

# Compact Answers to Temporal Regular Path Queries (Supplementary Material)

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## 1 INTRODUCTION

This document provides detailed definitions and proofs for the article *Compact answers to Temporal Regular Path Queries*, submitted at CIKM 2023.

As opposed to the structure adopted in the article, the result here are grouped by topic (inductive representation, finiteness, complexity, etc) rather than representation ( $\mathcal{U}^{[t]}$ ,  $\mathcal{U}^{[d]}$ , etc.). This allows us to emphasize which proofs differ from one representation to the other.

## 2 NOTATION

Let  $\langle o_1, o_2, \tau, \delta, b, e \rangle \in \mathcal{U}^{[t,d],b,e}$ , and let  $t \in \tau$ .

In the article, we defined the interval  $\delta_t$  for each  $t$  as

$$\delta_t [b_\delta + \max(0, b - t), e_{\delta_t} - \max(0, t - e)]_\delta$$

In this supplementary material, we will use  $\delta(t)$  instead of  $\delta_t$ . This notation will allow us to write  $\delta_1(t)$  when several tuples are involved. Note that the time points  $b$  and  $e$  in this notation are still omitted, for conciseness, because they should be clear from the context.

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### 3 INDUCTIVE REPRESENTATION

Let  $q$  be a TRPQ and  $G$  a TG.

Then  $\llbracket q \rrbracket_G$  is the set of answers to  $q$  over  $G$  (represented as tuples in  $\mathcal{U}$ ).

In this section, we provide the full definition of the four inductive representations of  $\llbracket q \rrbracket_G$  discussed in the article, in  $\mathcal{U}^{[t]}$ ,  $\mathcal{U}^{[d]}$ ,  $\mathcal{U}^{[t,d]}$  and  $\mathcal{U}^{[t,d],b,e}$  respectively, and prove that they are correct.

These representations are denoted as  $\langle q \rangle_G^{[t]}$ ,  $\langle q \rangle_G^{[d]}$ ,  $\langle q \rangle_G^{[t,d]}$  and  $\langle q \rangle_G^{[t,d],b,e}$  respectively.

#### 3.1 Grammar

For convenience, we reproduce here the syntax of TRPQs (already provided in the article).

*Definition 3.1 (TRPQ).* A TRPQ is an expression for the symbol path in the following grammar:

$$\begin{aligned} \text{path} &::= \text{test} \mid \text{axis} \mid (\text{path}/\text{path}) \mid (\text{path} + \text{path}) \mid \text{path}[m, n] \mid \text{path}[m, \_] \\ \text{test} &::= \text{pred} \mid (? \text{path}) \mid \text{test} \vee \text{test} \mid \text{test} \wedge \text{test} \mid \neg \text{test} \\ \text{axis} &::= F \mid B \mid T_\delta \end{aligned}$$

with  $\delta \in \text{intv}(\mathcal{T})$ ,  $m, n \in \mathbb{N}^+$  and  $m \leq n$ .

#### 3.2 In $\mathcal{U}$

For convenience still, we reproduce the full semantics of TRPQs, i.e. the inductive definition of the evaluation  $\llbracket q \rrbracket_G$  of a query  $q$  over a graph  $G$  in  $\mathcal{U}$  (already provided in the article).

$$\begin{aligned} \llbracket \text{pred} \rrbracket_G &= \{ \langle o, o, t, 0 \rangle \mid t \in \tau \text{ for some } \tau \in \text{val}(o, \text{pred}) \} \\ \llbracket T_\delta \rrbracket_G &= \{ \langle o, o, t, d \rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G \} \\ \llbracket F \rrbracket_G &= \{ \langle v, e, t, 0 \rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G \} \cup \{ \langle e, v, t, 0 \rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G \} \\ \llbracket B \rrbracket_G &= \{ \langle v, e, t, 0 \rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G \} \cup \{ \langle e, v, t, 0 \rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G \} \\ \llbracket ? \text{path} \rrbracket_G &= \{ \langle o, o, t, 0 \rangle \mid \langle o, o', t, t + d \rangle \in \llbracket \text{path} \rrbracket_G \text{ for some } o' \in N \cup E, d \in \mathcal{T} \} \\ \llbracket \text{test}_1 \vee \text{test}_2 \rrbracket_G &= \llbracket \text{test}_1 \rrbracket_G \cup \llbracket \text{test}_2 \rrbracket_G \\ \llbracket \text{test}_1 \wedge \text{test}_2 \rrbracket_G &= \llbracket \text{test}_1 \rrbracket_G \cap \llbracket \text{test}_2 \rrbracket_G \\ \llbracket \neg \text{test} \rrbracket_G &= (\{ \langle o, o \rangle \mid o \in N \cup E \} \times \mathcal{T}_G \times \{0\}) \setminus \llbracket \text{test} \rrbracket_G \\ \llbracket \text{path}_1 / \text{path}_2 \rrbracket_G &= \{ \langle o_1, o_3, t, d_1 + d_2 \rangle \mid \exists o_2: \langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G \wedge \langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G \} \\ \llbracket \text{path}_1 + \text{path}_2 \rrbracket_G &= \llbracket \text{path}_1 \rrbracket_G \cup \llbracket \text{path}_2 \rrbracket_G \\ \llbracket \text{path}[m, n] \rrbracket_G &= \bigcup_{k=m}^n \llbracket \text{path}^k \rrbracket_G \\ \llbracket \text{path}[m, \_] \rrbracket_G &= \bigcup_{k \geq m} \llbracket \text{path}^k \rrbracket_G \end{aligned}$$

#### 3.3 In $\mathcal{U}^{[t]}$

##### 3.3.1 Definition.

The full definition of  $\langle q \rangle_G^{[t]}$  is already provided in the article. We only reproduce it here for convenience.

$$\begin{aligned} \langle \text{pred} \rangle_G^{[t]} &= \{ \langle o, o, \tau, 0 \rangle \mid o \in (N \cup E), \tau \in \text{val}(o, \text{pred}) \} \\ \langle T_\delta \rangle_G^{[t]} &= \{ \langle o, o, [t_1, t_1], t_2 - t_1 \rangle \mid o \in (N \cup E), t_1 \in \mathcal{T}_G, t_2 \in (\delta + t_1) \cap \mathcal{T}_G \} \\ \langle F \rangle_G^{[t]} &= \{ \langle v, e, \mathcal{T}_G, 0 \rangle \mid \text{src}(e) = v \} \cup \{ \langle e, v, \mathcal{T}_G, 0 \rangle \mid \text{tgt}(e) = v \} \\ \langle B \rangle_G^{[t]} &= \{ \langle v, e, \mathcal{T}_G, 0 \rangle \mid \text{tgt}(e) = v \} \cup \{ \langle e, v, \mathcal{T}_G, 0 \rangle \mid \text{src}(e) = v \} \\ \langle (? \text{path}) \rangle_G^{[t]} &= \{ \langle o, o, \tau, 0 \rangle \mid \langle o, o', \tau, d \rangle \in \langle \text{path} \rangle_G^{[t]} \text{ for some } o' \in N \cup E, d \in \mathcal{T} \} \\ \langle \text{test}_1 \vee \text{test}_2 \rangle_G^{[t]} &= \langle \text{test}_1 \rangle_G^{[t]} \cup \langle \text{test}_2 \rangle_G^{[t]} \\ \langle \text{test}_1 \wedge \text{test}_2 \rangle_G^{[t]} &= \{ \langle o, o, \tau_1 \cap \tau_2, 0 \rangle \mid \langle o, o, \tau_1, 0 \rangle \in \langle \text{test}_1 \rangle_G^{[t]}, \langle o, o, \tau_2, 0 \rangle \in \langle \text{test}_2 \rangle_G^{[t]}, \tau_1 \cap \tau_2 \neq \emptyset \} \\ \langle \neg \text{test} \rangle_G^{[t]} &= \bigcup_{o \in N \cup E} \left\{ \langle o, o, \tau, 0 \rangle \mid \tau \in \text{compl} \left( \{ \tau' \mid \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]}, \mathcal{T}_G \} \right) \right\} \\ \langle \text{path}_1 / \text{path}_2 \rangle_G^{[t]} &= \left\{ \langle o_1, o_3, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2 \rangle \mid \exists o_2: \langle o_1, o_2, \tau_1, d_1 \rangle \in \langle \text{path}_1 \rangle_G^{[t]} \wedge \langle o_2, o_3, \tau_2, d_2 \rangle \in \langle \text{path}_2 \rangle_G^{[t]} \wedge (\tau_1 + d_1) \cap \tau_2 \neq \emptyset \right\} \\ \langle \text{path}_1 + \text{path}_2 \rangle_G^{[t]} &= \langle \text{path}_1 \rangle_G^{[t]} \cup \langle \text{path}_2 \rangle_G^{[t]} \\ \langle \text{path}[m, n] \rangle_G^{[t]} &= \bigcup_{k=m}^n \langle \text{path}^k \rangle_G^{[t]} \\ \langle \text{path}[m, \_] \rangle_G^{[t]} &= \bigcup_{k \geq m} \langle \text{path}^k \rangle_G^{[t]} \end{aligned}$$

We observe that when  $q$  is of the form  $(\text{path}_1 + \text{path}_2)$ ,  $(\text{path}[m, \_])$  and  $(\text{path}[m, n])$ , the definition of  $\langle q \rangle_G^{[t]}$  is nearly identical to the one of  $\llbracket q \rrbracket_G$ . This will also be the case for the three representations below.

