3-1. Find the poles and zeros of the following functions (including the ones at infinity, if any). Mark the finite poles with × and the finite zeros with o in the *s*-plane.

(a) $$G(s) = \frac{10(s+2)}{s^2(s+1)(s+10)}$$

(b) $$G(s) = \frac{10s(s+1)}{(s+2)(s^2+3s+2)}$$

(c) $$G(s) = \frac{10(s+2)}{s(s^2+2s+2)}$$

(d) $$G(s) = \frac{e^{-2s}}{10s(s+1)(s+2)}$$

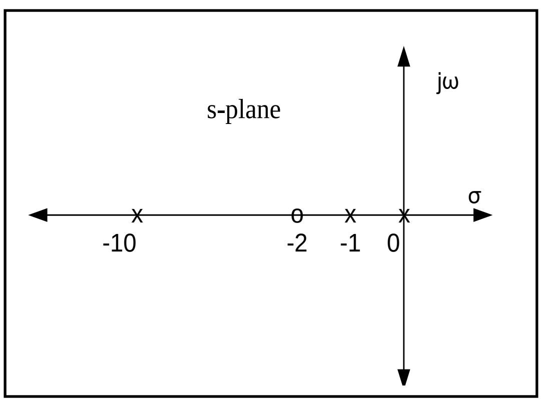
(a) Poles: s = 0, 0, –1, –10;

Zeros: s = –2, ∞, ∞, ∞.

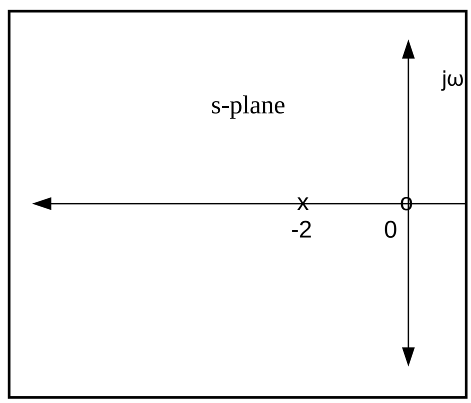
(b) Poles: s = –2, –2;

Zeros: s = 0.

The pole and zero at *s* = –1 cancel each other.

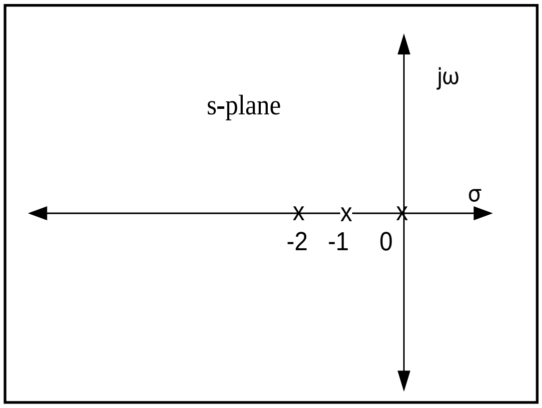


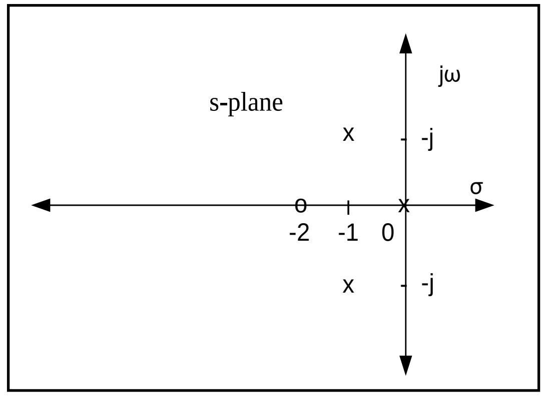
(c) Poles: 0, –1 s = + j, –1 – j;



(d) Poles: s = 0, –1, –2, ∞.

Zeros: s = –2.





3-10. Find the Laplace transform of the function in Fig. 3P-10.

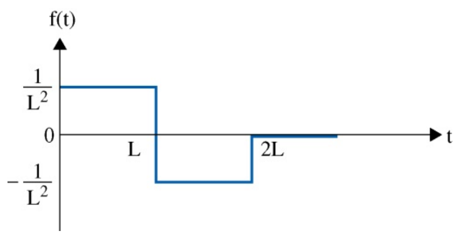


Figure 3P-10

$$\mathcal{L}\{f(t)\} = \int_0^T f(t)e^{-st}dt = \int_0^{\frac{T}{2}} e^{-st}dt + \int_{\frac{T}{2}}^{T} (-1)e^{-st}dt$$

$$= \frac{1 - e^{-\frac{Ts}{2}}}{s} + \frac{e^{-Ts} - e^{-\frac{Ts}{2}}}{s} = \frac{1}{s}\left[1 - e^{-\frac{Ts}{2}}\right]^2$$

3-11. The following differential equations represent linear time-invariant systems, where $r(t)$ denotes the input and $y(t)$ the output. Find the transfer function $Y(s)/R(s)$ for each of the systems. (Assume zero initial conditions.)

