Homework 5

Due on April 7, 2025

Instructions:

- Install `pdflatex`, `R`, and `RStudio` on your computer.
- Please edit the `HW5_First_Last.Rnw` file in `Rstudio` and compile with `knitr` instead of `Sweave`. Go to the menu `RStudio|Preferences...|Sweave` choose the `knitr` option, i.e., `Weave Rnw files using knitr?` You may have to install `knitr` and other necessary packages.
- Replace "First" and "Last" in the file-name with your first and last names, respectively. Complete your assignment by modifying and/or adding necessary R-code in the text below.
- You should submit both a **PDF file** and a **ZIP file** containing the data and the *HW5_First_Last.Rnw* file required to produce the PDF. The file should be named "HW5_First_Last.zip" and it should contain a single folder named "First_Last" with all the necessary data files, the `HW5_First_Last.Rnw` and `HW5_First_Last.pdf` file, which was obtained from compiling `HW5_First_Last.Rnw` with `knitr` and LaTeX.
- The GSI grader will annotate the PDF file you submit. However, they will also check if the code you provide in the ZIP file compiles correctly. If the file fails to compile due to errors other than missing packages, there will be an automatic 10% deduction to your score.

Problems:

1. Consider the Dow Jones data set provided in the file `DowJones30.csv` under the textbook data sets folder in Canvas. The data set is also available here.

   (a) Consider the vectors of daily log-returns of all 30 stocks

   $$r_k, \ k = 1, \ldots, 2529$$

   over the period of 10 years from 1990 through 2000. Model the joint distribution of the daily log-returns with a multivariate $t-$distribution with mean μ and covariance matrix Σ. More precisely,

   $$r_k \stackrel{d}{=} \mu + T, \quad \text{with } T = \frac{X}{\sqrt{Y/\nu}},$$

   where $X \sim \mathcal{N}(0, \Lambda = \Sigma \times (\nu - 2)/\nu)$ and $Y \sim \text{Gamma}(\nu/2, 1/2)$ are independent.

   Fit this model to the data. Assuming that $\nu > 2$, estimate Λ by using the sample covariance of the data and use the method of profile likelihood to estimate ν.

   (b) Using the method of profile likelihood, obtain an asymptotic 95%-confidence interval for the parameter ν.

   (c) Compute the daily Value-at-Risk at level $\alpha = 0.95$ for the equal-weighted portfolio including all 30 stocks and using the model-fit in part **(a)**, obtain $VaR_{0.95}$, so that

   $$\mathbb{P}(r < -VaR_{0.95}) = 0.05,$$

where r denotes log-return on this portfolio.

Hint: Use the package `mnormt` for the function `dmt`, which computes the density of a multivariate t-distribution with specified mean and shape matrix Λ. You can modify and complete the following code.

```
library("mnormt")
t.loglike = function(dat,nu.range){
  mu = apply(dat,2,mean);
  Sig = cov(dat);
  t.loglike = matrix(0,nrow=1,ncol=length(nu.range));
  for (i in 1:length(nu.range)){
   nu = nu.range[i];
   t.loglike[1,i] = sum(log(dmt(dat,mean=mu, S = Sig/(nu/(nu-2)), df=nu)));
  }
  return(t.loglike);
}

dat = read.csv("DowJones30.csv",head=T);
n = length(dat$AA);
x = unlist(dat);
x = matrix(as.numeric(x[-(1:n)]),nrow=n,ncol=30);
ret = log(x[-1,]/x[-n,]);
nu.range = seq(from=2.01,to=10,by=0.01);
## Complete or correct the above code.

log_likelihoods = t.loglike(ret, nu.range)

# Estimate nu using maximum likelihood
optimal_nu = nu.range[which.max(log_likelihoods)]
cat("Estimated nu:", optimal_nu, "\n")

## Estimated nu: 7.25

# Compute 95% confidence interval for nu using likelihood ratio test
logL_max = max(log_likelihoods)
threshold = logL_max - 1.92  # 95% confidence threshold

CI_nu = nu.range[log_likelihoods >= threshold]
cat("95% CI for nu: [", min(CI_nu), ",", max(CI_nu), "]\n")

## 95% CI for nu: [ 6.92 , 7.6 ]

# Compute Value-at-Risk at alpha = 0.95 for an equal-weighted portfolio
weights = rep(1 / ncol(ret), ncol(ret))  # Equal-weighted portfolio
mu = colMeans(ret)
```

```r
Sigma = cov(ret) * (optimal_nu - 2) / optimal_nu  # Shape matrix for t-distribution

# Portfolio mean and variance
port_mean = sum(weights * mu)
port_var = t(weights) %*% Sigma %*% weights

# Compute VaR
VaR_95 = qt(0.05, df = optimal_nu) * sqrt(port_var) + port_mean
cat("VaR at 95% confidence level:", -VaR_95, "\n")
```

```
## VaR at 95% confidence level: 0.01487999
```

