Compact Answers to Temporal Regular Path Queries (Supplementary Material)

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1 INTRODUCTION

This document provides detailed definitions and proofs for the article *Compact answers to Temporal Regular Path Queries*, submitted at CIKM 2023.

As opposed to the structure adopted in the article, the result here are grouped by topic (inductive representation, finiteness, complexity, etc) rather than representation ($\mathcal{U}^{[t]}$, $\mathcal{U}^{[d]}$, etc.). This allows us to emphasize which proofs differ from one representation to the other.

2 NOTATION

Let $\langle o_1, o_2, \tau, \delta, b, e \rangle \in \mathcal{U}^{[t,d],b,e}$, and let $t \in \tau$.
In the article, we defined the interval δ_t for each t as

$$_{\delta}\lfloor\ b_{\delta} + \max(0, b - t)\ ,\ e_{\delta_i} - \max(0, t - e)\ \rfloor_{\delta}$$

In this supplementary material, we will use $\delta(t)$ instead of δ_t. This notation will allow us to write $\delta_1(t)$ when several tuples are involved. Note that the time points b and e in this notaton are still omitted, for conciseness, because they should be clear from the context.

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3 INDUCTIVE REPRESENTATION

Let q be a TRPQ and G a TG.
Then $[\![q]\!]_G$ is the set of anwers to q over G (represented as tuples in $\mathcal{U}$).
In this section, we provide the full definition of the four inductive representations of $[\![q]\!]_G$ discussed in the article, in $\mathcal{U}^{[t]}$, $\mathcal{U}^{[d]}$, $\mathcal{U}^{[t,d]}$ and $\mathcal{U}^{[t,d],b,e}$ respectively, and prove that they are correct.
These representations are denoted as $(\!|q|\!)_G^{[t]}$, $(\!|q|\!)_G^{[d]}$, $(\!|q|\!)_G^{[t,d]}$ and $(\!|q|\!)_G^{[t,d],b,e}$ respectively.

3.1 Grammar

For convenience, we reproduce here the syntax of TRPQs (already provided in the article).
A TRPQ is an expression for the symbol path in the following grammar:

$$\begin{aligned}
\mathsf{path} &::= \mathsf{test} \mid \mathsf{axis} \mid (\mathsf{path}/\mathsf{path}) \mid (\mathsf{path} + \mathsf{path}) \mid \mathsf{path}[m,n] \mid \mathsf{path}[m,_] \\
\mathsf{test} &::= \mathit{pred} \mid (?\mathsf{path}) \mid \mathsf{test} \vee \mathsf{test} \mid \mathsf{test} \wedge \mathsf{test} \mid \neg\mathsf{test} \\
\mathsf{axis} &::= \mathrm{F} \mid \mathrm{B} \mid \mathrm{T}_\delta
\end{aligned}$$

with $\delta \in \mathsf{intv}(\mathcal{T})$, $m, n \in \mathbb{N}^+$ and $m \leq n$.

3.2 In $\mathcal{U}$

For convenience still, we reproduce the full semantics of TRPQs, i.e. the inductive definition of the evaluation $[\![q]\!]_G$ of a query q over a graph G in $\mathcal{U}$ (already provided in the article).

$$\begin{aligned}
[\![\mathit{pred}]\!]_G &= \{\langle o,o,t,0\rangle \mid \langle o,t\rangle \models \mathit{pred}\} \\
[\![\mathrm{T}_\delta]\!]_G &= \{\langle o,o,t,d\rangle \mid o \in (N \cup E), t \in \mathcal{T}_G, d \in \delta, t+d \in \mathcal{T}_G\} \\
[\![\mathrm{F}]\!]_G &= \{\langle v,e,t,0\rangle \mid \mathsf{src}(e)=v, t \in \mathcal{T}_G\} \cup \{\langle e,v,t,0\rangle \mid \mathsf{tgt}(e)=v, t \in \mathcal{T}_G\} \\
[\![\mathrm{B}]\!]_G &= \{\langle v,e,t,0\rangle \mid \mathsf{tgt}(e)=v, t \in \mathcal{T}_G\} \cup \{\langle e,v,t,0\rangle \mid \mathsf{src}(e)=v, t \in \mathcal{T}_G\} \\
[\![?\mathsf{path}]\!]_G &= \{\langle o,o,t,0\rangle \mid \langle o,o',t,t+d\rangle \in [\![\mathsf{path}]\!]_G \text{ for some } o' \in N \cup E, d \in \mathcal{T}\} \\
[\![\mathsf{test}_1 \vee \mathsf{test}_2]\!]_G &= [\![\mathsf{test}_1]\!]_G \cup [\![\mathsf{test}_2]\!]_G \\
[\![\mathsf{test}_1 \wedge \mathsf{test}_2]\!]_G &= [\![\mathsf{test}_1]\!]_G \cap [\![\mathsf{test}_2]\!]_G \\
[\![\neg\mathsf{test}]\!]_G &= (\{\langle o,o\rangle \mid o \in N \cup E\} \times \mathcal{T}_G \times \{0\}) \setminus [\![\mathsf{test}]\!]_G \\
[\![\mathsf{path}_1/\mathsf{path}_2]\!]_G &= \{\langle o_1,o_3,t,d_1+d_2\rangle \mid \exists o_2 \colon \langle o_1,o_2,t,d_1\rangle \in [\![\mathsf{path}_1]\!]_G \wedge \langle o_2,o_3,t+d_1,d_2\rangle \in [\![\mathsf{path}_2]\!]_G\} \\
[\![\mathsf{path}_1 + \mathsf{path}_2]\!]_G &= [\![\mathsf{path}_1]\!]_G \cup [\![\mathsf{path}_2]\!]_G \\
[\![\mathsf{path}[m,n]]\!]_G &= \bigcup_{k=m}^{n} [\![\mathsf{path}^k]\!]_G \\
[\![\mathsf{path}[m,_]]\!]_G &= \bigcup_{k \geq m} [\![\mathsf{path}^k]\!]_G
\end{aligned}$$

