# Towards Robust Network Design using Integer Linear Programming Techniques

Arie M. C. A. Koster
RWTH Aachen University
Lehrstuhl II für Mathematik
Wüllnerstr. 5b
D-52062 Aachen, Germany
Email: koster@math2.rwth-aachen.de

Manuel Kutschka
RWTH Aachen University
Lehrstuhl II für Mathematik
Wüllnerstr. 5b
D-52062 Aachen, Germany
Email: kutschka@math2.rwth-aachen.de

Christian Raack
Zuse Institute Berlin (ZIB)
Department Optimization
Takustr. 7
D-14195 Berlin, Germany
Email: raack@zib.de

***Abstract*—Traffic in communication networks fluctuates heavily over time. Thus, to avoid capacity bottlenecks, operators highly overestimate the traffic volume during network planning. In this paper we consider telecommunication network design under traffic uncertainty, adapting the robust optimization approach of [11]. We present three different mathematical formulations for this problem, provide valid inequalities, study the computational implications, and evaluate the realized robustness.**

**To enhance the performance of the mixed-integer programming solver we derive robust cutset inequalities generalizing their deterministic counterparts. Instead of a single cutset inequality for every network cut, we derive multiple valid inequalities by exploiting the extra variables available in the robust formulations.**

**For realistic networks and live traffic measurements we compare the formulations and report on the speed up by the valid inequalities. We study the "price of robustness" and evaluate the approach by analyzing the real network load. The results show that the robust optimization approach has the potential to support network planners better than present methods.**

***Index Terms*—network design, robust optimization, price of robustness, integer linear programming, cutset inequalities**

## I. Introduction

Dimensioning telecommunication networks is a complex task and it is crucial for the behavior and flexibility of the resulting network. Network design typically involves decisions about the network topology, link capacities, and traffic routing. In the classical network design problem integer capacities (corresponding to bandwidth batches) have to be installed on the network links at minimum cost such that all traffic demands can be realized by flow simultaneously without exceeding the link capacities. Assuming a given single traffic matrix, this problem has been studied extensively in the literature, see [6, 12, 13, 24, 31, 40] and the references therein.

In practice, telecommunication networks are typically designed without the knowledge of actual traffic. In most approaches each demand is estimated in the design process, e.g., by using traffic measurements or population statistics [14, 20]. To handle future changes in the traffic volume and distribution, these values (and consequently capacities) are (highly) overestimated. Obviously, this approach leads to a wastage of network capacities, investments, and energy. To create and operate more resource- and cost-efficient telecommunication networks the uncertainty of future traffic demand has to be taken into account already in the strategic capacity design process. A number of different approaches have been proposed in the literature based on stochastic optimization (e.g., [17, 35, 42]) and robust optimization (e.g., [9, 21, 22, 29]).

Robust optimization, first considered by Soyster [42], aims at finding solutions that are feasible for all realizations of data in a given (bounded) *uncertainty set*. Bertsimas and Sim [10, 11] introduced a way to describe uncertainty in linear programs which results in tractable robust counterparts preserving the linearity of the original problem. In addition, they introduce a parameter $\Gamma$ to control the *price of robustness*, the trade-off between the degree of uncertainty taking into account and the cost of this additional feature.

In telecommunication network design, there have been different approaches to incorporate uncertainty in the planning process. In multi-period (multi-hour) network design [43], an explicit set of demand matrices is given, and the network should be designed in such a way that each of the demand matrices can be routed non-simultaneously within the installed capacities. In this context Oriolo [38] introduces the concept of dominating demand matrices (i.e., $\mathcal{D}_1$ dominates $\mathcal{D}_2$ if every link capacity vector supporting $\mathcal{D}_1$ also supports $\mathcal{D}_2$).

Instead of describing demand matrices explicitly, Ben-Ameur and Kerivin [7, 8] consider the optimized routing of demands that may vary within a given polytope. For network design problems this concept has mainly been applied using the *hose model*, a polyhedral demand uncertainty set which has been introduced in the context of virtual private networks (VPNs) [19]. In its symmetric version, the hose model defines upper bounds on the sum of the incoming and outgoing node traffic for all network nodes. Hence the hose polytope is defined by one inequality for every network node. The hose model has attracted a lot of attention in recent years, in particular, due to its nice algorithmic properties assuming continuous capacities (e.g., polynomial solvable cases, see [15, 23, 26]).

Altin et al. [3] develop a compact integer linear programming model for virtual private network design with continuous capacities and single path routing using the hose model. Altin et al. [5] study the network design problem assuming splittable flow and integer capacities (also known as *network* *loading*) with a general polyhedral uncertainty set. They derive a reformulation along the lines of Soyster [42]. For the case of the hose model, a polyhedral investigation and computational evaluation of this model is carried out. Also using the hose model, Mattia [32] provides a branch-and-cut algorithm for robust network design with dynamic routing together with a computational comparison to the static version studied in [5]. Altin et al. [4] combine the hose model with interval restrictions for individual point-to-point demands.

