A Review of Recent Progress in Seismic Waves Propagation Modeling Using Machine Learning Based Methods

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https://github.com/oscar-rincon/Review-Seismic-Waves

Abstract

Numerical modeling has been crucial for addressing problems across various scientific and engineering disciplines involving partial differential equations. In particular, wave propagation modeling has seen significant development in scientific computation. Standard numerical modeling methods have demonstrated notable accuracy; however, their computational cost can be substantial. Recently, alternative methods based on machine learning have emerged, offering a promising balance between computational cost and accuracy when applied to wave propagation problems. In this work, we present a review of methods developed and used to model wave propagation, with a special emphasis on computational seismology. We discuss the fundamentals of wave propagation modeling, standard numerical methods, and recent advances in solving differential equations through these approaches. We conduct a systematic review of the literature to identify applications of these methods, either standalone or in hybrid approaches with standard numerical methods. The results of this review provide insights into the potential of machine learning techniques for wave propagation modeling and their impact on computational seismology.

Keywords: wave equations, numerical methods, machine learning, partial differential equations, computational seismology.

Introduction

Wave propagation is a physical phenomenon governed by partial differential equations, which hold significant importance across various applied sciences and engineering fields. However, analytical solutions are not always available in many practical situations and numerical methods are usually required to approximate the exact solutions. Consequently, these methods have been applied to solve the partial differential equations (Seriani and Oliveira, 2020).

In the field of wave propagation, numerous techniques address wave propagation challenges. Classical methods include finite-difference, finite-element and spectral-element methods (Moczo et al.; Virieux et al.; Igel; Komatitsch and Tromp; Chaljub et al., 2007; 2011; 2017; 1999; 2007). In these approaches, the spatial coordinates are discretized. In the context of mathematical modeling, the primary objective is to ensure that the solution methods are computationally efficient without sacrificing accuracy to capture the physical details inherent to the system that are required by the problem to be solved. However, standard numerical methods often encounter difficulties when addressing complex problems such as irregular geometries, material changes, and mixed boundary conditions. Therefore, the computational demand associated with many common models in computer sciences and engineering has increased the development of innovative strategies.

Research conducted with the use of machine learning has considerably grown in the late 2010s, owing to advancements in hardware, such as graphic processing units and data storage technologies and the growth of available data. Additionally, the discovery of better training practices for neural networks, and the availability of open-source packages like Tensorflow, PyTorch and JAX (Abadi et al.; Paszke et al.; Bradbury et al., 2016; 2019; 2018), as well as the availability of Automatic Differentiation in such packages (Paszke et al.; Baydin et al., 2017; 2017). Particularly, neural networks learning algorithms offer attractive approximation capabilities for any function by mapping the input features to the output targets in a data-driven manner. A version of the Universal Approximation Theorem conclusively demonstrates that neural networks have the capability to accurately approximate a wide variety of nonlinear functions without any dimensionality constraints (Barron, 1993). Therefore computational scientists have explored the potential of machine learning as a numerical tool to model systems governed by partial differential equations (Cuomo et al.; Karniadakis et al., 2022; 2021). Figures 1 shows the number of publications that related machine learning and standard numerical methods to the modeling of partial differential equations (A) and particularly the seismic wave equation (B). From those works, physics-informed neural networks (PINNs) is one of the methods that has gained more attention in the last years, being cited by over 10,000 publications (Raissi et al., 2019).

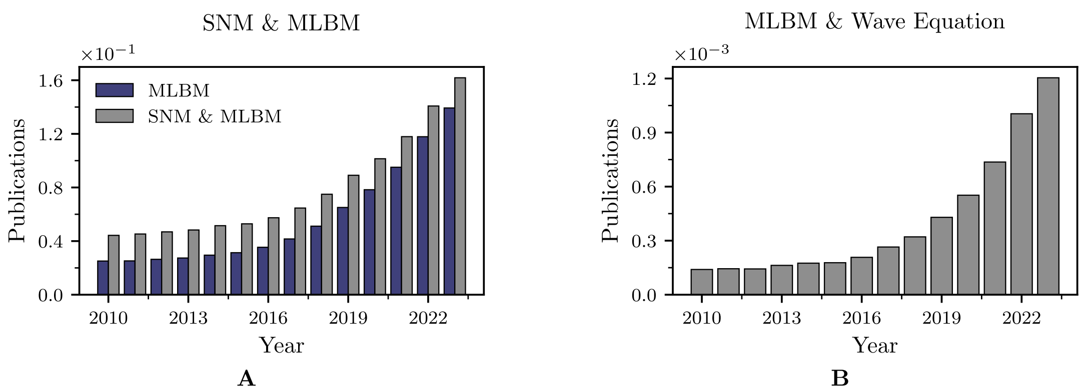

Figure 1. The growth of literature related to machine learning and wave propagation modeling is shown. The number of publications was retrieved from Scopus between 2010 and 2023. The relative number of publications is calculated as the number of publications containing the selected terms relative to the total number of publications in Scopus during the same period. The chosen terms were machine learning-based methods (MLBM) and standard numerical methods (SNM) (A), as well as MLBM specifically associated with wave propagation modeling (B).