2. Consider the data set from the previous problem. Assume that the vector of daily log-returns $r \overset{d}{=} \mu + T$, where $\mu = \overline{r}$ is the vector of sample means and T is distributed according to the t-distribution found in the previous problem.

(a) Given a target expected daily return level v, determine the portflio weights w_i, $i = 1, \dots, 30$ such that $\sum_{i=1}^{30} w_i = 1$, which *minimize* $\mathrm{VaR}_{0.95}$ of the portfolio return

$$r(w) := w_1 r_1 + \cdots + w_{30} r_{30},$$

where r_i is the daily log-return of the ith stock in DJIA. Write an R-function that returns the optimal portfolio weights given a value for the target expected return.

How do the optimal portfolio weights for a given target expected return depend on the VaR level α? Explain.

The optimal portfolio weights depend on the VaR level α through the quantile $q_{1-\alpha}(\nu)$ of the t-distribution. Specifically, under the model assumption that

$$r(w) \sim w^\top \mu + \sqrt{w^\top \Lambda w} \cdot T_\nu,$$

the Value-at-Risk at level α is given by:

$$\mathrm{VaR}_\alpha = -w^\top \mu - \sqrt{w^\top \Lambda w} \cdot q_{1-\alpha}(\nu),$$

where $q_{1-\alpha}(\nu)$ is the $(1-\alpha)$ quantile of the t-distribution with ν degrees of freedom.

As α increases (i.e., moving from $\alpha = 0.90$ to 0.95 or 0.99), the quantile $q_{1-\alpha}(\nu)$ increases. This places greater emphasis on the variance term $\sqrt{w^\top \Lambda w}$ in the optimization. Since the VaR objective penalizes portfolios with higher variance more heavily as α increases, the optimizer will naturally shift the portfolio weights toward lower-risk, more diversified portfolios.

Therefore, for a fixed target expected return, increasing α results in:

- Lower portfolio variance (due to higher penalization of risk),
- More conservative portfolio allocations,

- Reduced exposure to high-variance assets.

In summary, as α increases, the optimal portfolio weights become more conservative, prioritizing risk reduction more strongly.

```
library("quadprog");
optimization_weight = function(vs){
 for (i in (1:length(vs))){
 v = vs[i];
 Dmat = 2 * Lambda
 Amat = cbind(rep(1, 30), mu)
 bvec = c(1, v)
 dvec = rep(0, 30)
 res = solve.QP(Dmat=Dmat, Amat=Amat, dvec=dvec, bvec=bvec, meq=2)
 sig[i] = sqrt(res$value);
 weights[i,] = res$solution;
 }

 return(list(weights=weights, sig=sig))
}
```