In this paper, we consider the technology-independent robust network design problem following Bertsimas and Sim [11]. The corresponding polyhedral demand uncertainty set, which we call the $\Gamma$-model, provides a reasonable alternative to the hose model. For every origin-destination pair the user provides two values, the expected nominal demand and the maximum possible peak demand. Given that in realistic traffic scenarios it is unlikely to have all demands at their peak at the same time, the number of simultaneous peaks is restricted to a (small) nonnegative value $\Gamma$. Adjusting $\Gamma$ relates to adjusting the robustness and the level of conservatism of the solutions which provides additional flexibility. Recently, Klopfenstein and Nace [28] consider the same approach in the context of the bandwidth packing problem (without design decision).

We enhance in three different ways the classical flow formulation for network design to include demand uncertainties. First, we derive a straightforward exponential-size mixed-integer programming (MIP) formulation. Next, we use duality theory to obtain a compact formulation, and finally, we project out the flow variables.

To improve the performance of the MIP solver we study the robust counterpart of the well-known cutset polyhedron for network design. Instead of a single cutset inequality for every network cut, we derive multiple classes of facet-defining cut-based inequalities by exploiting the extra variables available in two of the robust formulations.

In computational studies we first compare the three formulations on their pros and cons. The robust cut-based inequalities significantly reduce the computation times. Since the compact formulation outperforms the alternatives, we exploit this formulation to determine the "price of robustness" for realistic networks and live traffic measurements. Finally, the real-life traffic matrices allow an evaluation of the "realized robustness" of the cost-optimal robust network designs.

This paper is structured in three parts: formulations, valid inequalities, and computations. In Section II we introduce three different formulations for robust network design using the $\Gamma$-model. Section III is devoted to cut-based valid inequalities to improve the formulations. In Section IV, we report on the computational comparison of the formulations and an evaluation of the robust network designs. We close with concluding remarks.

## II. $\Gamma$-Robust Network Design Formulations

The $\Gamma$-robust network design problem is described on an undirected graph $G = (V, E)$ representing the network topology. On each of the links $e \in E$ capacity can be installed in batches of $C > 0$ units and costs $\kappa$ per batch. For every commodity $k$ in a set $K$ of point-to-point demands, a routing has to be defined from source $s^k \in V$ to target $t^k \in V$. The traffic volume of a commodity $k$ varies, with mean $\bar{d}^k > 0$ and deviation $\hat{d}^k > 0$, i.e., the actual demand is known to be in $[0, \bar{d}^k + \hat{d}^k]$ for every $k \in K$. Given a parameter $\Gamma \in \{0, 1, \ldots, |K|\}$, the $\Gamma$-robust network design problem is to find a minimum-cost installation of capacities such that a (multi-path) routing exist which does not exceed the link capacities if at most $\Gamma$ commodities deviate from their mean value simultaneously. For every commodity a routing template is provided, that is, every realization of a demand $k$ has to use the same set of paths from $s^k$ to $t^k$ with the same percental splitting of flow. This principle is known as *oblivious routing*.

Setting $\Gamma = |K|$ means to design the network against the maximum demand matrix (having all demands at their **peak** value $\bar{d}^k + \hat{d}^k$). By varying $\Gamma$ we may adjust the robustness.

**Deterministic Network Design Model.** If $\Gamma = 0$, the network has to be designed for the mean values only and can be formulated as an integer linear program, where $N(i)$ denotes the set of neighboring nodes of $i$:

$$\min \sum_{e \in E} \kappa_e x_e \tag{1a}$$

$$s.t. \sum_{j \in N(i)} (f^k_{ij} - f^k_{ji}) = \begin{cases} 1 & i = s^k \\ -1 & i = t^k \\ 0 & \text{else} \end{cases}, \forall i \in V,\, k \in K \tag{1b}$$

$$\sum_{k \in K} \bar{d}^k f^k_e \leq C x_e, \qquad \forall e \in E \tag{1c}$$

$$f, x \geq 0 \tag{1d}$$

$$x \in \mathbb{Z}^{|E|} \tag{1e}$$

Capacity variables $x_e$ denote the number of batches installed on $e \in E$. The flow variables $f^k_{ij}$ ($f^k_{ji}$) denote the fraction of $\bar{d}^k$ routed on $e = \{i, j\}$ away from node $i$ ($j$) and $f^k_e := f^k_{ij} + f^k_{ji}$.

**Exponential $\Gamma$-Robust Model.** If $\Gamma > 0$, the capacity of a link has to be determined subject to at most $\Gamma$ commodities deviating from the mean demand value. For each deviating commodity, the peak value $\bar{d}^k + \hat{d}^k$ describes the worst-case (capacity-wise). Accordingly, the $\Gamma$-robust counterpart of formulation (1) only differs regarding the capacity constraints (1c). The straightforward, but nonlinear, robust capacity constraint for a given $\Gamma$ and edge $e \in E$ is:

$$\sum_{k \in K} \bar{d}^k f^k_e + \max_{Q \subseteq K, |Q| \leq \Gamma} \{ \sum_{k \in Q} \hat{d}^k f^k_e \} \leq C x_e, \tag{1c'}$$

which can be rewritten using exp. many linear constraints resulting in a first *exponential $\Gamma$-robust formulation*:

$$(1a), (1b), (1d), (1e)$$

$$\sum_{k \in K \setminus Q} \bar{d}^k f^k_e + \sum_{k \in Q} \left( \bar{d}^k + \hat{d}^k \right) f^k_e \leq C x_e, \tag{2a}$$

where (2a) is given for every link $e \in E$ and every commodity subset $Q \subseteq K$ with $|Q| \leq \Gamma$. This exponential set of inequalities can be considered implicitly. Given a solution

$(f, x)$, we can sort the commodities non-decreasingly w.r.t. the value $\hat{d}^k f_e^k$ for every link $e \in E$. The $\Gamma$ largest values determine the worst-case commodity subset $Q$. If inequality (2a) corresponding to link $e$ and $Q$ is violated we add it to the current formulation starting with formulation (1).