### 3.3.2 Correctness.

We start with a lemma:

LEMMA 3.2. *Let  $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$  be a TPG and let  $q$  an expression for the symbol test in the grammar of Definition 3.1. Then:*

- *each tuples in  $\llbracket q \rrbracket_G$  is of the form  $\langle o_1, o_2, t, 0 \rangle$  for some  $o_1, o_2$  and  $t$ ,*
- *each tuples in  $\langle q \rangle_G^{[t]}$  is of the form  $\langle o_1, o_2, \tau, 0 \rangle$  for some  $o_1, o_2$  and  $\tau$ .*

PROOF. Immediate from the definitions of  $\llbracket q \rrbracket_G$  and  $\langle q \rangle_G^{[t]}$ . □

The following result states that the representation  $\langle q \rangle_G^{[t]}$  is correct:

PROPOSITION 3.3. *Let  $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$  be a TPG and  $q$  a TRPQ. Then the unfolding of  $\langle q \rangle_G^{[t]}$  is  $\llbracket q \rrbracket_G$ .*

PROOF.

Let  $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$  be a TG, and let  $q$  be a TRPQ.

We show below that:

- (I) for any  $\langle o_1, o_2, t, d \rangle \in \llbracket q \rrbracket_G$ , there is a  $\tau \in \text{intv}(\mathcal{T})$  such that
  - (a)  $\langle o_1, o_2, \tau, d \rangle \in \langle q \rangle_G^{[t]}$ , and
  - (b)  $t \in \tau$ ,
- (II) for any  $\langle o_1, o_2, \tau, d \rangle \in \langle q \rangle_G^{[t]}$  for any  $t \in \tau$ ,
  - $\langle o_1, o_2, t, d \rangle$  is in  $\llbracket q \rrbracket_G$ .

We proceed by induction on the structure of  $q$ .

If  $q$  is of the form  $\text{pred}$ ,  $F$ ,  $B$ ,  $(\text{test} \vee \text{test})$ ,  $(\text{path} + \text{path})$ ,  $\text{path}[m, n]$  or  $\text{path}[m, \_]$ , then (I) and (II) immediately follow from the definitions of  $\llbracket q \rrbracket_G$  and  $\langle q \rangle_G^{[t]}$ .

So we focus below on the five remaining cases:

- $q = T_\delta$ .

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o, o, t, d \rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G \} \\ \langle q \rangle_G^{[t]} &= \{ \langle o, o, [t_1, t_1], t_2 - t_1 \rangle \mid o \in (N \cup E), t_1 \in \mathcal{T}_G, t_2 \in (\delta + t_1) \cap \mathcal{T}_G \} \end{aligned}$$

- For (I), let  $\mathbf{v} = \langle o, o, t, t + d \rangle \in \llbracket q \rrbracket_G$ .

And let  $\mathbf{u} = \langle o, o, [t, t], d \rangle$  in  $\mathcal{U}^{[t]}$ .

For (Ia) we show that  $\mathbf{u} \in \langle q \rangle_G^{[t]}$ .

From  $\mathbf{v} \in \llbracket q \rrbracket_G$ , we get  $o \in N \cup E$  and  $t \in \mathcal{T}_G$ .

Besides, because  $\mathbf{v} \in \llbracket q \rrbracket_G$  still,

$$t + d \in \mathcal{T}_G \tag{1}$$

and

$$d \in \delta \tag{2}$$

$$t + d \in t + \delta \tag{3}$$

So from (1) and (3)

$$t + d \in (\delta + t) \cap \mathcal{T}_G \tag{4}$$

So there is a  $t_2$  (namely  $t + d$ ) such that  $d = t_2 - t$  and  $t_2 \in t + \delta \cap \mathcal{T}_G$ .

Together with the definition of  $\langle q \rangle_G^{[t]}$ , this implies  $\mathbf{u} \in \langle q \rangle_G^{[t]}$ , which concludes the proof for (Ia).

And trivially,  $t \in [t, t]$ , so (Ib) is verified as well.

- For (II), let  $\mathbf{u} = \langle o, o, [t, t], d \rangle \in \llbracket q \rrbracket_G^{[t]}$ .

From  $\mathbf{u} \in \llbracket q \rrbracket_G^{[t]}$ , we get  $o \in N \cup E$  and  $t \in \mathcal{T}_G$ .

So to conclude the proof, it is sufficient to show that (i)  $d \in \delta$  and (ii)  $t + d \in \mathcal{T}_G$ .

Because  $\mathbf{u} \in \llbracket q \rrbracket_G^{[t]}$  still, we have

$$d = t_2 - t \text{ for some } t_2 \in (\delta + t) \cap \mathcal{T}_G \quad (5)$$

From (5), we get  $t_2 = t + d$ .

Therefore from (5) still,

$$t + d \in (\delta + t) \cap \mathcal{T}_G \quad (6)$$

which proves (ii).

And from (6), we also get

$$\begin{aligned} t + d &\in \delta + t \\ t + d - t &\in (\delta + t) - t \\ d &\in \delta \end{aligned}$$

which proves (i).

- $q = \text{test}_1 \wedge \text{test}_2$ .

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \llbracket \text{test}_1 \rrbracket_G \cap \llbracket \text{test}_2 \rrbracket_G \\ \llbracket q \rrbracket_G^{[t]} &= \{ \langle o, o, \tau_1 \cap \tau_2, 0 \rangle \mid \langle o, o, \tau_1, 0 \rangle \in \llbracket \text{test}_1 \rrbracket_G^{[t]}, \langle o, o, \tau_2, 0 \rangle \in \llbracket \text{test}_2 \rrbracket_G^{[t]}, \tau_1 \cap \tau_2 \neq \emptyset \} \end{aligned}$$

- For (I), let  $\mathbf{v} = \langle o, o, t, d \rangle \in \llbracket q \rrbracket_G$ .

From Lemma 3.2,  $d = 0$ .

And from the definition of  $\llbracket q \rrbracket_G$ ,  $\mathbf{v} \in \llbracket \text{test}_1 \rrbracket_G \cap \llbracket \text{test}_2 \rrbracket_G$ .

So by IH, there are intervals  $\tau_1$  and  $\tau_2$  s.t.  $\langle o, o, \tau_i, 0 \rangle \in \llbracket \text{test}_i \rrbracket_G^{[t]}$  for  $i \in \{1, 2\}$  and  $t \in \tau_1 \cap \tau_2$ .

Together with the definition of  $\llbracket q \rrbracket_G^{[t]}$ , this proves (I).

- For (II), let  $\langle o, o, \tau, d \rangle \in \llbracket q \rrbracket_G^{[t]}$ .

Then from Lemma 3.2,  $d = 0$ .

And from the definition of  $\llbracket q \rrbracket_G^{[t]}$ , there are two intervals  $\tau_1$  and  $\tau_2$  s.t.  $\tau = \tau_1 \cap \tau_2$  and  $\langle o, o, \tau_i, 0 \rangle \in \llbracket \text{test}_i \rrbracket_G^{[t]}$  for  $i \in \{1, 2\}$ .

Now take any  $t \in \tau$ .

Then  $t \in \tau_i$  for  $i \in \{1, 2\}$ .

So by IH,  $\langle o, o, t, 0 \rangle \in \llbracket \text{test}_i \rrbracket_G$  for each  $i \in \{1, 2\}$ .

Together with the definition of  $\llbracket q \rrbracket_G$ , this proves (II).

- $q = (? \text{path})$ .

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o, o, t, 0 \rangle \mid \langle o, o', t, t + d \rangle \in \llbracket \text{path} \rrbracket_G \text{ for some } o' \in N \cup E, d \in \mathcal{T} \} \\ \llbracket q \rrbracket_G^{[t]} &= \{ \langle o, o, \tau, 0 \rangle \mid \langle o, o', \tau, d \rangle \in \llbracket \text{path} \rrbracket_G^{[t]} \text{ for some } o' \in N \cup E, d \in \mathcal{T} \} \end{aligned}$$

- For (I), let  $\langle o, o, t, 0 \rangle \in \llbracket q \rrbracket_G$ .

From the definition of  $\llbracket q \rrbracket_G$ , there are  $o'$  and  $d$  such that  $\langle o, o', t, t + d \rangle \in \llbracket \text{path} \rrbracket_G$ .

So by IH, there is a  $\tau$  s.t.  $t \in \tau$  and  $\langle o, o', \tau, d \rangle \in \llbracket \text{path} \rrbracket_G^{[t]}$ .

Therefore  $\langle o, o, \tau, 0 \rangle \in \llbracket q \rrbracket_G^{[t]}$ , from the definition of  $\llbracket q \rrbracket_G^{[t]}$ .

- For (II), let  $\langle o, o, \tau, 0 \rangle \in \llbracket q \rrbracket_G^{[t]}$ .

From the definition of  $\llbracket q \rrbracket_G^{[t]}$ , there are  $o'$  and  $d$  s.t.  $\langle o, o', \tau, d \rangle \in \llbracket \text{path} \rrbracket_G^{[t]}$ .

Now take any  $t \in \tau$ .

By IH,  $\langle o, o', t, t + d \rangle \in \llbracket \text{path} \rrbracket_G$ .