3.3 In $\mathcal{U}^{[t]}$

3.3.1 *Definition.*

The full definition of $(\!|q|\!)_G^{[t]}$ is already provide in the article. We only reproduce it here for convenience.

$$
\begin{aligned}
(\!| \mathit{pred} |\!)_G^{[t]} &= \{\langle o, o, \tau, 0\rangle \mid \tau \in \mathsf{val}(o, \mathit{pred})\}\\
(\!| \mathrm{T}_\delta |\!)_G^{[t]} &= \{\langle o, o, [t_1, t_1], t_2 - t_1\rangle \mid t_1 \in \mathcal{T}_G, t_2 \in (\delta + t_1) \cap \mathcal{T}_G\}\\
(\!| \mathrm{F} |\!)_G^{[t]} &= \{\langle v, e, \mathcal{T}_G, 0\rangle \mid \mathsf{src}(e) = v\} \cup \{\langle e, v, \mathcal{T}_G, 0\rangle \mid \mathsf{tgt}(e) = v\}\\
(\!| \mathrm{B} |\!)_G^{[t]} &= \{\langle v, e, \mathcal{T}_G, 0\rangle \mid \mathsf{tgt}(e) = v\} \cup \{\langle e, v, \mathcal{T}_G, 0\rangle \mid \mathsf{src}(e) = v\}\\
(\!| (?\mathsf{path}) |\!)_G^{[t]} &= \{\langle o_1, o_1, \tau, 0\rangle \mid \exists o_2, d \colon \langle o_1, o_2, \tau, d\rangle \in (\!| \mathsf{path} |\!)_G^{[t]}\}\\
(\!| \mathsf{test}_1 \vee \mathsf{test}_2 |\!)_G^{[t]} &= (\!| \mathsf{test}_1 |\!)_G^{[t]} \cup (\!| \mathsf{test}_2 |\!)_G^{[t]}\\
(\!| \mathsf{test}_1 \wedge \mathsf{test}_2 |\!)_G^{[t]} &= \{\langle o, o, \tau_1 \cap \tau_2, 0\rangle \mid \langle o, o, \tau_1, 0\rangle \in (\!| \mathsf{test}_1 |\!)_G^{[t]}, \langle o, o, \tau_2, 0\rangle \in (\!| \mathsf{test}_2 |\!)_G^{[t]}, \tau_1 \cap \tau_2 \neq \emptyset\}\\
(\!| \neg\mathsf{test} |\!)_G^{[t]} &= \bigcup_{o \in N \cup E} \left\{\langle o, o, \tau, 0\rangle \mid \tau \in \mathsf{compl}\left(\{\tau' \mid \langle o, o, \tau', 0\rangle \in (\!| \mathsf{test} |\!)_G^{[t]}\}, \mathcal{T}_G\right)\right\}\\
(\!| \mathsf{path}_1/\mathsf{path}_2 |\!)_G^{[t]} &= \left\{\langle o_1, o_3, ((\tau_1 + d_1) \cap \tau_2) - d_1, d_1 + d_2\rangle \mid \exists o_2 \colon \langle o_1, o_2, \tau_1, d_1\rangle \in (\!| \mathsf{path}_1 |\!)_G^{[t]} \wedge \langle o_2, o_3, \tau_2, d_2\rangle \in (\!| \mathsf{path}_2 |\!)_G^{[t]} \wedge (\tau_1 + d_1) \cap \tau_2 \neq \emptyset\right\}\\
(\!| \mathsf{path}_1 + \mathsf{path}_2 |\!)_G^{[t]} &= (\!| \mathsf{path}_1 |\!)_G^{[t]} \cup (\!| \mathsf{path}_2 |\!)_G^{[t]}\\
(\!| \mathsf{path}[m, n] |\!)_G^{[t]} &= \bigcup_{k=m}^{n} (\!| \mathsf{path}^k |\!)_G^{[t]}\\
(\!| \mathsf{path}[m, \_] |\!)_G^{[t]} &= \bigcup_{k \geq m} (\!| \mathsf{path}^k |\!)_G^{[t]}
\end{aligned}
$$

We observe that for the operators ($\mathsf{path}_1 + \mathsf{path}_2$), ($\mathsf{path}[m, \_]$) and ($\mathsf{path}[m, n]$), the defnition of $(\!| q |\!)_G^{[t]}$ is nearly identical to the one of $\llbracket q \rrbracket_G$. It will also be the case for the three representations below.