**Compact $\Gamma$-Robust Model.** Bertsimas and Sim [11] proposed a compact reformulation using LP duality. Fixing the flow vector $f$, the subset $Q \subseteq K$ that maximizes the traffic on $e \in E$ can be computed by linear programming:

$$\max_{Q\subseteq K, |Q|\leq \Gamma} \{\sum_{k\in Q} \hat{d}^k f_e^k\}$$

$$\begin{aligned} = \max \sum_{k\in K} \hat{d}^k f_e^k z_e^k \quad &= \quad \min \sum_{k\in K} p_e^k + \pi_e \Gamma \\ \text{s.t.} \sum_{k\in K} z_e^k \leq \Gamma \quad & \quad \text{s.t.} \quad p_e^k + \pi_e \geq \hat{d}^k f_e^k, \ k \in K \\ 0 \leq z_e^k \leq 1 \quad & \quad\quad p_e^k, \pi_e \geq 0, \quad k \in K \end{aligned}$$

where the primal variables $z_e^k$ specify whether or not a commodity is part of the subset $Q$, the dual variable $\pi_e$ corresponds to the generalized upper bound constraint $\sum_{k\in K} z_e^k \leq \Gamma$, and the dual variables $p_e^k$ correspond to $z_e^k \leq 1$. Note that the primal linear program is integral, bounded, and feasible for all vectors $f$ and every edge $e \in E$. With this relation we can reformulate the set of feasible solutions of the *compact $\Gamma$-robust network design problem* as:

$$(1a), (1b), (1d), (1e)$$

$$\Gamma\pi_e + \sum_{k\in K} \bar{d}^k f_e^k + \sum_{k\in K} p_e^k \leq C x_e, \quad \forall e \in E \tag{3a}$$

$$-\pi_e + \hat{d}^k f_e^k - p_e^k \leq 0, \quad \forall e \in E, k \in K \tag{3b}$$

$$p, \pi \geq 0 \tag{3c}$$

Compared to the deterministic model (1), we have $|E|+|E||K|$ additional variables and $|E||K|$ additional constraints.

**Cutset $\Gamma$-Robust Model.** Altin et al. [5] propose a projection of their model to the space of the design and dual variables. This projection has been studied before by Mirchandani [34] in the context of deterministic network design and can be applied to the compact model (3) as well. For this, we introduce slack variables $q_e^k$ corresponding to inequalities (3b). Hence, flow variables $f_e^k$ in (3a) can be replaced by $\frac{1}{\hat{d}^k}(\pi_e + p_e^k - q_e^k)$. Now, the flow variables of different commodities are no longer bundled and the existence of a flow from $s^k$ to $t^k$ can be guaranteed by the max-flow-min-cut principle. Thus a minimum cut value of at least 1 for every $k \in K$ between source $s^k$ and target $t^k$ with respect to the edge weights $\frac{1}{\hat{d}^k}\left(\pi_e + p_e^k - q_e^k\right)$ is necessary and sufficient. The *Cutset $\Gamma$-robust network design model* then reads:

$$(1a), (1d), (1e)$$

$$\left(\Gamma + \sum_{k\in K} \frac{\bar{d}^k}{\hat{d}^k}\right)\pi_e + \sum_{k\in K}\left(\frac{\bar{d}^k + \hat{d}^k}{\hat{d}^k} p_e^k - \frac{\bar{d}^k}{\hat{d}^k} \bar{q}_e^k\right) \leq C x_e, \forall e \in E \tag{4a}$$

$$\sum_{e\in\delta(S)} \left(p_e^k + \pi_e - q_e^k\right) \geq \hat{d}^k \ \forall \ \begin{array}{l} k \in K, S \subset V: \\ s^k \in S, t^k \notin S \end{array} \tag{4b}$$

$$p, q, \pi \geq 0 \tag{4c}$$

Again, the exponential set of inequalities (4b) can be handled implicitly by separation using a max-flow-min-cut algorithm.

## III. Valid Inequalities

In deterministic network design, cutset inequalities have been proven to be of particular importance [6, 12, 16, 30, 40]. In this section, we generalize the well-known cutset inequality to robust network design and provide a complete description in a particular case. We use these generalized cutset inequalities as cutting planes to improve the solving performance for all three formulations.

**The Robust-$\Gamma$ Cutset Polyhedron.** We consider a proper and nonempty subset $S$ of the nodes $V$ and the corresponding cutset $\delta(S)$ and denote by $Q_S \subseteq K$ the subset of commodities with source $s^k$ and target $t^k$ not in the same shore of the cut. Since we may always reverse single demands without changing the model we may in this description assume $s^k \in S$ for all $k \in Q_S$. We assume that $Q_S \neq \emptyset$ and denote by $\bar{d}_S := \sum_{k\in Q_S} \bar{d}^k > 0$ the aggregated *mean cut-demand* with respect to $S$.

