} + \max(0, b_1 - t) + t \quad (64)$$

Then we have two further subcases:

(I) $t \geq b_1$, or

(II) $t < b_1$

In case (I):

$$t \geq b_1 \quad (65)$$

$$0 \geq b_1 - t \quad (66)$$

$$\max(0, b_1 - t) = 0 \quad (67)$$

$$a = b_{\delta_1} + t \quad \text{from (64)} \quad (68)$$

$$\max(0, b_2 - a) = \max(0, b_2 - b_{\delta_1} - t) \quad (69)$$

$$= b_2 - b_{\delta_1} - t \quad \text{from (54)} \quad (70)$$

$$= b_2 - a \quad \text{from (68)} \quad (71)$$

In case (II):

$$t < b_1 \quad (72)$$

$$0 < b_1 - t \quad (73)$$

$$\max(0, b_1 - t) = b_1 - t \quad (74)$$

$$a = b_{\delta_1} + b_1 - t + t \quad \text{from (64)} \quad (75)$$

$$= b_{\delta_1} + b_1 \quad (76)$$

$$\max(0, b_2 - a) = \max(0, b_2 - b_{\delta_1} - b_1) \quad (77)$$

$$= b_2 - b_{\delta_1} - b_1 \quad \text{from (51)} \quad (78)$$

$$= b_2 - a \quad \text{from (76)} \quad (79)$$

$$(80)$$

So in both cases (I) and (II), we get

$$\max(0, b_2 - a) = b_2 - a$$

Therefore from (62)

$$a' = b_{\delta_2} + b_2 - a + a \quad (81)$$

$$= b_{\delta_2} + b_2 \quad (82)$$

$$= t + b_{\delta_t} \quad \text{from (60)} \quad (83)$$

which concludes the proof for Case (1)- (i).

We continue with Case (1)- (ii).

From Case (ii):

$$t \geq b_2 - b_{\delta_1} \quad (84)$$

$$0 \geq b_2 - b_{\delta_1} - t \quad (85)$$

$$\max(0, b_2 - b_{\delta_1} - t) = 0 \quad (86)$$

Now from the definition of δ_t :

$$b_{\delta_t} = b_{\delta_1} + b_{\delta_2} + \max(0, b - t) \quad (87)$$

$$= b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_1 - t) \quad \text{from (48)} \quad (88)$$

$$= b_{\delta_1} + b_{\delta_2} \quad \text{from (86)} \quad (89)$$

$$b_{\delta_t} + t = b_{\delta_1} + b_{\delta_2} + t \quad (90)$$

Next, from Case (1) and Case (ii), by transitivity, we get

$$b_1 \leq t \quad (91)$$

$$\max(0, b_1 - t) = 0 \quad (92)$$

And from the definition of a

$$a = b_{\delta_1}(t) + t \quad (93)$$

$$= b_{\delta_1} + \max(0, b_1 - t) + t \quad (94)$$

$$= b_{\delta_1} + t \quad \text{from (92)} \quad (95)$$

$$\geq b_{\delta_1} + b_2 - b_{\delta_1} \quad \text{from Case (ii)} \quad (96)$$

$$\geq b_2 \quad (97)$$

$$0 \geq b_2 - a \quad (98)$$

$$\max(0, b_2 - a) = 0 \quad (99)$$

Therefore from (62) and (99)

$$a' = b_{\delta_2} + a \quad (100)$$

$$= b_{\delta_2} + b_{\delta_1} + t \quad \text{from (95)} \quad (101)$$

$$= b_{\delta_t} + t \quad \text{from (60)} \quad (102)$$

which concludes the proof for Case (1)- (ii).

We continue with Case (2).

In this case, we get

$$b_1 \geq b_2 - b_{\delta_1} \quad (103)$$

$$\max(b_1, b_2 - b_{\delta_1}) = b_1 \quad (104)$$

$$b = b_1 \quad \text{from the definition of } b \quad (105)$$

And from Case (2) still, we derive

$$b_1 \geq b_2 - b_{\delta_1} \quad (106)$$

$$0 \geq b_2 - b_{\delta_1} - b_1 \quad (107)$$

$$\max(0, -b_{\delta_1} - b_1) = 0 \quad (108)$$

As well as

$$b_1 \geq b_2 - b_{\delta_1} \quad (109)$$

$$b_1 + b_{\delta_1} \geq b_2 \quad (110)$$

Next, we distinguish two subcases, namely

(a) $t < b_1$ and

(b) $t \geq b_1$

We start with Case (a).