Therefore  $\langle o, o, t, 0 \rangle \in \llbracket q \rrbracket_G$ , from the definition of  $\llbracket q \rrbracket_G$ .

- $q = \neg \text{test}$ .

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= (\{\langle o, o \rangle \mid o \in N \cup E\} \times \mathcal{T}_G \times \{0\}) \setminus \llbracket \text{test} \rrbracket_G \\ \langle q \rangle_G^{[t]} &= \bigcup_{o \in N \cup E} \left\{ \langle o, o, \tau, 0 \rangle \mid \tau \in \text{compl} \left( \{\tau' \mid \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]} \}, \mathcal{T}_G \right) \right\} \end{aligned}$$

- For (I), let  $\mathbf{v} = \langle o, o, t, 0 \rangle \in \llbracket q \rrbracket_G$ .  
From the definition of  $\llbracket q \rrbracket_G$ ,  $\mathbf{v} \notin \llbracket \text{test} \rrbracket_G$ .  
So

$$t \notin \{\tau' \mid \langle o, o, \tau', 0 \rangle \in \llbracket \text{test} \rrbracket_G\} \quad (7)$$

Now by IH, together with Lemma 3.2, we get:

$$\langle o, o, t, 0 \rangle \in \llbracket \text{test} \rrbracket_G \text{ iff } t' \in \tau' \text{ for some } \tau' \text{ s.t. } \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]} \quad (8)$$

So from (7) and (8):

$$t \notin \bigcup \{\tau' \mid \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]}\}$$

Therefore

$$t \in \mathcal{T}_G \setminus \bigcup \{\tau' \mid \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]}\} \quad (9)$$

So  $t \in \tau$  for some  $\tau \in \text{compl} \left( \bigcup \{\tau' \mid \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]} \}, \mathcal{T}_G \right)$ .

And  $\langle o, o, \tau, 0 \rangle \in \langle q \rangle_G^{[t]}$ , from the definition of  $\langle q \rangle_G^{[t]}$ .

- For (II), let  $\langle o, o, \tau, 0 \rangle \in \langle q \rangle_G^{[t]}$ .  
And take any  $t \in \tau$ .

From the definition of  $\langle q \rangle_G^{[t]}$ :

$$t \in \mathcal{T}_G \setminus \bigcup \{\tau' \mid \langle o, o, \tau', 0 \rangle \in \langle \text{test} \rangle_G^{[t]}\}$$

Together with (8), this implies

$$\langle o, o, t, 0 \rangle \notin \llbracket \text{test} \rrbracket_G$$

Therefore  $\langle o, o, t, 0 \rangle \in \llbracket q \rrbracket_G$ , from the definition of  $\llbracket q \rrbracket_G$ .

- $q = \text{path}_1 / \text{path}_2$ .

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o_1, o_3, t, d_1 + d_2 \rangle \mid \exists o_2: \langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G \wedge \langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G \} \\ \langle q \rangle_G^{[t]} &= \left\{ \langle o_1, o_3, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2 \rangle \mid \exists o_2: \langle o_1, o_2, \tau_1, d_1 \rangle \in \langle \text{path}_1 \rangle_G^{[t]} \wedge \langle o_2, o_3, \tau_2, d_2 \rangle \in \langle \text{path}_2 \rangle_G^{[t]} \wedge \right. \\ &\quad \left. (\tau_1 + d_1) \cap \tau_2 \neq \emptyset \right\} \end{aligned}$$

- For (I), let  $\mathbf{v} = \langle o_1, o_3, t, d \rangle \in \llbracket q \rrbracket_G$ .  
From the definition of  $\llbracket q \rrbracket_G$ , there are  $o_2, d_1$  and  $d_2$  such that  $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$ ,  $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$  and  $d = d_1 + d_2$ .  
By IH, because  $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$ , there is a  $\tau_1$  such that  $t \in \tau_1$  and

$$\langle o_1, o_2, \tau_1, d_1 \rangle \in \langle \text{path}_1 \rangle_G^{[t]} \quad (10)$$

And similarly, because  $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$ , there is a  $\tau_2$  such that  $t + d_1 \in \tau_2$  and

$$\langle o_2, o_3, \tau_2, d_2 \rangle \in \langle \text{path}_2 \rangle_G^{[t]} \quad (11)$$

From  $t \in \tau_1$ , we get

$$t + d_1 \in \tau_1 + d_1 \quad (12)$$

Together with the fact that  $t + d_1 \in \tau_2$ , this implies

$$\tau_1 + d_1 \cap \tau_2 \neq \emptyset \quad (13)$$

So from (10), (11), (13) and the definition of  $\langle q \rangle_G^{[t]}$ ,

$$\langle o_1, o_2, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2 \rangle \in \langle q \rangle_G^{[t]}$$

which proves (Ia).

And in order to prove (Ib), we only need to show that

$$t \in ((\tau_1 + d_1) \cap \tau_2) - d_1$$

We know that  $t \in \tau_1$ , therefore

$$t + d_1 \in \tau_1 + d_1$$

Together with the fact that  $t + d_1 \in \tau_2$ , this yields

$$\begin{aligned} t + d_1 &\in (\tau_1 + d_1) \cap \tau_2 \\ t &\in ((\tau_1 + d_1) \cap \tau_2) - d_1 \end{aligned}$$

– For (II), let  $\mathbf{u} = \langle o_1, o_3, \tau, d \rangle \in \langle q \rangle_G^{[t]}$ , and let  $t \in \tau$ .

We show that  $\langle o_1, o_3, t, t + d \rangle \in \llbracket q \rrbracket_G$ .

Because  $\mathbf{u} \in \langle q \rangle_G^{[t]}$ , from the definition of  $\langle q \rangle_G^{[t]}$ , there are  $\tau_1, \tau_2, d_1, d_2$  and  $o_2$  s.t.:

- (i)  $d = d_1 + d_2$
- (ii)  $\tau = ((\tau_1 + d_1) \cap \tau_2) - d_1$
- (iii)  $\langle o_1, o_2, \tau_1, d_1 \rangle \in \langle \text{path}_1 \rangle_G^{[t]}$
- (iv)  $\langle o_2, o_3, \tau_2, d_2 \rangle \in \langle \text{path}_2 \rangle_G^{[t]}$

Since  $t \in \tau$ , from (ii), we have

$$t \in ((\tau_1 + d_1) \cap \tau_2) - d_1 \quad (14)$$

$$t + d_1 \in (((\tau_1 + d_1) \cap \tau_2) - d_1) + d_1 \quad (15)$$

$$t + d_1 \in (\tau_1 + d_1) \cap \tau_2 \quad (16)$$

$$t + d_1 \in \tau_1 + d_1 \quad (17)$$

$$t \in \tau_1 \quad (18)$$

From (iii), by IH, for any  $t' \in \tau_1$

$$\langle o_1, o_2, t' + d_1 \rangle \in \llbracket q \rrbracket_G$$

In particular, from (18)

$$\langle o_1, o_2, t, t + d_1 \rangle \in \llbracket q \rrbracket_G \quad (19)$$

And from (iv), by IH, for any  $t'' \in \tau_2$

$$\langle o_2, o_3, t'', t'' + d_2 \rangle \in \llbracket q \rrbracket_G$$

In particular, from (16)

$$\langle o_2, o_3, t + d_1, (t + d_1) + d_2 \rangle \in \llbracket q \rrbracket_G \quad (20)$$

So from (19), (20) and the definition of  $\llbracket q \rrbracket_G$

$$\langle o_1, o_3, t, t + d_1 + d_2 \rangle \in \llbracket q \rrbracket_G$$

□

### 3.4 In $\mathcal{U}^{[d]}$

#### 3.4.1 Definition.

We start with the case where  $q$  is an expression for the symbol test in the grammar of Definition 3.1.

The definitions of  $\langle \text{pred} \rangle_G^{[t]}$  and  $\langle \neg \text{test} \rangle_G^{[t]}$  are already provided in the article, we reproduce them here for completeness:

$$\begin{aligned} \langle \text{pred} \rangle_G^{[d]} &= \{ \langle o, o, t, [0, 0] \rangle \mid t \in \tau \text{ for some } \tau \in \text{val}(o, \text{pred}) \} \\ \langle \text{F} \rangle_G^{[d]} &= \{ \langle v, e, t, [0, 0] \rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G \} \cup \{ \langle e, v, t, [0, 0] \rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G \} \\ \langle \text{B} \rangle_G^{[d]} &= \{ \langle v, e, t, [0, 0] \rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G \} \cup \{ \langle e, v, t, [0, 0] \rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G \} \\ \langle (? \text{path}) \rangle_G^{[d]} &= \{ \langle o, o, t, [0, 0] \rangle \mid \exists o', \delta: \langle o, o', t, \delta \rangle \in \langle \text{path} \rangle_G^{[d]} \} \\ \langle \text{test}_1 \vee \text{test}_2 \rangle_G^{[d]} &= \langle \text{test}_1 \rangle_G^{[d]} \cup \langle \text{test}_2 \rangle_G^{[d]} \\ \langle \text{test}_1 \wedge \text{test}_2 \rangle_G^{[d]} &= \langle \text{test}_1 \rangle_G^{[d]} \cap \langle \text{test}_2 \rangle_G^{[d]} \\ \langle \neg \text{test} \rangle_G^{[d]} &= \left\{ \langle o, o, t, [0, 0] \rangle \mid o \in N \cup E, t \in \mathcal{T}_G \setminus \{ t' \mid \langle o, o, t', [0, 0] \rangle \in \langle \text{test} \rangle_G^{[d]} \} \right\} \end{aligned}$$

Next, we consider the operators  $(\text{path}_1 + \text{path}_2)$ ,  $(\text{path}[m, \_])$  and  $(\text{path}[m, n])$ .

