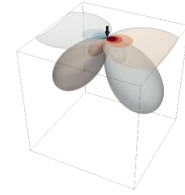




# A Review of Recent Progress in Seismic Waves Propagation Modeling Using Machine Learning Based Methods



Oscar Rincón-Cardeno<sup>1,2</sup>, , 

<sup>1</sup> School of Applied Sciences and Engineering, Universidad EAFIT, Medellín, Colombia - <sup>2</sup> Applied Mechanics Research Group from Universidad EAFIT

 <https://github.com/oscar-rincon/review-seismic-waves>

## Abstract

Numerical modeling has been crucial for addressing problems across various scientific and engineering disciplines that involve partial differential equations. In particular, wave propagation modeling is one of the areas where intensive scientific computation has been developed. Standard numerical modeling methods demonstrate notable accuracy. However, their computational cost can be significant. Alternative methods based on machine learning have recently emerged, showing an adequate balance between computational cost and accuracy when applied to wave propagation problems. In this work, we present a review of machine learning methods that have been developed and used to model wave propagation, with special emphasis on seismology. We discuss the fundamentals of wave propagation modeling, standard numerical methods, and the recent advances achieved through machine learning methods. We also discuss the challenges and opportunities that these methods present for future research in the field of wave propagation modeling.

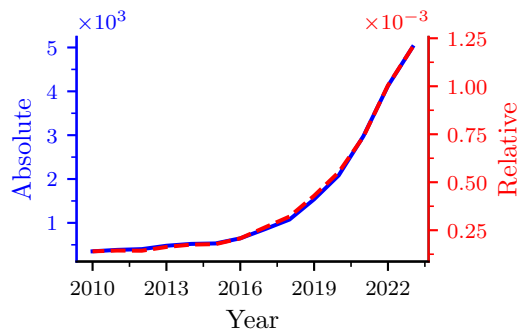
## Introduction

Wave propagation is a physical phenomenon governed by partial differential equations, which hold significant importance across various applied sciences and engineering fields. However, analytical solutions are not always available for most nonlinear systems, and numerical methods are usually required to approximate the exact solutions. Consequently, these methods have been applied to solve the partial differential equations, achieving the required accuracy (Seriani and Oliveira, 2020).

In the field of wave propagation, numerous techniques address wave propagation challenges. Classical methods include finite-difference, finite-element and spectral-element methods (Moczo et al.; Virieux et al.; Igel; Komatitsch and Tromp; Chaljub et al., 2007; 2011; 2017; 1999; 2007). In these approaches, the spatial coordinates are discretized. In the context of mathematical modeling, the primary objective is to ensure that the solution methods are computationally efficient without sacrificing accuracy to capture the physical details inherent to the system. However, standard numerical methods often encounter difficulties when addressing complex problems such as irregular geometries, material changes, and mixed boundary conditions. Therefore, the computational demand associated with many common models in computational mechanics has increased the development of innovative strategies.

Research conducted with the use of machine learning has considerably grown in the late 2010s, owing to advancements in hardware, such as graphic processing units and data storage technolo-

gies and the growth of available data. Additionally, the discovery of better training practices for neural networks, and the availability of open-source packages like Tensorflow, PyTorch and JAX (Abadi et al.; Paszke et al.; Bradbury et al., 2016; 2019; 2018), as well as the availability of Automatic Differentiation in such packages (Paszke et al.; Baydin et al., 2017; 2017). Particularly, deep learning algorithms offer attractive approximation capabilities for any function by mapping the input features to the output targets in a data-driven manner. A version of the Universal Approximation Theorem conclusively demonstrates that neural networks have the capability to accurately approximate a wide variety of nonlinear functions without any dimensionality constraints (Barron, 1993). Additionally, multiple physics-informed neural network methods have been recently proposed (Cuomo et al., 2022). As a consequence there has been a marked increase in the number of publications associated with neural networks and wave propagation modeling. Figure 1 shows the number of document published associated to wave propagation modeling and machine learning.



**Figure 1.** The growth of literature related to machine learning and wave propagation modeling. Number of publications according to Scopus between 2010 and 2023. The implemented query was: "machine learning" OR "deep learning" OR "neural networks" AND "wave propagation" OR "wave equation" AND (modeling OR modelling OR model OR simulation).

Remarkable reviews have been conducted to address the increasing use of machine learning algorithms across various engineering and scientific disciplines (Vadyala et al.; Deng et al.; Lino et al., 2022; 2023; 2023). Particularly, this review may be of help for researchers with interest into apply these emerging techniques in wave propagation modeling. Due to the extensive areas where wave equation is applied, our aim is to review the recent progress achieved in wave propagation modeling by employing these emerging techniques, specially on geophysics, and the publication time-span studied for those applications was 2014-2024.

This work provides a comprehensive analysis of the advancements made and their impact. While this area can be applied to a wide range of problems, our focus will be limited to the propagation of mechanical waves. The work is organized into the following sections: Section 1 describes general aspects about wave propagation modeling. Furthermore, in Sections 2 and 3, we identify existing standard and machine learning methods used to solve the differential equations and particularly the wave equation. Then, in Section 4 we systematically review the recent advances in wave propagation modeling achieved through these emerging methods and identify when they can be an alternative to traditional numerical methods in terms of computational time and accuracy.

# 1 Modeling of Wave Propagation

A dynamic model, such as the propagation of waves in a medium, describes how a system changes over time. This is different from a static model, which shows how a system is at equilibrium. Dynamic models typically use differential equations to describe how a system evolves. A general formulation of the governing equation of a physical problem can be:

$$D(u(x, t); \lambda) = f(x, t), \quad x \in \Omega, \quad t \in [0, T]$$

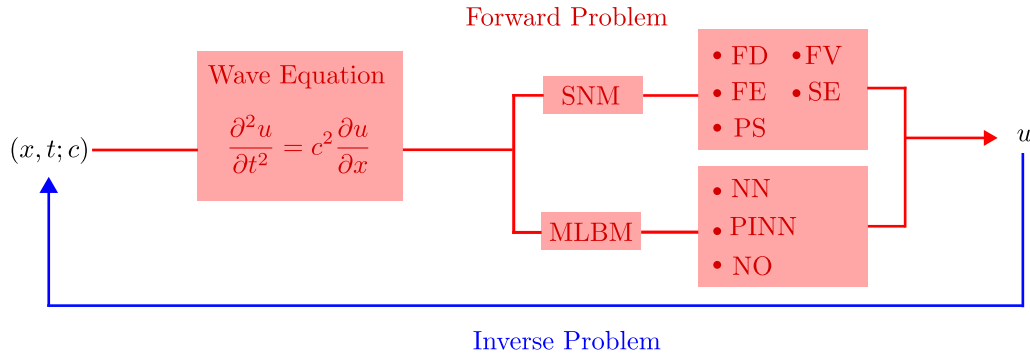
where  $D$  denotes the differential operator acting on the solution to the differential equation  $u(x, t)$  parameterized by  $\lambda$  and  $f(x, t)$  is a source term. While,  $\Omega$  and  $\partial\Omega$  denote the spatial domain and its boundary. Equation 1 can be used to model different systems. We denote the corresponding boundary conditions and the initial condition by:

$$B(u(x, t)) = g(x, t), \quad x \in \partial\Omega, \quad t \in [0, T]$$

and

$$u(x, 0) = h(x, 0), \quad x \in \Omega$$

The prescribed initial condition and boundary condition are characterized via  $h(x)$  and  $g(x, t)$ . Two general approaches when mathematically modeling a system are the inverse and forward problems. The process of determining the causes of a set of observations is known as the inverse problem (Tarantola, 2005), aiming to infer, for example, the properties of a medium based on its reaction to mechanical wave propagation. Since it begins with the effects and then calculates the causes, it is the opposite of a forward problem, which determines the effects based on the causes. In most cases, the inverse problem is formulated by iteratively performing forward modeling to infer the causes that produce a desired effect, making it computationally complex. An schematic representation of the forward and inverse model using numerical methods is shown in Figure 2.



**Figure 2.** Scheme of the forward and inverse problems encountered in solving partial differential equations (PDEs). In the forward scenario, the inputs  $(x, t; c)$  are employed to characterize a model across PDEs. Subsequently, the PDEs are resolved through either standard numerical methods (SNM) or neural networks based methods (MLBM) to derive a solution  $u$ . Standard numerical methods such as: finite differences (FD), finite elements (FE), pseudo-spectral (PS), finite volumes (FV), and spectral elements (SE). Also, deep learning techniques include, for example, Physics Informed Neural Networks (PINNs), Neural Operator (NO), and Neural Networks (NN). In the case of the inverse problem, the objective is to determine the parameters, for example, the wave speed,  $c$  starting from the solution  $u$ .

Inverse problems are closely tied to simulation, and solving them is crucial for many real-world tasks. Moreover, some complex physical problems require determining the properties of a physical system governed by partial differential equations from observational data, rather than solving them directly to obtain a function that satisfies them (Galiounas et al.; Ren et al.; McCann et al., 2022; 2024; 2017). The objective is to estimate a set of latent or unobserved parameters of a system based on real-world observations. Within the framework described by Equation 1, the task involves estimating  $\lambda$  given  $u$ . Inversion can be exceedingly challenging since often requires numerous forward simulations to align the predictions of the physical model with the set of observations.

Despite being the most elementary among mechanical wave equations, the scalar (acoustic) wave equation is widely used to study seismic waves and in medical applications (Moseley; Alkhadhr and Almekkawy, 2022; 2023). The second-order linear wave equation in a homogeneous medium can be expressed as (Carcione, 2002):

$$\frac{\partial^2 u(x_i, t)}{\partial t^2} - c^2 \nabla^2 u(x_i, t) = f(x_i, t) ,$$

where  $u(x_i, t)$  describes the pressure of the generated seismic waves, and  $f(x_i, t)$  is a source term that describes the strength and duration of the source.

Another common expression used to describe the propagation of seismic waves, for the case of a heterogeneous isotropic medium, is the elastic wave equation (Moseley et al.; Lehmann et al., 2018; 2023). This equation can be expressed as:

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla(\lambda(\nabla \cdot u)) + \nabla \mu [\nabla u + (\nabla u)^T] + (\lambda + 2\mu) \nabla(\nabla \cdot u) - \mu \nabla \times (\nabla \times u) ,$$

where  $\rho$  is the material density,  $u$  is the displacement vector, and  $\lambda, \mu$  are the Lamé parameters characterizing the material. These equations are fundamental for modeling the propagation of seismic waves in elastic media. The acoustic wave equation is a simplification that assumes the waves are longitudinal and the medium is homogeneous and isotropic. In contrast, the elastic wave equation accounts for the heterogeneous and anisotropic properties of the medium, allowing for the modeling of both longitudinal and transverse waves.

Besides the acoustic and elastic equations, there are other important variants of the wave equation used in different contexts of computational seismology. Viscoelastic Wave Equation is a variant that incorporates damping effects due to the viscosity of the medium. It is useful for modeling wave attenuation in real geological media that exhibit viscoelastic behavior.

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla(\lambda(\nabla \cdot u)) + \nabla \mu [\nabla u + (\nabla u)^T] + (\lambda + 2\mu) \nabla(\nabla \cdot u) - \mu \nabla \times (\nabla \times u) - \eta \frac{\partial u}{\partial t} ,$$

where  $\eta$  is the viscosity coefficient. Anisotropic Wave Equation describe propagation in anisotropic media, the elastic properties vary with direction. The wave equation is modified to include additional terms representing this anisotropy.

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla \cdot \sigma + f ,$$

where  $\sigma$  is the anisotropic stress tensor and  $f$  is a source term. Nonlinear Wave Equation are considered in situations where wave amplitudes are very large, linear approximations are insufficient, and nonlinear terms must be considered in the wave equation.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u + \beta \frac{\partial u^2}{\partial x^2} = f(x_i, t) ,$$

where  $\beta$  is a nonlinearity coefficient. These variants allow for more precise and realistic modeling of seismic wave propagation in different types of media and under various conditions. The choice of the appropriate wave equation depends on the characteristics of the medium and the seismic phenomenon being studied.

## 2 Standard Numerical Methods

In the past decades various numerical methods have been proposed to solve physics systems by partial differential equations such as the wave equation. The finite-difference method is among the most popular to solve partial differential equations, and particularly the wave equation. This possibly associated with its simplicity and efficiency. A complete review of the finite-differences method applied to wave propagation can be found the [Moczo et al. \(2014\)](#). Partial derivatives are approximated by discrete operators involving differences between adjacent grid points. The finite difference method suits for tackling issues related to simple geometric structures. In contrast, the finite element method offers more grid flexibility, facilitating the handling of intricate geometric boundaries.