In this case,

$$t < b_1 \quad (111)$$

$$0 < b_1 - t \quad (112)$$

$$\max(0, b_1 - t) = b_1 - t \quad (113)$$

And from the definition of δ_t :

$$b_{\delta_t} = b_{\delta_1} + b_{\delta_2} + \max(0, b - t) \quad (114)$$

$$= b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) \quad \text{from (105)} \quad (115)$$

$$= b_{\delta_1} + b_{\delta_2} + b_1 - t \quad \text{from (113)} \quad (116)$$

$$b_{\delta_t} + t = b_{\delta_1} + b_{\delta_2} + b_1 - t + t \quad (117)$$

$$= b_{\delta_1} + b_{\delta_2} + b_1 \quad (118)$$

Next, from the definition of a

$$a = b_{\delta_1(t)} + t \quad (119)$$

$$= \max(0, b_1 - t) + b_{\delta_1} + t \quad (120)$$

$$= b_1 - t + b_{\delta_1} + t \quad \text{from (113)} \quad (121)$$

$$= b_1 + b_{\delta_1} \quad (122)$$

So from (110)

$$a \geq b_2 \quad (123)$$

$$0 \geq b_2 - a \quad (124)$$

$$\max(0, b_2 - a) = 0 \quad (125)$$

$$b_{\delta_2} + \max(0, b_2 - a) = b_{\delta_2} \quad (126)$$

$$b_{\delta_2(a)} = b_{\delta_2} \quad (127)$$

$$b_{\delta_2(a)} + a = b_{\delta_2} + a \quad (128)$$

$$a' = b_{\delta_2} + a \quad \text{from the definition of } a' \quad (129)$$

$$a' = b_{\delta_2} + b_1 + b_{\delta_1} \quad \text{from (122)} \quad (130)$$

$$a' = b_{\delta_t} + t \quad \text{from (118)} \quad (131)$$

which concludes the proof for Case (2)- (a).

We end with Case (2)- (b). In this case,

$$t \geq b_1 \quad (132)$$

$$0 \geq b_1 - t \quad (133)$$

$$\max(0, b_1 - t) = 0 \quad (134)$$

And from the definition of δ_t :

$$b_{\delta_t} = b_{\delta_1} + b_{\delta_2} + \max(0, b - t) \quad (135)$$

$$= b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) \quad \text{from (105)} \quad (136)$$

$$= b_{\delta_1} + b_{\delta_2} \quad \text{from (135)} \quad (137)$$

$$b_{\delta_t} + t = b_{\delta_1} + b_{\delta_2} + t \quad (138)$$

Next, from the definition of a

$$a = b_{\delta_1(t)} + t \quad (139)$$

$$= \max(0, b_1 - t) + b_{\delta_1} + t \quad (140)$$

$$= b_{\delta_1} + t \quad \text{from (135)} \quad (141)$$

Now from Case (b)

$$b_1 + \leq t \quad (143)$$

$$b_1 + b_{\delta_1} \leq t + b_{\delta_1} \quad (144)$$

$$b_1 + b_{\delta_1} \leq a \quad \text{from (142)} \quad (145)$$

$$b_2 \leq a \quad \text{from (110), by transitivity} \quad (146)$$

$$b_2 - a \leq 0 \quad (147)$$

$$\max(0, b_2 - a) = 0 \quad (148)$$

$$b_{\delta_2} + \max(0, b_2 - a) = b_{\delta_2} \quad (149)$$

$$b_{\delta_2(a)} = b_{\delta_2} \quad (150)$$

$$b_{\delta_2(a)} + a = b_{\delta_2} + a \quad (151)$$

$$a' = b_{\delta_2} + a \quad \text{from the definition of } a' \quad (152)$$

$$= b_{\delta_2} + b_{\delta_1} + t \quad \text{from (142)} \quad (153)$$

$$= b_{\delta_t} + t \quad \text{from (139)} \quad (154)$$

which concludes the proof for Case (2)- (b). \square

The following result states that the representation $\langle q \rangle_G^{[t,d],b,e}$ is correct:

PROPOSITION 3.9. *Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$ be a TPG and q a TRPQ. Then the unfolding of $\langle q \rangle_G^{[t,d],b,e}$ is $\llbracket q \rrbracket_G$.*

PROOF. Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$ be a TG, and let q be a TRPQ.

To prove the result, it is sufficient to show that:

- (I) for any $\langle o_1, o_2, t, d \rangle \in \llbracket q \rrbracket_G$, there are $\tau, \delta \in \text{intv}(\mathcal{T})$ and $b, e \in \mathcal{T}$ such that
 - (a) $\langle o_1, o_2, \tau, \delta, b, e \rangle \in \langle q \rangle_G^{[t,d],b,e}$,
 - (b) $t \in \tau$, and
 - (c) $d \in \delta(t)$ (where $\delta(t)$ is defined in terms of t, δ, b and e , as explained above).
- (II) for any $\langle o_1, o_2, \tau, \delta, b, e \rangle \in \langle q \rangle_G^{[t,d],b,e}$ for any $t \in \tau$ and $d \in \delta(t)$, $\langle o_1, o_2, t, d \rangle$ is in $\llbracket q \rrbracket_G$.

Again, the proof is by induction on the structure of q .

If q is of the form *pred*, *F*, *B*, (*test* \vee *test*), (*path* $+$ *path*), *path*[m, n] or *path*[$m, _$], then (I) and (II) immediately follow from the definitions of $\llbracket q \rrbracket_G$ and $\langle q \rangle_G^{[t,d],b,e}$.

If q is of the form *test* \wedge *test*, \neg *test* or (*?path*), then the proof is nearly identical to the one already provided for $\langle q \rangle_G^{[t]}$.

And if q is of the form T_δ or $\text{path}_1/\text{path}_2$, then the proof is nearly identical to the one already provided for $\langle q \rangle_G^{[t,d]}$, using Lemma 3.8 instead of 3.6. \square