Contracting both shores of the cut $\delta(S)$, we consider the following $\Gamma$-robust two-node formulation corresponding to (3):

$$\sum_{\{i,j\}\in\delta(S)} (f_{ij}^k - f_{ji}^k) = 1 \quad \forall k \in Q_S \tag{5a}$$

$$\Gamma\pi_e + \sum_{k\in Q_S} \bar{d}^k f_e^k + \sum_{k\in Q_S} p_e^k \leq C x_e \forall e \in \delta(S) \tag{5b}$$

$$p_e^k + \pi_e - \hat{d}^k f_e^k \geq 0 \quad \forall e \in \delta(S), k \in Q_S \tag{5c}$$

$$x, f, p, \pi \geq 0 \tag{5d}$$

**Robust Cutset Inequalities.** Let $Q$ be an arbitrary but nonempty subset of the cut-commodities $Q_S$. From the flow-conservation constraints (5a) follows that

$$\sum_{k\in Q} \bar{d}^k f^k(\delta(S)) \geq \bar{d}(Q) \text{ and } \sum_{k\in Q} \hat{d}^k f^k(\delta(S)) \geq \hat{d}(Q). \tag{6}$$

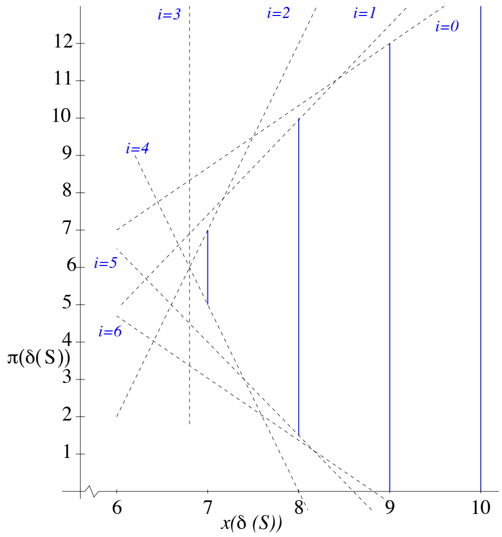
Aggregating all capacity constraints (5b), adding all constraints (5c) for $e \in \delta(S)$ and $k \in Q$, using (6), and relaxing the backward flow variables results in

$$Cx(\delta(S)) + (|Q| - \Gamma)\pi(\delta(S)) \geq \bar{d}_S + \hat{d}(Q) \ . \tag{7}$$

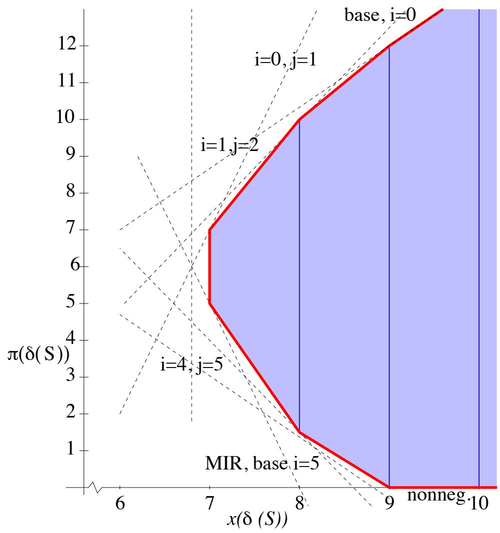
Obviously, among all subsets $Q$ with $|Q| = \lambda$ the one maximizing $\hat{d}(Q)$ gives the strongest inequality (7). Consequently, we have to consider the $\lambda$ largest $\hat{d}^k$ values. Let $\rho : Q_S \mapsto \{ 0, \ldots, |Q_S|\}$ be a permutation of the commodities $k \in Q_S$ such that $\hat{d}^{\rho^{-1}(1)} \geq \hat{d}^{\rho^{-1}(2)} \geq \ldots \geq \hat{d}^{\rho^{-1}(|Q_S|)}$.

Let $J = \{-\Gamma, \ldots, |Q_S| - \Gamma\}$. We define $Q_i := \{k \in Q_S : \rho(k) \leq i + \Gamma\}$ for $i \in J$ as the commodities corresponding to the $i + \Gamma$ largest $\hat{d}^k$. Hence, with $d_i := \bar{d}_S + \hat{d}(Q_i)$, inequality (7) reduces to

$$Cx(\delta(S)) + i\pi(\delta(S)) \geq d_i. \tag{8}$$




(a) base inequalities and feasible set




(b) convex envelope

Fig. 1. Example of $\operatorname{conv}(\bar{X})$ with $C = 5$, $\Gamma = 3$, $|Q_S| = 6$, $\bar{d}_S = 9$, $\hat{d} = (11, 8, 6, 6, 3, 1)$. The upper convex envelope inequalities (11) for $i = -3$, $j = -2$ ($2x(\delta(S)) - \pi(\delta(S)) \geq 6$) and for $i = -2$, $j = -1$ ($3x(\delta(S)) - \pi(\delta(S)) \geq 14$), the lower convex envelope inequalities (12) for $i = 1$, $j = 2$ ($7x(\delta(S)) + 2\pi(\delta(S)) \geq 59$), the MIR inequality (9) for $i = 0$ ($x(\delta(S)) \geq 3$), the MIR inequality (9) for $i = 2$ ($3x(\delta(S)) + 2\pi(\delta(S)) \geq 27$), the base inequality $Cx(\delta(S)) - \Gamma\pi(\delta(S)) \geq 9$, and $\pi(\delta(S)) \geq 0$, completely describe the convex envelope.