In wave propagation simulations, the partial differential equations are typically discretized on a staggered grid ([Madariaga; Virieux, 1976; 1986](#)). This approach facilitates the resolution of the rupture propagation problem. Particularly an approach was proposed in the work of [Zhou et al. \(2021\)](#) a finite-difference method with variable-length temporal and spatial operators was proposed to increase the stability and efficiency of the standard method. Also, [Liu et al. \(2023\)](#) combined a standard staggered-grid, finite-difference approach and the perfectly matched layer absorbing boundary to solve 3D first-order velocity-stress equations of acoustoelasticity to simulate wave propagating.

Finite-element methods are suitable for dealing with intricate shapes and diverse materials because they can use irregular grids. They permit flexibility in size, shape, and approximation order. Nevertheless, a drawback is their high demand for computing power. This methodology involves the transformation of the problem at hand into a system of linear equations utilizing the weak formulation of the pertinent differential equation. This transformation is facilitated by employing an interpolation basis comprised of polynomials defined over disjoint domains, commonly referred to as elements.

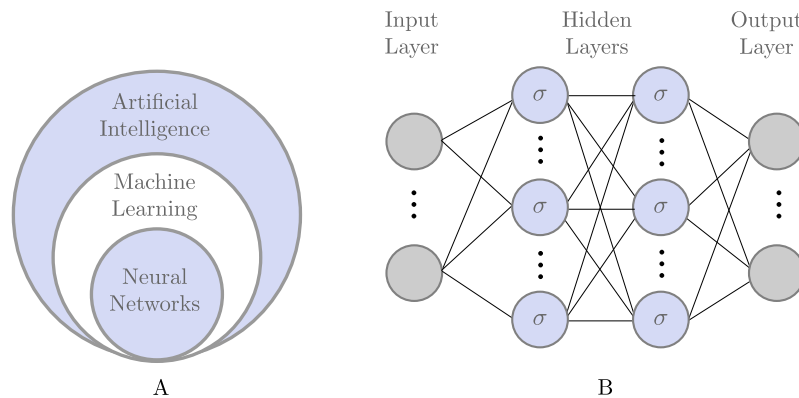
Popular open-source software is available to apply numerical methods for solving the wave equation. FEniCS, proposed by [Langtangen and Logg \(2016\)](#), offers a computing platform specifically designed for solving partial differential equations using the finite elements method. Specializing in seismic wave propagation and full waveform imaging, SPECFEM is widely used in simulations implemented in Fortran ([Komatitsch et al.; Komatitsch et al., 2023; 2024](#)). Similarly, but using finite differences, the modeling the fortran library SEISMIC\_CPML ([Komatitsch and Martin, 2007](#)) is also available. In general, these implementations of standard methods have enabled effective simulations of the wave equation.

A significant difficulty in using standard methods for wave propagation simulations is their computational cost. Their accuracy is achieved at the expense of the number of points in the grid. Modeling a complex domain can entail a huge amount of grid points, with the wavefield requiring iterative updates across the entire grid at each time step. Additionally, model evaluation and storage are significantly costly [Saloma \(1993\)](#), and their limited capacity to incorporate measured data into their predictions makes them less ideal for use in inverse problems.

### 3 Machine learning Methods

The realm of machine learning has recently displayed promise in its capacity to approximate predictions of physical phenomena. These methods can grasp highly nonlinear physics and offer significantly quicker inference times compared to conventional simulations. Therefore, as an alternative to standard methods deep learning have been employed to apply their capability as universal function approximator (Hornik, 1991).

Neural network based methods are a subset of machine learning, whose models are composed of an artificial neural network with a single or multiple processing layers (Figure 4.A). It have shown potential in overcoming the limitations of multiple approaches in various fields such as computer vision, natural language processing, and genomics (LeCun et al.; Goodfellow et al., 2015; 2016). The fundamental architecture of a neural network architecture is conformed by an input layer, an output layer, and an arbitrary number of hidden layers. Particularly, in a fully connected neural network, neurons in adjacent layers are connected with each other but neurons within a single layer share no connection (Figure 4.B). Furthermore, neural networks methods have emerged as an attractive tool to augment and complement conventional numerical solvers of partial differential equations, thereby enabling the tackling of challenges across multiple dimensions, scales, and parameterization with the promise of efficiency and precision (Blechschmidt and Ernst, 2021).



**Figure 3.** Artificial Intelligence subsets. (A) Deep learning as a subset of machine learning and artificial intelligence and (B) basic architecture of artificial neural networks.

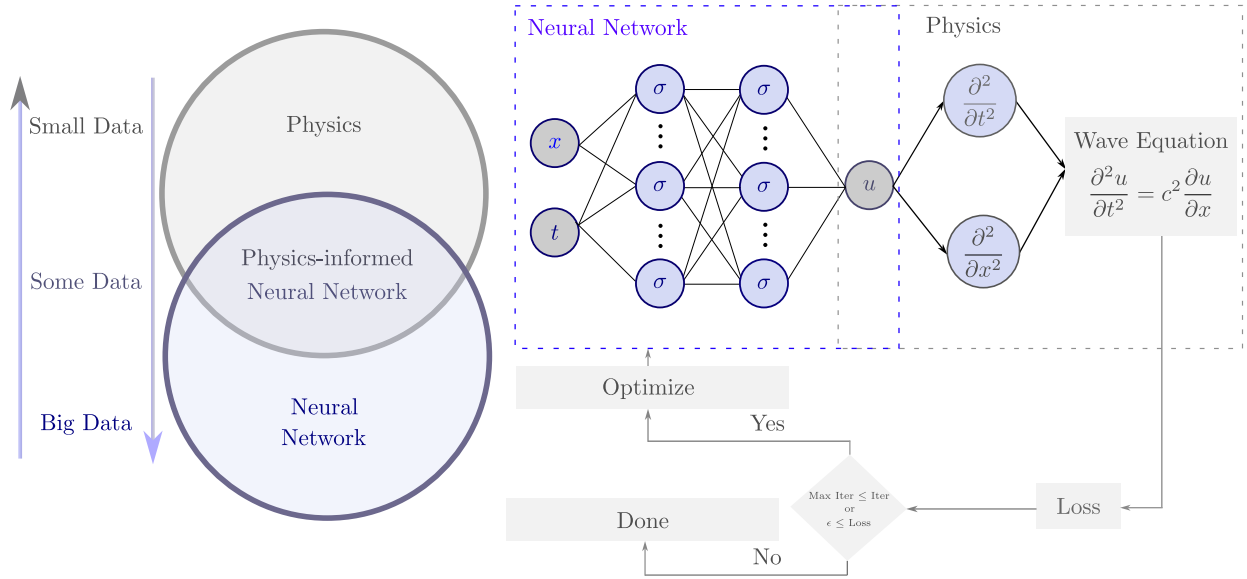
They essentially model the partial differential equation solution by a deep neural network and train the network's parameters to approximate the solution. Data-driven neural networks methods are capable of directly learning the trajectory of a system of partial differential equations from available data (Li et al.; Li et al., 2020; 2021).