For these cases,  $\langle q \rangle_G^{[t, d]}$  is once again defined analogously to  $\llbracket q \rrbracket_G$ , in terms of temporal join (a.k.a.  $\text{path}_1/\text{path}_2$ ) and set union.

We only write the definitions here for the sake of completeness:

$$\begin{aligned} \llbracket \text{path}_1 + \text{path}_2 \rrbracket_G^{[t,d]} &= \llbracket \text{path}_1 \rrbracket_G^{[t,d]} \cup \llbracket \text{path}_2 \rrbracket_G^{[t,d]} \\ \llbracket \text{path}[m, n] \rrbracket_G &= \bigcup_{k=m}^n \llbracket \text{path}^k \rrbracket_G^{[t,d]} \\ \llbracket \text{path}[m, \_] \rrbracket_G &= \bigcup_{k \geq m} \llbracket \text{path}^k \rrbracket_G^{[t,d]} \end{aligned}$$

The only remaining operators are temporal join ( $\text{path}_1/\text{path}_2$ ) and temporal navigation ( $\text{T}_\delta$ ), already defined in the article. We reproduce here these two definition for convenience:

$$\begin{aligned} \llbracket \text{path}_1/\text{path}_2 \rrbracket_G^{[d]} &= \left\{ \langle o_1, o_3, t_1, \delta_2 + t_2 - t_1 \rangle \mid \exists o_2: \langle o_1, o_2, t_1, \delta_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G^{[d]} \wedge \langle o_2, o_3, t_2, \delta_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G^{[d]} \wedge t_2 \in t_1 + \delta_1 \right\} \\ \llbracket \text{T}_\delta \rrbracket_G^{[d]} &= \left\{ \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t \rangle \mid o \in N \cup E, t \in \mathcal{T}_G, (\delta + t) \cap \mathcal{T}_G \neq \emptyset \right\} \end{aligned}$$

We also reproduce the alternative characterization of  $\llbracket \text{T}_\delta \rrbracket_G^{[d]}$  provided in the article, as a unary operator:

$$\llbracket q/\text{T}_\delta \rrbracket_G^{[d]} = \{ \langle o_1, o_2, t, (\delta' + \delta) \cap \mathcal{T}_G \rangle \mid \langle o_1, o_2, t, \delta' \rangle \in \llbracket q \rrbracket_G^{[d]}, (t + (\delta' + \delta)) \cap \mathcal{T}_G \neq \emptyset \}$$

### 3.4.2 Correctness.

The following result states that the representation  $\llbracket q \rrbracket_G^{[d]}$  is correct:

**PROPOSITION 3.4.** *Let  $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$  be a TPG and  $q$  a TRPQ. Then the unfolding of  $\llbracket q \rrbracket_G^{[d]}$  is  $\llbracket q \rrbracket_G$ .*

**PROOF.**

Let  $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$  be a TG, and let  $q$  be a TRPQ.

We show below that:

- (I) for any  $\langle o_1, o_2, t, d \rangle \in \llbracket q \rrbracket_G$ , there is a  $\delta \in \text{intv}(\mathcal{T})$  such that
  - (a)  $\langle o_1, o_2, t, \delta \rangle \in \llbracket q \rrbracket_G^{[d]}$ , and
  - (b)  $d \in \delta$ ,
- (II) for any  $\langle o_1, o_2, t, \delta \rangle \in \llbracket q \rrbracket_G^{[d]}$  for any  $d \in \delta$ ,  $\langle o_1, o_2, t, d \rangle$  is in  $\llbracket q \rrbracket_G$ .

We proceed once again by induction on the structure of  $q$ .

If  $q$  is of the form *pred*, F, B, (test  $\vee$  test), (test  $\wedge$  test),  $\neg$ test, (path + path), path[m, n] or path[m, \_], then (I) and (II) immediately follow from the definitions of  $\llbracket q \rrbracket_G$  and  $\llbracket q \rrbracket_G^{[d]}$ .

If  $q$  is of the form (?path), then the proof is nearly identical to one already provided for  $\llbracket (?path) \rrbracket_G^{[t]}$ .

So we focus below on the two remaining cases:

- $q = \text{T}_\delta$ .

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o, o, t, d \rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G \} \\ \llbracket q \rrbracket_G^{[d]} &= \{ \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t \rangle \mid o \in N \cup E, t \in \mathcal{T}_G, (\delta + t) \cap \mathcal{T}_G \neq \emptyset \} \end{aligned}$$

- For (I), let  $\mathbf{v} = \langle o, o, t, d \rangle \in \llbracket q \rrbracket_G$ .  
And let  $\mathbf{u} = \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t \rangle$  in  $\mathcal{U}^{[d]}$ .  
For (Ia) we show that  $\mathbf{u} \in \llbracket q \rrbracket_G^{[d]}$ .  
From  $\mathbf{v} \in \llbracket q \rrbracket_G$ , we get  $o \in N \cup E$  and  $t \in \mathcal{T}_G$ .  
Besides, because  $\mathbf{v} \in \llbracket q \rrbracket_G$  still,

$$t + d \in \mathcal{T}_G \tag{21}$$

and

$$d \in \delta \tag{22}$$

$$t + d \in t + \delta \tag{23}$$

So from (21) and (23)

$$t + d \in (\delta + t) \cap \mathcal{T}_G \tag{24}$$

$$(\delta + t) \cap \mathcal{T}_G \neq \emptyset \tag{25}$$

Together with the definition of  $\langle q \rangle_G^{[d]}$ , this implies  $\mathbf{u} \in \langle q \rangle_G^{[d]}$ , which concludes the proof for (Ia).  
Finally, from (24), we get

$$t + d - t \in ((\delta + t) \cap \mathcal{T}_G) - t \quad (26)$$

$$d \in ((\delta + t) \cap \mathcal{T}_G) - t \quad (27)$$

which proves (Ib).

- For (II), let  $\mathbf{u} = \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t \rangle \in \langle q \rangle_G^{[d]}$ , and let  $d \in ((\delta + t) \cap \mathcal{T}_G) - t$ .

From  $\mathbf{u} \in \langle q \rangle_G^{[d]}$ , we get  $o \in N \cup E$  and  $t \in \mathcal{T}_G$ .

So to conclude the proof, it is sufficient to show that (i)  $d \in \delta$  and (ii)  $t + d \in \mathcal{T}_G$ .

By assumption, we have

$$d \in ((\delta + t) \cap \mathcal{T}_G) - t \quad (28)$$

$$d + t \in (\delta + t) \cap \mathcal{T}_G \quad (29)$$

$$d + t \in \mathcal{T}_G \quad (30)$$

which proves (ii).

And from (29), we also get

$$d + t \in \delta + t$$

$$d + t - t \in (\delta + t) - t$$

$$d \in \delta$$

which proves (i).

- $q = \text{path}_1/\text{path}_2$ .

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o_1, o_3, t, d_1 + d_2 \rangle \mid \exists o_2: \langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G \wedge \langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G \} \\ \langle q \rangle_G^{[d]} &= \left\{ \langle o_1, o_3, t_1, \delta_2 + t_2 - t_1 \rangle \mid \exists o_2: \langle o_1, o_2, t_1, \delta_1 \rangle \in \langle \text{path}_1 \rangle_G^{[d]} \wedge \langle o_2, o_3, t_2, \delta_2 \rangle \in \langle \text{path}_2 \rangle_G^{[d]} \wedge t_2 \in t_1 + \delta_1 \right\} \end{aligned}$$

- For (I), let  $\mathbf{v} = \langle o_1, o_3, t, d \rangle \in \llbracket q \rrbracket_G$ .

From the definition of  $\llbracket q \rrbracket_G$ , there are  $o_2, d_1$  and  $d_2$  such that  $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$ ,  $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$  and  $d = d_1 + d_2$ .

By IH, because  $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$ , there is a  $\delta_1$  such that  $d_1 \in \delta_1$  and

$$\langle o_1, o_2, t, \delta_1 \rangle \in \langle \text{path}_1 \rangle_G^{[d]} \quad (31)$$

And similarly, because  $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$ , there is a  $\delta_2$  such that  $d_2 \in \delta_2$  and

$$\langle o_2, o_3, t + d_1, \delta_2 \rangle \in \langle \text{path}_2 \rangle_G^{[d]} \quad (32)$$

Next, since  $d \in \delta_1$

$$t + d_1 \in t + \delta_1 \quad (33)$$

So from (31), (32), (33) and the definition of  $\langle q \rangle_G^{[d]}$  (replacing  $t_1$  with  $t$  and  $t_2$  with  $t + d_1$ ), we get

$$\langle o_1, o_2, t, \delta_2 + (t + d_1) - t \rangle \in \langle q \rangle_G^{[d]}$$

which proves (Ia).