Remarkable reviews have been conducted to address the increasing use of machine learning algorithms across various engineering and scientific disciplines (Vadyala et al.; Deng et al.; Lino et al., 2022; 2023; 2023). Also emphasis has been placed on the application of neural networks to model seismic inversion problems (JingBo et al., 2023). However, there is uncertainty, given the rapid growth of the field, about what machine learning based methods have been applied and demostrated to be an efficient complement or alternative to standard numerical methods (Grossmann et al.; McGreivy and Hakim, 2023; 2024). In principle, machine learning methods have the potential to learn a surrogate model able to approximate the solution of a partial differential equation. This is particularly relevant in the context of computational seismology, where the complexity of the domain phenomena can be challenging to model. However, some methods can be more efficient than others according to the problem being solved. Also, despite the importance of this characteristic, it is not always reported in the literature. Moreover, already proposed methods may still haven't been fully explored in the context of seismic wave propagation modeling. Therefore the aim of this review is to provide insights into the already demostrated potential of machine learning methods for wave propagation modeling and their impact on computational seismology.

This work presents a systematic review, focusing on the advancements in modeling partial differential equations using machine learning techniques and their resulting impact. While this area can be applied to a wide range of problems, our focus will be limited to the propagation of seismic waves. Our aim is to assist researchers interested in applying these emerging techniques to wave propagation modeling. The work is organized into the following sections: First three sections provide a state of the art of seismic waves propagation modeling using machine learning based methods. Especifically, section 1 describes general aspects about wave propagation modeling. Furthermore, in sections 2 and 3, we identify existing standard and machine learning methods used to solve partial diferential equations. Then, in section 4 we systematically review the recent advances in wave propagation modeling achieved through these emerging methods and identify when they have demostrated to be an alternative to traditional numerical methods or in an hibrid way when they can improve the solver performance in terms of computational time. We aimed to answer the following research question:

> **What machine learning techniques have shown to be a complement or alternative to traditional numerical methods for modeling the wave equation in computational seismology?**

We considered a complementary approach as one in which both machine learning and traditional numerical methods are employed together for modeling, particularly when synthetic data generated by standard numerical methods is used to train the model and the computational time required to solve it is reduced. Or alternatively, when a physics informed machine learning methods have been used and also reported an improvement in the efficiency. Furthermore, our scope includes studies where this emerging methods have been applied to solve inverse problems, since they have shown to be an alternative in this particular context (Haghighat et al.; Raissi et al.; Hao et al., 2021; 2020; 2023). This is attributed to their capacity to handle varying amounts of data and also to incorporate physical laws directly into the model.

1 Modeling of Wave Propagation

A dynamic model, such as wave propagation in a medium, aims to describe through a function how a system changes over time. These models typically rely on differential equations to characterize the system's evolution. A general formulation of the governing equation for a physical problem can be expressed as:

$$D(u(x,t);\lambda) = f(x,t), \quad x \in \Omega, \quad t \in [0,T].$$

Here, D represents the differential operator acting on the solution $u(x,t)$ to the differential equation, which is parameterized by λ, and $f(x,t)$ is the source term. The symbols Ω and $\partial\Omega$ denote the spatial domain and its boundary, respectively. Equation 1 can be applied to model various systems. The corresponding boundary and initial conditions are given by:

$$B(u(x,t)) = g(x,t), \quad x \in \partial\Omega, \quad t \in [0,T]$$

and

$$u(x, 0) = h(x, 0), \quad x \in \Omega.$$

The prescribed initial and boundary conditions are characterized by $h(x)$ and $g(x, t)$, respectively. In mathematical modeling, two general approaches are commonly used: the forward and inverse problems. The inverse problem involves determining the causes of a set of observations (Röth and Tarantola; Tarantola, 1994; 2005), such as inferring the properties of a medium based on its response to wave propagation. This approach is the opposite of the forward problem, which calculates the effects based on known causes. Since the inverse problem starts with the effects and seeks to determine the causes, it typically requires iterative forward modeling, making it computationally complex. A schematic representation of forward and inverse modeling using numerical methods is shown in Figure 2.