One of the most popular types of deep neural networks is known as convolutional neural networks. A convolutional neural network convolves learned features with input data, and uses 2D convolutional layers, making this architecture well suited to processing 2D data, such as images.

All these approaches employ machine learning algorithms and others such as support vector machines, random forests, Gaussian processes have been also applied to model physical systems. However, they are implemented mainly as black-box tools. The constructed neural network can be thought of being ignorant of the mathematical description of the physical phenomenon. In order to overcome this limitation physics-informed neural networks architectures have been proposed. Where the activation and the loss functions are designed according to the context of the problem.

There has been an increasing interest in leveraging physics-informed neural networks to solve

forward and inverse problems where full or partial knowledge of the governing equations is known since the published works of [Raissi and Karniadakis \(2018\)](#), [Raissi et al. \(2018\)](#) and [Raissi et al. \(2019\)](#). Although similar ideas for constraining neural networks using physical laws have been explored in previous studies ([Lagaris et al., 1998](#)). The general principle of physics-informed neural networks is to integrate deep neural networks and physical laws to learn the underlying consistent dynamics from small or zero labeled data ([Karniadakis et al., 2021](#)). As universal approximators, neural networks have the potential to represent any partial differential equation. This capability eliminates the need for the discretization step, thereby avoiding discretization-based physics errors as well.



**Figure 4.** Physics-informed neural networks scheme applied to the wave equation.

Physics-informed neural networks aim to address physical systems governed by the equation

$$u_{tt} - D[u(t, x); \lambda] = 0,$$

where  $x \in \mathbb{R}^D$  and  $t \in \mathbb{R}$ . The expression  $N[u(t, x); \lambda]$  denotes an underlying differential operator that characterizes the physical system, parametrized by  $\lambda$ . The function  $u(t, x)$  represents the system's solution. The loss function is of the general form

$$L := \beta_{\text{pde}} L_{\text{pde}}(\sigma) + \beta_{\text{ic}}(\sigma) L_{\text{ic}} + \beta_{\text{bc}} L_{\text{bc}}(\sigma),$$

where



$$\begin{aligned}\mathcal{L}_{pde}(\boldsymbol{\sigma}) &= \frac{1}{n_{pde}} \sum_{i=1}^{n_{pde}} |u_{tt} - \mathcal{D}[\hat{u}(t, \mathbf{x}_i; \boldsymbol{\sigma})] - f(t, \mathbf{x}_i)|^2, \\ \mathcal{L}_{bc}(\boldsymbol{\sigma}) &= \frac{1}{n_{bc}} \sum_{i=1}^{n_{bc}} |\hat{u}(t, \mathbf{x}_i; \boldsymbol{\sigma}) - g(t, \mathbf{x}_i)|^2, \\ \mathcal{L}_{ic}(\boldsymbol{\sigma}) &= \frac{1}{n_{ic}} \sum_{i=1}^{n_{ic}} |\hat{u}(0, \mathbf{x}_i; \boldsymbol{\sigma}) - h(t, \mathbf{x}_i)|^2,\end{aligned}$$

and  $L_{pde}$  represents the residuals of the PDEs,  $L_{ic}$  represents the error at the collocation points at the initial time point, and  $L_{bc}$  represents the error at the collocation points on the boundaries. The coefficients  $\beta_{ic}$  and  $\beta_{bc}$  are training hyper-parameters.

One major drawback of these methods is the difficulty of transferring knowledge between different configurations. For example, when solving the wave equation, CNNs and PINNs are trained with a fixed velocity parameter and cannot predict anything for a different velocity value. One of the main challenges in numerically modeling mechanical is associated with the dimensional, given the computational complexity.

Tackling complex high-dimensional systems comes with significant challenges. Despite this, machine learning-based algorithms offer promising prospects for solving partial differential equations, as indicated by studies such as the one by [Blehschmidt and Ernst \(2021\)](#). Most of the applications are implemented in one dimensional or two-dimensional domains. In [Lehmann et al. \(2023\)](#) the Fourier Neural Operator method is applied to model seismic waves.

Emerging machine learning methods for solving partial differential equations can face difficulties in establishing fair comparison points with standard numerical methods. [McGreiv and Hakim](#) identified two common pitfalls. First, comparing the runtime of a less accurate machine learning method to a more accurate standard numerical method, whereas a fair approach would be to make the comparison under similar accuracy levels. Second, evaluating the standard numerical method that is not suitable for the partial differential equation being solved. These two criteria are essential for properly evaluating performance, but they are not always followed.

Different extensions of the classical work where PINNs was originally proposed have emerged. [Kharazmi et al. \(2019\)](#) proposed variational physics-informed neural networks which instead trained physics-informed neural networks using the variational form of the underlying differential equations. A neural network is still used to approximate the solution of the differential equation, but it is combined with a set of analytical test functions to compute the residual of the variational form of the equation in its physics loss term. Furthermore, they used quadrature points to estimate the corresponding integrals in the variational loss, rather than random collocation points. They found that the variational physics-informed neural networks was able to solve differential equations including Poisson's equation with similar or better accuracy to a physics-informed neural networks trained using the strong form, whilst requiring less collocation points to train. However, most of these extensions have not yet been applied to wave propagation modeling.

Various open-source frameworks are available for solving partial differential equations using emerging machine learning methods. Python packages such as NeuroDiffEq ([Chen et al., 2020](#)) and DeepXDE ([Lu et al., 2021](#)) facilitate the solving of both ordinary and partial differential equations using neural networks as function approximators. A similar implementation in the Julia programming language is NeuralPDE ([Zubov et al., 2021](#)). Additionally, PINNs-Torch ([Bafghi and Raissi, 2023](#)) enables the application of Physics-Informed Neural Networks using PyTorch, offering improved performance compared to the original model.