In the sequel we consider the set

$$X_\Gamma(S) = \{(x,\pi) \in \mathbb{Z}_+^{\delta(S)} \times \mathbb{R}_+^{\delta(S)} \mid (x,\pi) \text{ satisfies (8) } \forall i \in J\}$$

and the polyhedron defined by the convex hull of $X_\Gamma(S)$. Every valid inequality for $\operatorname{conv}(X_\Gamma(S))$ is also valid for the $\Gamma$-robust formulations (3) and (4).

We define $r(d,c) := d - c(\lceil \frac{d}{c} \rceil - 1)$ as the remainder of the division of $d$ by $c$ with $r(d,c) = c$ in case $c$ divides $d$. Setting $r_i := r(d_i, C)$ and applying MIR [36] to (7) results in

$$r_i x(\delta(S)) + \max(0,i)\pi(\delta(S)) \geq r_i \left\lceil \frac{d_i}{C} \right\rceil. \tag{9}$$

In particular, for $i = 0$ this inequality reduces to

$$x(\delta(S)) \geq \left\lceil \frac{d_0}{C} \right\rceil \tag{10}$$

which defines a facet of $\operatorname{conv}(X_\Gamma(S))$ if and only if $r_0 < C$. This generalizes the classical cutset inequality [30], stating that the capacity on the cut should be at least the mean cut demand plus the $\Gamma$ largest deviations among $Q_S$. Since no dual variables $\pi_e$ appear in this inequality, it is also valid for the exponential formulation (2). We use inequality (10) to tighten all three formulations during branch-and-cut.

Unfortunately, only one further MIR inequality (9) defines a facet (see below). To obtain stronger inequalities, we set $\bar{x} := x(\delta(S))$ and $\bar{\pi} := \pi(\delta(S))$ and study the two dimensional set (cf. Figure 1)

$$\bar{X} = \{(\bar{x}, \bar{\pi}) \in \mathbb{Z}_+ \times \mathbb{R}_+ \mid C\bar{x} + i\bar{\pi} \geq d_i \quad \text{for all } i \in J\}.$$

Note that for $|\delta(S)| = 1$, $\bar{X} = X_\Gamma(S)$. Introducing the index sets $J_- = \{-\Gamma, \ldots, -1\}$ and $J_+ := \{1, \ldots, |Q_S| - \Gamma\}$, we will consider two arbitrary base constraints corresponding to $i, j \in J_-$ (or $i, j \in J_+$) with $i < j$ in the following. The valid inequalities defined below cut off fractional points $(x, \pi)$ with $x \in [\lfloor b_{i,j} \rfloor, \lceil b_{i,j} \rceil]$ where $b_{i,j} := (jd_i - id_j)/((j-i)C)$ denotes the $x$-value at the intersection of the two base inequalities. We define $r_{i,j} := r(jd_i - id_j, (j-i)C)$.

*Lemma 3.1:* For $i, j \in J_-$ with $i < j$, the following inequality is valid for $\operatorname{conv}(\bar{X})$:

$$(-Cj + r_{i,j})x - ij\pi \quad \geq \quad r_{i,j} \lceil b_{i,j} \rceil - jd_i \tag{11}$$

*Proof:* We multiply the base constraints for $i$ and $j$ by $-j$ and $-i$, respectively:

$$-jCx - ji\pi \quad \geq \quad -jd_i \quad \text{and} \quad -iCx - ij\pi \quad \geq \quad -id_j.$$

Introducing the slack $s_i := -jCx - ji\pi + jd_i \geq 0$ for the first constraint and combining gives

$$(j-i)Cx + s_i \quad \geq \quad jd_i - id_j,$$

Applying MIR and resubstituting results in (11). ■

*Lemma 3.2:* For $i, j \in J_+$ with $i < j$, the following inequality is valid for $\operatorname{conv}(\bar{X})$:

$$(Ci + r_{i,j})x + ij\pi \quad \geq \quad r_{i,j} \lceil b_{i,j} \rceil + id_j \tag{12}$$

*Proof:* Analog to the proof of Lemma 3.1. ■

Notice that (11) and (12) are closely related to the mixing MIR inequalities of Günlük and Pochet [25]. Also note that if $(j-i)C \mid d_{i,j}$ then inequality (12) (resp. (11)) reduces to the base inequality for $i$ (resp. $j$).

Only linearly many of the inequalities (11) and (12) are non-redundant. We define the function

$$\pi(k, x) := \frac{d_k - Cx}{k} \qquad \text{for all } k \in J_- \cup J_+ \text{ and } x \in \mathbb{R}_+.$$

*Lemma 3.3:* Let $a \in \mathbb{Z}_+$ and $i = \arg\min_{k \in J_-} \pi(k, a+1)$ and $j = \arg\min_{k \in J_-} \pi(k, a)$. If $i \neq j$, the inequality (11) for $i, j$ dominates all other inequalities (11) on $[a, a+1]$. If $i = j$, the base inequality (8) dominates all inequalities (11) on $[a, a+1]$.

