Compact Answers to Temporal Regular Path Queries (Supplementary Material)

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1 INTRODUCTION

This document provides detailed definitions and proofs for the article *Compact answers to Temporal Regular Path Queries*, submitted at CIKM 2023.

2 NOTATION

Let $\langle o_1, o_2, \tau, \delta, b, e \rangle \in \mathcal{U}^{[t,d],b,e}$, and let $t \in \tau$.
In the article, we defined the interval δ_t for each t as

$$_{\delta}\lfloor\ b_{\delta} + \max(0, b - t)\ ,\ e_{\delta_i} - \max(0, t - e)\ \rfloor_{\delta}$$

In this supplementary material, we will use $\delta(t)$ instead of δ_t. This notation will allow us to write $\delta_1(t)$ when several tuples are involved. Note that the time points b and e in this notaton are still omitted, for conciseness, because they should be clear from the context.

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3 INDUCTIVE REPRESENTATION

Let q be a TRPQ and G a TG.
Then $[\![q]\!]_G$ is the set of anwers to q over G (represented as tuples in $\mathcal{U}$).
In this section, we provide the full definition of the four inductive representations of $[\![q]\!]_G$ discussed in the article, in $\mathcal{U}^{[t]}$, $\mathcal{U}^{[d]}$, $\mathcal{U}^{[t,d]}$ and $\mathcal{U}^{[t,d],b,e}$ respectively, and prove that they are correct.
These representations are denoted as $(\!|q|\!)_G^{[t]}$, $(\!|q|\!)_G^{[d]}$, $(\!|q|\!)_G^{[t,d]}$ and $(\!|q|\!)_G^{[t,d],b,e}$ respectively.

3.1 Grammar

For convenience, we reproduce here the syntax of TRPQs (already provided in the article).

Definition 3.1 (TRPQ). A TRPQ is an expression for the symbol path in the following grammar:

$$\begin{aligned}
\text{path} &::= \text{test} \mid \text{axis} \mid (\text{path}/\text{path}) \mid (\text{path} + \text{path}) \mid \text{path}[m,n] \mid \text{path}[m,_] \\
\text{test} &::= \mathit{pred} \mid (?\text{path}) \mid \text{test} \vee \text{test} \mid \text{test} \wedge \text{test} \mid \neg\text{test} \\
\text{axis} &::= \mathrm{F} \mid \mathrm{B} \mid \mathrm{T}_\delta
\end{aligned}$$

with $\delta \in \text{intv}(\mathcal{T})$, $m, n \in \mathbb{N}^+$ and $m \leq n$.

3.2 In $\mathcal{U}$

For convenience still, we reproduce the full semantics of TRPQs, i.e. the inductive definition of the evaluation $[\![q]\!]_G$ of a query q over a graph G in $\mathcal{U}$ (already provided in the article).

$$\begin{aligned}
[\![\mathit{pred}]\!]_G &= \{\langle o,o,t,0\rangle \mid t \in \tau \text{ for some } \tau \in \text{val}(o,\mathit{pred})\} \\
[\![\mathrm{T}_\delta]\!]_G &= \{\langle o,o,t,d\rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t+d \in \mathcal{T}_G\} \\
[\![\mathrm{F}]\!]_G &= \{\langle v,e,t,0\rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e,v,t,0\rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G\} \\
[\![\mathrm{B}]\!]_G &= \{\langle v,e,t,0\rangle \mid \text{tgt}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e,v,t,0\rangle \mid \text{src}(e) = v, t \in \mathcal{T}_G\} \\
[\![?\text{path}]\!]_G &= \{\langle o,o,t,0\rangle \mid \langle o,o',t,t+d\rangle \in [\![\text{path}]\!]_G \text{ for some } o' \in N \cup E, d \in \mathcal{T}\} \\
[\![\text{test}_1 \vee \text{test}_2]\!]_G &= [\![\text{test}_1]\!]_G \cup [\![\text{test}_2]\!]_G \\
[\![\text{test}_1 \wedge \text{test}_2]\!]_G &= [\![\text{test}_1]\!]_G \cap [\![\text{test}_2]\!]_G \\
[\![\neg\text{test}]\!]_G &= (\{\langle o,o\rangle \mid o \in N \cup E\} \times \mathcal{T}_G \times \{0\}) \setminus [\![\text{test}]\!]_G \\
[\![\text{path}_1/\text{path}_2]\!]_G &= \{\langle o_1,o_3,t,d_1+d_2\rangle \mid \exists o_2 \colon \langle o_1,o_2,t,d_1\rangle \in [\![\text{path}_1]\!]_G \wedge \langle o_2,o_3,t+d_1,d_2\rangle \in [\![\text{path}_2]\!]_G\} \\
[\![\text{path}_1 + \text{path}_2]\!]_G &= [\![\text{path}_1]\!]_G \cup [\![\text{path}_2]\!]_G \\
[\![\text{path}[m,n]]\!]_G &= \bigcup_{k=m}^{n} [\![\text{path}^k]\!]_G \\
[\![\text{path}[m,_]]\!]_G &= \bigcup_{k \geq m} [\![\text{path}^k]\!]_G
\end{aligned}$$

3.3 In $\mathcal{U}^{[t]}$

3.3.1 *Definition.*

The full definition of $(\!|q|\!)_G^{[t]}$ is already provided in the article. We only reproduce it here for convenience.

$$\begin{aligned}
(\!|\mathit{pred}|\!)_G^{[t]} &= \{\langle o,o,\tau,0\rangle \mid o \in (N \cup E), \tau \in \text{val}(o,\mathit{pred})\} \\
(\!|\mathrm{T}_\delta|\!)_G^{[t]} &= \{\langle o,o,[t_1,t_1],t_2-t_1\rangle \mid o \in (N \cup E), t_1 \in \mathcal{T}_G, t_2 \in (\delta + t_1) \cap \mathcal{T}_G\} \\
(\!|\mathrm{F}|\!)_G^{[t]} &= \{\langle v,e,\mathcal{T}_G,0\rangle \mid \text{src}(e) = v\} \cup \{\langle e,v,\mathcal{T}_G,0\rangle \mid \text{tgt}(e) = v\} \\
(\!|\mathrm{B}|\!)_G^{[t]} &= \{\langle v,e,\mathcal{T}_G,0\rangle \mid \text{tgt}(e) = v\} \cup \{\langle e,v,\mathcal{T}_G,0\rangle \mid \text{src}(e) = v\} \\
(\!|(?\text{path})|\!)_G^{[t]} &= \{\langle o,o,\tau,0\rangle \mid \langle o,o',\tau,d\rangle \in (\!|\text{path}|\!)_G^{[t]} \text{ for some } o' \in N \cup E, d \in \mathcal{T}\} \\
(\!|\text{test}_1 \vee \text{test}_2|\!)_G^{[t]} &= (\!|\text{test}_1|\!)_G^{[t]} \cup (\!|\text{test}_2|\!)_G^{[t]} \\
(\!|\text{test}_1 \wedge \text{test}_2|\!)_G^{[t]} &= \{\langle o,o,\tau_1 \cap \tau_2,0\rangle \mid \langle o,o,\tau_1,0\rangle \in (\!|\text{test}_1|\!)_G^{[t]}, \langle o,o,\tau_2,0\rangle \in (\!|\text{test}_2|\!)_G^{[t]}, \tau_1 \cap \tau_2 \neq \emptyset\} \\
(\!|\neg\text{test}|\!)_G^{[t]} &= \bigcup_{o \in N \cup E} \left\{\langle o,o,\tau,0\rangle \mid \tau \in \text{compl}\left(\{\tau' \mid \langle o,o,\tau',0\rangle \in (\!|\text{test}|\!)_G^{[t]}\}, \mathcal{T}_G\right)\right\} \\
(\!|\text{path}_1/\text{path}_2|\!)_G^{[t]} &= \left\{\langle o_1,o_3,((\tau_1+d_1) \cap \tau_2) - d_1, d_1+d_2\rangle \mid \exists o_2 \colon \langle o_1,o_2,\tau_1,d_1\rangle \in (\!|\text{path}_1|\!)_G^{[t]} \wedge \langle o_2,o_3,\tau_2,d_2\rangle \in (\!|\text{path}_2|\!)_G^{[t]} \wedge (\tau_1+d_1) \cap \tau_2 \neq \emptyset\right\} \\
(\!|\text{path}_1 + \text{path}_2|\!)_G^{[t]} &= (\!|\text{path}_1|\!)_G^{[t]} \cup (\!|\text{path}_2|\!)_G^{[t]} \\
(\!|\text{path}[m,n]|\!)_G^{[t]} &= \bigcup_{k=m}^{n} (\!|\text{path}^k|\!)_G^{[t]} \\
(\!|\text{path}[m,_]|\!)_G^{[t]} &= \bigcup_{k \geq m} (\!|\text{path}^k|\!)_G^{[t]}
\end{aligned}$$

We observe that when q is of the form $(\text{path}_1 + \text{path}_2)$, $(\text{path}[m, \_])$ and $(\text{path}[m, n])$, the defnition of $(\!|q|\!)^{[t]}_G$ is nearly identical to the one of $[\![q]\!]_G$. This will also be the case for the three representations below.