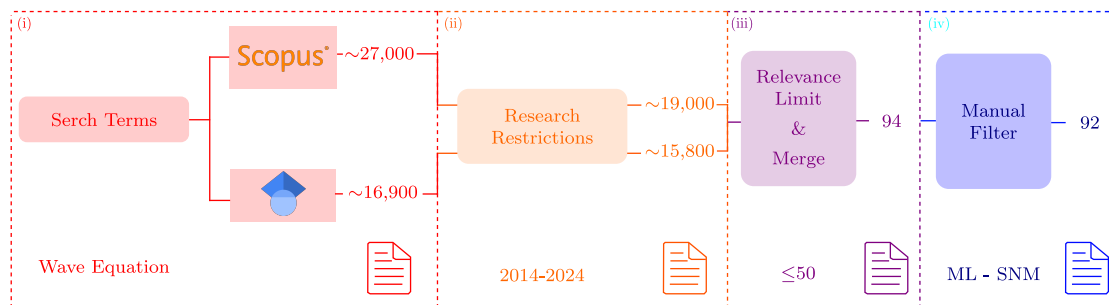
## 4 Applications

This section provides a systematic review of the literature on applying machine learning methods to model wave propagation. We aimed to answer the following research question: In which computational seismology applications have machine learning methods demonstrated superior or comparable performance to standard numerical methods in terms of computational time and accuracy?

In order to answer this question, we conducted a systematic review of the literature. The search was conducted on the Scopus and Google Scholar databases. A search strategy was implemented to identify relevant publications that permit a comparison between machine learning methods and standard numerical methods in the context of wave propagation. Therefore, initially, our search focused on how machine learning has been used for modeling wave propagation. The search was primarily conducted using the following query on the Scopus website as an initial filter:

"machine learning" OR "deep learning" OR "neural networks"  
AND "wave propagation" OR "wave equation" AND  
(modeling OR modelling OR model OR simulation)

The search aimed to find significant contributions and advancements in modeling wave propagation using machine learning techniques over the past decade (2014-2024), limited to English-language publications in the Scopus database. Additionally, the search was complemented by Google Scholar using the same terms, time interval, and language. The resulting list was then sorted by relevance, with the analysis limited to the first 50 results from both databases, which were then merged, and duplicate works removed. The final list was manually filtered by reading the titles and abstracts, with relevance to the chosen field as the inclusion criterion. Additionally, the list was restricted to works that provided a comparative analysis between machine learning methods and standard numerical methods in the abstract. The flowchart and total number of works that met these conditions are shown in Figure 5.



**Figure 5.** Search flowchart and number of publications after each step. During the systematic review process, Scopus and Google Scholar were utilized with the relevant search terms (i), and the research was restricted to works in English and within the time frame of 2013-2024 (ii). The resulting lists were then sorted by relevance and limited to a maximum of 50 entries, with duplicates removed (iii). Finally (iv), a manual filter was applied by reading the titles and abstracts to ensure the publications were pertinent to our chosen field.

We identified 92 articles that passed the manual filtering process. From these, we extracted relevant information to address the research question. The articles were classified based on several criteria: the type of partial differential wave equation employed, the number of spatial dimensions considered, whether forward modeling, inverse modeling, or both approaches were utilized,

and the specific application within computational seismology. This classification allowed us to systematically analyze the advancements and contributions of machine learning methods in various aspects of wave propagation modeling.

The introduction of physics-informed neural networks has generated a large amount of work related. For example the work of [Karimpouli and Tahmasebi \(2020\)](#) explores the application of deep learning in geosciences, specifically solving the 1-dimensional time-dependent seismic wave equation. Comparing Gaussian process and physics-informed neural networks, the research shows that these meshless methods, requiring smaller datasets, effectively incorporate physics laws, with the Gaussian process excelling in solution prediction and the neural network proving superior in velocity and density inversion. Their significant potential lies in addressing surrogate modeling and inverse analysis, striking a balance between accuracy and computational efficiency ([Song and Alkhalifah, 2022](#)). [Rasht-Behesht et al. \(2022\)](#) took a similar approach, but extending to 2D heterogeneous acoustic media, showing that PINNs could invert for ellipsoidal and checkerboard velocity models given the seismic response from sources placed within these models.

In [Moseley \(2022\)](#), physics-informed neural networks were successfully applied to the 2D acoustic wave equation, demonstrating satisfactory outcomes in forward wave propagation and full waveform inversions, despite the efficiency of standard methods. Additionally, physics-informed neural networks showcased efficient performance in solving the inverse problem. To train the model, the results of a finite difference model are used. Spatial and temporal coordinates are used as inputs and intra-inputs of the network and their respective wave field as output, which provides uniqueness to the solution. While restricting the solution obtained to the equation used. To test the model, 3 different speed models are used: homogeneous, layered and an irregular model of the Earth's subsoil.

Similarly, in [Ren et al. \(2024\)](#), physics-informed neural networks were employed for modeling mechanical wave propagation in semi-infinite domains, showcasing their versatility in forward modeling for seismic wave propagation, without requiring labeled data. These studies collectively highlight the effectiveness of physics-informed neural networks across different wave propagation scenarios.

## 5 Conclusions

It's important to notice that deep learning methods shouldn't be seen as substitutes for standard numerical techniques used in solving partial differential equations. These established methods have been refined over decades and often fulfill the robustness and computational efficiency criteria needed in real-world applications. Also, we focus this review on geophysical applications but the methods revised here can be implemented on other fields, whose physical system modeling require the use of the wave equation.

## References

- G. Seriani and S. P. Oliveira. Numerical modeling of mechanical wave propagation, 9 2020. ISSN 18269850.
- Peter Moczo, Johan O.A. Robertsson, and Leo Eisner. The finite-difference time-domain method for modeling of seismic wave propagation. In Ru-Shan Wu, Valerie Maupin, and Renata Dmowska, editors, *Advances in Wave Propagation in Heterogenous Earth*, volume 48 of *Advances in*