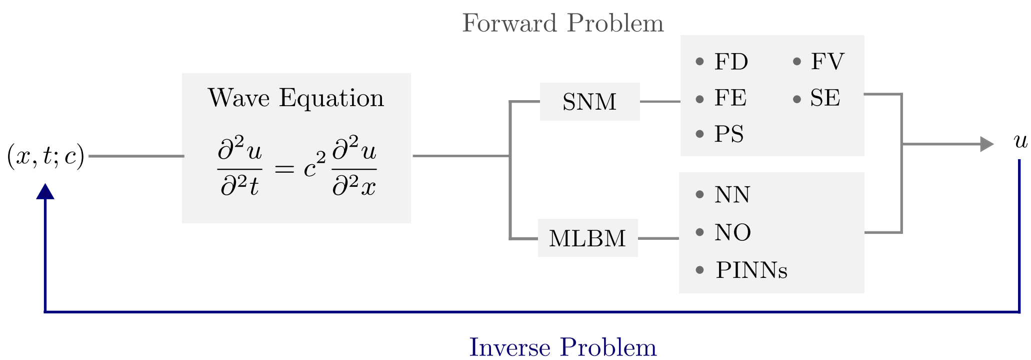

Figure 2. Scheme of the forward and inverse problems encountered in solving partial differential equations. In the forward scenario, the inputs $(x, t; c)$ are employed to characterize a model across PDEs. Subsequently, the PDEs are resolved through either standard numerical methods (SNM) or neural networks based methods (MLBM) to derive a solution u. Standard numerical methods such as: finite differences (FD), finite elements (FE), pseudo-spectral (PS), finite volumes (FV), and spectral elements (SE). Also, deep learning techniques include, for example, Physics Informed Neural Networks (PINNs), Neural Operator (NO), and Neural Networks (NN). In the case of the inverse problem, the objective is to determine the parameters, for example, the wave speed c starting from the solution u.

Inverse problems are closely tied to computational modelling, and solving them is crucial for many real-world tasks. Moreover, some complex physical problems require determining the properties of a physical system governed by partial differential equations from observational data, rather than solving them directly to obtain a function that satisfies them (Galiounas et al.; Ren et al.; McCann et al., 2022; 2024; 2017). The objective is to estimate a set of latent or unobserved parameters of a system based on real-world observations. Within the framework described by Equation 1, the task involves estimating λ given u. Inversion can be exceedingly challenging since often requires numerous forward simulations to align the predictions of the physical model with the set of observations.

Despite being the most elementary among mechanical wave equations, the scalar (acoustic) wave equation is widely used to study seismic waves and in medical applications (Moseley; Alkhadhr and Almekkawy, 2022; 2023). The second-order linear wave equation in a homogeneous medium can be expressed as (Carcione, 2002):

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \nabla^2 u(x,t) = f(x,t) \; ,$$

where $\nabla^2 = \sum_{i=1}^{d} = \frac{\partial}{\partial^2 x_i}$, $u(x,t)$ describes the pressure of the generated waves, and $f(x,t)$ is a source term that describes the strength and duration of the source.

Another common expression used to describe the propagation of seismic waves, for the case of a heterogeneous isotropic medium, is the elastic wave equation (Moseley et al.; Lehmann et al., 2018; 2023). This equation can be expressed as:

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla(\lambda(\nabla \cdot u)) + \nabla \mu \left[\nabla u + (\nabla u)^T \right] + (\lambda + 2\mu)\nabla(\nabla \cdot u) - \mu \nabla \times (\nabla \times u) \; ,$$

where ρ is the material density, u is the displacement vector, and λ, μ are the Lamé parameters characterizing the material. These equations are fundamental for modeling the propagation of seismic waves in elastic media. The acoustic wave equation is a simplification that assumes the waves are longitudinal and the medium is homogeneous and isotropic. In contrast, the elastic wave equation accounts for the heterogeneous and anisotropic properties of the medium, allowing for the modeling of both longitudinal and transverse waves.