3.4 In $\mathcal{U}^{[d]}$

3.4.1 Definition.

We start with the case where q is an expression for the symbol test in the grammar of Section 3.1.
The definitions of $(\!| \mathit{pred} |\!)_G^{[t]}$ and $(\!| \neg\mathsf{test} |\!)_G^{[t]}$ are already provided in the article, we reproduce them here for completeness:

$$
\begin{aligned}
(\!| \mathit{pred} |\!)_G^{[d]} &= \{\langle o, o, t, [0, 0]\rangle \mid \tau \in \mathsf{val}(o, \mathit{pred}) \text{ and } t \in \tau\}\\
(\!| \mathrm{F} |\!)_G^{[d]} &= \{\langle v, e, t, [0, 0]\rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, [0, 0]\rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\}\\
(\!| \mathrm{B} |\!)_G^{[d]} &= \{\langle v, e, t, [0, 0]\rangle \mid \mathsf{tgt}(e) = v, t \in \mathcal{T}_G\} \cup \{\langle e, v, t, [0, 0]\rangle \mid \mathsf{src}(e) = v, t \in \mathcal{T}_G\}\\
(\!| (?\mathsf{path}) |\!)_G^{[d]} &= \{\langle o_1, o_1, t, [0, 0]\rangle \mid \exists o_2, \delta \colon \langle o_1, o_2, t, \delta\rangle \in (\!| \mathsf{path} |\!)_G^{[d]}\}\\
(\!| \mathsf{test}_1 \vee \mathsf{test}_2 |\!)_G^{[d]} &= (\!| \mathsf{test}_1 |\!)_G^{[d]} \cup (\!| \mathsf{test}_2 |\!)_G^{[d]}\\
(\!| \mathsf{test}_1 \wedge \mathsf{test}_2 |\!)_G^{[d]} &= (\!| \mathsf{test}_1 |\!)_G^{[d]} \cap (\!| \mathsf{test}_2 |\!)_G^{[d]}\\
(\!| \neg\mathsf{test} |\!)_G^{[d]} &= \left\{\langle o, o, t, [0, 0]\rangle \mid o \in N \cup E, t \in \mathcal{T}_G \setminus \{t' \mid \langle o, o, t', [0, 0]\rangle \in (\!| \mathsf{test} |\!)_G^{[d]}\}\right\}
\end{aligned}
$$

Next, we consider the operators ($\mathsf{path}_1 + \mathsf{path}_2$), ($\mathsf{path}[m, \_]$) and ($\mathsf{path}[m, n]$).
For these cases, $(\!| q |\!)_G^{[t,d]}$ is once again defined analogously to $\llbracket q \rrbracket_G$, in terms of temporal join (a.k.a. $\mathsf{path}_1/\mathsf{path}_2$) and set union.
We only write the definitions here for the sake of completeness:

$$
\begin{aligned}
(\!| \mathsf{path}_1 + \mathsf{path}_2 |\!)_G^{[t,d]} &= (\!| \mathsf{path}_1 |\!)_G^{[t,d]} \cup (\!| \mathsf{path}_2 |\!)_G^{[t,d]}\\
\llbracket \mathsf{path}[m, n] \rrbracket_G &= \bigcup_{k=m}^{n} (\!| \mathsf{path}^k |\!)_G^{[t,d]}\\
\llbracket \mathsf{path}[m, \_] \rrbracket_G &= \bigcup_{k \geq m} (\!| \mathsf{path}^k |\!)_G^{[t,d]}
\end{aligned}
$$

The only remaining operators are temporal join ($\mathsf{path}_1/\mathsf{path}_2$) and temporal navigation (T_δ), already defined in the article.
We reproduce here these two definition for convenience:

$$
\begin{aligned}
(\!| \mathsf{path}_1/\mathsf{path}_2 |\!)_G^{[d]} &= \left\{\langle o_1, o_3, t_1, \delta_2 + t_2 - t_1\rangle \mid \exists o_2 \colon \langle o_1, o_2, t_1, \delta_1\rangle \in (\!| \mathsf{path}_1 |\!)_G^{[d]} \wedge \langle o_2, o_3, t_2, \delta_2\rangle \in (\!| \mathsf{path}_2 |\!)_G^{[d]} \wedge t_2 \in t_1 + \delta_1\right\}\\
(\!| \mathrm{T}_\delta |\!)_G^{[d]} &= \{\langle o, o, t, (\delta + t) \cap \mathcal{T}_G\rangle \mid o \in N \cup E, t \in \mathcal{T}_G\}
\end{aligned}
$$