- Geophysics*, pages 421–516. Elsevier, 2007. doi: [https://doi.org/10.1016/S0065-2687\(06\)48008-0](https://doi.org/10.1016/S0065-2687(06)48008-0). URL <https://www.sciencedirect.com/science/article/pii/S0065268706480080>.
- Jean Virieux, Henri Calandra, and René-Edouard Plessix. A review of the spectral, pseudo-spectral, finite-difference and finite-element modelling techniques for geophysical imaging. *Geophysical Prospecting*, 59(5):794–813, 2011. doi: <https://doi.org/10.1111/j.1365-2478.2011.00967.x>. URL <https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1365-2478.2011.00967.x>.
- Heiner Igel. *Computational seismology: a practical introduction*. Oxford University Press, 2017.
- Dimitri Komatitsch and Jeroen Tromp. Introduction to the spectral element method for three-dimensional seismic wave propagation. *Geophysical Journal International*, 139(3):806–822, December 1999. ISSN 0956-540X. doi: [10.1046/j.1365-246x.1999.00967.x](https://doi.org/10.1046/j.1365-246x.1999.00967.x). URL <https://doi.org/10.1046/j.1365-246x.1999.00967.x>.
- Emmanuel Chaljub, Dimitri Komatitsch, Jean-Pierre Vilotte, Yann Capdeville, Bernard Valette, and Gaetano Festa. Spectral-element analysis in seismology. In Ru-Shan Wu, Valerie Maupin, and Renata Dmowska, editors, *Advances in Geophysics*, volume 48 of *Advances in Wave Propagation in Heterogeneous Earth*, pages 365–419. Elsevier, January 2007. doi: [10.1016/S0065-2687\(06\)48007-9](https://doi.org/10.1016/S0065-2687(06)48007-9). URL <https://www.sciencedirect.com/science/article/pii/S0065268706480079>.
- Martin Abadi, Paul Barham, Jianmin Chen, Zhifeng Chen, Andy Davis, Jeffrey Dean, Matthieu Devin, Sanjay Ghemawat, Geoffrey Irving, Michael Isard, Manjunath Kudlur, Josh Levenberg, Rajat Monga, Sherry Moore, Derek G. Murray, Benoit Steiner, Paul Tucker, Vijay Vasudevan, Pete Warden, Martin Wicke, Yuan Yu, and Xiaoqiang Zheng. {TensorFlow}: A System for {Large-Scale} Machine Learning. pages 265–283, 2016. ISBN 978-1-931971-33-1. URL <https://www.usenix.org/conference/osdi16/technical-sessions/presentation/abadi>.
- Adam Paszke, Sam Gross, Francisco Massa, Adam Lerer, James Bradbury, Gregory Chanan, Trevor Killeen, Zeming Lin, Natalia Gimelshein, Luca Antiga, Alban Desmaison, Andreas Kopf, Edward Yang, Zachary DeVito, Martin Raison, Alykhan Tejani, Sasank Chilamkurthy, Benoit Steiner, Lu Fang, Junjie Bai, and Soumith Chintala. PyTorch: An Imperative Style, High-Performance Deep Learning Library. In *Advances in Neural Information Processing Systems*, volume 32. Curran Associates, Inc., 2019. URL [https://proceedings.neurips.cc/paper\\_files/paper/2019/hash/bdbca288fee7f92f2bfa9f7012727740-Abstract.html](https://proceedings.neurips.cc/paper_files/paper/2019/hash/bdbca288fee7f92f2bfa9f7012727740-Abstract.html).
- James Bradbury, Roy Frostig, Peter Hawkins, Matthew James Johnson, Chris Leary, Dougal Maclaurin, George Necula, Adam Paszke, Jake VanderPlas, Skye Wanderman-Milne, and Qiao Zhang. JAX: composable transformations of Python+NumPy programs, 2018. URL <http://github.com/google/jax>.
- Adam Paszke, Sam Gross, Soumith Chintala, Gregory Chanan, Edward Yang, Zachary DeVito, Zeming Lin, Alban Desmaison, Luca Antiga, and Adam Lerer. Automatic differentiation in PyTorch. 2017.
- Atilim Günes Baydin, Barak A. Pearlmutter, Alexey Andreyevich Radul, and Jeffrey Mark Siskind. Automatic differentiation in machine learning: a survey. *The Journal of Machine Learning Research*, 18(1):5595–5637, January 2017. ISSN 1532-4435.
- A.R. Barron. Universal approximation bounds for superpositions of a sigmoidal function. *IEEE Transactions on Information Theory*, 39(3):930–945, May 1993. ISSN 1557-9654. doi: [10.1109/18.1109](https://doi.org/10.1109/18.1109).

- 18.256500. URL <https://ieeexplore.ieee.org/abstract/document/256500>. Conference Name: IEEE Transactions on Information Theory.
- Salvatore Cuomo, Vincenzo Schiano Di Cola, Fabio Giampaolo, Gianluigi Rozza, Maziar Raissi, and Francesco Piccialli. Scientific Machine Learning Through Physics-Informed Neural Networks: Where we are and What's Next. *Journal of Scientific Computing*, 92(3):88, July 2022. ISSN 1573-7691. doi: 10.1007/s10915-022-01939-z. URL <https://doi.org/10.1007/s10915-022-01939-z>.
- Shashank Reddy Vadyala, Sai Nethra Betgeri, John C. Matthews, and Elizabeth Matthews. A review of physics-based machine learning in civil engineering. *Results in Engineering*, 13: 100316, March 2022. ISSN 2590-1230. doi: 10.1016/j.rineng.2021.100316. URL <https://www.sciencedirect.com/science/article/pii/S2590123021001171>.
- Weikun Deng, Khanh T. P. Nguyen, Kamal Medjaher, Christian Gogu, and Jérôme Morio. Physics-informed machine learning in prognostics and health management: State of the art and challenges. *Applied Mathematical Modelling*, 124:325–352, December 2023. ISSN 0307-904X. doi: 10.1016/j.apm.2023.07.011. URL <https://www.sciencedirect.com/science/article/pii/S0307904X23003086>.
- Mario Lino, Stathi Fotiadis, Anil A. Bharath, and Chris D. Cantwell. Current and emerging deep-learning methods for the simulation of fluid dynamics. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 479(2275):20230058, July 2023. doi: 10.1098/rspa.2023.0058. URL <https://royalsocietypublishing.org/doi/full/10.1098/rspa.2023.0058>. Publisher: Royal Society.
- Albert Tarantola. *Inverse Problem Theory and Methods for Model Parameter Estimation*. Society for Industrial and Applied Mathematics, 2005. doi: 10.1137/1.9780898717921. URL <https://epubs.siam.org/doi/abs/10.1137/1.9780898717921>.
- Elias Galionas, Tom G. Tranter, Rhodri E. Owen, James B. Robinson, Paul R. Shearing, and Dan J. L. Brett. Battery state-of-charge estimation using machine learning analysis of ultrasonic signatures. *Energy and AI*, 10:100188, November 2022. ISSN 2666-5468. doi: 10.1016/j.egyai.2022.100188. URL <https://www.sciencedirect.com/science/article/pii/S2666546822000374>.
- Pu Ren, Chengping Rao, Su Chen, Jian-Xun Wang, Hao Sun, and Yang Liu. SeismicNet: Physics-informed neural networks for seismic wave modeling in semi-infinite domain. *Computer Physics Communications*, 295:109010, February 2024. ISSN 0010-4655. doi: 10.1016/j.cpc.2023.109010. URL <https://www.sciencedirect.com/science/article/pii/S0010465523003557>.
- Michael T. McCann, Kyong Hwan Jin, and Michael Unser. Convolutional Neural Networks for Inverse Problems in Imaging: A Review. *IEEE Signal Processing Magazine*, 34(6):85–95, November 2017. ISSN 1558-0792. doi: 10.1109/MSP.2017.2739299. URL <https://ieeexplore.ieee.org/document/8103129>. Conference Name: IEEE Signal Processing Magazine.
- B. Moseley. *Physics-informed machine learning: from concepts to real-world applications*. PhD thesis, University of Oxford, 2022. URL <https://ora.ox.ac.uk/objects/uuid:b790477c-771f-4926-99c6-d2f9d248cb23>.
- S. Alkhadhr and M. Almekkawy. Wave Equation Modeling via Physics-Informed Neural Networks: Models of Soft and Hard Constraints for Initial and Boundary Conditions. *Sensors*, 23(5), 2023. ISSN 1424-8220. doi: 10.3390/s23052792.

