CSCI 480, Winter 2016
Math Exercises # 4

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Due date:

- Build deterministic finite automata and/or regular expressions (as requested) for each of the languages in questions 1 to 6. Create simple, meaningful automata and regular expressions (rather than, *e.g.*, using the algorithm to create a regular expression from a DFA) and explain how they work. In all cases the alphabet is $\Sigma = \{0, 1\}$.
- DFAs should be specified with pictures, preferably typeset with **TikZ**, not tables (as tables are very hard to read). Try to typeset them so that the labels on the arcs are clear, *etc.*
- If you are having a difficult time with **TikZ**, clear, legible hand-drawn figures (or figures created with a drawing program) are acceptable as graphics inclusions into your LaTeX documents.

For solutions to problems 1 6, see figures 1, 2, and 3 at the very bottom.

1. The language $\{100, 10, 011\}$. Regular expression:
2. The language $\{100, 10, 011\}$. DFA:
3. The set of all strings that begin or end with a doubled digit, either 11 or 00. Regular expression:
4. The set of all strings that begin or end with a doubled digit, either 11 or 00. DFA:
5. The set of all strings that have exactly one doubled digit in them. In other words, either 11 or 00 occurs in the string, but not both, and it only occurs once. Regular expression:
6. The set of all strings that have exactly one doubled digit in them. In other words, either 11 or 00 occurs in the string, but not both, and it only occurs once. DFA:
7. Use the pumping lemma to show that the language that consists of all *palindromes* over $\Sigma = \{0, 1\}$ is not regular. A palindrome is a string that reads the same backwards and forwards, for example 11011, 01010, 0, and 0110.

   $$L = \{w \in \{0, 1\}^* \mid w = w^R\}$$

   1) Suppose that L is regular. L then, must satisfy the pumping lemma.

   2) $\exists p$, the pumping length, such that:

   3) $\exists S \in L$, $|S| \geq p$

   4) $S = xyz$, $y \neq \epsilon$

   5) $\exists i \geq 0 \mid xy^iz \notin L$

   6) $|xy| < p$

   Suppose $S = 0^p1^p0^p$, since $|xy| < p$, we know that the substring xy must fall into the first p symbols of the string.

   This means that for the string S, xy will consist of only 0's, with z being the rest of the string.

   If $i = 0$, then $xy^0z = xz$.

$xz = 0^{p-|y|}1^p0^p$

After y is pumped to $i = 0$ times, the string no longer has an even number of 0's on both sides of the 1's in the middle. String S fails to satisfy the palindrome requirement of the language, and thus is not in the language.

By contradiction then, L is not a regular language.

8. Use the pumping lemma to show that the following language is not regular. For examples, 00110000, 01110000 and 00 are in the language, but 110 is not.

$$L = \{0^i1^j0^{i+j} \mid i, j \in \{0, 1, 2, \ldots\}\}$$

1) Suppose that L is regular. L then, must satisfy the pumping lemma.

2) $\exists p$, the pumping length, such that:

3) $\exists S \in L$, $|S| \geq p$

4) $S = xyz$, $y \neq \epsilon$

5) $\exists i \geq 0 \mid xy^iz \notin L$

6) $|xy| < p$

Suppose $S = 0^p1^{p+1}0^{2p+1}$. Since $|xy| < p$, then the substring xy must fall into the first p symbols of the input string. In this case, xy would be comprised of only 0's from the first 0^p zeroes. z comprises the rest of the string.

Suppose that we pump $y, i = 0$ times. $xy^0z = xz$,

$xz = 0^{p-|y|}1^{p+1}0^{2p+1}$

By contradiction, the language L is not regular because the string xz does not follow the rule $0^i1^j0^{i+j}$ and is not in the language as a result.

9. Give an example of a regular language R and a nonregular language N such that $R \cup N$ is regular. Describe all three languages in English and either prove they are regular/nonregular, or show that they are instances of languages with known regularity.

