Math 400 Fall 2019 Computing Project 1 due : Friday, Nov. 1

Project A : Differentiation matrices

In homework 1 we saw the 2nd-order accurate centered difference approximation

$$f'(x) \approx D_0 f(x) = \frac{f(x+h) - f(x-h)}{2h},$$

which we used to approximate $f'(x)$ at a specific point $x = x_0$. In this problem we consider the similar problem of needing to approximate $f'(x)$ across a range of x values associated with a spatial domain.

1. Consider the 2π-periodic function $f(x) = e^{\sin x}$. Discretize the interval $[0, 4\pi)$ by $x_n = \{x_1, x_2, \ldots, x_{n-1}, x_n\}$ into n equal subintervals of length $h = 4\pi/n$, with $x_1 = 0$ and $x_n = 4\pi - h$. Let w_i stand for the approximation of $f'(x)$ at $x = x_i$. Apply the centered difference formula to find $w_i \approx f'(x_i)$ (hint: since $f(x)$ is periodic, $f(x_0) = f(x_n)$ and $f(x_{n+1}) = f(x_1)$):

$$\begin{aligned} w_1 &= \frac{1}{2h}(f(x_2) - f(x_n)) \\ w_2 &= \frac{1}{2h}(f(x_3) - f(x_1)) \\ \vdots &= \ \vdots \\ w_n &= \frac{1}{2h}(f(x_1) - f(x_{n-1})) \end{aligned}$$

2. Let $\boldsymbol{y} = [f(x_1), f(x_2), \ldots, f(x_n)]^T$ and $\boldsymbol{w} = [w_1, w_2, \ldots, w_n]^T$. Show that the system of equations in part (a) can be written as $\boldsymbol{w} = D_0 \boldsymbol{y}$, where the matrix D_0 is

$$D_0 = \frac{1}{2h} \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -1 & 0 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ 0 & & & -1 & 0 & 1 & & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{bmatrix}.$$

3. For $f(x) = e^{\sin x}$, plot the graph of $\boldsymbol{w}$ and the exact derivative $f'(x) = \cos x e^{\sin x}$ on the interval $x \in [0, 4\pi)$ using $n = 10$, 20, and 80. Notice that most of the entries in D_0 are zero; thus, D_0 is an example of a *sparse matrix*. Use the Matlab tool `diag` to construct D_0.

4. Repeat part the previous section for the following functions on the interval $[0, 6\pi)$.

   (a) $g(x) = \sin x + 2 \sin 3x \cos x$

   (b) $h(x) = \sin x + \sin 10x$

(c) $y(x) = e^{\sin x \cos x}$

5. The matrix D_0 approximates the first derivative of a function given n values of that function; it seems reasonable to expect that $(D_0)^2\boldsymbol{y} = D_0 D_0 \boldsymbol{y}$ might approximate the second derivative. Test this idea on the function $f(x) = e^{\sin x}$. Compare the structure of $(D_0)^2$ to D_0 and comment on the number of nonzero diagonals in $(D_0)^2$ compared to D_0.

6. Explore the connection between the operation of matrix multiplication and differentiation further by considering $(D_0)^n$ as a proxy for $f^{(n)}$, the nth derivative of a function. Find the first 4 derivatives of $f(x) = \sin x + 0.3 \sin 2x - 0.4 \sin 3x$. Write a Matlab program that computes the first four derivatives' approximations $(D_0)^k f(x)$ for $k = 2, 3, 4$ and $x \in [0, 6\pi)$. Plot the exact derivatives on the same axes with their approximations using $n = 100$ grid points. Discuss the connection between the step size h and the accuracy of the approximation as the order of differentiation increases (note: this will require several values of n). Support your discussion with quantitative results (tables, Taylor series analysis, or log-log plots).

Project B : Two-point boundary value problems

Use the methods discussed in class for the finite difference scheme and LU solver. Note that your code should not construct the entire matrix; instead, use linear arrays containing the nonzero matrix elements.

1. Consider the 2-point BVP, $-\epsilon y'' + y = 2x + 1$ on the domain $0 \le x \le 1$, with boundary conditions $y(0) = 0,\ y(1) = 0$ and $\epsilon = 10^{-3}$.

   Show that the exact solution is $y(x) = 2x+1-\left(\sinh\dfrac{1-x}{\sqrt{\epsilon}} + 3\sinh\dfrac{x}{\sqrt{\epsilon}}\right)\left(\sinh\dfrac{1}{\sqrt{\epsilon}}\right)^{-1}$.

   Plot the exact solution and the numerical solution for $h = \dfrac{1}{32}$.

2. A wooden beam of square cross section is supported at both ends and is carrying a distributed lateral load of uniform intensity $w = 20$lb/ft and axial tension load $T = 100$lb. The deflection, $u(x)$, of the beam's satisfies the equation

$$u'' - \frac{T}{EI}u = -\frac{w}{2EI}x(L-x), \qquad u(0) = u(L) = 0,$$

   where $L = 6$ ft is the length, $E = 1.3 \times 10^6$ lb/in^2 is the modulus of the elasticity and $I = s^4$ is the moment of inertia of the beam. The side length of the square cross section is $s = 4$ inches. Determine the deflection the beam at 1-inch intervals. Also calculate two additional cases: 2-inch and 4-inch intervals. Plot all three cases on the same axes (use difference symbols to distinguish the cases). Give the maximum beam deflection computed in each case.