3.3.2 *Correctness.*

We start with a lemma:

Lemma 3.2. *Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val}\rangle$ be a TPG and let q an expression for the symbol* test *in the grammar of Definition 3.1. Then:*

- *each tuples in $[\![q]\!]_G$ is of the form $\langle o_1, o_2, t, 0\rangle$ for some o_1, o_2 and t,*
- *each tuples in $(\!|q|\!)^{[t]}_G$ is of the form $\langle o_1, o_2, \tau, 0\rangle$ for some o_1, o_2 and τ.*

Proof. Immediate from the definitions of $[\![q]\!]_G$ and $(\!|q|\!)^{[t]}_G$. □

The following result states that the representation $(\!|q|\!)^{[t]}_G$ is correct:

Proposition 3.3. *Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val}\rangle$ be a TPG and q a TRPQ. Then the unfolding of $(\!|q|\!)^{[t]}_G$ is $[\![q]\!]_G$.*

Proof.
Let $G = \langle N, E, \text{conn}, \mathcal{T}_G, \text{val}\rangle$ be a TG, and let q be a TRPQ.
We show below that:

(I) for any $\langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$, there is a $\tau \in \text{intv}(\mathcal{T})$ such that
  (a) $\langle o_1, o_2, \tau, d\rangle \in (\!|q|\!)^{[t]}_G$, and
  (b) $t \in \tau$,
(II) for any $\langle o_1, o_2, \tau, d\rangle \in (\!|q|\!)^{[t]}_G$ for any $t \in \tau$,
  $\langle o_1, o_2, t, d\rangle$ is in $[\![q]\!]_G$.

We proceed by induction on the structure of q.
If q is of the form *pred*, F, B, (test ∨ test), (path + path), path$[m, n]$ or path$[m, _]$, then (I) and (II) immediately follow from the definitions of $[\![q]\!]_G$ and $(\!|q|\!)^{[t]}_G$.
So we focus below on the five remaining cases:

- $q = \text{T}_\delta$.
  From the above definitions, we have:

$$\begin{aligned}
[\![q]\!]_G &= \{\langle o, o, t, d\rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G\} \\
(\!|q|\!)^{[t]}_G &= \{\langle o, o, [t_1, t_1], t_2 - t_1\rangle \mid o \in (N \cup E), t_1 \in \mathcal{T}_G, t_2 \in (\delta + t_1) \cap \mathcal{T}_G\}
\end{aligned}$$

  – For (I), let $\mathbf{v} = \langle o, o, t, t + d\rangle \in [\![q]\!]_G$.
    And let $\mathbf{u} = \langle o, o, [t, t], d\rangle$ in $\mathcal{U}^{[t]}$.
    For (Ia) we show that $\mathbf{u} \in (\!|q|\!)^{[t]}_G$.
    From $\mathbf{v} \in [\![q]\!]_G$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.
    Besides, because $\mathbf{v} \in [\![q]\!]_G$ still,

$$t + d \in \mathcal{T}_G \tag{1}$$

    and

$$d \in \delta \tag{2}$$

$$t + d \in t + \delta \tag{3}$$

    So from (1) and (3)

$$t + d \in (\delta + t) \cap \mathcal{T}_G \tag{4}$$

    So there is a t_2 (namely $t + d$) such that $d = t_2 - t$ and $t_2 \in t + \delta \cap \mathcal{T}_G$.
    Together with the definition of $(\!|q|\!)^{[t]}_G$, this implies $\mathbf{u} \in (\!|q|\!)^{[t]}_G$, which concludes the proof for (Ia).
    And trivially, $t \in [t, t]$, so (Ib) is verified as well.

- For (II), let $\mathbf{u} = \langle o, o, [t, t], d \rangle \in (\!|q|\!)^{[t]}_G$.
  From $\mathbf{u} \in (\!|q|\!)^{[t]}_G G$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.
  So to conclude the proof, it is sufficient to show that *(i)* $d \in \delta$ and *(ii)* $t + d \in \mathcal{T}_G$.
  Because $\mathbf{u} \in (\!|q|\!)^{[t]}_G G$ still, we have
  $$d = t_2 - t \text{ for some } t_2 \in (\delta + t) \cap \mathcal{T}_G \tag{5}$$
  From (5), we get $t_2 = t + d$.
  Therefore from (5) still,
  $$t + d \in (\delta + t) \cap \mathcal{T}_G \tag{6}$$
  which proves *(ii)* .
  And from (6), we also get
  $$\begin{aligned} t + d &\in \delta + t \\ t + d - t &\in (\delta + t) - t \\ d &\in \delta \end{aligned}$$
  which proves *(i)* .

- $q = \text{test}_1 \wedge \text{test}_2$.
  From the above definitions, we have:
  $$\begin{aligned} [\![q]\!]_G &= [\![\text{test}_1]\!]_G \cap [\![\text{test}_2]\!]_G \\ (\!|q|\!)^{[t]}_G &= \{\langle o, o, \tau_1 \cap \tau_2, 0 \rangle \mid \langle o, o, \tau_1, 0 \rangle \in (\!|\text{test}_1|\!)^{[t]}_G, \langle o, o, \tau_2, 0 \rangle \in (\!|\text{test}_2|\!)^{[t]}_G, \tau_1 \cap \tau_2 \neq \emptyset\} \end{aligned}$$

  - For (I), let $\mathbf{v} = \langle o, o, t, d \rangle \in [\![q]\!]_G$.
    From Lemma 3.2, $d = 0$.
    And from the definition of $[\![q]\!]_G$, $\mathbf{v} \in [\![\text{test}_1]\!]_G \cap [\![\text{test}_2]\!]_G$.
    So by IH, there are intervals τ_1 and τ_2 s.t. $\langle o, o, \tau_i, 0 \rangle \in (\!|\text{test}_i|\!)^{[t]}_G$ for $i \in \{1, 2\}$ and $t \in \tau_1 \cap \tau_2$.
    Together with the definition of $(\!|q|\!)^{[t]}_G$, this proves (I).
  - For (II), let $\langle o, o, \tau, d \rangle \in (\!|q|\!)^{[t]}_G$.
    Then from Lemma 3.2, $d = 0$.
    And from the definition of $(\!|q|\!)^{[t]}_G$, there are two intervals τ_1 and τ_2 s.t. $\tau = \tau_1 \cap \tau_2$ and $\langle o, o, \tau_i, 0 \rangle \in (\!|\text{test}_i|\!)^{[t]}_G$ for $i \in \{1, 2\}$.
    Now take any $t \in \tau$.
    Then $t \in \tau_i$ for $i \in \{1, 2\}$.
    So by IH, $\langle o, o, t, 0 \rangle \in [\![\text{test}_i]\!]_G$ for each $i \in \{1, 2\}$.
    Together with the defintiion of $[\![q]\!]_G$, this proves (II).

- $q = (?\text{path})$.
  From the above definitions, we have:
  $$\begin{aligned} [\![q]\!]_G &= \{\langle o, o, t, 0 \rangle \mid \langle o, o', t, t + d \rangle \in [\![\text{path}]\!]_G \text{ for some } o' \in N \cup E, d \in \mathcal{T}\} \\ (\!|q|\!)^{[t]}_G &= \{\langle o, o, \tau, 0 \rangle \mid \langle o, o', \tau, d \rangle \in (\!|\text{path}|\!)^{[t]}_G \text{ for some } o' \in N \cup E, d \in \mathcal{T}\} \end{aligned}$$

  - For (I), let $\langle o, o, t, 0 \rangle \in [\![q]\!]_G$.
    From the definition of $[\![q]\!]_G$, there are o' and d such that $\langle o, o', t, t + d \rangle \in [\![\text{path}]\!]_G$.
    So by IH, there is a τ s.t. $t \in \tau$ and $\langle o, o', \tau, d \rangle \in (\!|\text{path}|\!)^{[t]}_G$.
    Therefore $\langle o, o, \tau, 0 \rangle \in (\!|q|\!)^{[t]}_G$, from the definition of $(\!|q|\!)^{[t]}_G$.
  - For (II), let $\langle o, o, \tau, 0 \rangle \in (\!|q|\!)^{[t]}_G$.
    From the definition of $(\!|q|\!)^{[t]}_G$, there are o' and d s.t. $\langle o, o', \tau, d \rangle \in (\!|\text{path}|\!)^{[t]}_G$.
    Now take any $t \in \tau$.
    By IH, $\langle o, o', t, t + d \rangle \in [\![\text{path}]\!]_G$.
    Therefore $\langle o, o, t, 0 \rangle \in [\![q]\!]_G$, from the definition of $[\![q]\!]_G$.