- José Carcione. *Wave Propagation in Anisotropic, Anelastic, Porous and Electromagnetic Media*. 2002.
- Benjamin Moseley, Andrew Markham, and Tarje Nissen-Meyer. Fast approximate simulation of seismic waves with deep learning. July 2018. URL <http://arxiv.org/abs/1807.06873>.
- Fanny Lehmann, Filippo Gatti, Michaël Bertin, and Didier Clouteau. Fourier Neural Operator Surrogate Model to Predict 3D Seismic Waves Propagation. pages 297–310, 2023. doi: 10.7712/120223.10339.20362. URL <http://arxiv.org/abs/2304.10242>. arXiv:2304.10242 [physics].
- Peter Moczo, Jozef Kristek, and Martin Gális. *The Finite-Difference Modelling of Earthquake Motions : Waves and Ruptures*. 2014. ISBN 9781107028814.
- Raul Madariaga. Dynamics of an expanding circular fault. *Bulletin of the Seismological Society of America*, 66(3):639–666, June 1976. ISSN 0037-1106. doi: 10.1785/BSSA0660030639. URL <https://doi.org/10.1785/BSSA0660030639>.
- Jean Virieux. p-sv wave propagation in heterogeneous media: Velocity-stress finite-difference method, 1986. URL <http://library.seg.org/>.
- Hongyu Zhou, Yang Liu, and Jing Wang. Acoustic finite-difference modeling beyond conventional courant-friedrichs-lewy stability limit: Approach based on variable-length temporal and spatial operators. *Earthquake Science*, 34:123–236, 2021. doi: 10.29382/eqs-2021.
- Siqin Liu, Zhusheng Zhou, and Weizu Zeng. Simulation of Elastic Wave Propagation Based on Meshless Generalized Finite Difference Method with Uniform Random Nodes and Damping Boundary Condition. *Applied Sciences (Switzerland)*, 13(3), February 2023. doi: 10.3390/app13031312. Publisher: MDPI.
- Hans Petter Langtangen and Anders Logg. Solving pdes in python the fenics tutorial i, 2016. URL <http://www.springer.com/series/13548>.
- Dimitri Komatitsch, Jeroen Tromp, Hom Nath Gharti, Daniel Peter, Eduardo Valero Cano, Etienne Bachmann, Alexis Bottero, Quentin Brissaud, Bryant Chow, Paul Cristini, Congyue Cui, Rene Gassmoeller, Michael Gineste, Felix Halpaap, Eric Heien, Jesus Labarta, Matthieu Lefebvre, Nicolas Le Goff, Pieyre Le Loher, Qiancheng Liu, Qinya Liu, Youshan Liu, Zhaolun Liu, David Luet, Roland Martin, Rene Matzen, Ryan Modrak, Christina Morency, Masaru Nagaso, Eric Rosenkrantz, Herurisa Rusmanugroho, Elliott Sales de Andrade, Carl Tape, Jean-Pierre Vilotte, Zhinan Xie, and Zhendong Zhang. Specfem/specfem2d: Specfem2d v8.1.0, December 2023. URL <https://doi.org/10.5281/zenodo.10415228>.
- Dimitri Komatitsch, Jeroen Tromp, Rahul Garg, Hom Nath Gharti, Masaru Nagaso, Elif Oral, Daniel Peter, Michael Afanasiev, Rafael Almada, Jean-Paul Ampuero, Etienne Bachmann, Kangchen Bai, Piero Basini, Stephen Beller, Jordan Bishop, Francois Bissey, Celine Blitz, Alexis Bottero, Ebru Bozdogan, Emanuele Casarotti, Joseph Charles, Min Chen, Paul Cristini, Clement Durochat, Percy Galvez Barron, Rene Gassmoeller, Dominik Goeddeke, Leopold Grinberg, Aakash Gupta, Eric Heien, Vala Hjoerleifsdottir, Foivos Karakostas, Sue Kientz, Jesus Labarta, Nicolas Le Goff, Pieyre Le Loher, Matthieu Lefebvre, Qinya Liu, Youshan Liu, David Luet, Yang Luo, Alessia Maggi, Federica Magnoni, Roland Martin, Rene Matzen, G. D. McBain, Dennis McRitchie, Matthias Meschede, Peter Messmer, David Michea, David Miller, Ryan Modrak, Vadim Monteiller, Christina Morency, Surendra Nadh Somala, Tarje Nissen-Meyer, Kevin Pouget, Max Rietmann, Elliott Sales de Andrade, Brian Savage, Bernhard Schuberth, Anne