*Proof:* Either inequality (8) or (11) connects the point $(a+1, \pi(i, a+1))$ with $(a, \pi(j, a))$. By definition of $i$ and $j$ all other inequalities (11) are above this one on the interval $[a, a+1]$. ■

*Lemma 3.4:* Let $a \in \mathbb{Z}_+$ and $i = \arg\max_{k \in J_+} \pi(k, a)$ and $j = \arg\max_{k \in J_+} \pi(k, a+1)$. If $i \neq j$, the inequality (12) for $i, j$ dominates all other inequalities (12) on $[a, a+1]$. If $i = j$, the base inequality (8) dominates all inequalities (12) on $[a, a+1]$.

*Proof:* Analog to the proof of Lemma 3.3. ■

For all intervals $[a, a+1]$ it is thus sufficient to consider only the dominating inequalities (8), (11), and (12) on those intervals. We define $J_-^*$ as the sequence of indices in $J_-$ taking the minimum for $a = \lfloor d_{-\Gamma} \rfloor, \lfloor d_{-\Gamma} \rfloor - 1, \ldots, \lceil d_0 \rceil$. Similarly, we define $J_+^*$ as the sequence of indices in $J_+$ taking the maximum for $a = \lceil d_0 \rceil, \lceil d_0 \rceil + 1, \ldots, \lceil d_{|Q_S| - \Gamma} \rceil - 1$. Then (8) and (11) (with $J_-$ replaced by $J_-^*$) describe the *upper convex envelope* of $\operatorname{conv}(\bar{X})$. Analogously, (8), (12) (with $J_+$ replaced by $J_+^*$) together with the additional MIR inequality (9) for the last index in $J_+^*$ describe the *lower convex envelope*. The MIR inequality (9) for $i = 0$ connects the two parts, see Figure 1 for an illustration.

Using a result of Miller and Wolsey [33], it is not difficult to see that the above inequalities form a complete and non-redundant description of the two variable set $\operatorname{conv}(\bar{X})$. Moreover, it can be shown that these inequalities are strong (facet-defining) for the general robust network design polytope under mild conditions (analog to the deterministic case, cf. [40]).

## IV. Computations

**Setting** To achieve realistic results we consider live traffic data from two sources: the U.S. Internet2 Network (abilene) [1] and the national research backbone network operated by the German DFN-Verein [18] mapped on the network (ger17) defined by the NOBEL project [37], and in addition mapped on a larger network (ger50) [39]. The Abilene network consists of 12 nodes, 15 links, and 66 demands. The ger17 network consists of 17 nodes, 26 links, and 136 demands. The ger50 network consists of 50 nodes, 89 links, and 1044 demands. For all networks we used traffic data given by the network providers in 5-minutes intervals over one specific day (abilene: 2004/05/12, ger{17,50}-DFN: 2005/02/15), 288 traffic matrices in total per instance. For each network we determined the mean and peak demand values for each demand for each network using these traffic matrices. Next, we scaled these values such that the maximum total demand of each network equals 1 Tbps. This yields three realistic test instances.

We implemented formulations (2), (3), and (4) of the Γ-robust network design problem in C++ using SCIP 1.2.0 [2, 41] as branch-and-cut framework and IBM ILOG CPLEX 12.1 [27] as underlying LP solver. For the formulations (2) and (4) violated model constraints of type (2a) resp. (4b) are separated on-the-fly as so-called lazy constraints using the constraint handler concept of SCIP (cf. Section II). Using the callback functionality of SCIP, we added a separator for the inequalities (10) for all formulations, and inequalities (9) (for $\max J_+^*$), (11), and (12) for formulations (3) and (4). The separators are called at every node of the branch-and-cut tree. Violated inequalities are separated for all cuts $\delta(S)$ with $|\delta(S)| = 1$ as well as the cuts resulting from a shrinking heuristic dating back to [13, 24] and used by [40] for the deterministic model (1). The aim of the shrinking heuristic is to minimize the sum of weights $w_e$ for edges $e$ on the cut. Given the solution of the current LP relaxation, we use $w_e := s_e - |\omega_e|$, where $s_e$ ($\omega_e$) denotes the sum of the slack (dual value) of the capacity constraint (3a) for edge $e$ and the slacks (dual values) of the constraints (3b) for edge $e$ and $k$. By contracting edges with large slack we shrink the network until 5 nodes are left and enumerate all cuts in the remaining network. For every network cut we separate the violated inequalities.

The computations were carried out on a Linux machine with 2.93 GHz Intel Xeon W3540 CPU and 12 GB RAM. A time limit of 4 hours was set for solving each problem instance. All other solver settings were left at their defaults. As we observed a significant speed-up of 12.0% (22.6%) on average for the abilene (ger17) network by separating violated robust cutset inequalities, these inequalities are separated in the three studies presented below.