We also reproduce the alternative characterization of $(\!| \mathrm{T}_\delta |\!)_G^{[d]}$ provided in the article, as a unary operator:

$$
(\!| q/\mathrm{T}_\delta |\!)_G^{[d]} = \{\langle o_1, o_2, t, (\delta' + \delta) \cap \mathcal{T}_G\rangle \mid \langle o_1, o_2, t, \delta'\rangle \in (\!| q |\!)_G^{[d]}\}
$$

3.5 In $\mathcal{U}^{[t,d]}$

3.5.1 *Definition.*

If q is an expression for the symbol test in the grammar of Section 3.1, then all tuples in $[\![q]\!]_G$ must have distance 0. As a result, $(\!|q|\!)_G^{[t,d]}$ can be trivially defined out of $(\!|q|\!)_G^{[t]}$ by replacing the distance 0 with the interval $[0,0]$, i.e.

$$(\!|\mathsf{test}|\!)_G^{[t,d]} = \{\langle o, o, \tau, [0,0]\rangle \mid \{\langle o, o, \tau, 0\rangle \in (\!|\mathsf{test}|\!)_G^{[t]}\}$$

Again, for the operators $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m,\_])$ and $(\mathsf{path}[m,n])$, the definition of $(\!|q|\!)_G^{[t,d]}$ is nearly identical to the one of $[\![q]\!]_G$:

$$\begin{aligned}
(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t,d]} &= (\!|\mathsf{path}_1|\!)_G^{[t,d]} \cup (\!|\mathsf{path}_2|\!)_G^{[t,d]}\\
[\![\mathsf{path}[m,n]]\!]_G &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t,d]}\\
[\![\mathsf{path}[m,_]]\!]_G &= \bigcup_{k\geq m} (\!|\mathsf{path}^k|\!)_G^{[t,d]}
\end{aligned}$$

The only remaining operators are temporal join $(\mathsf{path}_1/\mathsf{path}_2)$ and temporal navigation (T_δ), already defined in the article, and reproduced here for convenience:

$$\begin{aligned}
(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[t,d]} &= \bigcup\{\mathbf{u}_1 \mathbin{\overline{\bowtie}} \mathbf{u}_2 \mid \mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)_G^{[t,d]}, \mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)_G^{[t,d]}\}\\
(\!|\mathrm{T}_\delta|\!)_G^{[t,d]} &= \bigcup_{o \in N \cup E} \{\langle o, o, \mathcal{T}_G, \delta\rangle \mathbin{\overline{\bowtie}} \langle o, o, \mathcal{T}_G, [0,0]\rangle\}
\end{aligned}$$

where $\mathbf{u}_1 \mathbin{\overline{\bowtie}} \mathbf{u}_2$ is defined as follows.
Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1\rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2\rangle$.
If $o_2 \neq o_3$, then $\mathbf{u}_1 \mathbin{\overline{\bowtie}} \mathbf{u}_2 = \emptyset$.
Otherwise, each tuple in $\mathbf{u}_1 \mathbin{\overline{\bowtie}} \mathbf{u}_2$ is of the form $\langle o_1, o_4, \tau, \delta\rangle$ for some τ and δ.
For $i \in \{1,2\}$, let R_i be the binary relation over $\mathcal{T}$ specified by the time points and distances in $\mathbf{u}_i$, i.e., $R_i = \{(t, t+d) \mid t \in \tau_i, d \in \delta_i\}$.
And let $R_1 \bowtie R_2$ denote $\{(t_1, t_3) \mid (t_1, t_2) \in R_1 \text{ and } (t_2, t_3) \in R_2 \text{ for some } t_2\}$. Then the intervals in the set $\mathbf{u}_1 \mathbin{\overline{\bowtie}} \mathbf{u}_2$ should intuitively represent this relation $R_1 \bowtie R_2$.
For each time point $t \in \mathsf{dom}(R_1 \bowtie R_2)$, let $\delta(t)$ denote the maximal interval s.t. $(\delta(t) + t) \subseteq \tau_2$.
We define $\mathbf{u}_1 \bowtie \mathbf{u}_2$ as

$$\{\langle o_1, o_4, [t,t], \delta(t) + \delta_2\rangle \mid t \in \mathsf{dom}(R_1 \bowtie R_2)\}$$

3.6 In $\mathcal{U}^{[t,d],b,e}$

3.6.1 *Definition.*

If q is an expression for the symbol test in the grammar of Section 3.1, then the definition of $(\!|q|\!)_G^{[t,d],b,e}$ is nearly identical to the one of $(\!|q|\!)_G^{[t,d]}$, extending each tuple $\{\langle o, o, \tau, [0,0]\rangle$ with b_τ and e_τ, i.e.

$$(\!|\mathsf{test}|\!)_G^{[t,d],b,e} = \{\langle o, o, \tau, [0,0], b_\tau, e_\tau\rangle \mid \{\langle o, o, \tau, [0,0]\rangle \in (\!|\mathsf{test}|\!)_G^{[t,d]}\}$$