The collection of regular languages over an alphabet is defined recursively as follows:

1) The empty language $\emptyset$ is a regular language. 2) For each a   (a belongs to), the singleton language a is a regular language. 3) If A and B are regular languages, then A B (union), A B (concatenation), and A* (Kleene star) are regular languages. 4) No other languages over are regular. -Wikipedia page on regular languages

All finite languages are regular. This means that the smallest regular languages are: $\emptyset, \{\epsilon\}, \{0\}, \{1\}, etc.$ Note that $\sum^*, (where \sum = \{0, 1\}), \{0\}, \{1\}, etc.$ is also a regular language. Any language that can be represented via a regular expression is also regular.

Suppose there is a language $L_1 = \{w \in \sum |w = 0^n1^n, n \geq 0\}$, which is non-regular (because it has an infinite and uncountable number of substrings for all n).

Additionally, there is a language $L_2 = \{0^n1^m|n, m \geq 0\}$. L_2 is regular because n and m do not have to be equal, and thus can satisfy the pumping lemma.

$L_1 \cup L_2 = L_2$. The reason for this is that L_1 is merely a subset of L_2 where the variables m and n happen to be equal. The resulting language is regular.

Proof for $L_1 = \{w \in \sum |w = 0^n1^n, n \geq 0\}$:

By pumping lemma:

Suppose L_1 is regular, $S \in L_1$ and $|S| \geq p$. $S = xyz, i \geq 0$ and $y \neq \epsilon$. $S = xyz = 0^p1^p$, since the P.L. states that $|xy| < p$, then xy must fall into the first p symbols of the string, making xy comprised of only 0's in this string.

If we pump $y^i, i = 0$ times, then the string $xy^0z = xz = 0^{p-|y|}1^p$

Then xz no longer satisfies the rule of the language 0^n1^n, and is therefore not in the language.

By contradiction, L_1 cannot be a regular language.

Proof for $L_2 = \{0^n1^m | n, m \geq 0\}$:

By pumping lemma:

Suppose L_2 is regular, $S \in L_2$ and $|S| \geq p$. $S = xyz, i \geq 0$ and $y \neq \epsilon$. $S = xyz = 0^p1^p$, since the P.L. states that $|xy| < p$, then xy must fall into the first p symbols of the string, making xy comprised of only 0's in this string.

If we pump $y^i, i = 0$ times, then the string $xy^0z = xz = 0^{p-|y|}1^p$

The substring xz still satisfies the rule of the language 0^n1^m, so long as both n and m are greater than zero, and therefore is still in the language. Merely satisfying the pumping lemma does not make a language regular though.

However, because L_2 can be represented by the regular expression 0^*1^*, it is in fact a regular language.

10. Give an example of a regular language R and a nonregular language N such that $R \cup N$ is nonregular. Describe all three languages in English and either prove they are regular/nonregular, or show that they are instances of languages with known regularity.

All finite languages are regular. This means that the smallest regular languages are: $\emptyset, \{\epsilon\}, \{0\}, \{1\}, etc.$ Note that $\sum^*, (where \sum = \{0,1\}), \{0\}, \{1\}, etc.$ is also a regular language. Any language that can be represented via a regular expression is also regular.