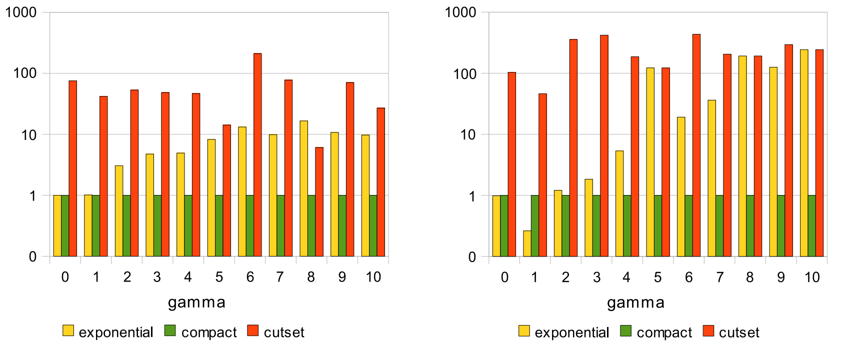
**Comparison of the Formulations** For each of the three test instances we ran tests for $\Gamma = 0, 1, \ldots, 10$ for each of the three formulations (2)–(4) to study the performance of the different formulations.

Figure 2 shows the solving time for the tested instances that could be solved within the time limit. For each Γ-value the solving times are normalized to the time it took to solve the problem using the compact formulation (3). The vertical axis is logarithmically scaled. Note that using the cutset formulation (4) the design problem for the ger17 network could not be solved within the 4 hours time limit.

It is clearly shown, that the compact Γ-robust formulation (3) performs best. The solving time for the abilene (ger17) instance ranges from 0.2 (33.1) to 7.3 (311.4) seconds with a geometric mean of 1.4 (72.1) seconds. The computations using the exponential Γ-robust formulation (2) are 5.5 (12.8) times slower w.r.t. the geometric means. The tests using the cutset Γ-robust formulation (4) perform worst and are 43.7 (at least 200) times slower than the compact formulation for the abilene (ger17) instance. Note that in the tests runs for formulation (2) and (4) most time (> 99%) is spent by SCIP itself, especially by dual simplex routines, branching, and primal heuristics.

In the following, we will therefore restrict on formulation (3) and study its robustness behavior.

**Price of Robustness** Figure 3 shows the "price of robustness" i. e., the additional price/cost that has to be paid to obtain level Γ of robustness. For each network the cost of an optimal solution for $\Gamma = 0, 1, \ldots, \min(100, |K|)$ is normalized to the cost of an optimal non-robust ($\Gamma = 0$) solution. The ger50




(a) abilene

(b) ger17

Fig. 2. Comparison of the formulations (2) (exponential), (3) (compact), and (4) (cutset)

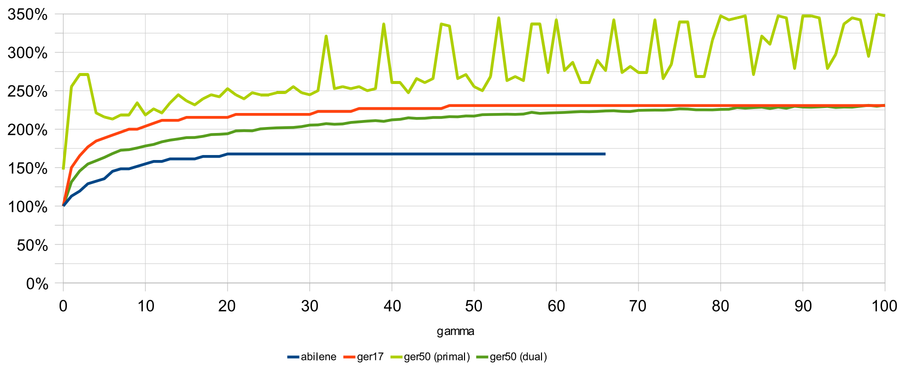



Fig. 3. Price of Robustness and maximum link load of optimal designs

network could not be solved optimally. Hence, the best dual and primal bounds are given as lower and upper bounds on the actual costs. Both the best primal and dual bounds are normalized to the best dual bound for $\Gamma = 0$. Hence, the difference shows the relative gap.

After a relatively steep increase in cost for small values of $\Gamma$, the cost becomes almost constant for larger values of $\Gamma$. For example, robustness of $\Gamma = 1$ (10/50/100) increases the cost by 13% (55%/68%/68%) for abilene, by 50% (104%/131%/131%) for ger17, and by at least 31% (78%/117%/131%) for ger50.

**Evaluation of Realized Robustness** To assess the actual robustness we have to analyze the link load in the real traffic matrices used to compute $\bar{d}^k$ and $\hat{d}^k$. For the abilene (ger17) network the results are shown in Figure 4 (Figure 5). For each network the average percentage of overloaded links (a), the percentage of traffic matrices that cannot be provided by the optimal network design (b), and the inverse distribution of link loads (c) are shown for $\Gamma = 0, 1, \ldots, 10$.