Besides the acoustic and elastic equations, there are other important variants of the wave equation used in different contexts of computational seismology. Viscoelastic Wave Equation is a variant tha incorporates damping effects due to the viscosity of the medium. It is useful for modeling wave attenuation in real geological media that exhibit viscoelastic behavior and can be expressed as:

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla(\lambda(\nabla \cdot u)) + \nabla \mu \left[\nabla u + (\nabla u)^T \right] + (\lambda + 2\mu)\nabla(\nabla \cdot u) - \mu \nabla \times (\nabla \times u) - \eta \frac{\partial u}{\partial t} \; ,$$

where η is the viscosity coefficient. Anisotropic Wave Equation describe propagation in anisotropic media, the elastic properties vary with direction. The wave equation is modified to include additional terms representing this anisotropy. It can be described as:

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla \cdot \sigma + f \; ,$$

where σ is the anisotropic stress tensor and f is a source term. Nonlinear Wave Equation are considered in situations where wave amplitudes are very large, linear approximations are insufficient, and nonlinear terms must be considered in the wave equation.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u + \beta \frac{\partial u^2}{\partial x^2} = f(x_i, t) \; ,$$

where β is a nonlinearity coefficient. These variants allow for more precise and realistic modeling of seismic wave propagation in different types of media and under various conditions. The choice of the appropriate wave equation depends on the characteristics of the medium and the seismic phenomenon being studied. Tradicionally, the wave equation and its applications to computational seismology have been solved using numerical methods (Igel, 2017).

2 Standard Numerical Methods

In the past decades various numerical methods have been proposed to solve physics systems by partial differential equations such as the wave equation. The finite-difference method is among the most popular to solve partial differential equations, and particularly the wave equation. This possibly associated with its straightforward concept and easy implementation. A complete review of the finite-differences method applied to wave propagation can be found the Moczo et al. (2014). Partial derivatives are approximated by discrete operators involving differences between adjacent grid points. The finite difference method suits for tackling issues related to simple geometric structures. In contrast, other methods such as the finite element offers more grid flexibility, facilitating the handling of intricate geometric boundaries.

In wave propagation simulations, the partial differential equations are typically discretized on a staggered grid (Madariaga; Virieux, 1976; 1986). This approach facilitates the resolution of the rupture propagation problem. Particularly an approach was proposed in the work of Zhou et al. (2021) a finite-difference method with variable-length temporal and spatial operators was proposed to increase the stability and efficiency of the standard method. Also, Liu et al. (2023) combined a standard staggered-grid, finite-difference approach and the perfectly matched layer absorbing boundary to solve 3D first-order velocity-stress equations of acoustoelasticity to simulate wave propagating.

Finite-element methods are suitable for dealing with intricate shapes and diverse materials because they can use irregular grids. They permit flexibility in size, shape, and approximation order. Nevertheless, a drawback is their high demand for computing power. This methodology involves the transformation of the problem at hand into a system of linear equations utilizing the weak formulation of the pertinent differential equation. This transformation is facilitated by employing an interpolation basis comprised of polynomials defined over disjoint domains, commonly referred to as elements.

Open-source software is available for applying numerical methods to solve the wave equation. For example, FEniCS and DUNE (Langtangen and Logg; Sander, 2016; 2020), offer computing frameworks designed for solving partial differential equations using the finite element method. SPECFEM, which specializes in seismic wave propagation, is widely used in simulations implemented in Fortran (Komatitsch et al.; Komatitsch et al., 2023; 2024). Similarly, SEISMIC_CPML (Komatitsch and Martin, 2007) uses finite differences for modeling. These implementations of standard methods have enabled effective simulations of the wave equation.

A significant difficulty in using standard methods for wave propagation simulations is their computational cost. Their accuracy is achieved at the expense of the number of points in the grid. Modeling a complex domain can entail a huge amount of grid points, with the wavefield requiring iterative updates across the entire grid at each time step. Associated with the required discretization is the challenge when dealing with high-dimensional systems. The curse of dimensionality can lead to a rapid increase in computational cost as the number of dimensions grows. Additionally, model evaluation and storage could be significantly costly (Saloma, 1993), and their limited capacity to incorporate measured data into their predictions makes them less ideal for use in inverse problems. There is considerable scientific interest in employing machine learning techniques to address these challenges.

3 Machine learning Based Methods

The field of machine learning has recently shown significant promise in approximating predictions of physical phenomenas. These methods are capable of capturing highly nonlinear physics and provide substantially faster inference times compared to traditional simulations. Consequently, machine learning has been employed as an alternative to conventional methods, leveraging its capability as a universal function approximator (Hornik, 1991). For example, support vector machines have been used to solve ordinary and partial differential equations. Although this method was originally designed for classification tasks, an extention to the method that apply least square to the objetive function has been proposed to solve differential equations (Mehrkanoon et al.; Mehrkanoon and Suykens, 2012; 2015).