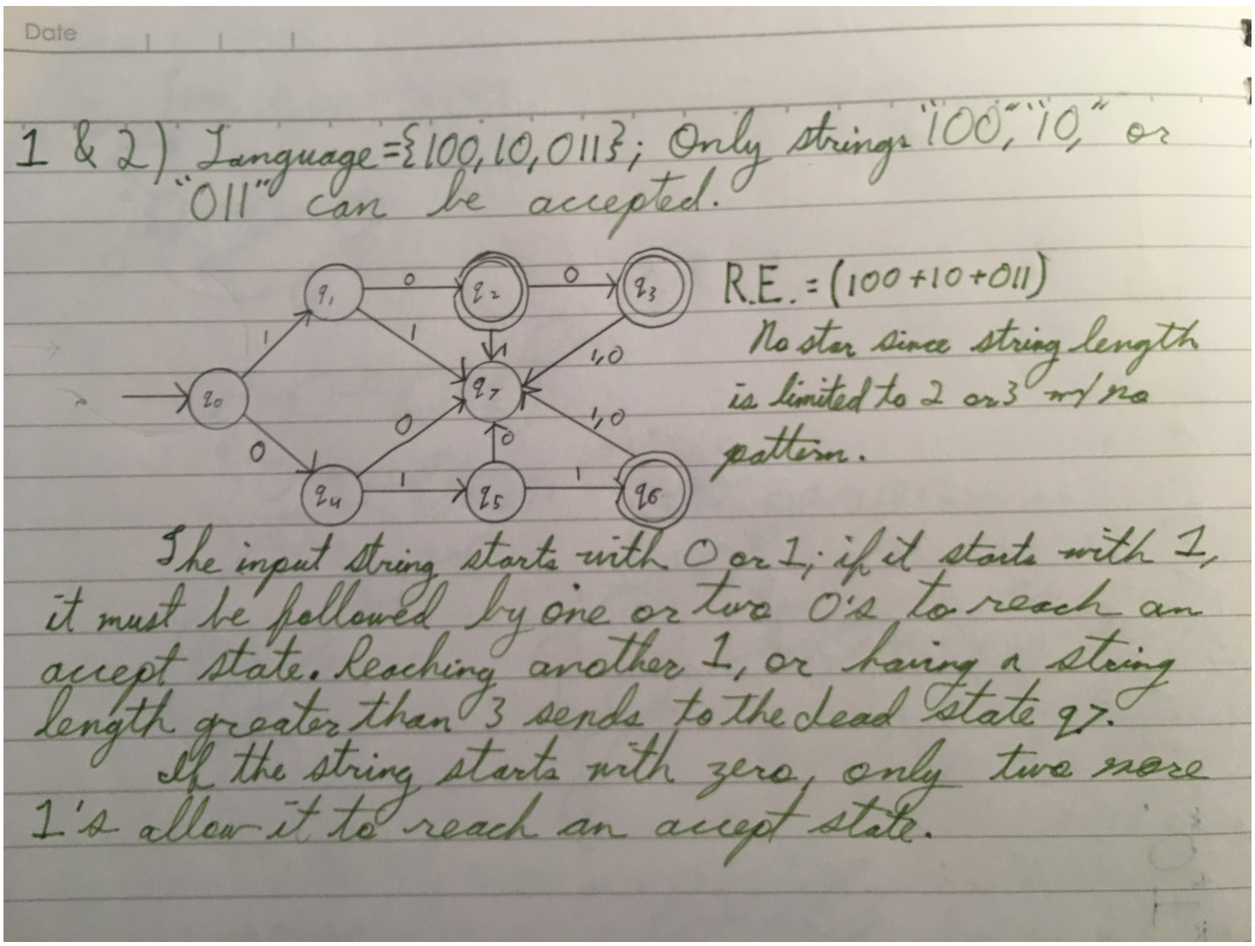
Suppose there is a language $L_1 = \{w \in \sum | w = 0^n1^n, n \geq 0\}$, which is non-regular (because it has an infinite and uncountable number of substrings for all n). This is the same language used in 9), the proof is there.

There is another language, $L_2 = \{0,1\}$ that is regular (because it is finite and countable).

$L_1 \cup L_2 = \{0,1\}L_1$ The resulting language is non-regular, because even with the concatenation of $\{0,1\}$, the L_1 portion is still non-regular by definition, making the language resulting from the union also non-regular.

Another example would be to simply replace L_2 with the empty set.

$L_1 \cup \emptyset = L_1$, which still remains non-regular.



1 & 2) Language = {100, 10, 011}; Only strings "100", "10," or "011" can be accepted.

