HW2 Rubric

Q1. <u>Proof</u>. $n^2+3n+4=(n+1)(n+2)+2$

<u>Case 1</u>. n is even. then $n+1$ is odd and $n+2$ is even.

3pts

Thus $(n+1)(n+2)$ is even.

Therefore $(n+1)(n+2)+2$ is even.

That is ~~Therefore~~ n^2+3n+4 is even.

<u>Case 2</u>. n is odd. then $n+1$ is even and $n+2$ is odd.

3pts

Thus $(n+1)(n+2)$ is even.

Thus $(n+1)(n+2)+2$ is even.

Thus n^2+3n+4 is even.

Therefore n^2+3n+4 is even when $n \in \mathbb{Z}$.

Q2. <u>Proof</u>. Suppose x, y are positive real numbers

2pts Then $x+y>0$

Suppose $x<y$.

2pts then $x-y<0$

Thus $(x+y)(x-y)<0$

2pts That is $x^2-y^2<0$

Therefore $x^2<y^2$.

Q3. Proof. $n^2+n = n(n+1)$

4pts | Case 1 n is even & $n \in \mathbb{N}$.

(Similar to Q1) $n(n+1)$ is an even natural number

Case 2. n is an odd natural number

4pts | (Similar to Q1) $n(n+1)$ is an even natural number.

Thus $n(n+1) = 2a$ where $a \in \mathbb{N}$

Therefore $2 \mid n(n+1)$.

Q4. Proof (By contrapositive)

2pts | Suppose $x \mid y$ or $x \mid z$. (and $x \neq 0$) $x, y, z \in \mathbb{Z}$

Show $x \mid yz$.

Case 1 Suppose $x \mid y$

2pts | We have $y = xa$ for some int a.

then $yz = xaz$.

So $x \mid yz$.

Case 2 Suppose $x \mid z$

2pts | We have $z = xa$ for some int a

then $yz = xya$

So $x \mid yz$.

Done.

Q5. Proof (by contradiction).

Suppose both a and b are odd, and $a^2+b^2=c^2$,

2pts

then $a=2m+1$ for some int m.

$b=2n+1$ for some int n.

then $a^2+b^2=(2m+1)^2+(2n+1)^2$

$=4m^2+4m+1+4n^2+4n+1$

$=4(m^2+m+n^2+n)+2$

Case 1 c is even.

2pts

then $4 \mid c^2$

Since $a^2+b^2=c^2$

We have $4 \mid a^2+b^2$

that is $4 \mid 4(m^2+m+n^2+n)+2$

Contradiction!

Case 2 c is odd. < then c^2 is odd.

2pts

but $a^2+b^2=4(m^2+m+n^2+n)+2$ is even.

Contradiction!

Q6. Proof (by contradiction)

3pts → Suppose $a, b \in \mathbb{Z}^+$ and $a^2+b^2=1$

3 pts → Since $a, b \in \mathbb{Z}^+$, we have

$a^2 \geq 1$ & $b^2 \geq 1$

Thus $a^2+b^2 \geq 1+1=2$

2pts → Thus $a^2+b^2 \neq 1$

Contradiction!

Q7. Proof. ~~Suppose~~ let $\gcd(a,b)=m$, we have

part 1

$a = ml$

and $b = mk$ for some integers l and k.

4pts

Then $a-bq_0 = ml - mkq_0 = m(l-kq_0)$

where $l-kq_0$ is an integer.

Thus $m \mid (a-bq_0)$

thus $m \leq \gcd(b, a-bq_0)$.

part 2. let $\gcd(b, a-bq_0)=m$, we have

$b = ml$ and

$a-bq_0 = mk$ for some integers l and k.

Thus $bq_e = mlq_e$

4 pts

Thus $a - bq_e + bq_e = mk + mlq_e$

That is $a = m(k + lq_e)$

That means $m | a$.

Thus $m \leq \gcd(a,b)$

That is $\gcd(a,b) \leq \gcd(b, a-bq_e)$ and

$\gcd(a,b) \geq \gcd(b, a-bq_e)$.

We have $\gcd(a,b) = \gcd(b, a-bq_e)$.

Q8. Proof

part 1. suppose $y = x^2$ or $y = -x$, show $x^3 + x^2 y = y^2 + xy$

Proof: $LHS = x^3 + x^2 y = x^2(x+y)$

$RHS \quad y^2 + xy = y(y+x) = y(x+y)$

2pts

Case 1 $y = -x$, then we have

$x + y = 0$. Therefore $x^2(x+y) = y(x+y)$.

Therefore $x^3 + x^2 y = y^2 + xy$.

Case 2 $y = x^2$ then we have

$x^3 + x^2 y = x^2(x+y) = y(x+y) = y^2 + xy$.

Part 2. Suppose $x^3+x^2y=y^2+xy$
Show $y=x^2$ or $y=-x$.

2pts

Proof: Suppose $x^3+x^2y=y^2+xy$

We have $x^2(x+y)=y(y+x)$

~~That~~ is $x^2(x+y)-y(x+y)=0$

That is $(x+y)(x^2-y)=0$

Thus $x+y=0$ or $x^2-y=0$

Therefore $y=-x$ or $y=x^2$

Q9: $X=\mathbb{N}\cup\{\mathbb{N}\}$

3pts

$=\cancel{\{1,2,3}\ \{\mathbb{N},1,2,3,\cdots\}$