(b) Using the function in part **(a)** compute the efficient frontier of portfolios corresponding to several target daily log-return levels $v = 0.001/365$ to $v = 0.65/365$. Graph the points $(SD(r(w)), \mathbb{E}(r(w)))$, corresponding to these portfolia. That is, produce a risk-reward diagram for the portfolios, where $SD(r(w))$ and $\mathbb{E}(r(w))$ denote the standard deviation and expectation of the returns, respectively. Assuming that the **yearly** risk-free interest rate is $\mu_F = 0.03$, determine the "tangency" portfolio, i.e., the one maximizing the Sharpe ratio $(\mathbb{E}r(w) - \mu_F)/SD(r(w))$. (Note: remember to work in the correct units – i.e. determine the **daily** risk-free rate. You may also have to find an appropriate range of target expected returns in order to find the global maximum of the Sharpe ratio.)

Hint: In part **(a)** observe that

$$\begin{aligned} r(w) &\overset{d}{=} w^\top \mu + w^\top X/\sqrt{Y/\nu} \\ &\overset{d}{=} w^\top \mu + (w^\top \Lambda w)^{1/2} T_\nu, \end{aligned}$$

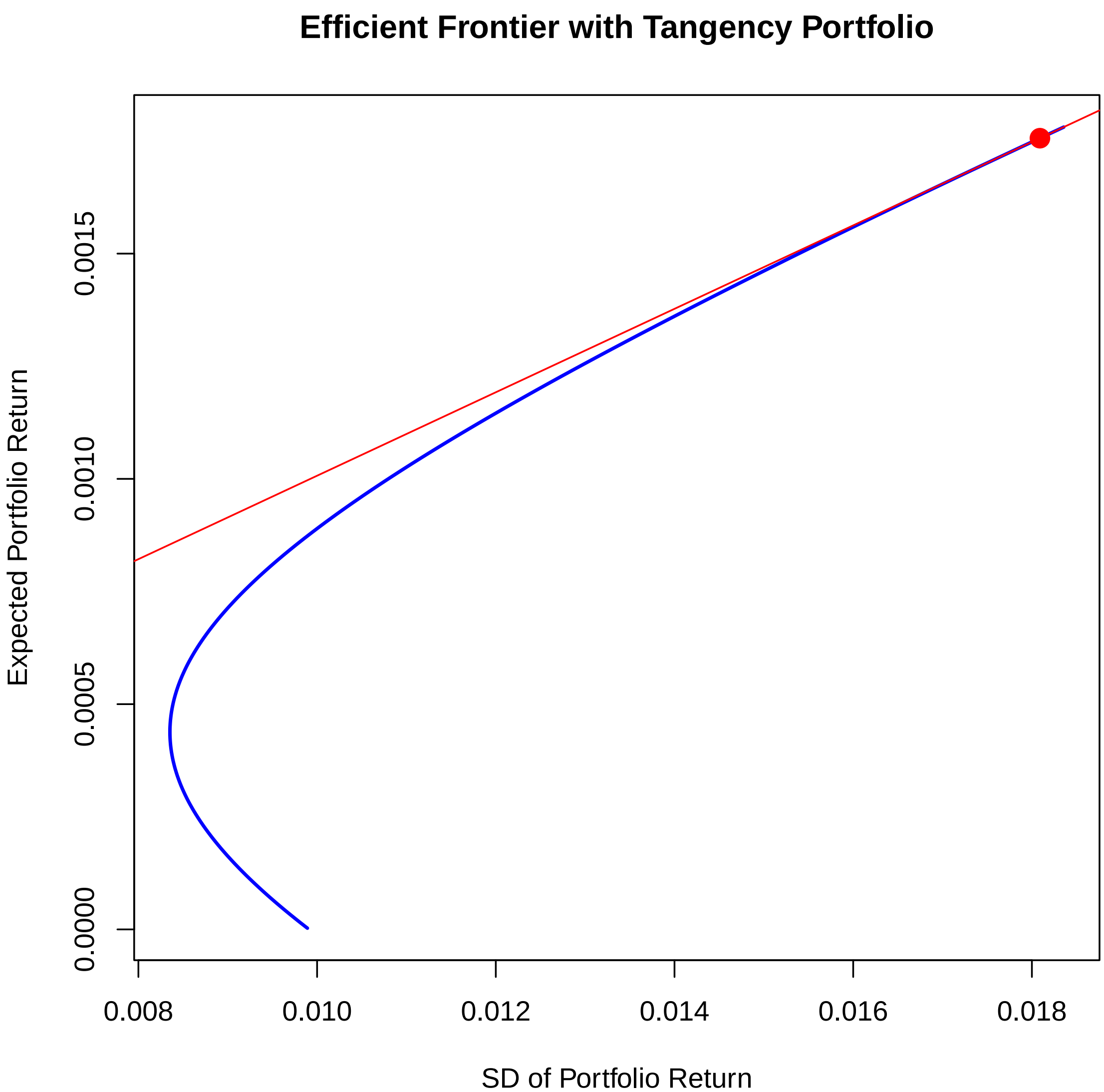
where T_ν is a t-distributed scalar random variable with ν degrees of freedom. Use this fact to show that