- $q = \neg\text{test}$.
  From the above definitions, we have:

$$\llbracket q \rrbracket_G = (\{\langle o, o\rangle \mid o \in N \cup E\} \times \mathcal{T}_G \times \{0\}) \setminus \llbracket \text{test} \rrbracket_G$$
$$(\!| q |\!)_G^{[t]} = \bigcup_{o \in N \cup E} \left\{ \langle o, o, \tau, 0\rangle \mid \tau \in \text{compl}\left(\{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!| \text{test} |\!)_G^{[t]}\}, \mathcal{T}_G \right) \right\}$$

  - For (I), let $\mathbf{v} = \langle o, o, t, 0\rangle \in \llbracket q \rrbracket_G$.
    From the definition of $\llbracket q \rrbracket_G$, $\mathbf{v} \notin \llbracket \text{test} \rrbracket_G$.
    So
    $$t \notin \{t' \mid \langle o, o, t', 0\rangle \in \llbracket \text{test} \rrbracket_G\} \tag{7}$$
    Now by IH, together with Lemma 3.2, we get:
    $$\langle o, o, t', 0\rangle \in \llbracket \text{test} \rrbracket_G \text{ iff } t' \in \tau' \text{ for some } \tau' \text{ s.t. } \langle o, o, \tau', 0\rangle \in (\!| \text{test} |\!)_G^{[t]} \tag{8}$$
    So from (7) and (8):
    $$t \notin \bigcup \{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!| \text{test} |\!)_G^{[t]}\}$$
    Therefore
    $$t \in \mathcal{T}_G \setminus \bigcup \{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!| \text{test} |\!)_G^{[t]}\} \tag{9}$$
    So $t \in \tau$ for some $\tau \in \text{compl}\left(\bigcup \{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!| \text{test} |\!)_G^{[t]}\}, \mathcal{T}_G \right)$.
    And $\langle o, o, \tau, 0\rangle \in (\!| q |\!)_G^{[t]}$, from the definition of $(\!| q |\!)_G^{[t]}$.
  - For (II), let $\langle o, o, \tau, 0\rangle \in (\!| q |\!)_G^{[t]}$.
    And take any $t \in \tau$.
    From the definition of $(\!| q |\!)_G^{[t]}$:
    $$t \in \mathcal{T}_G \setminus \bigcup \{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!| \text{test} |\!)_G^{[t]}\}$$
    Together with (8), this implies
    $$\langle o, o, t, 0\rangle \notin \llbracket \text{test} \rrbracket_G$$
    Therefore $\langle o, o, t, 0\rangle \in \llbracket q \rrbracket_G$, from the definition of $\llbracket q \rrbracket_G$.

- $q = \text{path}_1/\text{path}_2$.
  From the above definitions, we have:

$$\llbracket q \rrbracket_G = \{\langle o_1, o_3, t, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, t, d_1\rangle \in \llbracket \text{path}_1 \rrbracket_G \wedge \langle o_2, o_3, t + d_1, d_2\rangle \in \llbracket \text{path}_2 \rrbracket_G\}$$
$$(\!| q |\!)_G^{[t]} = \Big\{\langle o_1, o_3, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, \tau_1, d_1\rangle \in (\!| \text{path}_1 |\!)_G^{[t]} \wedge \langle o_2, o_3, \tau_2, d_2\rangle \in (\!| \text{path}_2 |\!)_G^{[t]} \wedge (\tau_1 + d_1) \cap \tau_2 \neq \emptyset \Big\}$$

  - For (I), let $\mathbf{v} = \langle o_1, o_3, t, d\rangle \in \llbracket q \rrbracket_G$.
    Fom the definition of $\llbracket q \rrbracket_G$, there are o_2, d_1 and d_2 such that $\langle o_1, o_2, t, d_1\rangle \in \llbracket \text{path}_1 \rrbracket_G$, $\langle o_2, o_3, t + d_1, d_2\rangle \in \llbracket \text{path}_2 \rrbracket_G$ and $d = d_1 + d_2$.
    By IH, because $\langle o_1, o_2, t, d_1\rangle \in \llbracket \text{path}_1 \rrbracket_G$, there is a τ_1 such that $t \in \tau_1$ and
    $$\langle o_1, o_2, \tau_1, d_1\rangle \in (\!| \text{path}_1 |\!)_G^{[t]} \tag{10}$$
    And similarly, because $\langle o_2, o_3, t + d_1, d_2\rangle \in \llbracket \text{path}_2 \rrbracket_G$, there is a τ_2 such that $t + d_1 \in \tau_2$ and
    $$\langle o_2, o_3, \tau_2, d_2\rangle \in (\!| \text{path}_2 |\!)_G^{[t]} \tag{11}$$
    From $t \in \tau_1$, we get
    $$t + d_1 \in \tau_1 + d_1 \tag{12}$$
    Together with the fact that $t + d_1 \in \tau_2$, this implies
    $$\tau_1 + d_1 \cap \tau_2 \neq \emptyset \tag{13}$$
    So from (10), (11), (13) and the definition of $(\!| q |\!)_G^{[t]}$,
    $$\langle o_1, o_2, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2\rangle \in (\!| q |\!)_G^{[t]}$$
    which proves (Ia).
    And in order to prove (Ib), we only need to show that
    $$t \in ((\tau_1 + d_1) \cap \tau_2) - d_1$$

We know that $t \in \tau_1$, therefore

$$t + d_1 \in \tau_1 + d_1$$

Together with the fact that $t + d_1 \in \tau_2$, this yields

$$\begin{aligned} t + d_1 &\in (\tau_1 + d_1) \cap \tau_2 \\ t &\in ((\tau_1 + d_1) \cap \tau_2) - d_1 \end{aligned}$$

– For (II), let $\mathbf{u} = \langle o_1, o_3, \tau, d \rangle \in (\!|q|\!)_G^{[t]}$, and let $t \in \tau$.
We show that $\langle o_1, o_3, t, t + d \rangle \in [\![q]\!]_G$.
Because $\mathbf{u} \in (\!|q|\!)_G^{[t]}$, from the definition of $(\!|q|\!)_G^{[t]}$, there are τ_1, τ_2, d_1, d_2 and o_2 s.t.:
(i) $d = d_1 + d_2$
(ii) $\tau = ((\tau_1 + d_1) \cap \tau_2) - d_1$
(iii) $\langle o_1, o_2, \tau_1, d_1 \rangle \in (\!|\mathsf{path}_1|\!)_G^{[t]}$
(iv) $\langle o_2, o_3, \tau_2, d_2 \rangle \in (\!|\mathsf{path}_2|\!)_G^{[t]}$
Since $t \in \tau$, from (ii), we have

$$t \in ((\tau_1 + d_1 \cap \tau_2) - d_1 \tag{14}$$

$$t + d_1 \in (((\tau_1 + d_1 \cap \tau_2) - d_1) + d_1 \tag{15}$$

$$t + d_1 \in (\tau_1 + d_1) \cap \tau_2 \tag{16}$$

$$t + d_1 \in \tau_1 + d_1 \tag{17}$$

$$t \in \tau_1 \tag{18}$$

From (iii), by IH, for any $t' \in \tau_1$

$$\langle o_1, o_2, t' + d_1 \rangle \in [\![q]\!]_G$$

In particular, from (18)

$$\langle o_1, o_2, t, t + d_1 \rangle \in [\![q]\!]_G \tag{19}$$

And from (iv), by IH, for any $t'' \in \tau_2$

$$\langle o_2, o_3, t'', t'' + d_2 \rangle \in [\![q]\!]_G$$

In particular, from (16)