Next, for the operators, $(\mathsf{path}_1 + \mathsf{path}_2)$, $(\mathsf{path}[m,\_])$ and $(\mathsf{path}[m,n])$, the definition of $(\!|q|\!)_G^{[t,d]}$ is once again nearly identical to the one of $[\![q]\!]_G$:

$$\begin{aligned}
(\!|\mathsf{path}_1 + \mathsf{path}_2|\!)_G^{[t,d],b,e} &= (\!|\mathsf{path}_1|\!)_G^{[t,d],b,e} \cup (\!|\mathsf{path}_2|\!)_G^{[t,d],b,e}\\
[\![\mathsf{path}[m,n]]\!]_G &= \bigcup_{k=m}^{n} (\!|\mathsf{path}^k|\!)_G^{[t,d],b,e}\\
[\![\mathsf{path}[m,_]]\!]_G &= \bigcup_{k\geq m} (\!|\mathsf{path}^k|\!)_G^{[t,d],b,e}
\end{aligned}$$

So the only remaining operator are temporal join $(\mathsf{path}_1/\mathsf{path}_2)$ and temporal navigation (T_δ), already defined in the article. We reproduce here these two definition for convenience:

$$\begin{aligned}
(\!|\mathsf{path}_1/\mathsf{path}_2|\!)_G^{[t,d],b,e} &= \{\mathbf{u}_1 \mathbin{\overline{\bowtie}} \mathbf{u}_2 \mid \mathbf{u}_1 \in (\!|\mathsf{path}_1|\!)_G^{[t,d],b,e}, \mathbf{u}_2 \in (\!|\mathsf{path}_2|\!)_G^{[t,d],b,e}, \mathbf{u}_1 \sim \mathbf{u}_2\}\\
(\!|\mathrm{T}_\delta|\!)_G^{[t,d],b,e} &= \{\langle o, o, \mathcal{T}_G, \delta, b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \mathbin{\overline{\bowtie}} \langle o, o, \mathcal{T}_G, [0,0], b_{\mathcal{T}_G}, e_{\mathcal{T}_G}\rangle \mid o \in N \cup E\}
\end{aligned}$$

where $\mathbf{u}_1 \sim \mathbf{u}_2$ and $\mathbf{u}_1 \mathbin{\overline{\bowtie}} \mathbf{u}_2$ are defined by:

$$\langle o_1, o_2, \tau_1, \delta_1, b_1, e_1\rangle \sim \langle o_3, o_4, \tau_2, \delta_2, e_2, b_2\rangle \text{ iff } o_2 = o_3$$

$$\begin{aligned}
\langle o_1, o_3, \tau_1, \delta_1, b_1, e_1\rangle \mathbin{\overline{\bowtie}} \langle o_3, o_4, \tau_2, \delta_2, e_2, b_2\rangle &= \langle o_1, o_4, \tau, \delta_1 + \delta_2, b, e\rangle\\
&\text{with}\\
\tau &= (((\tau_1 + \delta_1) \cap \tau_2) \ominus \delta_1) \cap \tau_1\\
b &= \max(b_1, b_2 - b_{\delta_1})\\
e &= \min(e_1, e_2 - e_{\delta_1})
\end{aligned}$$

3.6.2 *Correctness.*

Operator $\mathsf{path}_1/\mathsf{path}_2$**.**
Let $\mathbf{u}_1 = \langle o_1, o_2, \tau_1, \delta_1\rangle$ and $\mathbf{u}_2 = \langle o_3, o_4, \tau_2, \delta_2\rangle$, with $o_2 = o_3$.
Let also $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2 = \langle o_1, o_4, \tau_1', \delta_1 + \delta_2, b, e\rangle$, with

$$\tau = (((\tau_1 + \delta_1) \cap \tau_2) \ominus \delta_1) \cap \tau_1$$
$$b = \max(b_1, b_2 - b_{\delta_1})$$
$$e = \min(e_1, e_2 - e_{\delta_1})$$

As explained in Section 2, for $i \in \{1, 2\}$ and $t \in \tau_i$, we use $\delta_i(t)$ for the interval

$${}_{\delta_i}\lfloor\, b_{\delta_i} + \max(0, b_i - t)\ ,\ e_{\delta_i} - \max(0, t - e_i)\, \rfloor_{\delta_i}$$

And similarly to what we did for $\mathcal{U}^{[t,d]}$, we use R_i for be the binary relation over $\mathcal{T}$ specified by the time points and distances in $\mathbf{u}_i$, i.e. $R_i = \{(t, t + d) \mid t \in \tau_i, d \in \delta_i(t)\}$.
Then the intervals in the set $\mathbf{u}_1 \,\overline{\bowtie}\, \mathbf{u}_2$ should intuitively represent this relation $R_1 \bowtie R_2$, i.e. we need to prove that *(i)* $\tau = \mathrm{dom}(R_1 \bowtie R_2)$ and that *(ii)* for each $t \in \tau$,

$$t + \delta(t) = \{t' \mid (t, t') \in R_1 \bowtie R_2\}$$

For *(ii)* , let $t \in \tau$. We use

- a for the least value s.t. $(t, a) \in \mathrm{range}(R_1) \cap \mathrm{dom}(R_2)$, and
- a' for the least value s.t. $(a, a') \in R_2$

Then a' is also the least value s.t. $(t, a') \in R_1 \bowtie R_2$.
Analogously, we use z for the greatest value s.t. $(t, z) \in \mathrm{range}(R_1) \cap \mathrm{dom}(R_2)$, and z' for the greatest value s.t. $(z, z') \in R_2$. Then z' is also the greatest value s.t. $(t, z') \in R_1 \bowtie R_2$.