$$VaR_\alpha = -w^\top \mu - (w^\top \Lambda w)^{1/2} q_{1-\alpha}(\nu), \tag{1}$$

where $q_{1-\alpha}(\nu)$ is the $1-\alpha$ quantile of the $t-$distribution with ν degrees of freedom, i.e., $\mathbb{P}(T_\nu \leq q_{1-\alpha}(\nu)) = 1 - \alpha$.

Formulate the above optimization problem as a quadratic program with linear constraints. Use the function `solve.QP` in package `quadprog` to obtain a numerical solution.

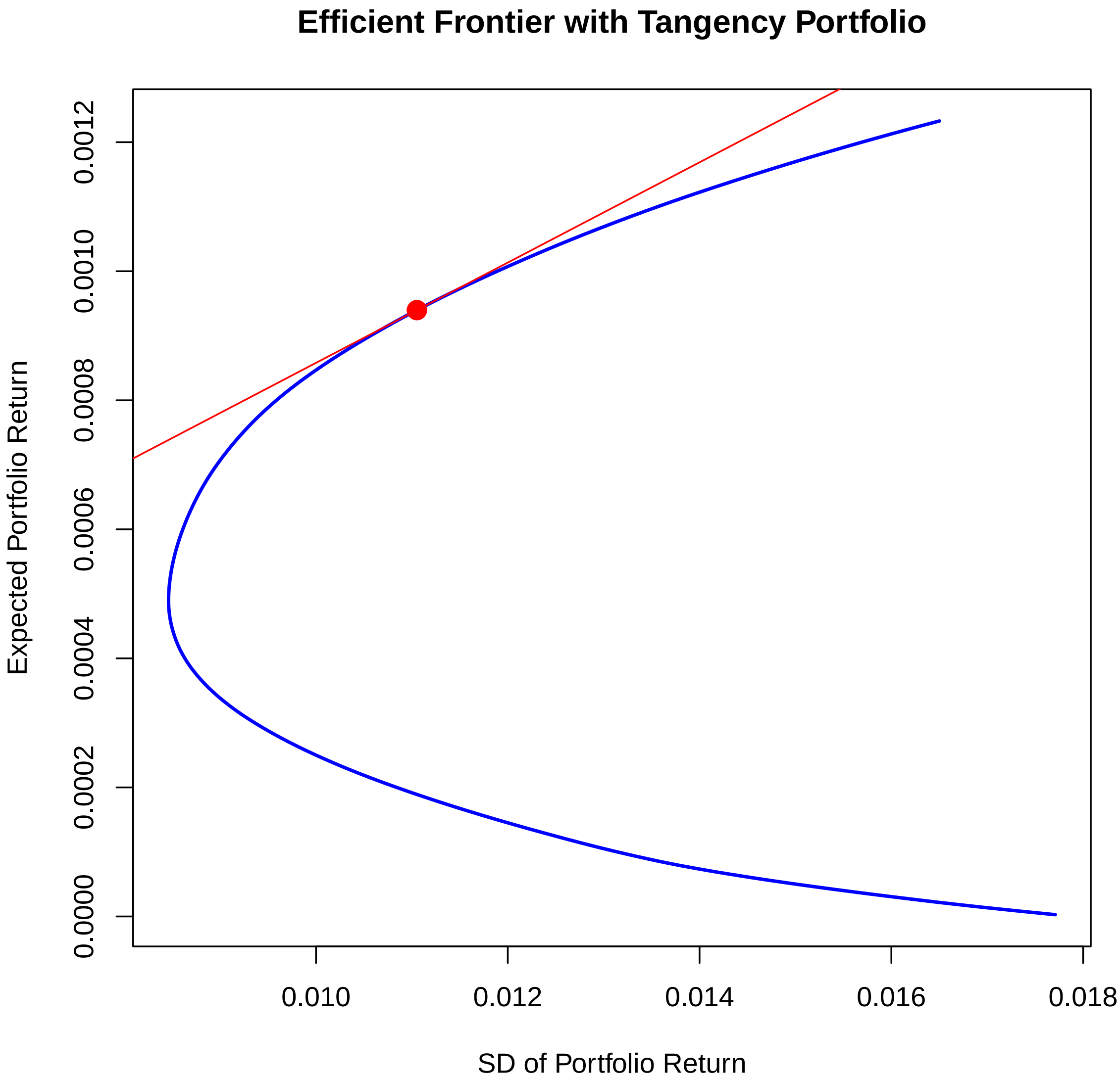
```r
mu <- matrix(apply(ret,2,mean),nrow=30,ncol=1);
Lambda <- cov(ret);
q <- qt(0.05,7.25)
mu_f = log(1.03)/365;
library("quadprog");
vs = seq(from=0.001,to=0.65,by=0.001)/365;
sig = matrix(0,nrow=length(vs),ncol=1);
weights = matrix(0,nrow=length(vs),ncol=30);

result = optimization_weight(vs)

plot(result$sig, vs, type="l", col="blue", lwd=2,
     xlab="SD of Portfolio Return", ylab="Expected Portfolio Return",
     main="Efficient Frontier with Tangency Portfolio")
Sharpe = (vs - mu_f) / result$sig
idx_tan = which.max(Sharpe)
abline(a=mu_f, b=Sharpe[idx_tan], col="red")
points(result$sig[idx_tan], vs[idx_tan], col="red", pch=19, cex=1.5)
```




```
print(result$weights[idx_tan,])

##  [1]  0.142825006 -0.001979842 -0.334841403 -0.156349923  0.029709663
##  [6]  0.205248003  0.104662387 -0.047953880 -0.159448268  0.093609266
## [11]  0.248987119 -0.162983437  0.007303358  0.179626310  0.123520835
## [16]  0.186787485 -0.140561241 -0.205908943  0.081405763  0.119285777
## [21]  0.045289316  0.149812710  0.163041268  0.107800438  0.044438997
## [26] -0.121893014  0.163756279  0.216363794 -0.038304446 -0.043249377
```