And in order to prove (Ib), we only need to show that  $d \in \delta_2 + (t + d_1) - t$ , or in other words that

$$d \in \delta_2 + d_1$$

We know that

$$d_2 \in \delta_2 \quad (34)$$

$$d_2 + d_1 \in \delta_2 + d_1 \quad (35)$$

Together with the fact that  $d = d_1 + d_2$ , this concludes the proof for (Ib).

- For (II), let  $\mathbf{u} = \langle o_1, o_3, t_1, \delta \rangle \in \langle q \rangle_G^{[d]}$ , and let  $d \in \delta$ .

Because  $\mathbf{u} \in \langle q \rangle_G^{[d]}$ , from the definition of  $\langle q \rangle_G^{[d]}$ , there are  $\delta_1, \delta_2, t_2$  and  $o_2$  s.t.:

$$(i) \quad \delta = \delta_2 + t_2 - t_1$$

$$(ii) \quad t_2 \in t_1 + \delta_1$$

$$(iii) \quad \langle o_1, o_2, t_1, \delta_1 \rangle \in \langle \text{path}_1 \rangle_G^{[d]}$$



- (iv)  $\langle o_2, o_3, t_2, \delta_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G^{[d]}$   
 From (i) and (ii), we get

$$\begin{aligned}\delta &= \delta_2 + (t_1 + \delta_1) - t_1 \\ &= \delta_2 + \delta_1\end{aligned}$$

Together with  $d \in \delta$ , this implies that there are  $d_1 \in \delta_1$  and  $d_2 \in \delta_2$  such that  $d = d_1 + d_2$ .  
 Next, because  $d_1 \in \delta_1$ , from (iii), by IH

$$\langle o_1, o_2, t_1, t_1 + d_1 \rangle \in \llbracket q \rrbracket_G \quad (36)$$

And similarly, because  $d_2 \in \delta_2$ , from (iv)

$$\langle o_2, o_3, t_2, t_2 + d_2 \rangle \in \llbracket q \rrbracket_G \quad (37)$$

So from (36), (37) and the definition of  $\llbracket q \rrbracket_G$

$$\langle o_1, o_3, t_1, d_1 + d_2 \rangle \in \llbracket q \rrbracket_G \quad (38)$$

Together with the fact that  $d = d_1 + d_2$ , this concludes the proof for (II).  $\square$

### 3.5 In $\mathcal{U}^{[t,d]}$

#### 3.5.1 Definition.

We start with the case where  $q$  is an expression for the symbol test in the grammar of Definition 3.1.

As a consequence of Lemma 3.2,  $\llbracket q \rrbracket_G^{[t,d]}$  can be trivially defined out of  $\llbracket q \rrbracket_G^{[t]}$  by replacing the distance 0 with the interval  $[0, 0]$ , i.e.

$$\llbracket \text{test} \rrbracket_G^{[t,d]} = \{ \langle o, o, \tau, [0, 0] \rangle \mid \langle o, o, \tau, 0 \rangle \in \llbracket \text{test} \rrbracket_G^{[t]} \}$$

Next, if  $q$  is of the form  $(\text{path}_1 + \text{path}_2)$ ,  $(\text{path}[m, \_])$  or  $(\text{path}[m, n])$ , then the definition of  $\llbracket q \rrbracket_G^{[t,d]}$  is once again nearly identical to the one of  $\llbracket q \rrbracket_G$ :

$$\begin{aligned}\llbracket \text{path}_1 + \text{path}_2 \rrbracket_G^{[t,d]} &= \llbracket \text{path}_1 \rrbracket_G^{[t,d]} \cup \llbracket \text{path}_2 \rrbracket_G^{[t,d]} \\ \llbracket \text{path}[m, n] \rrbracket_G &= \bigcup_{k=m}^n \llbracket \text{path}^k \rrbracket_G^{[t,d]} \\ \llbracket \text{path}[m, \_] \rrbracket_G &= \bigcup_{k \geq m} \llbracket \text{path}^k \rrbracket_G^{[t,d]}\end{aligned}$$

The only remaining operators are temporal join  $(\text{path}_1 / \text{path}_2)$  and temporal navigation  $(T_\delta)$ , already defined in the article, and reproduced here for convenience:

$$\begin{aligned}\llbracket \text{path}_1 / \text{path}_2 \rrbracket_G^{[t,d]} &= \bigcup \{ \mathbf{u}_1 \bowtie \mathbf{u}_2 \mid \mathbf{u}_1 \in \llbracket \text{path}_1 \rrbracket_G^{[t,d]}, \mathbf{u}_2 \in \llbracket \text{path}_2 \rrbracket_G^{[t,d]} \} \\ \llbracket T_\delta \rrbracket_G^{[t,d]} &= \bigcup_{o \in N \cup E} \{ \langle o, o, \mathcal{T}_G, \delta \rangle \bowtie \langle o, o, \mathcal{T}_G, [0, 0] \rangle \}\end{aligned}$$

where  $\mathbf{u}_1 \bowtie \mathbf{u}_2$  is defined as follows.

Let  $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$  and  $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2 \rangle$ .

If  $o_2 \neq o_3$ , then  $\mathbf{u}_1 \bowtie \mathbf{u}_2 = \emptyset$ .

Otherwise, let:

$$\begin{aligned}\tau &= (((\tau_1 + \delta_1) \cap \tau_2) \ominus \delta_1) \cap \tau_1 \\ b &= b_{\tau_2} - b_{\tau_1} \\ e &= e_{\tau_2} - e_{\tau_1}\end{aligned}$$

And for every  $t \in \tau$ , let

$$\delta(t) = \delta_1 \downarrow b_{\delta_1} + \max(0, b - t), \quad e_{\delta_1} - \max(0, t - e) \downarrow_{\delta_1}$$

Then

$$\mathbf{u}_1 \bowtie \mathbf{u}_2 = \{ \langle o_1, o_4, [t, t], \delta(t) + \delta_2 \rangle \mid t \in \tau \}$$

### 3.5.2 Correctness.

We start with a lemma

LEMMA 3.5. *Let  $\alpha, \beta \in \text{intv}(\mathcal{T})$ . Then*

$$\beta \ominus \alpha = \{t \mid (t + \alpha) \cap \beta \neq \emptyset\}$$

Next, if  $\mathbf{u} = \langle o_1, o_2, \tau, \delta \rangle \in \mathcal{U}^{[t,d]}$ , we call *temporal relation induced by  $\mathbf{u}$*  the set  $\{(t, t + d) \mid t \in \tau, d \in \delta\}$ .

We also define the binary operator  $\bowtie$ :  $(\mathcal{T} \times \mathcal{T}) \times (\mathcal{T} \times \mathcal{T}) \rightarrow (\mathcal{T} \times \mathcal{T})$  as in the article, i.e.

$$R_1 \bowtie R_2 = \{t_1, t_3 \mid (t_1, t_2) \in R_1 \text{ and } (t_2, t_3) \in R_2 \text{ for some } t_2\}$$

We can now formulate the following lemma:

LEMMA 3.6. *Let  $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}^{[t,d]}$ , and for  $i \in \{1, 2\}$ , let  $R_i$  denote the temporal relation induced by  $\mathbf{u}_i$ . If  $\mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2 \neq \emptyset$ , then*

$$R_1 \bowtie R_2 = \bigcup_{\langle o_1, o_2, \tau, \delta \rangle \in \mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2} \{(t, t + d) \mid t \in \tau, d \in \delta\}$$

The following result states that the representation  $\langle q \rangle_G^{[t,d]}$  is correct:

PROPOSITION 3.7. *Let  $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$  be a TPG and  $q$  a TRPQ. Then the unfolding of  $\langle q \rangle_G^{[t,d]}$  is  $\llbracket q \rrbracket_G$ .*

PROOF.

Let  $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$  be a TG, and let  $q$  be a TRPQ.

We show below that:

- (I) for any  $\langle o_1, o_2, t, d \rangle \in \llbracket q \rrbracket_G$ , there are  $\tau, \delta \in \text{intv}(\mathcal{T})$  such that
  - (a)  $\langle o_1, o_2, \tau, \delta \rangle \in \langle q \rangle_G^{[t,d]}$ ,
  - (b)  $t \in \tau$ , and
  - (c)  $d \in \delta$ .
- (II) for any  $\langle o_1, o_2, \tau, \delta \rangle \in \langle q \rangle_G^{[t,d]}$  for any  $(t, d) \in \tau \times \delta$ ,  $\langle o_1, o_2, t, d \rangle$  is in  $\llbracket q \rrbracket_G$ .

We proceed once again by induction on the structure of  $q$ .

If  $q$  is of the form *pred*, *F*, *B*, (*test*  $\vee$  *test*), (*path* + *path*), *path*[ $m, n$ ] or *path*[ $m, \_$ ], then (I) and (II) immediately follow from the definitions of  $\llbracket q \rrbracket_G$  and  $\langle q \rangle_G^{[t,d]}$ .

If  $q$  is of the form *test*  $\wedge$  *test*,  $\neg$ *test* or (*?path*), then the proof is nearly identical to the one already provided for  $\langle q \rangle_G^{[t]}$ .

So we focus below on the two remaining cases:

- $q = \text{path}_1 / \text{path}_2$ .