From Lemma **??**:

- $\{t\} \times (t + [a, z]) \subseteq R_1$, and
- $[a, b] \times [a', z'] \subseteq R_2$

Therefore $[a', z'] = \{c \mid (t, c) \in R_1 \bowtie R_2\}$.

To conclude the proof, we show that $t + \delta_t = [a, z]$.
We only prove that $t + b_{\delta_t} = a$ (the proof that $t + e_{\delta_t} = z$ is symmetric).
Following the definition of b, we consider 2 cases:

(1) $b_1 < b_2 - b_{\delta_1}$
(2) $b_1 \geq b_2 - b_{\delta_1}$

For Case (1), we get

$$b_1 < b_2 - b_{\delta_1} \tag{1}$$
$$\max(b_1, b_2 - b_{\delta_1}) = b_2 - b_{\delta_1} \tag{2}$$
$$b = b_2 - b_{\delta_1} \quad \text{from the definition of } b \tag{3}$$

And in Case (1) still, we get:

$$b_1 < b_2 - b_{\delta_1} \tag{4}$$
$$0 < b_2 - b_{\delta_1} - b_1 \tag{5}$$
$$\max(0, b_2 - b_{\delta_1} - b_1) = b_2 - b_{\delta_1} - b_1 \tag{6}$$

Next, we consider two subcases:

(i) $t < b_2 - b_{\delta_1}$
(ii) $t \geq b_2 - b_{\delta_1}$

In Case (i), we get

$$t < b_2 - b_{\delta_1} \tag{7}$$
$$0 < b_2 - b_{\delta_1} - t \tag{8}$$
$$\max(0, b_2 - b_{\delta_1} - t) = b_2 - b_{\delta_1} - t \tag{9}$$

Now from the definition of δ_t,

$$\begin{aligned}
b_{\delta_t} &= b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (10)\\
&= b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_{\delta_1} - t) && \text{from (3)} && (11)\\
&= b_{\delta_1} + b_{\delta_2} + b_2 - b_{\delta_1} - t && \text{from (9)} && (12)\\
&= b_{\delta_2} + b_2 - t && && (13)\\
b_{\delta_t} + t &= b_{\delta_2} + b_2 - t + t && && (14)\\
&= b_{\delta_2} + b_2 && && (15)
\end{aligned}$$

Next, from the definition of a'

$$\begin{aligned}
a' &= b_{\delta_2(a)} + a && (16)\\
&= b_{\delta_2} + \max(0, b_2 - a) + a && (17)
\end{aligned}$$

And, from the definition of a

$$\begin{aligned}
a &= b_{\delta_1(t)} + t && (18)\\
&= b_{\delta_1} + \max(0, b_1 - t) + t && (19)
\end{aligned}$$

Then we have two further subcases:

(I) $t \geq b_1$, or
(II) $t < b_1$

In case (I):

$$\begin{aligned}
t &\geq b_1 && && (20)\\
0 &\geq b_1 - t && && (21)\\
\max(0, b_1 - t) &= 0 && && (22)\\
a &= b_{\delta_1} + t && \text{from (19)} && (23)\\
\max(0, b_2 - a) &= \max(0, b_2 - b_{\delta_1} - t) && && (24)\\
&= b_2 - b_{\delta_1} - t && \text{from (9)} && (25)\\
&= b_2 - a && \text{from (23)} && (26)
\end{aligned}$$

In case (II):

$$\begin{aligned}
t &< b_1 && && (27)\\
0 &< b_1 - t && && (28)\\
\max(0, b_1 - t) &= b_1 - t && && (29)\\
a &= b_{\delta_1} + b_1 - t + t && \text{from (19)} && (30)\\
&= b_{\delta_1} + b_1 && && (31)\\
\max(0, b_2 - a) &= \max(0, b_2 - b_{\delta_1} - b_1) && && (32)\\
&= b_2 - b_{\delta_1} - b_1 && \text{from (6)} && (33)\\
&= b_2 - a && \text{from (31)} && (34)\\
& && && (35)
\end{aligned}$$

So in both cases (I) and (II), we get

$$\max(0, b_2 - a) = b_2 - a$$

Thefore from (17)

$$\begin{aligned}
a' &= b_{\delta_2} + b_2 - a + a && && (36)\\
&= b_{\delta_2} + b_2 && && (37)\\
&= t + b_{\delta_t} && \text{from (15)} && (38)
\end{aligned}$$

which concludes the proof for Case (1)- (i).