Neural network based methods are a subset of machine learning, whose models are composed of an artificial neural network with a single or multiple processing layers (Figure 3.A). It have shown potential in overcoming the limitations of multiple approaches in various fields such as computer vision, natural language processing, and genomics (LeCun et al.; Goodfellow et al., 2015; 2016). The fundamental architecture of a neural network architecture is conformed by an input layer, an output layer, and an arbitrary number of hidden layers. Particularly, in a fully connected neural network, neurons in adjacent layers are connected with each other but neurons within a single layer share no connection (Figure 3.B). Furthermore, neural networks methods have emerged as an attractive tool to augment and complement conventional numerical solvers of partial differential equations, thereby enabling the tackling of challenges across multiple dimensions, scales, and parameterization with the promise of efficiency and precision.

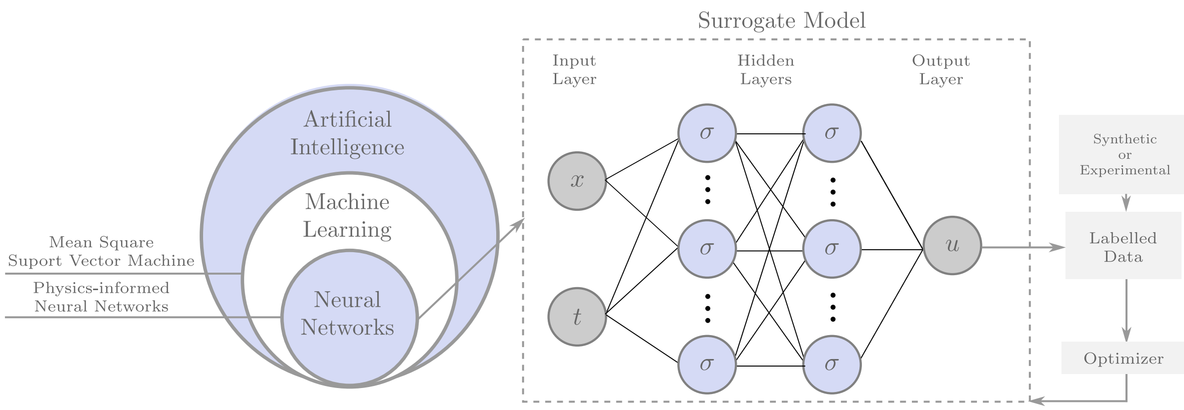

Figure 3. Artificial Intelligence subsets and artifical neural networks. (A) Deep learning as a subset of machine learning and artificial intelligence and (B) basic architecture of artificial neural networks.

They essentially model the partial differential equation solution by a deep neural network and train the network's parameters to approximate the solution. Data-driven neural networks methods are capable of directly learning the trajectory of a system of partial differential equations from available data (Li et al.; Li et al., 2020; 2021). Alternatively, syntetic data generated by standard numerical methods can be used to train the neural network. Therefore, a surrogate model can be used to predict the solution of the partial differential equation at a reduced computational cost. While keeping an acceptable level of accuracy. For example, one of the most popular types of deep neural networks is known as convolutional neural networks. A convolutional neural network convolves learned features with input data, and uses 2D convolutional layers, making this architecture well suited to processing 2D data, such as images.

All these approaches employ machine learning algorithms and others such as support vector machines, random forests, Gaussian processes have been also applied to model physical systems. However, they are implemented mainly as black-box tools. The constructed neural network can be thought of being ignorant of the mathematical description of the physical phenomenon. In order to overcome this limitation PINNs architectures have been proposed. Where the activation and the loss functions are designed according to the context of the problem.

There has been an increasing interest in leveraging PINNs to solve forward and inverse problems where full or partial knowledge of the governing equations is known since the published works of Raissi and Karniadakis (2018), Raissi et al. (2018) and Raissi et al. (2019). The core concept of PINNs is to minimize an energy functional that represents the residual of the PDE along with its initial and boundary conditions. Although similar ideas for constraining neural networks using physical laws have been explored in previous studies (Lagaris et al., 1998). The general principle of PINNs is to integrate deep neural networks and physical laws to learn the underlying consistent dynamics from small or zero labeled data (Karniadakis et al., 2021). As universal approximators, neural networks have the potential to represent any partial differential equation. They make use of the powerful tool that is automatic differentiation. This capability eliminates the need for the discretization step, thereby avoiding discretization-based physics errors as well. Instead a random sampling of the domain is implemented. PINNs aim to address physical systems governed by the equation

