Assignment 3

Q1. Proof (by Induction)

Basis: $n=1$. $5^{2n}-1=5^2-1=24=3*8$

2pts

The statement is true.

Induction: Suppose $3 \mid 5^{2k}-1$ for some integer k.

2pts Show $3 \mid 5^{2(k+1)}-1$

6pts

Since $3 \mid 5^{2k}-1$, that means

$5^{2k}-1=3m$ for some integer m

4pts

Thus $5^{2k}\cdot 5^2-5^2=3m\times 5^2$ (by multiplying 5^2 on both sides)

That is $5^{2(k+1)}-25=3m\times 25$

That is $5^{2(k+1)}-1=3m\times 25-24$

$=3(25m-8)$

Therefore $3 \mid 5^{2(k+1)}-1$.

Done

Q2. Proof (by Induction)

Basis: $n=2$

$$\sum_{K=1}^{2} \frac{1}{\sqrt{K}} = 1 + \frac{1}{\sqrt{2}}$$

Assume $1 + \frac{1}{\sqrt{2}} \leq \sqrt{2}$

That is $\sqrt{2} + 1 \leq 2$ (by multiplying $\sqrt{2}$ on both sides)

3pts

Contradiction!

Thus $1 + \frac{1}{\sqrt{2}} > \sqrt{2}$.

The statement is true.

Induction: Assume $\sum_{K=1}^{m} \frac{1}{\sqrt{K}} > \sqrt{m}$ for some integer m.

2pts Show $\sum_{K=1}^{m+1} \frac{1}{\sqrt{K}} > \sqrt{m+1}$.

5pts

$$\sum_{K=1}^{m+1} \frac{1}{\sqrt{K}} = \left(\sum_{K=1}^{m} \frac{1}{\sqrt{K}}\right) + \frac{1}{\sqrt{m+1}}$$

3pts

$$> \sqrt{m} + \frac{1}{\sqrt{m+1}}$$ (by applying Inductive hypothesis)

Assume $\sqrt{m} + \frac{1}{\sqrt{m+1}} \leq \sqrt{m+1}$

We have $\sqrt{m(m+1)} + 1 \leq m+1$ (multiply $\sqrt{m+1}$ on both sides)

That is $\sqrt{m(m+1)} \leq m$

Contradiction!

Thus $\sqrt{m} + \frac{1}{\sqrt{m+1}} > \sqrt{m+1}$

Therefore $\sum_{k=1}^{m+1} \frac{1}{\sqrt{k}} > \sqrt{m} + \frac{1}{\sqrt{m+1}} > \sqrt{m+1}$.

Done.

Q3. a. $\{(a,a), (b,b), (c,c), (a,b), (b,a), (b,c), (c,b)\}$

3pts per Question

b. ϕ

c. $\{(a,b), (b,a)\}$

Q4. Proof

- Reflexive: for every integer k, $k^2 \equiv k^2 \pmod 4$.

  2pts

  Thus all $(k,k) \in R$.

- Symmetric. Assume $(m,n) \in R$,

  3pts

  that means $m^2 \equiv n^2 \pmod 4$

  Thus $n^2 \equiv m^2 \pmod 4$

  thus $(n,m) \in R$.

- transitive: Assume $(p,q) \in R$ and $(q,r) \in R$.

  3pts

  That means $p^2 \equiv q^2 \pmod 4$ and

  $q^2 \equiv r^2 \pmod 4$

Thus $p^2 \equiv r^2 \pmod 4$

Thus $(p, r) \in R$.

Q5. Show R is ~~injective~~ reflexive, symmetric and transitive.

· reflexive. for any $k \in \mathbb{Z}$

2pts $k^2 + k^2 = 2k^2$ is even.

Thus $(k, k) \in R$.

Therefore, R is reflexive.

· symmetric: Assume $(x, y) \in R$

That means $x^2 + y^2 = 2k$

2pts for some int k.

Then $y^2 + x^2 = 2k$

that means $(y, x) \in R$.

Therefore, R is symmetric.

· transitive: Assume $(x, y) \in R$ and $(y, z) \in R$

That means $x^2 + y^2 = 2m$ and

2pts $y^2 + z^2 = 2n$ for some ints m & n.

Thus $x^2 + 2y^2 + z^2 = 2m + 2n$

Thus $x^2 + z^2 = 2(m + n - y^2)$

$= 2p$ for some int p.

Thus $(x, z) \in R$. R is transitive.

~~Q5~~ Since R is reflexive, symmetric and transitive. R is an equivalence relation.

3pts

Equivalence classes:

$[0] = \{0, 2, -2, 4, -4, 6, -6, \cdots\}$

$[1] = \{1, 3, -3, 5, -5, 7, -7, \cdots\}$

Q6. $f(x) = x^2 + 3x - 2$ is not ~~reflexive~~ injective. 2pts

Since ~~for~~ there exist $x \neq y \in R$, $f(x) = f(y)$

let $x = 0$, $y = -3$

$f(0) = f(-3) = -2$

3pts

Q7. Proof (by induction).

2pts

Basis: $n = 1$, $f_1 = 2 \geq 2^1$ ✓

$n = 2$, $f_2 = 4 \geq 2^2$ ✓

Induction: Assume $f_k \geq 2^k$ for all smaller ~~ks~~ ns. ~~ie~~ $k = 1, 2, \cdots, n-1$.

6pts

2pts

Show $f_k \geq 2^k$ for $k = n$

4pts

$f_k = f_{k-1} + 2 f_{k-2}$

$\geq 2^{k-1} + 2 \cdot 2^{k-2}$

$= 2^{k-2}(2+2)$

$= 2^k$

Thus $f_k \geq 2^k$ for $k = n$.

Done.