(c) Repeat part **(b)** with a suitably chosen range of target expected returns under the condition that no short-selling is allowed (i.e., the weights are all non-negative). Comment on the optimal portfolio weights in the two cases, i.e., with and without short-selling. Is the expected return in part **(b)** sustainable/reasonable?

```
mu <- matrix(apply(ret,2,mean),nrow=30,ncol=1);
Lambda <- cov(ret);
q <- qt(0.05,7.25)
mu_f = log(1.03)/365;
vs = seq(from=0.001,to=0.45,by=0.001)/365;
sig_noss = matrix(0,nrow=length(vs),ncol=1);
weights_noss = matrix(0,nrow=length(vs),ncol=30);

 for (i in (1:length(vs))){
 v = vs[i]
 Dmat_noss = 2 * Lambda
 Amat_noss = cbind(rep(1, 30), as.vector(mu), diag(30))
 bvec_noss = c(1, v, rep(0,30))
 dvec_noss = rep(0, 30)
 res = solve.QP(Dmat=Dmat_noss, Amat=Amat_noss, dvec=dvec_noss, bvec=bvec_noss, meq=2)
 sig_noss[i] = sqrt(res$value);
 weights_noss[i,] = res$solution;
 }


plot(sig_noss, vs, type="l", col="blue", lwd=2,
     xlab="SD of Portfolio Return", ylab="Expected Portfolio Return",
     main="Efficient Frontier with Tangency Portfolio")
Sharpe = (vs - mu_f) / sig_noss
idx_tan_noss = which.max(Sharpe)
abline(a=mu_f, b=Sharpe[idx_tan_noss], col="red")
points(sig_noss[idx_tan_noss], vs[idx_tan_noss], col="red", pch=19, cex=1.5)
```




```
print(weights_noss[idx_tan_noss,])

##  [1]  3.221948e-02 -5.531230e-18 -1.733313e-17 -4.819552e-19  6.411709e-18
##  [6]  1.114488e-01  4.372148e-02  6.465959e-18 -1.291230e-17  6.859540e-02
## [11]  8.038469e-02 -5.314772e-17 -7.271598e-18  8.090552e-02  3.741027e-02
## [16]  9.014440e-02 -4.420575e-17  4.592033e-17  3.238938e-02  8.203230e-02
## [21]  2.312375e-03  6.915790e-02  9.158119e-02  0.000000e+00  3.532387e-03
## [26]  8.076418e-18  5.664347e-02  1.175210e-01 -3.675669e-18 -1.223397e-17
```

3. Problem 6 on page 513 of the textbook (link to the textbook).

   (a) What is the beta of Stock A?

The beta is calculated as:

$$\beta_A = \frac{\text{Cov}(R_A, R_m)}{\sigma_m^2} = \frac{165}{121} \approx 1.36$$

(b) What is the expected return on Stock A?

Using the Capital Asset Pricing Model (CAPM):

$$E(R_A) = r_f + \beta_A \cdot (E(R_m) - r_f) = 4\% + 1.36 \cdot (12\% - 4\%) = 4\% + 10.88\% = 14.88\%$$

(c) What percentage of the variance is due to market risk?

First, compute the systematic (market) portion of the variance:

$$\beta_A^2 \cdot \sigma_m^2 = (1.36)^2 \cdot 121 \approx 1.8496 \cdot 121 \approx 223.8$$

Then, compute the percentage:

$$\frac{223.8}{320} \times 100\% \approx 69.94\%$$

So, approximately **69.94%** of the variance is due to market (systematic) risk.

4. Problem 7 on page 513 of the textbook.

(a) What is the beta of an equally weighted portfolio of these three assets?

The portfolio beta is the average of the individual betas:

$$\beta_p = \frac{1}{3}(0.9 + 1.1 + 0.6) = \frac{2.6}{3} \approx \boxed{0.867}$$

(b) What is the variance of the excess return on the equally weighted portfolio?

The variance of the portfolio's excess return includes the systematic and idiosyncratic components:

$$\text{Var}(R_p) = \beta_p^2 \cdot \sigma_M^2 + \frac{1}{n^2} \sum_{j=1}^{n} \sigma_{\varepsilon_j}^2$$

Compute each term:

$$\beta_p^2 \cdot \sigma_M^2 = (0.867)^2 \cdot 0.014 \approx 0.01052$$

$$\frac{1}{3^2}(0.010 + 0.015 + 0.011) = \frac{0.036}{9} = 0.004$$

So the total variance is:

$$\text{Var}(R_p) \approx 0.01052 + 0.004 = \boxed{0.01452}$$

(c) What proportion of the total risk of asset 1 is due to market risk?