$$D[u(t,x);\lambda] - f(x,t) = 0,$$

where $x \in \mathbb{R}^D$ and $t \in \mathbb{R}$. The expression $N[u(t,x);\lambda]$ denotes an underlying differential operator that characterizes the physical system, parametrized by λ. The function $u(t,x)$ represents the system's solution. The loss function is of the general form

$$L := \beta_{\text{pde}} L_{\text{pde}}(\sigma) + \beta_{\text{ic}}(\sigma) L_{\text{ic}} + \beta_{\text{bc}} L_{\text{bc}}(\sigma),$$

where

$$\begin{aligned}
\mathcal{L}_{pde}(\boldsymbol{\sigma}) &= \frac{1}{n_{pde}} \sum_{i=1}^{n_{pde}} \left| u_{tt} - \mathcal{D}\left[\hat{u}\left(t, \boldsymbol{x}_i; \boldsymbol{\sigma}\right)\right] - f\left(t, \boldsymbol{x}_i\right)\right|^2, \\
\mathcal{L}_{bc}(\boldsymbol{\sigma}) &= \frac{1}{n_{bc}} \sum_{i=1}^{n_{bc}} \left|\hat{u}\left(t, \boldsymbol{x}_i; \boldsymbol{\sigma}\right) - g\left(t, \boldsymbol{x}_i\right)\right|^2, \\
\mathcal{L}_{ic}(\boldsymbol{\sigma}) &= \frac{1}{n_{ic}} \sum_{i=1}^{n_{ic}} \left|\hat{u}\left(0, \boldsymbol{x}_i; \boldsymbol{\sigma}\right) - h\left(t, \boldsymbol{x}_i\right)\right|^2,
\end{aligned}$$

and $\mathcal{L}_{\text{pde}}$ represents the residuals of the PDEs, $\mathcal{L}_{\text{ic}}$ represents the error at the collocation points at the initial time point, and $\mathcal{L}_{\text{bc}}$ represents the error at the collocation points on the boundaries. The terms n_{pde}, n_{bc}, and n_{ic} denote the number of collocation points used for the PDE residuals, boundary conditions, and initial conditions, respectively. The coefficients β_{ic} and β_{bc} are training hyper-parameters. Figure 4 illustrates the application of PINNs to the wave equation.

One major drawback of these methods is the difficulty of transferring knowledge between different configurations. For example, when solving the wave equation, CNNs and PINNs are trained with a fixed velocity parameter and cannot predict anything for a different velocity value.

One of the main challenges in numerically modeling mechanical is associated with the dimensionality, given the computational complexity. Tackling complex high-dimensional systems comes with significant challenges. Despite this, machine learning-based algorithms offer promising prospects for solving partial differential equations, as indicated by studies such as the one by Blechschmidt and Ernst (2021). Most of the applications are implemented in one dimensional or two-dimensional domains. In Lehmann et al. (2023) the Fourier Neural Operator method is applied to model sesimic waves.