We continue with Case (1)- (ii).
From Case (ii):

$$\begin{aligned} t \geq&\ b_2 - b_{\delta_1} && (39)\\ 0 \geq&\ b_2 - b_{\delta_1} - t && (40)\\ \max(0, b_2 - b_{\delta_1} - t) =&\ 0 && (41) \end{aligned}$$

Now from the definition of δ_t:

$$\begin{aligned} b_{\delta_t} =&\ b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (42)\\ =&\ b_{\delta_1} + b_{\delta_2} + \max(0, b_2 - b_1 - t) && \text{from (3)} && (43)\\ =&\ b_{\delta_1} + b_{\delta_2} && \text{from (41)} && (44)\\ b_{\delta_t} + t =&\ b_{\delta_1} + b_{\delta_2} + t && && (45) \end{aligned}$$

Next, from Case (1) and Case (ii), by transitivity, we get

$$\begin{aligned} b_1 \leq&\ t && (46)\\ \max(0, b_1 - t) =&\ 0 && (47) \end{aligned}$$

And from the definition of a

$$\begin{aligned} a =&\ b_{\delta_1(t)} + t && && (48)\\ =&\ b_{\delta_1} + \max(0, b_1 - t) + t && && (49)\\ =&\ b_{\delta_1} + t && \text{from (47)} && (50)\\ \geq&\ b_{\delta_1} + b_2 - b_{\delta_1} && \text{from Case (ii)} && (51)\\ \geq&\ b_2 && && (52)\\ 0 \geq&\ b_2 - a && && (53)\\ \max(0, b_2 - a) =&\ 0 && && (54) \end{aligned}$$

Therefore from (17) and (54)

$$\begin{aligned} a' =&\ b_{\delta_2} + a && && (55)\\ =&\ b_{\delta_2} + b_{\delta_1} + t && \text{from (50)} && (56)\\ =&\ b_{\delta_t} + t && \text{from (15)} && (57) \end{aligned}$$

which concludes the proof for Case (1)- (ii).

We continute with Case (2).
In this case, we get

$$\begin{aligned} b_1 \geq&\ b_2 - b_{\delta_1} && && (58)\\ \max(b_1, b_2 - b_{\delta_1}) =&\ b_1 && && (59)\\ b =&\ b_1 && \text{from the definition of } b && (60) \end{aligned}$$

And from Case (2) still, we derive

$$\begin{aligned} b_1 \geq&\ b_2 - b_{\delta_1} && (61)\\ 0 \geq&\ b_2 - b_{\delta_1} - b_1 && (62)\\ \max(0, -b_{\delta_1} - b_1) =&\ 0 && (63) \end{aligned}$$

As well as

$$\begin{aligned} b_1 \geq&\ b_2 - b_{\delta_1} && (64)\\ b_1 + b_{\delta_1} \geq&\ b_2 && (65) \end{aligned}$$

Next, we distinguish two subcases, namely

(a) $t < b_1$ and
(b) $t \geq b_1$

We start with Case (a).
In this case,

$$\begin{aligned} t <\ & b_1 && (66)\\ 0 <\ & b_1 - t && (67)\\ \max(0, b_1 - t) =\ & b_1 - t && (68) \end{aligned}$$

And from the definition of δ_t:

$$\begin{aligned} b_{\delta_t} =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (69)\\ =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) && \text{from (60)} && (70)\\ =\ & b_{\delta_1} + b_{\delta_2} + b_1 - t && \text{from (68)} && (71)\\ b_{\delta_t} + t =\ & b_{\delta_1} + b_{\delta_2} + b_1 - t + t && && (72)\\ =\ & b_{\delta_1} + b_{\delta_2} + b_1 && && (73) \end{aligned}$$

Next, from the definition of a

$$\begin{aligned} a =\ & b_{\delta_1(t)} + t && && (74)\\ =\ & \max(0, b_1 - t) + b_{\delta_1} + t && && (75)\\ =\ & b_1 - t + b_{\delta_1} + t && \text{from (68)} && (76)\\ =\ & b_1 + b_{\delta_1} && && (77) \end{aligned}$$

So from (65) (78)

$$\begin{aligned} a \geq\ & b_2 && && (79)\\ 0 \geq\ & b_2 - a && && (80)\\ \max(0, b_2 - a) =\ & 0 && && (81)\\ b_{\delta_2} + \max(0, b_2 - a) =\ & b_{\delta_2} && && (82)\\ b_{\delta_2(a)} =\ & b_{\delta_2} && && (83)\\ b_{\delta_2(a)} + a =\ & b_{\delta_2} + a && && (84)\\ a' =\ & b_{\delta_2} + a && \text{from the defintiion of } a' && (85)\\ a' =\ & b_{\delta_2} + b_1 + b_{\delta_1} && \text{from (77)} && (86)\\ a' =\ & b_{\delta_t} + t && \text{from (73)} && (87) \end{aligned}$$

which concludes the proof for Case (2)- (a).