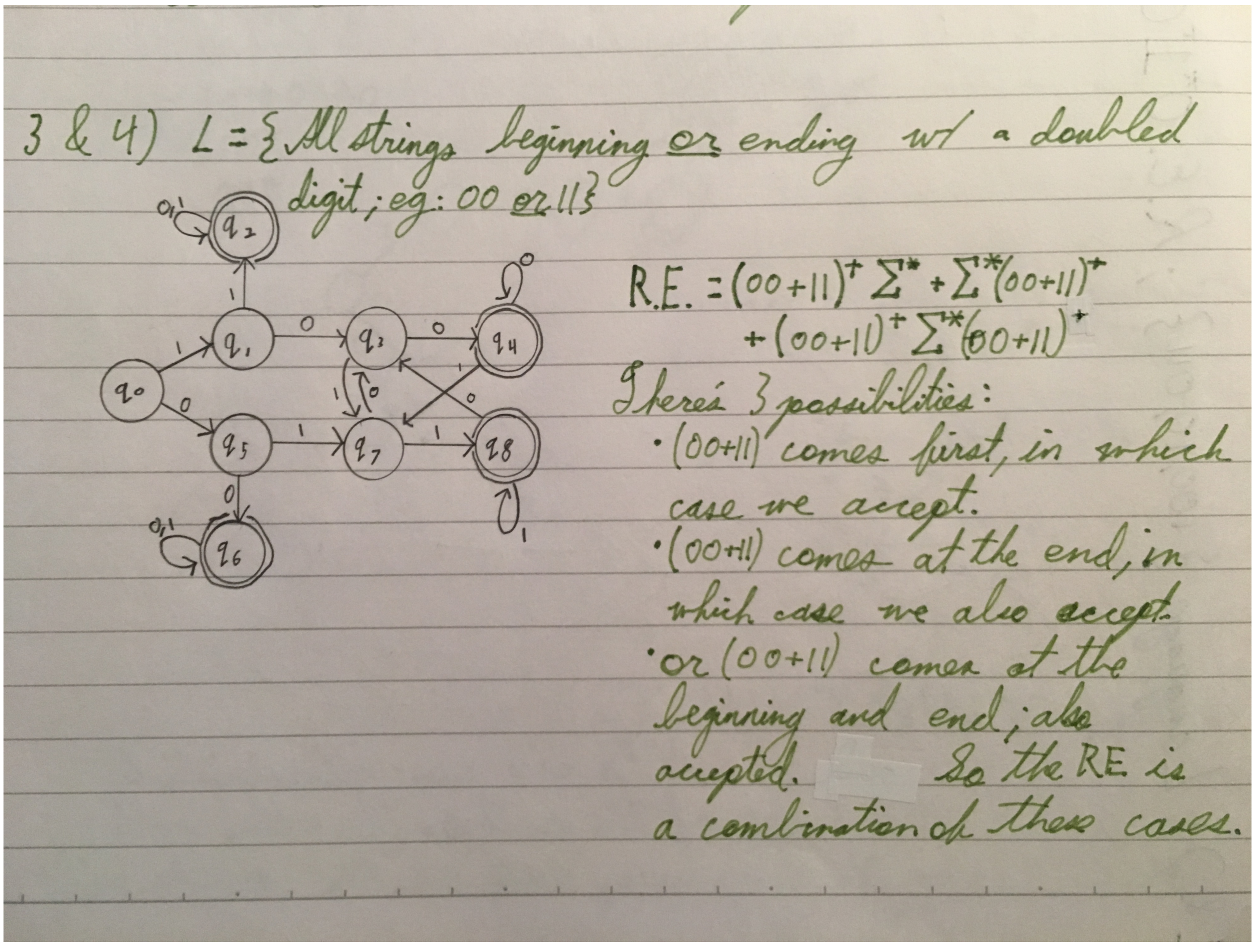
R.E. = (100 + 10 + 011)

No star since string length is limited to 2 or 3 w/ no pattern.

The input string starts with 0 or 1; if it starts with 1, it must be followed by one or two 0's to reach an accept state. Reaching another 1, or having a string length greater than 3 sends to the dead state q7.

If the string starts with zero, only two more 1's allow it to reach an accept state.

Figure 1: Figure 1 for questions 1 & 2



3 & 4) L = {All strings beginning or ending w/ a doubled digit; eg: 00 or 11}

R.E. $= (00+11)^+ \Sigma^* + \Sigma^*(00+11)^+ + (00+11)^+ \Sigma^* (00+11)^+$

There's 3 possibilities:

- (00+11) comes first, in which case we accept.
- (00+11) comes at the end, in which case we also accept.
- or (00+11) comes at the beginning and end; also accepted. So the RE is a combination of these cases.