From the above definitions, we have:

$$\begin{aligned} \llbracket q \rrbracket_G &= \{ \langle o_1, o_3, t, d_1 + d_2 \rangle \mid \exists o_2 : \langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G \wedge \langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G \} \\ \langle \text{path}_1 / \text{path}_2 \rangle_G^{[t,d]} &= \bigcup \{ \mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2 \mid \mathbf{u}_1 \in \langle \text{path}_1 \rangle_G^{[t,d]}, \mathbf{u}_2 \in \langle \text{path}_2 \rangle_G^{[t,d]} \} \end{aligned}$$

- For (I), let  $\mathbf{v} = \langle o_1, o_3, t, d \rangle \in \llbracket q \rrbracket_G$ .

From the definition of  $\llbracket q \rrbracket_G$ , there are  $o_2, d_1$  and  $d_2$  such that  $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$ ,  $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$  and  $d = d_1 + d_2$ .

By IH, because  $\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G$ , there are  $\tau_1$  and  $\delta_1$  such that  $t \in \tau_1$ ,  $d_1 \in \delta_1$  and

$$\langle o_1, o_2, \tau_1, \delta_1 \rangle \in \langle \text{path}_1 \rangle_G^{[t,d]} \quad (39)$$

Let  $R_1$  be the temporal relation induced by this tuple  $\langle o_1, o_2, \tau_1, \delta_1 \rangle$ .

Since  $t \in \tau_1$  and  $d_1 \in \delta_1$ , we have

$$(t, t + d_1) \in R_1 \quad (40)$$

Similarly, because  $\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G$ , there are  $\tau_2$  and  $\delta_2$  such that  $t + d_1 \in \tau_2$ ,  $d_2 \in \delta_2$  and

$$\langle o_2, o_3, \tau_2, \delta_2 \rangle \in \langle \text{path}_2 \rangle_G^{[t,d]} \quad (41)$$

Let  $R_2$  be the temporal relation induced by this tuple  $\langle o_2, o_3, \tau_2, \delta_2 \rangle$ .

Since  $t + d_1 \in \tau_2$  and  $d_2 \in \delta_2$ , we have

$$(t + d_1, t + d_1 + d_2) \in R_2 \quad (42)$$

So from (40), (42) and Lemma 3.6, there are  $\tau$  and  $\delta$  such that  $\langle o_1, o_3, \tau, \delta \rangle \in \mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2$ ,  $t \in \tau$  and  $d_1 + d_2 = d \in \delta$ , which concludes the proof for (I).

- For (II), let  $\mathbf{u} = \langle o_1, o_3, t_1, \delta \rangle \in \llbracket q \rrbracket_G^{[t,d]}$ , and let  $(t, d) \in \tau \times \delta$ .  
Because  $\mathbf{u} \in \llbracket q \rrbracket_G^{[t,d]}$ , from the definition of  $\llbracket q \rrbracket_G^{[t,d]}$ , there are  $\mathbf{u}_1$  and  $\mathbf{u}_2$  s.t.:

- (i)  $\mathbf{u} \in \mathbf{u}_1 \bowtie \mathbf{u}_2$
- (ii)  $\mathbf{u}_1 \in \llbracket \text{path}_1 \rrbracket_G^{[t,d]}$
- (iii)  $\mathbf{u}_2 \in \llbracket \text{path}_2 \rrbracket_G^{[t,d]}$

Let  $R_i$  be the temporal relation induced by  $u_i$  for  $i \in \{1, 2\}$ .

From (i), and Lemma 3.6,

$$(t, t + d) \in R_1 \bowtie R_2 \quad (43)$$

Now let  $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$  and  $\mathbf{u}_2 = \langle o_2, o_3, \tau_2, \delta_2 \rangle$  for some  $o_2, \tau_1, \tau_2, \delta_1$  and  $\delta_2$ .

From (43) and the definition of  $\bowtie$ , there must be  $d_1$  and  $d_2$  s.t.  $d = d_1 + d_2$ ,  $t \in \tau_1$ ,  $d_1 \in \delta_1$ ,  $t + d_1 \in \tau_2$  and  $d_2 \in \delta_2$ .

So from ii, and iii, by IH

$$\langle o_1, o_2, t, d_1 \rangle \in \llbracket \text{path}_1 \rrbracket_G \quad (44)$$

$$\langle o_2, o_3, t + d_1, d_2 \rangle \in \llbracket \text{path}_2 \rrbracket_G \quad (45)$$

So from (44), (45) and the definition of  $\llbracket q \rrbracket_G$

$$\langle o_1, o_3, t, d_1 + d_2 \rangle \in \llbracket q \rrbracket_G,$$

which concludes the proof for (II). □

### 3.6 In $\mathcal{U}^{[t,d],b,e}$

#### 3.6.1 Definition.

If  $q$  is an expression for the symbol test in the grammar of Definition ??, then the definition of  $\llbracket q \rrbracket_G^{[t,d],b,e}$  is nearly identical to the one of  $\llbracket q \rrbracket_G^{[t,d]}$ , extending each tuple  $\{\langle o, o, \tau, [0, 0] \rangle\}$  with  $b_\tau$  and  $e_\tau$ , i.e.

$$\llbracket \text{test} \rrbracket_G^{[t,d],b,e} = \{ \langle o, o, \tau, [0, 0] \rangle, b_\tau, e_\tau \mid \langle o, o, \tau, [0, 0] \rangle \in \llbracket \text{test} \rrbracket_G^{[t,d]} \}$$

Next, if  $q$  is of the form  $(\text{path}_1 + \text{path}_2)$ ,  $(\text{path}[m, \_])$  or  $(\text{path}[m, n])$ , then the definition of  $\llbracket q \rrbracket_G^{[t,d]}$  is once again nearly identical to the one of  $\llbracket q \rrbracket_G$ :

$$\begin{aligned} \llbracket \text{path}_1 + \text{path}_2 \rrbracket_G^{[t,d],b,e} &= \llbracket \text{path}_1 \rrbracket_G^{[t,d],b,e} \cup \llbracket \text{path}_2 \rrbracket_G^{[t,d],b,e} \\ \llbracket \text{path}[m, n] \rrbracket_G &= \bigcup_{k=m}^n \llbracket \text{path}^k \rrbracket_G^{[t,d],b,e} \\ \llbracket \text{path}[m, \_] \rrbracket_G &= \bigcup_{k \geq m} \llbracket \text{path}^k \rrbracket_G^{[t,d],b,e} \end{aligned}$$

So the only remaining operator are temporal join  $(\text{path}_1/\text{path}_2)$  and temporal navigation  $(T_\delta)$ , already defined in the article. We reproduce here these two definition for convenience:

$$\begin{aligned} \llbracket \text{path}_1/\text{path}_2 \rrbracket_G^{[t,d],b,e} &= \{ \mathbf{u}_1 \bowtie \mathbf{u}_2 \mid \mathbf{u}_1 \in \llbracket \text{path}_1 \rrbracket_G^{[t,d],b,e}, \mathbf{u}_2 \in \llbracket \text{path}_2 \rrbracket_G^{[t,d],b,e}, \mathbf{u}_1 \sim \mathbf{u}_2 \} \\ \llbracket T_\delta \rrbracket_G^{[t,d],b,e} &= \{ \langle o, o, \mathcal{T}_G, \delta, b_{\mathcal{T}_G}, e_{\mathcal{T}_G} \rangle \bowtie \langle o, o, \mathcal{T}_G, [0, 0], b_{\mathcal{T}_G}, e_{\mathcal{T}_G} \rangle \mid o \in N \cup E \} \end{aligned}$$

where  $\mathbf{u}_1 \sim \mathbf{u}_2$  and  $\mathbf{u}_1 \bowtie \mathbf{u}_2$  are defined as follows.

Let  $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1, b_1, e_1 \rangle$  and  $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2, b_2, e_2 \rangle$ .

Define

$$\delta'_1 = \delta_1 \sqcup b_{\delta_1} + \max(0, b_1 - b_{\tau_1}), e_{\delta_1} - \max(0, e_{\tau_1} - e_1) \sqcup_{\delta_1}$$

and

$$\tau = (((\tau_1 + \delta'_1) \cap \tau_2) \ominus \delta'_1) \cap \tau_1$$

Then  $\mathbf{u}_1 \sim \mathbf{u}_2$  iff  $o_2 = o_3$  and  $\tau \neq \emptyset$ . If  $\mathbf{u}_1 \sim \mathbf{u}_2$ , then  $\mathbf{u}_1 \bowtie \mathbf{u}_2 = \langle o_1, o_4, \tau, \delta_1 + \delta_2, b, e \rangle$ , with

$$\begin{aligned} b &= \max(b_1, b_2 - b_{\delta_1}) \\ e &= \min(e_1, e_2 - e_{\delta_1}) \end{aligned}$$

3.6.2 *Correctness.* Similarly to what we did above for  $\mathcal{U}^{[t,d]}$ , if  $\mathbf{u} = \langle o_1, o_2, \text{tau}, \delta, b, e \rangle \in \mathcal{U}^{[t,d],b,e}$ , we call *temporal relation induced by  $\mathbf{u}$*  the set  $\{(t, t+d) \mid t \in \tau, d \in \delta(t)\}$ .