$$\langle o_2, o_3, t + d_1, (t + d_1) + d_2 \rangle \in [\![q]\!]_G \tag{20}$$

So from (19), (20) and the definition of $[\![q]\!]_G$

$$\langle o_1, o_3, t, t + d_1 + d_2 \rangle \in [\![q]\!]_G$$

□

3.4 In $\mathcal{U}^{[d]}$

3.4.1 *Definition.*

We start with the case where q is an expression for the symbol test in the grammar of Definition 3.1.
The definitions of $(\!|pred|\!)_G^{[t]}$ and $(\!|\neg\mathsf{test}|\!)_G^{[t]}$ are already provided in the article, we reproduce them here for completeness:

$$\begin{aligned}
(\!|pred|\!)_G^{[d]} &= \{\langle o, o, t, [0,0] \rangle \mid t \in \tau \text{ for some } \tau \in \mathsf{val}(o, pred)\} \\
(\!|\mathrm{F}|\!)_G^{[d]} &= \{\langle v, e, t, [0,0] \rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, [0,0] \rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\} \\
(\!|\mathrm{B}|\!)_G^{[d]} &= \{\langle v, e, t, [0,0] \rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, [0,0] \rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \\
(\!|(?\mathsf{path})|\!)_G^{[d]} &= \{\langle o, o, t, [0,0] \rangle \mid \exists o', \delta \colon \langle o, o', t, \delta \rangle \in (\!|\mathsf{path}|\!)_G^{[d]}\} \\
(\!|\mathsf{test}_1 \vee \mathsf{test}_2|\!)_G^{[d]} &= (\!|\mathsf{test}_1|\!)_G^{[d]} \cup (\!|\mathsf{test}_2|\!)_G^{[d]} \\
(\!|\mathsf{test}_1 \wedge \mathsf{test}_2|\!)_G^{[d]} &= (\!|\mathsf{test}_1|\!)_G^{[d]} \cap (\!|\mathsf{test}_2|\!)_G^{[d]} \\
(\!|\neg\mathsf{test}|\!)_G^{[d]} &= \left\{\langle o, o, t, [0,0] \rangle \mid o \in N \cup E, t \in \mathcal{T}_G \setminus \{t' \mid \langle o, o, t', [0,0] \rangle \in (\!|\mathsf{test}|\!)_G^{[d]}\}\right\}
\end{aligned}$$

Next, we consider the operators $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m, \_])$ and $(\mathsf{path}[m, n])$.
For these cases, $(\!|q|\!)_G^{[t,d]}$ is once again defined analogously to $[\![q]\!]_G$, in terms of temporal join (a.k.a. $\mathsf{path}_1/\mathsf{path}_2$) and set union.

We only write the definitions here for the sake of completeness:

$$\begin{aligned}
(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t,d]} &= (\!|\mathsf{path}_1|\!)_G^{[t,d]} \cup (\!|\mathsf{path}_2|\!)_G^{[t,d]}\\
[\![\mathsf{path}[m,n]]\!]_G &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t,d]}\\
[\![\mathsf{path}[m,_]]\!]_G &= \bigcup_{k\geq m} (\!|\mathsf{path}^k|\!)_G^{[t,d]}
\end{aligned}$$

The only remaining operators are temporal join ($\mathsf{path}_1/\mathsf{path}_2$) and temporal navigation (T_δ), already defined in the article.
We reproduce here these two definition for convenience:

$$\begin{aligned}
(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[d]} &= \left\{\langle o_1, o_3, t_1, \delta_2 + t_2 - t_1\rangle \mid \exists o_2 \colon \langle o_1, o_2, t_1, \delta_1\rangle \in (\!|\mathsf{path}_1|\!)_G^{[d]} \wedge \langle o_2, o_3, t_2, \delta_2\rangle \in (\!|\mathsf{path}_2|\!)_G^{[d]} \wedge t_2 \in t_1 + \delta_1\right\}\\
(\!|\mathrm{T}_\delta|\!)_G^{[d]} &= \{\langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t\rangle \mid o \in N \cup E, t \in \mathcal{T}_G, (\delta + t) \cap \mathcal{T}_G \neq \emptyset\}
\end{aligned}$$

We also reproduce the alternative characterization of $(\!|\mathrm{T}_\delta|\!)_G^{[d]}$ provided in the article, as a unary operator:

$$(\!|q/\mathrm{T}_\delta|\!)_G^{[d]} = \{\langle o_1, o_2, t, (\delta' + \delta) \cap \mathcal{T}_G\rangle \mid \langle o_1, o_2, t, \delta'\rangle \in (\!|q|\!)_G^{[d]}, (t + (\delta' + \delta)) \cap \mathcal{T}_G \neq \emptyset\}$$

3.4.2 *Correctness.*

The following result states that the representation $(\!|q|\!)_G^{[d]}$ is correct:

Proposition 3.4. *Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TPG and q a TRPQ. Then the unfolding of $(\!|q|\!)_G^{[d]}$ is $[\![q]\!]_G$.*

Proof.
Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TG, and let q be a TRPQ.
We show below that:

(I) for any $\langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$, there is a $\delta \in \mathsf{intv}(\mathcal{T})$ such that
  (a) $\langle o_1, o_2, t, \delta\rangle \in (\!|q|\!)_G^{[d]}$, and
  (b) $d \in \delta$,

(II) for any $\langle o_1, o_2, t, \delta\rangle \in (\!|q|\!)_G^{[d]}$ for any $d \in \delta$,
$\langle o_1, o_2, t, d\rangle$ is in $[\![q]\!]_G$.

We proceed once again by induction on the structure of q.
If q is of the form *pred*, F, B, (test $\vee$ test), (test $\wedge$ test), $\neg$test, (path + path), path$[m, n]$ or path$[m, _]$, then (I) and (II) immediately follow from the definitions of $[\![q]\!]_G$ and $(\!|q|\!)_G^{[d]}$.
If q is of the form (?path), then the proof is nearly identical to one already provided for $(\!|(?\mathsf{path})|\!)_G^{[t]}$.
So we focus below on the two remaining cases:

- $q = \mathrm{T}_\delta$.
  From the above definitions, we have:

$$\begin{aligned}
[\![q]\!]_G &= \{\langle o, o, t, d\rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t + d \in \mathcal{T}_G\}\\
(\!|q|\!)_G^{[d]} &= \{\langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t\rangle \mid o \in N \cup E, t \in \mathcal{T}_G, (\delta + t) \cap \mathcal{T}_G \neq \emptyset\}
\end{aligned}$$

  - For (I), let $\mathbf{v} = \langle o, o, t, d\rangle \in [\![q]\!]_G$.
    And let $\mathbf{u} = \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t\rangle$ in $\mathcal{U}^{[d]}$.
    For (Ia) we show that $\mathbf{u} \in (\!|q|\!)_G^{[d]}$.
    From $\mathbf{v} \in [\![q]\!]_G$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.
    Besides, because $\mathbf{v} \in [\![q]\!]_G$ still,

$$t + d \in \mathcal{T}_G \tag{21}$$

    and

$$d \in \delta \tag{22}$$

$$t + d \in t + \delta \tag{23}$$

    So from (21) and (23)

$$t + d \in (\delta + t) \cap \mathcal{T}_G \tag{24}$$

$$(\delta + t) \cap \mathcal{T}_G \neq \emptyset \tag{25}$$

Together with the definition of $(\!|q|\!)_G^{[d]}$, this implies $\mathbf{u} \in (\!|q|\!)_G^{[d]}$, which concludes the proof for (Ia).
Finally, from (24), we get

$$t + d - t \in ((\delta + t) \cap \mathcal{T}_G) - t \tag{26}$$
$$d \in ((\delta + t) \cap \mathcal{T}_G) - t \tag{27}$$

which proves (Ib).

– For (II), let $\mathbf{u} = \langle o, o, t, ((\delta + t) \cap \mathcal{T}_G) - t\rangle \in (\!|q|\!)_G^{[d]}$, and let $d \in ((\delta + t) \cap \mathcal{T}_G) - t$.
From $\mathbf{u} \in (\!|q|\!)_G^{[d]} G$, we get $o \in N \cup E$ and $t \in \mathcal{T}_G$.
So to conclude the proof, it is sufficient to show that *(i)* $d \in \delta$ and *(ii)* $t + d \in \mathcal{T}_G$.
By assumption, we have

$$d \in ((\delta + t) \cap \mathcal{T}_G) - t \tag{28}$$
$$d + t \in (\delta + t) \cap \mathcal{T}_G \tag{29}$$
$$d + t \in \mathcal{T}_G \tag{30}$$

which proves *(ii)* .
And from (29), we also get

$$d + t \in \delta + t$$
$$d + t - t \in (\delta + t) - t$$
$$d \in \delta$$

which proves *(i)* .