Figure 2: Figure 2 for questions 3 & 4

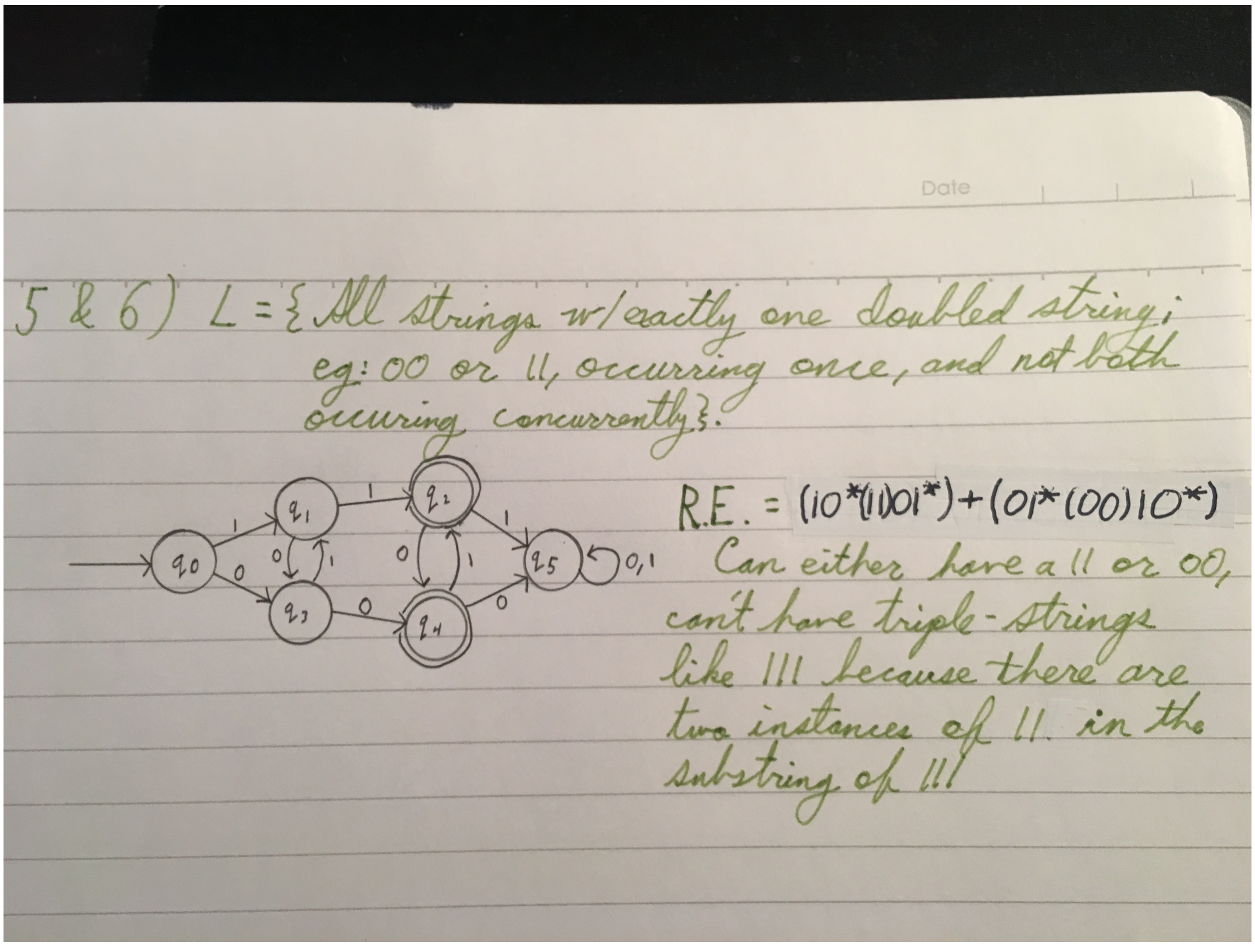


Figure 3: Figure 3 for questions 5 & 6