First, compute the total variance of asset 1:

$$\text{Var}(R_1) = \beta_1^2 \cdot \sigma_M^2 + \sigma_{\varepsilon_1}^2 = (0.9)^2 \cdot 0.014 + 0.010 = 0.01134 + 0.010 = 0.02134$$

Now compute the proportion due to market (systematic) risk:

$$\text{Proportion} = \frac{0.01134}{0.02134} \approx \boxed{53.13\%}$$

5. Problem 2 on page 356 in the textbook (read Problem 1 terein on how to get the data).

```
library(Ecdat)

## Warning:  package 'Ecdat' was built under R version 4.3.3
## Loading required package:  Ecfun
## Warning:  package 'Ecfun' was built under R version 4.3.3
##
## Attaching package:  'Ecfun'
## The following object is masked from 'package:base':
##
##     sign
##
## Attaching package:  'Ecdat'
## The following object is masked from 'package:datasets':
##
##     Orange

data(CRSPday)
crsp = CRSPday[,7]

fit_ar1 <- arima(crsp, order = c(1, 0, 0))
fit_ar2 <- arima(crsp, order = c(2, 0, 0))

AIC(fit_ar1)

## [1] -17406.37

AIC(fit_ar2)

## [1] -17404.87

phi_hat <- fit_ar1$coef[1]
phi_se <- sqrt(fit_ar1$var.coef[1,1])

ci_lower <- phi_hat - 1.96 * phi_se
ci_upper <- phi_hat + 1.96 * phi_se

ci_table <- data.frame(lower = ci_lower, upper = ci_upper)
kable(ci_table)
```

	lower	upper
ar1	0.0464615	0.1241421

Since lower AIC values indicate better models, AR(1) has a slightly better AIC than AR(2). The difference between the two AICs is only 1.5, which is quite small.

6. **Problem 4 on page 356 in the textbook. 4. Suppose that $Y_1, Y_2, \ldots$ is an AR(1) process with $\mu = 0.5$, $\phi = 0.4$, and $\sigma_\epsilon^2 = 1.2$.**

   (a) The variance of a stationary AR(1) process is given by:

$$\mathbf{Var}(Y_t) = \frac{\sigma_\epsilon^2}{1-\phi^2} = \frac{1.2}{1-0.4^2} = \frac{1.2}{0.84} \approx 1.4286$$

   (b) The autocovariance at lag h for an AR(1) process is:

$$\gamma(h) = \phi^{|h|} \cdot \gamma(0)$$

   where $\gamma(0) = \mathbf{Var}(Y_t) \approx 1.4286$. Thus:

$$\mathbf{Cov}(Y_1, Y_2) = \gamma(1) = 0.4 \times 1.4286 \approx 0.5714$$

$$\mathbf{Cov}(Y_1, Y_3) = \gamma(2) = 0.4^2 \times 1.4286 = 0.16 \times 1.4286 \approx 0.2286$$

   (c) Let $Z = \frac{Y_1+Y_2+Y_3}{2}$. Then:

$$\mathbf{Var}(Z) = \frac{1}{4} \cdot \mathbf{Var}(Y_1 + Y_2 + Y_3)$$

   We compute:

$$\mathbf{Var}(Y_1 + Y_2 + Y_3) = 3\gamma(0) + 4\gamma(1) + 2\gamma(2)$$

$$= 3(1.4286) + 4(0.5714) + 2(0.2286) = 4.2858 + 2.2856 + 0.4572 = 7.0286$$

$$\Rightarrow \mathbf{Var}(Z) = \frac{1}{4} \cdot 7.0286 \approx 1.7571$$