- $q = \text{path}_1/\text{path}_2$.
From the above definitions, we have:

$$\begin{aligned} [\![q]\!]_G &= \{\langle o_1, o_3, t, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, t, d_1\rangle \in [\![\text{path}_1]\!]_G \wedge \langle o_2, o_3, t + d_1, d_2\rangle \in [\![\text{path}_2]\!]_G\} \\ (\!|q|\!)_G^{[d]} &= \left\{\langle o_1, o_3, t_1, \delta_2 + t_2 - t_1\rangle \mid \exists o_2 \colon \langle o_1, o_2, t_1, \delta_1\rangle \in (\!|\text{path}_1|\!)_G^{[d]} \wedge \langle o_2, o_3, t_2, \delta_2\rangle \in (\!|\text{path}_2|\!)_G^{[d]} \wedge t_2 \in t_1 + \delta_1\right\} \end{aligned}$$

– For (I), let $\mathbf{v} = \langle o_1, o_3, t, d\rangle \in [\![q]\!]_G$.
Fom the definition of $[\![q]\!]_G$, there are o_2, d_1 and d_2 such that $\langle o_1, o_2, t, d_1\rangle \in [\![\text{path}_1]\!]_G$, $\langle o_2, o_3, t + d_1, d_2\rangle \in [\![\text{path}_2]\!]_G$ and $d = d_1 + d_2$.
By IH, because $\langle o_1, o_2, t, d_1\rangle \in [\![\text{path}_1]\!]_G$, there is a δ_1 such that $d_1 \in \delta_1$ and

$$\langle o_1, o_2, t, \delta_1\rangle \in (\!|\text{path}_1|\!)_G^{[d]} \tag{31}$$

And similarly, because $\langle o_2, o_3, t + d_1, d_2\rangle \in [\![\text{path}_2]\!]_G$, there is a δ_2 such that $d_2 \in \delta_2$ and

$$\langle o_2, o_3, t + d_1, \delta_2\rangle \in (\!|\text{path}_2|\!)_G^{[d]} \tag{32}$$

Next, since $d \in \delta_1$

$$t + d_1 \in t + \delta_1 \tag{33}$$

So from (31), (32), (33) and the definition of $(\!|q|\!)_G^{[d]}$ (replacing t_1 with t and t_2 with $t + d_1$), we get

$$\langle o_1, o_2, t, \delta_2 + (t + d_1) - t\rangle \in (\!|q|\!)_G^{[d]}$$

which proves (Ia).
And in order to prove (Ib), we only need to show that $d \in \delta_2 + (t + d_1) - t$, or in other words that

$$d \in \delta_2 + d_1$$

We know that

$$d_2 \in \delta_2 \tag{34}$$
$$d_2 + d_1 \in \delta_2 + d_1 \tag{35}$$

Together with the fact that $d = d_1 + d_2$, this concludes the proof for (Ib).

– For (II), let $\mathbf{u} = \langle o_1, o_3, t_1, \delta\rangle \in (\!|q|\!)_G^{[d]}$, and let $d \in \delta$.

Because $\mathbf{u} \in (\!|q|\!)_G^{[d]}$, from the definition of $(\!|q|\!)_G^{[d]}$, there are δ_1, δ_2, t_2 and o_2 s.t.:
(i) $\delta = \delta_2 + t_2 - t_1$
(ii) $t_2 \in t_1 + \delta_1$
(iii) $\langle o_1, o_2, t_1, \delta_1\rangle \in (\!|\text{path}_1|\!)_G^{[d]}$

(iv) $\langle o_2, o_3, t_2, \delta_2 \rangle \in (\!|\mathsf{path}_2|\!)_G^{[d]}$
From (i) and (ii), we get

$$\begin{aligned}\delta &= \delta_2 + (t_1 + \delta_1) - t_1 \\ &= \delta_2 + \delta_1\end{aligned}$$

Together with $d \in \delta$, this implies that there are $d_1 \in \delta_1$ and $d_2 \in \delta_2$ such that $d = d_1 + d_2$.
Next, because $d_1 \in \delta_1$, from (iii), by IH

$$\langle o_1, o_2, t_1, t_1 + d_1 \rangle \in [\![q]\!]_G \tag{36}$$

And similarly, because $d_2 \in \delta_2$, from (iv)

$$\langle o_2, o_3, t_2, t_2 + d_2 \rangle \in [\![q]\!]_G \tag{37}$$

So from (36), (37) and the definition of $[\![q]\!]_G$

$$\langle o_1, o_3, t_1, d_1 + d_2 \rangle \in [\![q]\!]_G \tag{38}$$

Together with the fact that $d = d_1 + d_2$, this concludes the proof for (II).

□

3.5 In $\mathcal{U}^{[t,d]}$

3.5.1 Definition.
We start with the case where q is an expression for the symbol test in the grammar of Definition 3.1.
As a consequence of Lemma 3.2, $(\!|q|\!)_G^{[t,d]}$ can be trivially defined out of $(\!|q|\!)_G^{[t]}$ by replacing the distance 0 with the interval $[0, 0]$, i.e.

$$(\!|\mathsf{test}|\!)_G^{[t,d]} = \{\langle o, o, \tau, [0,0] \rangle \mid \{\langle o, o, \tau, 0 \rangle \in (\!|\mathsf{test}|\!)_G^{[t]}\}$$

Next, if q is of the form $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m, \_])$ or $(\mathsf{path}[m, n])$, then the definition of $(\!|q|\!)_G^{[t,d]}$ is once again nearly identical to the one of $[\![q]\!]_G$:

$$\begin{aligned}(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t,d]} &= (\!|\mathsf{path}_1|\!)_G^{[t,d]} \cup (\!|\mathsf{path}_2|\!)_G^{[t,d]} \\ [\![\mathsf{path}[m,n]]\!]_G &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t,d]} \\ [\![\mathsf{path}[m,\_]]\!]_G &= \bigcup_{k \geq m} (\!|\mathsf{path}^k|\!)_G^{[t,d]}\end{aligned}$$

The only remaining operators are temporal join $(\mathsf{path}_1/\mathsf{path}_2)$ and temporal navigation (T_δ), already defined in the article, and reproduced here for convenience:

$$\begin{aligned}(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[t,d]} &= \bigcup\{\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 \mid \mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)_G^{[t,d]}, \mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)_G^{[t,d]}\} \\ (\!|\mathrm{T}_\delta|\!)_G^{[t,d]} &= \bigcup_{o \in N \cup E} \{\langle o, o, \mathcal{T}_G, \delta \rangle \,\overline{\bowtie}\, \langle o, o, \mathcal{T}_G, [0,0] \rangle\}\end{aligned}$$

where $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2$ is defined as follows.
Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2 \rangle$.
Define τ_2' as

$$\tau_2' = (\tau_1 + \delta_1) \cap \tau_2$$

If $o_2 \neq o_3$ or $tau_2' = \emptyset$, then $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 = \emptyset$.
Otherwise, let:

$$\begin{aligned}\tau &= (\tau_2' \ominus \delta_1) \cap \tau_1 \\ b &= b_{\tau_2'} - b_{\delta_1} \\ e &= e_{\tau_2'} - e_{\delta_1}\end{aligned}$$

And for every $t \in \tau$, let

$$\delta(t) = {}_{\delta_1}\lfloor\, b_{\delta_1} + \max(0, b - t),\ e_{\delta_1} - \max(0, t - e)\, \rfloor_{\delta_1}$$

Then

$$\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 = \{\langle o_1, o_4, [t,t], \delta(t) + \delta_2 \rangle \mid t \in \tau\}$$

3.5.2 Correctness.
We start with a lemma

Lemma 3.5. *Let $\alpha, \beta \in \mathrm{intv}(\mathcal{T})$. Then*
$$\beta \ominus \alpha = \{t \mid (t + \alpha) \cap \beta \neq \emptyset\}$$

Next, if $\mathbf{u} = \langle o_1, o_2, \tau, \delta \rangle \in \mathcal{U}^{[t,d]}$, we call *temporal relation induced by* $\mathbf{u}$ the set $\{(t, t + d) \mid t \in \tau, d \in \delta\}$.
We also define the binary operator $\bowtie$: $(\mathcal{T} \times \mathcal{T}) \times (\mathcal{T} \times \mathcal{T}) \to (\mathcal{T} \times \mathcal{T})$ as in the article, i.e.
$$R_1 \bowtie R_2 = \{t_1, t_3 \mid (t_1, t_2) \in R_1 \text{ and } (t_2, t_3) \in R_2 \text{ for some } t_2\}$$

We can now formulate the following lemma:

Lemma 3.6. *Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2 \rangle$ be two tuples in $\mathcal{U}^{[t,d]}$ such that $o_2 = o_3$. And for $i \in \{1, 2\}$, let R_i denote the temporal relation induced by $\mathbf{u}_i$. Then*
$$R_1 \bowtie R_2 = \bigcup_{\langle o_1, o_2, \tau, \delta \rangle \in \mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2} \{(t, t + d) \mid t \in \tau, d \in \delta\}$$

The following result states that the representation $(\!|q|\!)_G^{[t,d]}$ is correct:

Proposition 3.7. *Let $G = \langle N, E, \mathrm{conn}, \mathcal{T}_G, \mathrm{val} \rangle$ be a TPG and q a TRPQ. Then the unfolding of $(\!|q|\!)_G^{[t,d]}$ is $[\![q]\!]_G$.*

Proof.
Let $G = \langle N, E, \mathrm{conn}, \mathcal{T}_G, \mathrm{val} \rangle$ be a TG, and let q be a TRPQ.
We show below that:

(I) for any $\langle o_1, o_2, t, d \rangle \in [\![q]\!]_G$, there are $\tau, \delta \in \mathrm{intv}(\mathcal{T})$ such that
  (a) $\langle o_1, o_2, \tau, \delta \rangle \in (\!|q|\!)_G^{[t,d]}$,
  (b) $t \in \tau$, and
  (c) $d \in \delta$.

(II) for any $\langle o_1, o_2, \tau, \delta \rangle \in (\!|q|\!)_G^{[t,d]}$ for any $(t, d) \in \tau \times \delta$,
  $\langle o_1, o_2, t, d \rangle$ is in $[\![q]\!]_G$.

We proceed once again by induction on the structure of q.
If q is of the form *pred*, F, B, (test $\vee$ test), (path + path), path$[m, n]$ or path$[m, _]$, then (I) and (II) immediately follow from the definitions of $[\![q]\!]_G$ and $(\!|q|\!)_G^{[t,d]}$.
If q is of the form test $\wedge$ test, $\neg$test or (?path), then the proof is nearly identical to the one already provided for $(\!|q|\!)_G^{[t]}$.
So we focus below on the two remaining cases:

- $q = \mathrm{path}_1/\mathrm{path}_2$.
  From the above definitions, we have:
  $$\begin{aligned} [\![q]\!]_G &= \{\langle o_1, o_3, t, d_1 + d_2 \rangle \mid \exists o_2 \colon \langle o_1, o_2, t, d_1 \rangle \in [\![\mathrm{path}_1]\!]_G \wedge \langle o_2, o_3, t + d_1, d_2 \rangle \in [\![\mathrm{path}_2]\!]_G\} \\ (\!|\mathrm{path}_1/\mathrm{path}_2|\!)_G^{[t,d]} &= \bigcup\{\mathbf{u}_1 \overline{\bowtie} \mathbf{u}_2 \mid \mathbf{u}_1 \in (\!|\mathrm{path}_1|\!)_G^{[t,d]}, \mathbf{u}_2 \in (\!|\mathrm{path}_2|\!)_G^{[t,d]}\} \end{aligned}$$
  - For (I), let $\mathbf{v} = \langle o_1, o_3, t, d \rangle \in [\![q]\!]_G$.
    Fom the definition of $[\![q]\!]_G$, there are o_2, d_1 and d_2 such that $\langle o_1, o_2, t, d_1 \rangle \in [\![\mathrm{path}_1]\!]_G$, $\langle o_2, o_3, t + d_1, d_2 \rangle \in [\![\mathrm{path}_2]\!]_G$ and $d = d_1 + d_2$.
    By IH, because $\langle o_1, o_2, t, d_1 \rangle \in [\![\mathrm{path}_1]\!]_G$, there are τ_1 and δ_1 such that $t \in \tau_1$, $d_1 \in \delta_1$ and
    $$\langle o_1, o_2, \tau_1, \delta_1 \rangle \in (\!|\mathrm{path}_1|\!)_G^{[t,d]} \tag{39}$$
    Let R_1 be the temporal relation induced by this tuple $\langle o_1, o_2, \tau_1, \delta_1 \rangle$.
    Since $t \in \tau_1$ and $d_1 \in \delta_1$, we have
    $$(t, t + d_1) \in R_1 \tag{40}$$
    Similarly, because $\langle o_2, o_3, t + d_1, d_2 \rangle \in [\![\mathrm{path}_2]\!]_G$, there are τ_2 and δ_2 such that $t + d_1 \in \tau_2$, $d_2 \in \delta_2$ and
    $$\langle o_2, o_3, \tau_2, \delta_2 \rangle \in (\!|\mathrm{path}_2|\!)_G^{[t,d]} \tag{41}$$
    Let R_2 be the temporal relation induced by this tuple $\langle o_2, o_3, \tau_2, \delta_2 \rangle$.
    Since $t + d_1 \in \tau_2$ and $d_2 \in \delta_2$, we have
    $$(t + d_1, t + d_1 + d_2) \in R_2 \tag{42}$$
    So from (40), (42) and Lemma 3.6, there are τ and δ such that $\langle o_1, o_3, \tau, \delta \rangle \in u_1 \overline{\bowtie} u_2$, $t \in \tau$ and $d_1 + d_2 = d \in \delta$, which concludes the proof for (I).

– For (II), let $\mathbf{u} = \langle o_1, o_3, t_1, \delta\rangle \in (\!|q|\!)_G^{[t,d]}$, and let $(t, d) \in \tau \times \delta$.
Because $\mathbf{u} \in (\!|q|\!)_G^{[t,d]}$, from the definition of $(\!|q|\!)_G^{[t,d]}$, there are $\mathbf{u}_1$ and $\mathbf{u}_2$ s.t.:
(i) $\mathbf{u} \in \mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2$
(ii) $\mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)_G^{[t,d]}$
(iii) $\mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)_G^{[t,d]}$
Let R_i be the temporal relation induced by u_i for $i \in \{1, 2\}$.
From (i), and Lemma 3.6,

$$(t, t + d) \in R_1 \bowtie R_2 \tag{43}$$

Now let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1\rangle$ and $\mathbf{u}_2 = \langle o_2, o_3, \tau_2, \delta_2\rangle$ for some $o_2, \tau_1, \tau_2, \delta_1$ and δ_2.
From (43) and the definition of $\bowtie$, there must be d_1 and d_2 s.t. $d = d_1 + d_2$, $t \in \tau_1$, $d_1 \in \delta_1$, $t + d_1 \in \tau_2$ and $d_2 \in \delta_2$.
So from (ii), and (iii), by IH

$$\langle o_1, o_2, t, d_1\rangle \in [\![\mathsf{path}_1]\!]_G \tag{44}$$

$$\langle o_2, o_3, t + d_1, d_2\rangle \in [\![\mathsf{path}_2]\!]_G \tag{45}$$

So from (44), (45) and the definition of $[\![q]\!]_G$

$$\langle o_1, o_3, t, d_1 + d_2\rangle \in [\![q]\!]_G,$$

which concludes the proof for (II).

□

3.6 In $\mathcal{U}^{[t,d],b,e}$

3.6.1 Definition.

If q is an expression for the symbol test in the grammar of Definition **??**, then the definition of $(\!|q|\!)_G^{[t,d],b,e}$ is nearly identical to the one of $(\!|q|\!)_G^{[t,d]}$, extending each tuple $\{\langle o, o, \tau, [0, 0]\rangle$ with b_τ and e_τ, i.e.

$$(\!|\mathsf{test}|\!)_G^{[t,d],b,e} = \{\langle o, o, \tau, [0, 0], b_\tau, e_\tau\rangle \mid \{\langle o, o, \tau, [0, 0]\rangle \in (\!|\mathsf{test}|\!)_G^{[t,d]}\}$$

Next, if q is of the form $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m, \_])$ or $(\mathsf{path}[m, n])$, then the definition of $(\!|q|\!)_G^{[t,d]}$ is once again nearly identical to the one of $[\![q]\!]_G$:

$$\begin{aligned}
(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t,d],b,e} &= (\!|\mathsf{path}_1|\!)_G^{[t,d],b,e} \cup (\!|\mathsf{path}_2|\!)_G^{[t,d],b,e}\\
[\![\mathsf{path}[m, n]]\!]_G &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t,d],b,e}\\
[\![\mathsf{path}[m, _]]\!]_G &= \bigcup_{k\geq m} (\!|\mathsf{path}^k|\!)_G^{[t,d],b,e}
\end{aligned}$$