We can now formulate a lemma analogous to Lemma 3.6:

LEMMA 3.8. *Let  $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}^{[t,d],b,e}$ , and for  $i \in \{1, 2\}$ , let  $R_i$  denote the temporal relation induced by  $\mathbf{u}_i$ . If  $\mathbf{u}_1 \sim \mathbf{u}_2$  and  $\mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2 = \langle o_1, o_3, \tau, \delta, b, e \rangle$ , then*

$$R_1 \bowtie R_2 = \{(t, t+d) \mid t \in \tau, d \in \delta(t)\}$$

PROOF. Let  $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$  and  $\mathbf{u}_2 = \langle o_2, o_3, \tau_2, \delta_2 \rangle$ . As explained in Section 2, for  $i \in \{1, 2\}$  and  $t \in \tau_i$ , we use  $\delta_i(t)$  for the interval

$$\delta_i \downarrow b_{\delta_i} + \max(0, b_i - t), e_{\delta_i} - \max(0, t - e_i) \downarrow \delta_i$$

We need to prove that (i)  $\tau = \text{dom}(R_1 \bowtie R_2)$  and that (ii) for each  $t \in \tau$ ,

$$t + \delta(t) = \{t' \mid (t, t') \in R_1 \bowtie R_2\}$$

The proof of (i) is nearly identical to the one provided above for Lemma 3.6.

For (ii), let  $t \in \tau$ . We use

- $a$  for the least value s.t.  $(t, a) \in \text{range}(R_1) \cap \text{dom}(R_2)$ , and
- $a'$  for the least value s.t.  $(a, a') \in R_2$

Then  $a'$  is also the least value s.t.  $(t, a') \in R_1 \bowtie R_2$ .

Analogously, we use  $z$  for the greatest value s.t.  $(t, z) \in \text{range}(R_1) \cap \text{dom}(R_2)$ , and  $z'$  for the greatest value s.t.  $(z, z') \in R_2$ .

Then  $z'$  is also the greatest value s.t.  $(t, z') \in R_1 \bowtie R_2$ .

From Lemma ??:

- $\{t\} \times (t + [a, z]) \subseteq R_1$ , and
- $[a, b] \times [a', z'] \subseteq R_2$

Therefore  $[a', z'] = \{c \mid (t, c) \in R_1 \bowtie R_2\}$ .

To conclude the proof, we show that  $t + \delta_t = [a, z]$ .

We only prove that  $t + b_{\delta_t} = a$  (the proof that  $t + e_{\delta_t} = z$  is symmetric).

Following the definition of  $b$ , we consider 2 cases:

- (1)  $b_1 < b_2 - b_{\delta_1}$
- (2)  $b_1 \geq b_2 - b_{\delta_1}$

For Case (1), we get

$$b_1 < b_2 - b_{\delta_1} \tag{46}$$

$$\max(b_1, b_2 - b_{\delta_1}) = b_2 - b_{\delta_1} \tag{47}$$

$$b = b_2 - b_{\delta_1} \quad \text{from the definition of } b \tag{48}$$

And in Case (1) still, we get:

$$b_1 < b_2 - b_{\delta_1} \tag{49}$$

$$0 < b_2 - b_{\delta_1} - b_1 \tag{50}$$

$$\max(0, b_2 - b_{\delta_1} - b_1) = b_2 - b_{\delta_1} - b_1 \tag{51}$$

Next, we consider two subcases:

- (i)  $t < b_2 - b_{\delta_1}$
- (ii)  $t \geq b_2 - b_{\delta_1}$

In Case (i), we get

$$t < b_2 - b_{\delta_1} \tag{52}$$

$$0 < b_2 - b_{\delta_1} - t \tag{53}$$

$$\max(0, b_2 - b_{\delta_1} - t) = b_2 - b_{\delta_1} - t \tag{54}$$

Now from the definition of  $\delta_t$ ,

$$b_{\delta_t} = b_{\delta_1} + b_{\delta_2} + \max(0, b - t) \quad (55)$$

$$= b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_{\delta_1} - t) \quad \text{from (48)} \quad (56)$$

$$= b_{\delta_1} + b_{\delta_2} + b_2 - b_{\delta_1} - t \quad \text{from (54)} \quad (57)$$

$$= b_{\delta_2} + b_2 - t \quad (58)$$

$$b_{\delta_t} + t = b_{\delta_2} + b_2 - t + t \quad (59)$$

$$= b_{\delta_2} + b_2 \quad (60)$$

Next, from the definition of  $a'$

$$a' = b_{\delta_2(a)} + a \quad (61)$$

$$= b_{\delta_2} + \max(0, b_2 - a) + a \quad (62)$$

And, from the definition of  $a$

$$a = b_{\delta_1(t)} + t \quad (63)$$

$$= b_{\delta_1} + \max(0, b_1 - t) + t \quad (64)$$

Then we have two further subcases:

(I)  $t \geq b_1$ , or

(II)  $t < b_1$

In case (I):

$$t \geq b_1 \quad (65)$$

$$0 \geq b_1 - t \quad (66)$$

$$\max(0, b_1 - t) = 0 \quad (67)$$

$$a = b_{\delta_1} + t \quad \text{from (64)} \quad (68)$$

$$\max(0, b_2 - a) = \max(0, b_2 - b_{\delta_1} - t) \quad (69)$$

$$= b_2 - b_{\delta_1} - t \quad \text{from (54)} \quad (70)$$

$$= b_2 - a \quad \text{from (68)} \quad (71)$$

In case (II):

$$t < b_1 \quad (72)$$

$$0 < b_1 - t \quad (73)$$

$$\max(0, b_1 - t) = b_1 - t \quad (74)$$

$$a = b_{\delta_1} + b_1 - t + t \quad \text{from (64)} \quad (75)$$

$$= b_{\delta_1} + b_1 \quad (76)$$

$$\max(0, b_2 - a) = \max(0, b_2 - b_{\delta_1} - b_1) \quad (77)$$

$$= b_2 - b_{\delta_1} - b_1 \quad \text{from (51)} \quad (78)$$

$$= b_2 - a \quad \text{from (76)} \quad (79)$$

$$(80)$$

So in both cases (I) and (II), we get

$$\max(0, b_2 - a) = b_2 - a$$

Therefore from (62)

$$a' = b_{\delta_2} + b_2 - a + a \quad (81)$$

$$= b_{\delta_2} + b_2 \quad (82)$$

$$= t + b_{\delta_t} \quad \text{from (60)} \quad (83)$$

which concludes the proof for Case (1)- (i).

We continue with Case (1)- (ii).

From Case (ii):

$$t \geq b_2 - b_{\delta_1} \quad (84)$$

$$0 \geq b_2 - b_{\delta_1} - t \quad (85)$$

$$\max(0, b_2 - b_{\delta_1} - t) = 0 \quad (86)$$

Now from the definition of  $\delta_t$ :

$$b_{\delta_t} = b_{\delta_1} + b_{\delta_2} + \max(0, b - t) \quad (87)$$

$$= b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_1 - t) \quad \text{from (48)} \quad (88)$$

$$= b_{\delta_1} + b_{\delta_2} \quad \text{from (86)} \quad (89)$$

$$b_{\delta_t} + t = b_{\delta_1} + b_{\delta_2} + t \quad (90)$$

Next, from Case (1) and Case (ii), by transitivity, we get

$$b_1 \leq t \quad (91)$$

$$\max(0, b_1 - t) = 0 \quad (92)$$

And from the definition of  $a$

$$a = b_{\delta_1(t)} + t \quad (93)$$

$$= b_{\delta_1} + \max(0, b_1 - t) + t \quad (94)$$

$$= b_{\delta_1} + t \quad \text{from (92)} \quad (95)$$

$$\geq b_{\delta_1} + b_2 - b_{\delta_1} \quad \text{from Case (ii)} \quad (96)$$

$$\geq b_2 \quad (97)$$

$$0 \geq b_2 - a \quad (98)$$

$$\max(0, b_2 - a) = 0 \quad (99)$$

Therefore from (62) and (99)

$$a' = b_{\delta_2} + a \quad (100)$$

$$= b_{\delta_2} + b_{\delta_1} + t \quad \text{from (95)} \quad (101)$$

$$= b_{\delta_t} + t \quad \text{from (60)} \quad (102)$$

which concludes the proof for Case (1)- (ii).

We continue with Case (2).

In this case, we get

$$b_1 \geq b_2 - b_{\delta_1} \quad (103)$$

$$\max(b_1, b_2 - b_{\delta_1}) = b_1 \quad (104)$$

$$b = b_1 \quad \text{from the definition of } b \quad (105)$$

And from Case (2) still, we derive

$$b_1 \geq b_2 - b_{\delta_1} \quad (106)$$

$$0 \geq b_2 - b_{\delta_1} - b_1 \quad (107)$$

$$\max(0, -b_{\delta_1} - b_1) = 0 \quad (108)$$

As well as

$$b_1 \geq b_2 - b_{\delta_1} \quad (109)$$

$$b_1 + b_{\delta_1} \geq b_2 \quad (110)$$

Next, we distinguish two subcases, namely

(a)  $t < b_1$  and

(b)  $t \geq b_1$

We start with Case (a).