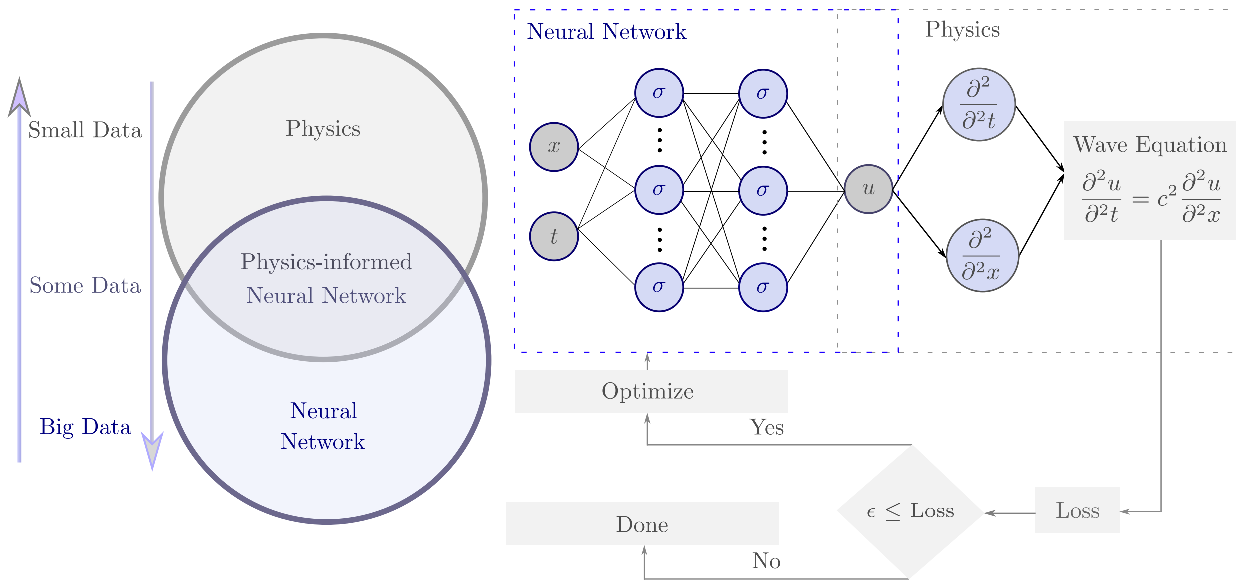

Figure 4. Physics-informed neural networks scheme applied to the wave equation.

Emerging machine learning methods for solving partial differential equations can face difficulties in establishing fair comparison points with standard numerical methods. McGreivy and Hakim identified two common pitfalls. First, comparing the runtime of a less accurate machine learning method to a more accurate standard numerical method, whereas a fair approach would be to make the comparison under similar accuracy levels. Second, evaluating the standard numerical method that is not suitable for the partial differential equation being solved. These two criteria are essential for properly evaluating performance, but they are not always followed.

Different extentions of the classical work where PINNs was originally proposed have emerged. Kharazmi et al. (2019) proposed variational PINNs which instead trained PINNs using the variational form of the underlying differential equations. A neural network is still used to approximate the solution of the differential equation, but it is combined with a set of analytical test functions to compute the residual of the variational form of the equation in its physics loss term. Furthermore, they used quadrature points to estimate the corresponding integrals in the variational loss, rather than random collocation points. They found that the variational PINNs was able to solve differential equations including Poisson's equation with similar or better accuracy to a PINNs trained using the strong form, whilst requiring less collocation points to train. However, most of these extensions have not yet been applied to wave propagation modeling.

Various open-source frameworks are available for solving partial differential equations using emerging machine learning methods. Python packages such as NeuroDiffEq (Chen et al., 2020) and DeepXDE (Lu et al., 2021) facilitate the solving of both ordinary and partial differential equations using neural networks as function approximators. A similar implementation in the Julia programming language is NeuralPDE (Zubov et al., 2021). Additionally, PINNs-Torch (Bafghi and Raissi, 2023) enables the application of PINNs using PyTorch, offering improved performance compared to the original model.

4 Applications

This section presents a systematic review of the literature on the application of machine learning methods to model wave propagation. We inicially describe the inclusion and exclusion criteria used to select the articles for the review. Then, we present the search strategy employed to identify the most relevant publications. Finally, we summarize the main findings from the reviewed articles, focusing into answering the research question outlined in the introduction.

Inclusion and Exclusion Criteria

Given the wide range of machine learning applications, we focused our systematic review specifically on their potential within computational seismology. Although seismological research can be approached in both data-driven and physics-informed ways—where the former uses machine learning to analyze seismic data and make predictions—our research was limited to methods that incorporate descriptions of physical phenomena through partial differential equations. This, given the research expertise of the authors.

In order to answer the research question, we considered articles that reported a quantitative or a supported qualitative comparison of the implemented model computational efficiency relative to a standard numerical methods, in a similar way to the criteria stablished by McGreivy and Hakim (2024). Therefore we excluded studies that compare their results to other machine learning methods or that do not provide a comparison at all. Also, those research articles that compared accuracies but not computational times. We considered studies that compared the computational time required to solve the models, but restricted our focus to those within our field of interest. We also included works that were applied to solve inverse problems.

Search Strategy

A search strategy was developed based on the research question and the inclusion and exclusion criteria to identify the most relevant publications. Figure 5 presents the flowchart and the total number of studies that met these criteria.