So the only remaining operator are temporal join $(\mathsf{path}_1/\mathsf{path}_2)$ and temporal navigation (T_δ), already defined in the article. We reproduce here these two definition for convenience:

$$\begin{aligned}
(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[t,d],b,e} &= \{\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 \mid \mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)_G^{[t,d],b,e}, \mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)_G^{[t,d],b,e}, \mathbf{u}_1 \sim \mathbf{u}_2\}\\
(\!|\mathrm{T}_\delta|\!)_G^{[t,d],b,e} &= \{\langle o, o, \mathcal{T}_G, \delta, b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \,\overline{\bowtie}\, \langle o, o, \mathcal{T}_G, [0, 0], b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \mid o \in N \cup E\}
\end{aligned}$$

where $\mathbf{u}_1 \sim \mathbf{u}_2$ and $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2$ are defined by:

$$\langle o_1, o_2, \tau_1, \delta_1, b_1, e_1\rangle \sim \langle o_3, o_4, \tau_2, \delta_2, e_2, b_2\rangle \text{ iff } o_2 = o_3$$

$$\begin{aligned}
\langle o_1, o_3, \tau_1, \delta_1, b_1, e_1\rangle \,\overline{\bowtie}\, \langle o_3, o_4, \tau_2, \delta_2, e_2, b_2\rangle &= \langle o_1, o_4, \tau, \delta_1 + \delta_2, b, e\rangle\\
&\text{with}\\
\tau &= (((\tau_1 + \delta_1) \cap \tau_2) \ominus \delta_1) \cap \tau_1\\
b &= \max(b_1, b_2 - b_{\delta_1})\\
e &= \min(e_1, e_2 - e_{\delta_1})
\end{aligned}$$

3.6.2 Correctness. Similarly to what we did above for $\mathcal{U}^{[t,d]}$, if $\mathbf{u} = \langle o_1, o_2, tau, \delta, b, e\rangle \in \mathcal{U}^{[t,d],b,e}$, we call *temporal relation induced by* $\mathbf{u}$ the set $\{(t, t + d) \mid t \in \tau, d \in \delta(t)\}$.

We can now formulate a lemma analogous to Lemma 3.6:

Lemma 3.8. *Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}^{[t,d],b,e}$, and for $i \in \{1, 2\}$, let R_i denote the temporal relation induced by $\mathbf{u}_i$. If $\mathbf{u}_1 \sim \mathbf{u}_2$ and $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 = \langle o_1, o_3, \tau, \delta, b, e\rangle$, then*

$$R_1 \bowtie R_2 = \{(t, t + d) \mid t \in \tau, d \in \delta(t)\}$$

Proof. Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1 \rangle$ and $\mathbf{u}_2 = \langle o_2, o_3, \tau_2, \delta_2 \rangle$. As explained in Section 2, for $i \in \{1, 2\}$ and $t \in \tau_i$, we use $\delta_i(t)$ for the interval

$$_{\delta_i}\lfloor\ b_{\delta_i} + \max(0, b_i - t)\ ,\ e_{\delta_i} - \max(0, t - e_i)\ \rfloor_{\delta_i}$$

We need to prove that *(i)* $\tau = \text{dom}(R_1 \bowtie R_2)$ and that *(ii)* for each $t \in \tau$,

$$t + \delta(t) = \{t' \mid (t, t') \in R_1 \bowtie R_2\}$$

The proof or *(i)* is nearly identical to the one provided above for Lemma 3.6.
For *(ii)* , let $t \in \tau$. We use

- a for the least value s.t. $(t, a) \in \text{range}(R_1) \cap \text{dom}(R_2)$, and
- a' for the least value s.t. $(a, a') \in R_2$

Then a' is also the least value s.t. $(t, a') \in R_1 \bowtie R_2$.
Analogously, we use z for the greatest value s.t. $(t, z) \in \text{range}(R_1) \cap \text{dom}(R_2)$, and z' for the greatest value s.t. $(z, z') \in R_2$.
Then z' is also the greatest value s.t. $(t, z') \in R_1 \bowtie R_2$.

From Lemma **??**:

- $\{t\} \times (t + [a, z]) \subseteq R_1$, and
- $[a, b] \times [a', z'] \subseteq R_2$

Therefore $[a', z'] = \{c \mid (t, c) \in R_1 \bowtie R_2\}$.

To conclude the proof, we show that $t + \delta_t = [a, z]$.
We only prove that $t + b_{\delta_t} = a$ (the proof that $t + e_{\delta_t} = z$ is symmetric).
Following the definition of b, we consider 2 cases:

(1) $b_1 < b_2 - b_{\delta_1}$
(2) $b_1 \geq b_2 - b_{\delta_1}$

For Case (1), we get

$$\begin{aligned} b_1 <\ & b_2 - b_{\delta_1} && (46)\\ \max(b_1, b_2 - b_{\delta_1}) =\ & b_2 - b_{\delta_1} && (47)\\ b =\ & b_2 - b_{\delta_1} \quad \text{from the definition of } b && (48) \end{aligned}$$

And in Case (1) still, we get:

$$\begin{aligned} b_1 <\ & b_2 - b_{\delta_1} && (49)\\ 0 <\ & b_2 - b_{\delta_1} - b_1 && (50)\\ \max(0, b_2 - b_{\delta_1} - b_1) =\ & b_2 - b_{\delta_1} - b_1 && (51) \end{aligned}$$

Next, we consider two subcases:

(i) $t < b_2 - b_{\delta_1}$
(ii) $t \geq b_2 - b_{\delta_1}$

In Case (i), we get

$$\begin{aligned} t <\ & b_2 - b_{\delta_1} && (52)\\ 0 <\ & b_2 - b_{\delta_1} - t && (53)\\ \max(0, b_2 - b_{\delta_1} - t) =\ & b_2 - b_{\delta_1} - t && (54) \end{aligned}$$

Now from the definition of δ_t,

$$\begin{aligned} b_{\delta_t} =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && (55)\\ =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_{\delta_1} - t) \quad \text{from (48)} && (56)\\ =\ & b_{\delta_1} + b_{\delta_2} + b_2 - b_{\delta_1} - t \quad \text{from (54)} && (57)\\ =\ & b_{\delta_2} + b_2 - t && (58)\\ b_{\delta_t} + t =\ & b_{\delta_2} + b_2 - t + t && (59)\\ =\ & b_{\delta_2} + b_2 && (60) \end{aligned}$$

Next, from the definition of a'

$$\begin{aligned} a' &= b_{\delta_2(a)} + a && (61)\\ &= b_{\delta_2} + \max(0, b_2 - a) + a && (62)\end{aligned}$$

And, from the definition of a

$$\begin{aligned} a &= b_{\delta_1(t)} + t && (63)\\ &= b_{\delta_1} + \max(0, b_1 - t) + t && (64)\end{aligned}$$

Then we have two further subcases:

(I) $t \geq b_1$, or
(II) $t < b_1$

In case (I):

$$\begin{aligned} t \geq&\ b_1 && (65)\\ 0 \geq&\ b_1 - t && (66)\\ \max(0, b_1 - t) =&\ 0 && (67)\\ a =&\ b_{\delta_1} + t && \text{from (64)} && (68)\\ \max(0, b_2 - a) =&\ \max(0, b_2 - b_{\delta_1} - t) && (69)\\ =&\ b_2 - b_{\delta_1} - t && \text{from (54)} && (70)\\ =&\ b_2 - a && \text{from (68)} && (71)\end{aligned}$$

In case (II):

$$\begin{aligned} t <&\ b_1 && (72)\\ 0 <&\ b_1 - t && (73)\\ \max(0, b_1 - t) =&\ b_1 - t && (74)\\ a =&\ b_{\delta_1} + b_1 - t + t && \text{from (64)} && (75)\\ =&\ b_{\delta_1} + b_1 && (76)\\ \max(0, b_2 - a) =&\ \max(0, b_2 - b_{\delta_1} - b_1) && (77)\\ =&\ b_2 - b_{\delta_1} - b_1 && \text{from (51)} && (78)\\ =&\ b_2 - a && \text{from (76)} && (79)\\ & && (80)\end{aligned}$$

So in both cases (I) and (II), we get

$$\max(0, b_2 - a) = b_2 - a$$

Thefore from (62)

$$\begin{aligned} a' =&\ b_{\delta_2} + b_2 - a + a && (81)\\ =&\ b_{\delta_2} + b_2 && (82)\\ =&\ t + b_{\delta_t} && \text{from (60)} && (83)\end{aligned}$$

which concludes the proof for Case (1)- (i).