In this case,

$$t < b_1 \quad (111)$$

$$0 < b_1 - t \quad (112)$$

$$\max(0, b_1 - t) = b_1 - t \quad (113)$$

And from the definition of  $\delta_t$ :

$$b_{\delta_t} = b_{\delta_1} + b_{\delta_2} + \max(0, b - t) \quad (114)$$

$$= b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) \quad \text{from (105)} \quad (115)$$

$$= b_{\delta_1} + b_{\delta_2} + b_1 - t \quad \text{from (113)} \quad (116)$$

$$b_{\delta_t} + t = b_{\delta_1} + b_{\delta_2} + b_1 - t + t \quad (117)$$

$$= b_{\delta_1} + b_{\delta_2} + b_1 \quad (118)$$

Next, from the definition of  $a$

$$a = b_{\delta_1}(t) + t \quad (119)$$

$$= \max(0, b_1 - t) + b_{\delta_1} + t \quad (120)$$

$$= b_1 - t + b_{\delta_1} + t \quad \text{from (113)} \quad (121)$$

$$= b_1 + b_{\delta_1} \quad (122)$$

So from (110)

$$a \geq b_2 \quad (123)$$

$$0 \geq b_2 - a \quad (124)$$

$$\max(0, b_2 - a) = 0 \quad (125)$$

$$b_{\delta_2} + \max(0, b_2 - a) = b_{\delta_2} \quad (126)$$

$$b_{\delta_2(a)} = b_{\delta_2} \quad (127)$$

$$b_{\delta_2(a)} + a = b_{\delta_2} + a \quad (128)$$

$$a' = b_{\delta_2} + a \quad \text{from the definition of } a' \quad (129)$$

$$a' = b_{\delta_2} + b_1 + b_{\delta_1} \quad \text{from (122)} \quad (130)$$

$$a' = b_{\delta_t} + t \quad \text{from (118)} \quad (131)$$

which concludes the proof for Case (2)- (a).

We end with Case (2)- (b). In this case,

$$t \geq b_1 \quad (132)$$

$$0 \geq b_1 - t \quad (133)$$

$$\max(0, b_1 - t) = 0 \quad (134)$$

And from the definition of  $\delta_t$ :

$$b_{\delta_t} = b_{\delta_1} + b_{\delta_2} + \max(0, b - t) \quad (135)$$

$$= b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) \quad \text{from (105)} \quad (136)$$

$$= b_{\delta_1} + b_{\delta_2} \quad \text{from (135)} \quad (137)$$

$$b_{\delta_t} + t = b_{\delta_1} + b_{\delta_2} + t \quad (138)$$

Next, from the definition of  $a$

$$a = b_{\delta_1}(t) + t \quad (139)$$

$$= \max(0, b_1 - t) + b_{\delta_1} + t \quad (140)$$

$$= b_{\delta_1} + t \quad \text{from (135)} \quad (141)$$

Now from Case (b)

$$b_1 + \leq t \quad (143)$$

$$b_1 + b_{\delta_1} \leq t + b_{\delta_1} \quad (144)$$

$$b_1 + b_{\delta_1} \leq a \quad \text{from (142)} \quad (145)$$

$$b_2 \leq a \quad \text{from (110), by transitivity} \quad (146)$$

$$b_2 - a \leq 0 \quad (147)$$

$$\max(0, b_2 - a) = 0 \quad (148)$$

$$b_{\delta_2} + \max(0, b_2 - a) = b_{\delta_2} \quad (149)$$

$$b_{\delta_2(a)} = b_{\delta_2} \quad (150)$$

$$b_{\delta_2(a)} + a = b_{\delta_2} + a \quad (151)$$

$$a' = b_{\delta_2} + a \quad \text{from the definition of } a' \quad (152)$$

$$= b_{\delta_2} + b_{\delta_1} + t \quad \text{from (142)} \quad (153)$$

$$= b_{\delta_t} + t \quad \text{from (139)} \quad (154)$$

which concludes the proof for Case (2)- (b).  $\square$

The following result states that the representation  $\langle q \rangle_G^{[t,d],b,e}$  is correct:

**PROPOSITION 3.9.** *Let  $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$  be a TPG and  $q$  a TRPQ. Then the unfolding of  $\langle q \rangle_G^{[t,d],b,e}$  is  $\llbracket q \rrbracket_G$ .*

**PROOF.** Let  $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val} \rangle$  be a TG, and let  $q$  be a TRPQ.

To prove the result, it is sufficient to show that:

- (I) for any  $\langle o_1, o_2, t, d \rangle \in \llbracket q \rrbracket_G$ , there are  $\tau, \delta \in \text{intv}(\mathcal{T})$  and  $b, e \in \mathcal{T}$  such that
  - (a)  $\langle o_1, o_2, \tau, \delta, b, e \rangle \in \langle q \rangle_G^{[t,d],b,e}$ ,
  - (b)  $t \in \tau$ , and
  - (c)  $d \in \delta(t)$  (where  $\delta(t)$  is defined in terms of  $t, \delta, b$  and  $e$ , as explained above).
- (II) for any  $\langle o_1, o_2, \tau, \delta, b, e \rangle \in \langle q \rangle_G^{[t,d],b,e}$  for any  $t \in \tau$  and  $d \in \delta(t)$ ,
  - $\langle o_1, o_2, t, d \rangle$  is in  $\llbracket q \rrbracket_G$ .

Again, the proof is by induction on the structure of  $q$ .

If  $q$  is of the form  $\text{pred}$ , F, B,  $(\text{test} \vee \text{test})$ ,  $(\text{path} + \text{path})$ ,  $\text{path}[m, n]$  or  $\text{path}[m, \_]$ , then (I) and (II) immediately follow from the definitions of  $\llbracket q \rrbracket_G$  and  $\langle q \rangle_G^{[t,d],b,e}$ .

If  $q$  is of the form  $\text{test} \wedge \text{test}$ ,  $\neg \text{test}$  or  $(? \text{path})$ , then the proof is nearly identical to the one already provided for  $\langle q \rangle_G^{[t]}$ .

And if  $q$  is of the form  $T_\delta$  or  $\text{path}_1/\text{path}_2$ , then the proof is nearly identical to the one already provided for  $\langle q \rangle_G^{[t,d]}$ , using Lemma 3.8 instead of 3.6.  $\square$

## 4 FINITENESS

## 5 COMPLEXITY OF QUERY ANSWERING

### 5.1 Problem

We define in this section a decision problem for each representation, similar to the problem **COMPACT ANSWER**<sup>[t]</sup> defined in the article.

First, we define a (possibly partial) order over tuples of each representation, in the expected way, i.e.:

- over  $\mathcal{U}^{[t]}$ :  $\langle o_1, o_2, \tau_1, d_1 \rangle \sqsubseteq_{[t]} \langle o_3, o_4, \tau_2, d_2 \rangle$  iff  $\langle o_1, o_2, d_1 \rangle = \langle o_3, o_4, d_2 \rangle$  and  $\tau_1 \subseteq \tau_2$ ,
- over  $\mathcal{U}^{[d]}$ :  $\langle o_1, o_2, t_1, \delta_1 \rangle \sqsubseteq_{[d]} \langle o_3, o_4, t_2, \delta_2 \rangle$  iff  $\langle o_1, o_2, t_1 \rangle = \langle o_3, o_4, t_2 \rangle$  and  $\delta_1 \subseteq \delta_2$ ,
- over  $\mathcal{U}^{[t,d]}$ :  $\langle o_1, o_2, \tau_1, \delta_1 \rangle \sqsubseteq_{[t,d]} \langle o_3, o_4, \tau_2, \delta_2 \rangle$  iff  $\langle o_1, o_2 \rangle = \langle o_3, o_4 \rangle$ ,  $\tau_1 \subseteq \tau_2$  and  $\delta_1 \subseteq \delta_2$ ,
- over  $\mathcal{U}^{[t,d],b,e}$ :  $\langle o_1, o_2, \tau_1, \delta_1, b_1, b_2 \rangle \sqsubseteq_{[t,d],b,e} \langle o_3, o_4, \tau_2, \delta_2, b_2, e_2 \rangle$  iff  $\langle o_1, o_2 \rangle = \langle o_3, o_4 \rangle$ ,  $\tau_1 \subseteq \tau_2$  and  $\delta_1(t) \subseteq \delta_2(t)$  for all  $t \in \tau_1 \cap \tau_2$  (the notation of  $\delta_i(t)$  is explained above, in Section 2).

Now let  $x$  be one of  $[t]$ ,  $[d]$ ,  $[t, d]$  or  $[t, d, b, e]$ .

We decline the notion of compact answer defined in Section XXX in four flavors, as follows:

**Definition 5.1.** Let  $G$  be a TG, let  $q$  be a TRPQ and let  $\mathbf{u} \in \mathcal{U}^x$ .

We say that  $\mathbf{u}$  is a *compact answer* to  $q$  over  $G$  (in  $\mathcal{U}^x$ ) if  $\mathbf{u} \in \max_{\sqsubseteq_x} \{\mathbf{u}' \in \mathcal{U}^x \mid \text{unfold}(\mathbf{u}') = \llbracket q \rrbracket_G\}$



And similarly, we get four decision problems:

COMPACT ANSWER<sup>x</sup>

**Input:** TG  $G$ , TRPQ  $q$ , tuple  $\mathbf{u} \in \mathcal{U}^x$

**Decide:**  $\mathbf{u}$  is a compact answer to  $q$  over  $G$  (in  $\mathcal{U}^x$ )

## 5.2 Hardness

## 5.3 Membership

## 6 MINIMIZATION

## 7 SIZE OF COMPACT ANSWERS