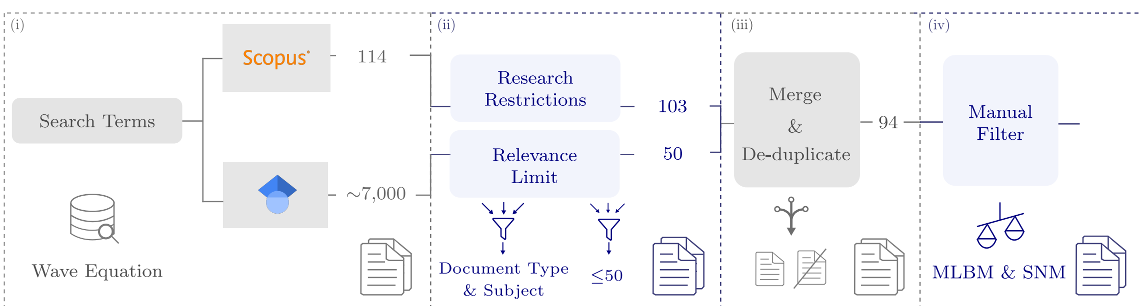

Figure 5. Search flowchart and number of publications after each step. During the systematic review process, Scopus and Google Scholar were utilized with the relevant search terms (i), and the research was restricted to works in English and within the time frame of 2014-2024 (ii). The resulting lists were then sorted by relevance and limited to a maximum of 50 entries, with duplicates removed (iii). Finally (iv), a manual filter was applied by reading the titles and abstracts to ensure the publications were pertinent to our chosen field.

Initially, our search focused on how machine learning has been applied to modeling seismic wave propagation. The search was conducted using the following query as an initial filter:

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("machine learning" OR "deep learning" OR "neural networks") AND
("seismic" OR "seismology") AND "wave equation" AND (modeling OR
modelling OR model OR simulation)
```

The search was conducted using the Scopus and Google Scholar databases. Initially, Google Scholar served to verify the comprehensiveness of the results obtained from Scopus. However, it also proved invaluable for its inclusion of references from both indexed journals and preprint platforms, facilitating the identification of the most recent literature in this rapidly evolving field. The filtering tools available in each database were employed to refine the search results. In Scopus, the search was restricted to documents containing the relevant terms in the title, abstract, or keywords, and further narrowed to include only articles, conference papers, and book chapters, with non-English documents excluded. In Google Scholar, the same search terms were used, and the results were sorted by relevance, with the first 50 articles selected. The lists generated from both databases were then merged, and duplicate entries were removed. A manual screening process was conducted by reviewing the abstracts and conclusions to ensure that the publications met the inclusion criteria. Additionally, the references cited in the manually filtered articles were reviewed to identify any further relevant articles that aligned with the search criteria. Figure _ shows an author map generated from the compiled bibliography using VOSViewer.

Content Analysis

From the publications that successfully passed the inclusion criteria we analysed the full content and summarized the information. For methods that have improved the computational efficiency relative to a standard numerical method, we extracted the follwing aspects: the type of wave equation modeled, the specific machine learning method employed, the type of standard method used to compare and the outcome from the comparison (Table 1).

Publication	Equation(s)	MLBM	SNM	Outcome
Ji et al. (2024)	3-D Acoustic and Elastic	RCNN	FD	can be significantly reduced to 1/110
Zou et al. (2024)	2-D Acoustic	VE-PINN	FD	significantly enhances efficiency in solving the acoustic wave equation
Roncoroni et al. (2021)	1-D Acoustic	RNN	FD	prediction time is much lower than classical methods
Moseley et al. (2020)	2-D Acoustic	WaveNet and Conditional Autoencoder	FD	Both networks are 20–500 times faster than FD modelling

Table 1. Summary of the reviewed publications that enhance efficiency of prediction relative to a traditional method.

For methods applied to solving inverse problems, we included: the type of wave equation modeled, the specific machine learning method employed, and the proposed application (Table 2).

Publication	Equation	MLBM	Application
Xiong and Yong (2022)	1-D Poroelastic	DNN	predict synthetic data as well as real logging data of shale reservoirs
Moseley et al. (2020)	2-D Acoustic	WaveNet	seismic inversion in the horizontally layered media
Huang et al. (2020)	1-D Acoustic	RNN	FWI
Karimpouli and Tahmasebi (2020)	1-D Acoustic	GP and PINNs	velocity (P- and S-wave) and density inversion

Table 2. Summary of the reviewed publications that solved an inverse problem.

Discussion

5 Conclusions

In this review, we have discussed the advancements in wave propagation modeling achieved through machine learning methods, with a focus on computational seismology. We have provided an overview of the fundamentals of wave propagation modeling, standard numerical methods, and the recent advances in machine learning methods to solve differential equations. A systematic review of the literature was conducted to identify the applications where machine learning methods have demonstrated to improve the computational performance to standard numerical methods in terms of computational time and accuracy. It is important to recognize that deep learning methods should complement, rather than replace, standard numerical techniques for solving partial differential equations. Traditional methods have been refined over decades to meet robustness and computational efficiency criteria in real-world applications. While this review focuses on computational seismology applications, the discussed methods can be applied to other fields where the wave equation is relevant. Future research should aim to integrate the strengths of both machine learning and traditional numerical methods, exploring hybrid approaches that can leverage the advantages of each technique.

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