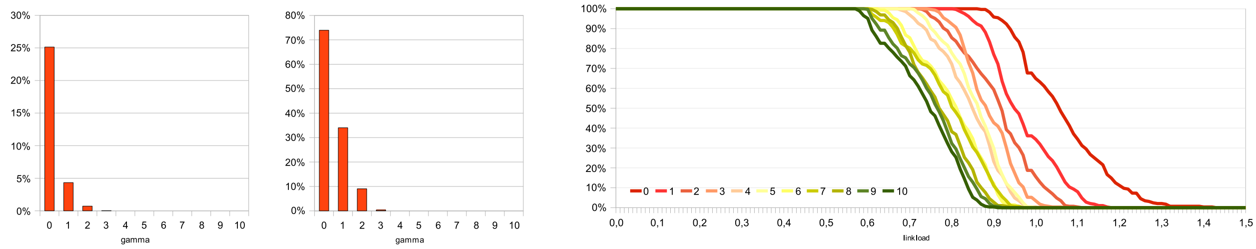
For the abilene network, Figures 4(a) and 4(b) show that the non-robust network design cannot host the traffic of 74% of all traffic matrices, resulting in 25% of all links being overloaded on average. For the ger17 network, 61% of all traffic matrices fail in the non-robust design, giving rise to 14% of all links being overloaded on average.

For both networks the percentage of overloaded links and failed traffic matrices drops significantly with the increase of $\Gamma$. The $\Gamma$-robust network design of abilene (ger17) satisfies all 288 traffic matrices for $\Gamma = 4$ ($\Gamma = 8$). Hence, the price of total robustness is 32% (100%) higher than the non-robust scenario. This cost can be reduced if the robustness requirements are relaxed. For example, if on average 5% overloaded links are acceptable, a corresponding robust network design costs only 13% (77%) more for abilene (ger17).

Figure 4(c) (5(c)) shows the inverse link load distribution of the optimal $\Gamma$-robust network designs for $\Gamma = 0, 1, \ldots, 10$, i.e., given a certain relative link load it shows the percentage of traffic matrices where a link with at least this load exists. Clearly, an increase of $\Gamma$ tends to decrease the maximal link loads and avoids overloads (i.e. loads $> 1.0$).

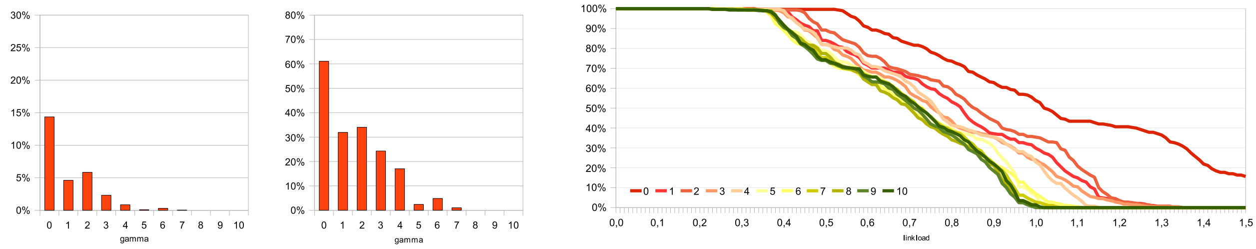
For the abilene (ger17) network it shows that $\Gamma = 4$ ($\Gamma = 8$) suffices to host all traffic matrices. The maximal link load is between 0.63 and 0.97 (resp. 0.20 and 0.99).

In summary, for both networks it is sufficient to choose a relative small $\Gamma$-value ($\Gamma = 4$ resp. $\Gamma = 8$) to obtain a total robustness for all 288 traffic matrices. The abilene (ger17) network data consists of 66 (136) demands. Hence for both networks it is sufficient to protect against the largest 6% of the demands to be at their peak values simultaneously to obtain robust designs.




(a) average % overloaded links (b) % failed traffic matrices (c) inverse load distribution

Fig. 4. Evaluation of realized robustness for the abilene network




(a) average % overloaded links (b) % failed traffic matrices (c) inverse load distribution

Fig. 5. Evaluation of realized robustness for the ger17 network

## V. Concluding Remarks

In this paper we have presented different mathematical formulations for a robust network design problem using the approach of Bertsimas and Sim [10, 11]. This generalizes the classical network design model for a single demand matrix. It lays a foundation for more elaborate technology-dependent design problems. In a polyhedral study, we have derived strong cut-based valid inequalities describing the projected robust cut-set polyhedron of a single edge. In our computational studies we have used realistic networks and life traffic measurements to obtain a meaningful test set. The separation of robust cutset inequalities speeds up the solving process of the robust network design problem significantly: a speed-up of 12.0 resp. 22.6% could be achieved for the considered networks.

Further, we have presented the results of computational studies comparing three different mathematical formulations to solve the robust network design problem, addressing the "price of robustness" and evaluating the realized robustness by analyzing 288 measured life traffic matrices. The results show that the compact $\Gamma$-robust model outperforms the alternatives by a factor of at least 5. In addition, we have shown that already small values of $\Gamma$ (in the region of about 6% of the number of demands) suffice to provide a total robustness with respect to all considered traffic matrices over time. The price of robustness can be adjusted by the decision makers using the parameter $\Gamma$ determining the desired trade-off between the level of robustness and the capital expenditures.

As our results for the ger50 network show, the solving process still requires improvements, i.e., the gap between the lower and upper bound on the cost of the robust network design has to be reduced faster. Presumably this can be tackled by a further exploitation of the (polyhedral) problem structure.

As a final remark we want to note that it is not a trivial task to decide about the mean and peak values in the presented formulations. Is it most promising to use the actual measured maximum value of a demand as its peak or should some statistical considerations be made? A more accurate modeling of the traffic fluctuations could be the topic of future research as well.

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