We continue with Case (1)- (ii).
From Case (ii):

$$\begin{aligned} t \geq&\ b_2 - b_{\delta_1} && (84)\\ 0 \geq&\ b_2 - b_{\delta_1} - t && (85)\\ \max(0, b_2 - b_{\delta_1} - t) =&\ 0 && (86)\end{aligned}$$

Now from the definition of δ_t:

$$\begin{aligned} b_{\delta_t} &= b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (87)\\ &= b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_1 - t) && \text{from (48)} && (88)\\ &= b_{\delta_1} + b_{\delta_2} && \text{from (86)} && (89)\\ b_{\delta_t} + t &= b_{\delta_1} + b_{\delta_2} + t && && (90) \end{aligned}$$

Next, from Case (1) and Case (ii), by transitivity, we get

$$\begin{aligned} b_1 &\leq t && (91)\\ \max(0, b_1 - t) &= 0 && (92) \end{aligned}$$

And from the definition of a

$$\begin{aligned} a &= b_{\delta_1(t)} + t && && (93)\\ &= b_{\delta_1} + \max(0, b_1 - t) + t && && (94)\\ &= b_{\delta_1} + t && \text{from (92)} && (95)\\ &\geq b_{\delta_1} + b_2 - b_{\delta_1} && \text{from Case (ii)} && (96)\\ &\geq b_2 && && (97)\\ 0 &\geq b_2 - a && && (98)\\ \max(0, b_2 - a) &= 0 && && (99) \end{aligned}$$

Therefore from (62) and (99)

$$\begin{aligned} a' &= b_{\delta_2} + a && && (100)\\ &= b_{\delta_2} + b_{\delta_1} + t && \text{from (95)} && (101)\\ &= b_{\delta_t} + t && \text{from (60)} && (102) \end{aligned}$$

which concludes the proof for Case (1)- (ii).

We continute with Case (2).
In this case, we get

$$\begin{aligned} b_1 &\geq b_2 - b_{\delta_1} && && (103)\\ \max(b_1, b_2 - b_{\delta_1}) &= b_1 && && (104)\\ b &= b_1 && \text{from the definition of } b && (105) \end{aligned}$$

And from Case (2) still, we derive

$$\begin{aligned} b_1 &\geq b_2 - b_{\delta_1} && (106)\\ 0 &\geq b_2 - b_{\delta_1} - b_1 && (107)\\ \max(0, -b_{\delta_1} - b_1) &= 0 && (108) \end{aligned}$$

As well as

$$\begin{aligned} b_1 &\geq b_2 - b_{\delta_1} && (109)\\ b_1 + b_{\delta_1} &\geq b_2 && (110) \end{aligned}$$

Next, we distinguish two subcases, namely

(a) $t < b_1$ and
(b) $t \geq b_1$

We start with Case (a).
In this case,

$$\begin{aligned} t &< b_1 && (111)\\ 0 &< b_1 - t && (112)\\ \max(0, b_1 - t) &= b_1 - t && (113) \end{aligned}$$

And from the definition of δ_t:

$$\begin{aligned}
b_{\delta_t} &= b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (114)\\
&= b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) && \text{from (105)} && (115)\\
&= b_{\delta_1} + b_{\delta_2} + b_1 - t && \text{from (113)} && (116)\\
b_{\delta_t} + t &= b_{\delta_1} + b_{\delta_2} + b_1 - t + t && && (117)\\
&= b_{\delta_1} + b_{\delta_2} + b_1 && && (118)
\end{aligned}$$

Next, from the definition of a

$$\begin{aligned}
a &= b_{\delta_1(t)} + t && && (119)\\
&= \max(0, b_1 - t) + b_{\delta_1} + t && && (120)\\
&= b_1 - t + b_{\delta_1} + t && \text{from (113)} && (121)\\
&= b_1 + b_{\delta_1} && && (122)\\
&\text{So from (110)} && && (123)\\
a &\geq b_2 && && (124)\\
0 &\geq b_2 - a && && (125)\\
\max(0, b_2 - a) &= 0 && && (126)\\
b_{\delta_2} + \max(0, b_2 - a) &= b_{\delta_2} && && (127)\\
b_{\delta_2(a)} &= b_{\delta_2} && && (128)\\
b_{\delta_2(a)} + a &= b_{\delta_2} + a && && (129)\\
a' &= b_{\delta_2} + a && \text{from the defintiion of } a' && (130)\\
a' &= b_{\delta_2} + b_1 + b_{\delta_1} && \text{from (122)} && (131)\\
a' &= b_{\delta_t} + t && \text{from (118)} && (132)
\end{aligned}$$

which concludes the proof for Case (2)- (a).

We end with Case (2)- (b). In this case,

$$\begin{aligned}
t &\geq b_1 && (133)\\
0 &\geq b_1 - t && (134)\\
\max(0, b_1 - t) &= 0 && (135)
\end{aligned}$$

And from the definition of δ_t:

$$\begin{aligned}
b_{\delta_t} &= b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (136)\\
&= b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) && \text{from (105)} && (137)\\
&= b_{\delta_1} + b_{\delta_2} && \text{from (135)} && (138)\\
b_{\delta_t} + t &= b_{\delta_1} + b_{\delta_2} + t && && (139)
\end{aligned}$$

Next, from the definition of a

$$\begin{aligned}
a &= b_{\delta_1(t)} + t && && (140)\\
&= \max(0, b_1 - t) + b_{\delta_1} + t && && (141)\\
&= b_{\delta_1} + t && \text{from (135)} && (142)
\end{aligned}$$

Now from Case (b)

$$
\begin{aligned}
b_1 + \leq\ & t && && (143)\\
b_1 + b_{\delta_1} \leq\ & t + b_{\delta_1} && && (144)\\
b_1 + b_{\delta_1} \leq\ & a && \text{from (142)} && (145)\\
b_2 \leq\ & a && \text{from (110), by transitivity} && (146)\\
b_2 - a \leq\ & 0 && && (147)\\
\max(0, b_2 - a) =\ & 0 && && (148)\\
b_{\delta_2} + \max(0, b_2 - a) =\ & b_{\delta_2} && && (149)\\
b_{\delta_2(a)} =\ & b_{\delta_2} && && (150)\\
b_{\delta_2(a)} + a =\ & b_{\delta_2} + a && && (151)\\
a' =\ & b_{\delta_2} + a && \text{from the defintiion of } a' && (152)\\
=\ & b_{\delta_2} + b_{\delta_1} + t && \text{from (142)} && (153)\\
=\ & b_{\delta_t} + t && \text{from (139)} && (154)
\end{aligned}
$$

which concludes the proof for Case (2)- (b). □

The following result states that the representation $(\!|q|\!)_G^{[t,d],b,e}$ is correct:

Proposition 3.9. *Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TPG and q a TRPQ. Then the unfolding of $(\!|q|\!)_G^{[t,d],b,e}$ is $[\![q]\!]_G$.*

Proof. Let $G = \langle N, E, \mathsf{conn}, \mathcal{T}_G, \mathsf{val}\rangle$ be a TG, and let q be a TRPQ.
To prove the result, it is sufficient to show that:

- (I) for any $\langle o_1, o_2, t, d\rangle \in [\![q]\!]_G$, there are $\tau, \delta \in \mathsf{intv}(\mathcal{T})$ and $b, e \in \mathcal{T}$ such that
  - (a) $\langle o_1, o_2, \tau, \delta, b, e\rangle \in (\!|q|\!)_G^{[t,d],b,e}$,
  - (b) $t \in \tau$, and
  - (c) $d \in \delta(t)$ (where $\delta(t)$ is defined in terms of t, δ, b and e, as explained above).
- (II) for any $\langle o_1, o_2, \tau, \delta, b, e\rangle \in (\!|q|\!)_G^{[t,d],b,e}$ for any $t \in \tau$ and $d \in \delta(t)$,
  $\langle o_1, o_2, t, d\rangle$ is in $[\![q]\!]_G$.

Again, the proof is by induction on the structure of q.
If q is of the form *pred*, F, B, (test $\vee$ test), (path + path), path$[m, n]$ or path$[m, _]$, then (I) and (II) immediately follow from the definitions of $[\![q]\!]_G$ and $(\!|q|\!)_G^{[t,d],b,e}$.
If q is of the form test $\wedge$ test, $\neg$test or (?path), then the proof is nearly identical to the one already provided for $(\!|q|\!)_G^{[t]}$.
And if q is of the form T_δ or path$_1$/path$_2$, then the proof is nearly identical to the one already provided for $(\!|q|\!)_G^{[t,d]}$, using Lemma 3.8 instead of 3.6. □