We end with Case (2)- (b). In this case,

$$\begin{aligned} t \geq\ & b_1 && (88)\\ 0 \geq\ & b_1 - t && (89)\\ \max(0, b_1 - t) =\ & 0 && (90) \end{aligned}$$

And from the definition of δ_t:

$$\begin{aligned} b_{\delta_t} =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b - t) && && (91)\\ =\ & b_{\delta_1} + b_{\delta_2} + \max(0, b_1 - t) && \text{from (60)} && (92)\\ =\ & b_{\delta_1} + b_{\delta_2} && \text{from (90)} && (93)\\ b_{\delta_t} + t =\ & b_{\delta_1} + b_{\delta_2} + t && && (94) \end{aligned}$$

Next, from the definition of a

$$\begin{aligned} a =\ & b_{\delta_1(t)} + t && && (95)\\ =\ & \max(0, b_1 - t) + b_{\delta_1} + t && && (96)\\ =\ & b_{\delta_1} + t && \text{from (90)} && (97) \end{aligned}$$

Now from Case (b)

$$
\begin{aligned}
b_1 + \leq \ & t && && (98)\\
b_1 + b_{\delta_1} \leq \ & t + b_{\delta_1} && && (99)\\
b_1 + b_{\delta_1} \leq \ & a && \text{from (97)} && (100)\\
b_2 \leq \ & a && \text{from (65), by transitivity} && (101)\\
b_2 - a \leq \ & 0 && && (102)\\
\max(0, b_2 - a) = \ & 0 && && (103)\\
b_{\delta_2} + \max(0, b_2 - a) = \ & b_{\delta_2} && && (104)\\
b_{\delta_2(a)} = \ & b_{\delta_2} && && (105)\\
b_{\delta_2(a)} + a = \ & b_{\delta_2} + a && && (106)\\
a' = \ & b_{\delta_2} + a && \text{from the defintiion of } a' && (107)\\
= \ & b_{\delta_2} + b_{\delta_1} + t && \text{from (97)} && (108)\\
= \ & b_{\delta_t} + t && \text{from (94)} && (109)
\end{aligned}
$$

which concludes the proof for Case (2)- (b). □

4 FINITENESS

5 COMPLEXITY OF QUERY ANSWERING

5.1 Problem

We define in this section a decision problem for each representation, similar to the problem Compact Answer$^{[t]}$ defined in the article. First, we define a (possibly partial) order over tuples of each representation, in the expected way, i.e.:

- over $\mathcal{U}^{[t]}$: $\langle o_1, o_2, \tau_1, d_1 \rangle \sqsubseteq_{[t]} \langle o_3, o_4, \tau_2, d_2 \rangle$ iff $\langle o_1, o_2, d_1 \rangle = \langle o_3, o_4, d_2 \rangle$ and $\tau_1 \subseteq \tau_2$,
- over $\mathcal{U}^{[d]}$: $\langle o_1, o_2, t_1, \delta_1 \rangle \sqsubseteq_{[d]} \langle o_3, o_4, t_2, \delta_1 \rangle$ iff $\langle o_1, o_2, t_1 \rangle = \langle o_3, o_4, t_2 \rangle$ and $\delta_1 \subseteq \delta_2$,
- over $\mathcal{U}^{[t,d]}$: $\langle o_1, o_2, \tau_1, \delta_1 \rangle \sqsubseteq_{[t,d]} \langle o_3, o_4, \tau_2, \delta_2 \rangle$ iff $\langle o_1, o_2 \rangle = \langle o_3, o_4 \rangle$, $\tau_1 \subseteq \tau_2$ and $\delta_1 \subseteq \delta_2$,
- over $\mathcal{U}^{[t,d],b,e}$: $\langle o_1, o_2, \tau_1, \delta_1, b_1, b_2 \rangle \sqsubseteq_{[t,d],b,e} \langle o_3, o_4, \tau_2, \delta_2, b_2, e_2 \rangle$ iff $\langle o_1, o_2 \rangle = \langle o_3, o_4 \rangle$, $\tau_1 \subseteq \tau_2$ and $\delta_1(t) \subseteq \delta_2(t)$ for all $t \in \tau_1 \cap \tau_2$ (the notation of $\delta_i(t)$ is explained above, in Section 2).

Now let x be one of $[t]$, $[d]$, $[t,d]$ or $([t,d], b, e)$.
We decline the notion of compact answer defined in Section XXX in four flavors, as follows:

Definition 5.1. Let G be a TG, let q be a TRPQ and let $\mathbf{u} \in \mathcal{U}^x$.
We say that $\mathbf{u}$ is a *compact answer* to q over G (in $\mathcal{U}^x$) if $\mathbf{u} \in \max_{\sqsubseteq_x}\{\mathbf{u}' \in \mathcal{U}^x \mid \mathsf{unfold}(\mathbf{u}') = [\![q]\!]_G\}$

And similary, we get four decision problems:

Compact Answer^x	
Input:	TG G, TRPQ q, tuple $\mathbf{u} \in \mathcal{U}^x$
Decide:	$\mathbf{u}$ is a compact answer to q over G (in $\mathcal{U}^x$)

5.2 Hardness

5.3 Membership

6 MINIMIZATION

7 SIZE OF COMPACT ANSWERS