- Sieminski, James Smith, Leif Strand, Carl Tape, Eduardo Valero Cano, Brice Videau, Jean-Pierre Vilotte, Huihui Weng, Zhinan Xie, Chang-Hua Zhang, and Hejun Zhu. Specfem/specfem3d: Specfem3d v4.1.1, March 2024. URL <https://doi.org/10.5281/zenodo.10823181>.
- Dimitri Komatitsch and Roland Martin. An unsplit convolutional perfectly matched layer improved at grazing incidence for the seismic wave equation. *GEOPHYSICS*, 72(5):SM155–SM167, September 2007. ISSN 0016-8033. doi: 10.1190/1.2757586. URL <https://library.seg.org/doi/10.1190/1.2757586>. Publisher: Society of Exploration Geophysicists.
- Caesar Saloma. Computational complexity and the observation of physical signals. *Journal of Applied Physics*, 74(9):5314–5319, November 1993. ISSN 0021-8979. doi: 10.1063/1.354232. URL <https://doi.org/10.1063/1.354232>.
- Kurt Hornik. Approximation capabilities of multilayer feedforward networks. *Neural Networks*, 4(2):251–257, January 1991. ISSN 0893-6080. doi: 10.1016/0893-6080(91)90009-T. URL <https://www.sciencedirect.com/science/article/pii/089360809190009T>.
- Yann LeCun, Yoshua Bengio, and Geoffrey Hinton. Deep learning. *Nature*, 521(7553):436–444, May 2015. ISSN 1476-4687. doi: 10.1038/nature14539. URL <https://doi.org/10.1038/nature14539>.
- Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep Learning*. MIT Press, November 2016. ISBN 978-0-262-33737-3. Google-Books-ID: omivDQAAQBAJ.
- Jan Blechschmidt and Oliver G. Ernst. Three ways to solve partial differential equations with neural networks — A review. *GAMM-Mitteilungen*, 44(2):e202100006, 2021. ISSN 1522-2608. doi: 10.1002/gamm.202100006. URL <https://onlinelibrary.wiley.com/doi/abs/10.1002/gamm.202100006>. \_eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1002/gamm.202100006>.
- Zongyi Li, Nikola Kovachki, Kamyar Azizzadenesheli, Burigede Liu, Kaushik Bhattacharya, Andrew Stuart, and Anima Anandkumar. Neural Operator: Graph Kernel Network for Partial Differential Equations, March 2020. URL <http://arxiv.org/abs/2003.03485>. arXiv:2003.03485 [cs, math, stat].
- Zongyi Li, Nikola Kovachki, Kamyar Azizzadenesheli, Burigede Liu, Kaushik Bhattacharya, Andrew Stuart, and Anima Anandkumar. Fourier Neural Operator for Parametric Partial Differential Equations, May 2021. URL <http://arxiv.org/abs/2010.08895>. arXiv:2010.08895 [cs, math].
- Maziar Raissi and George Em Karniadakis. Hidden physics models: Machine learning of nonlinear partial differential equations. *Journal of Computational Physics*, 357:125–141, March 2018. ISSN 0021-9991. doi: 10.1016/j.jcp.2017.11.039. URL <https://www.sciencedirect.com/science/article/pii/S0021999117309014>.
- Maziar Raissi, Paris Perdikaris, and George Em Karniadakis. Numerical Gaussian Processes for Time-Dependent and Nonlinear Partial Differential Equations. *SIAM Journal on Scientific Computing*, 40(1):A172–A198, January 2018. ISSN 1064-8275. doi: 10.1137/17M1120762. URL <https://epubs.siam.org/doi/abs/10.1137/17M1120762>. Publisher: Society for Industrial and Applied Mathematics.

- M. Raissi, P. Perdikaris, and G. E. Karniadakis. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics*, 378:686–707, 2 2019. ISSN 0021-9991. doi: 10.1016/J.JCP.2018.10.045.
- I.E. Lagaris, A. Likas, and D.I. Fotiadis. Artificial neural networks for solving ordinary and partial differential equations. *IEEE Transactions on Neural Networks*, 9(5):987–1000, September 1998. ISSN 1941-0093. doi: 10.1109/72.712178. URL <https://ieeexplore.ieee.org/document/712178>. Conference Name: IEEE Transactions on Neural Networks.
- George Em Karniadakis, Ioannis G. Kevrekidis, Lu Lu, Paris Perdikaris, Sifan Wang, and Liu Yang. Physics-informed machine learning. *Nature Reviews Physics*, 3(6):422–440, June 2021. ISSN 2522-5820. doi: 10.1038/s42254-021-00314-5. URL <https://www.nature.com/articles/s42254-021-00314-5>. Number: 6 Publisher: Nature Publishing Group.
- Nick McGreivy and Ammar Hakim. Weak baselines and reporting biases lead to overoptimism in machine learning for fluid-related partial differential equations, July 2024. URL <http://arxiv.org/abs/2407.07218>. arXiv:2407.07218 [physics].
- E. Kharazmi, Z. Zhang, and G. E. Karniadakis. Variational Physics-Informed Neural Networks For Solving Partial Differential Equations, November 2019. URL <http://arxiv.org/abs/1912.00873>. arXiv:1912.00873 [physics, stat].
- Feiyu Chen, David Sondak, Pavlos Protopapas, Marios Mattheakis, Shuheng Liu, Devansh Agarwal, and Marco Di Giovanni. Neurodiffee: A python package for solving differential equations with neural networks. *Journal of Open Source Software*, 5(46):1931, 2020.
- Lu Lu, Xuhui Meng, Zhiping Mao, and George Em Karniadakis. DeepXDE: A deep learning library for solving differential equations. *SIAM Review*, 63(1):208–228, 2021. doi: 10.1137/19M1274067.
- Kirill Zubov, Zoe McCarthy, Yingbo Ma, Francesco Calisto, Valerio Pagliarino, Simone Azeglio, Luca Bottero, Emmanuel Luján, Valentin Sulzer, Ashutosh Bharambe, Nand Vinchhi, Kaushik Balakrishnan, Devesh Upadhyay, and Chris Rackauckas. Neuralpde: Automating physics-informed neural networks (pinns) with error approximations, 2021. URL <https://arxiv.org/abs/2107.09443>.
- Reza Akbarian Bafghi and Maziar Raissi. PINNs-Torch: Enhancing Speed and Usability of Physics-Informed Neural Networks with PyTorch. October 2023. URL <https://openreview.net/forum?id=nl1ZzdHpab>.
- Sadegh Karimpouli and Pejman Tahmasebi. Physics informed machine learning: Seismic wave equation. *Geoscience Frontiers*, 11(6):1993–2001, November 2020. ISSN 1674-9871. doi: 10.1016/j.gsf.2020.07.007. URL <https://www.sciencedirect.com/science/article/pii/S1674987120301717>.
- Chao Song and Tariq A. Alkhalifah. Wavefield reconstruction inversion via physics-informed neural networks. *IEEE Transactions on Geoscience and Remote Sensing*, 60, 2022. ISSN 15580644. doi: 10.1109/TGRS.2021.3123122.
- Majid Rasht-Behesht, Christian Huber, Khemraj Shukla, and George Em Karniadakis. Physics-Informed Neural Networks (PINNs) for Wave Propagation and Full Waveform Inversions. *Journal*



*of Geophysical Research: Solid Earth*, 127(5):e2021JB023120, 2022. ISSN 2169-9356. doi: 10.1029/2021JB023120. URL <https://onlinelibrary.wiley.com/doi/abs/10.1029/2021JB023120>.