(a) $$\frac{d^3y(t)}{dt^3}+2\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=3\frac{dr(t)}{dt}+r(t)$$

(b) $$\frac{d^4y(t)}{dt^4}+10\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+5y(t)=5r(t)$$

(c) $$\frac{d^3y(t)}{dt^3}+10\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+y(t)+2\int_0^t y(\tau)\,d\tau=\frac{dr(t)}{dt}+2r(t)$$

(d) $$2\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+5y(t)=r(t)+2r(t-1)$$

(e) $$\frac{d^2y(t+1)}{dt^2}+4\frac{dy(t+1)}{dt}+5y(t+1)=\frac{dr(t)}{dt}+2r(t)+2\int_{-\infty}^t r(\tau)\,d\tau$$

(f) $$\frac{d^3y(t)}{dt^2}+2\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+2y(t)+2\int_{-\infty}^t y(\tau)\,d\tau=\frac{dr(t-2)}{dt}+2r(t-2)$$

Solution:

(a)

$$\frac{Y(s)}{R(s)}=\frac{3s+1}{s^3+2s^2+5s+6}$$

(b)

$$\frac{Y(s)}{R(s)}=\frac{5}{s^4+10s^2+s+5}$$

(c)

$$\frac{Y(s)}{R(s)}=\frac{s(s+2)}{s^4+10s^3+2s^2+s+2}$$

(d)

$$\frac{Y(s)}{R(s)}=\frac{1+2e^{-s}}{2s^2+s+5}$$

e) $x(t)=y(t+1)$

$$\Rightarrow \frac{d^2x(t)}{dt^2}+4\frac{dx(t)}{dt}+5x(t)=\frac{dr(t)}{dt}+2r(t)+2\int_{-\infty}^t r(\tau)\,d\tau$$

By using Laplace transform, we have:

$$s^2X(s)+4sX(s)+5X(s)=sR(s)+2R(s)+\frac{R(s)}{s}$$

As $X(s)=e^{-s}Y(s)$, then

$$(s^2 + 4s + t)e^{-s}Y(s) = \frac{s^2 + 2s + 1}{s} R(s)$$

Then:

$$\frac{Y(s)}{R(s)} = \frac{(s+1)^2 e^s}{s(s^2+4s+s)}$$

f) By using Laplace transform we have:

$$\left(s^3 + 2s^2 + s + 2 + \frac{2}{s}\right) Y(s) = se^{-s}R(s) + 2e^{-s}R(s)$$

As a result:

$$\frac{Y(s)}{R(s)} = \frac{s(s+2)e^{-s}}{s^4 + 2s^3 + s^2 + 2s + 2}$$

3-13. Find the inverse Laplace transforms of the following functions. First, perform partial-fraction expansion on $G(s)$; then, use the Laplace transform table.

(a) $$G(s) = \frac{1}{s(s+2)(s+3)}$$

(b) $$G(s) = \frac{10}{(s+1)^2(s+3)}$$

(c) $$G(s) = \frac{100(s+2)}{s(s^2+4)(s+1)} e^{-s}$$

(d) $$G(s) = \frac{2(s+1)}{s(s^2+s+2)}$$

(e) $$G(s) = \frac{1}{(s+1)^3}$$

(f) $$G(s) = \frac{2(s^2+s+1)}{s(s+1.5)(s^2+5s+5)}$$

(g) $$G(s) = \frac{2+2se^{-s}+4e^{-2s}}{s^2+3s+2}$$

Solution:

(a)

$$G(s)=\frac{1}{3s}-\frac{1}{2(s+2)}+\frac{1}{3(s+3)} \qquad g(t)=\frac{1}{3}-\frac{1}{2}e^{-2t}+\frac{1}{3}e^{-3t} \qquad t\ge 0$$

(b)

$$G(s)=\frac{-2.5}{s+1}+\frac{5}{(s+1)^2}+\frac{2.5}{s+3} \qquad g(t)=-2.5e^{-t}+5te^{-t}+2.5e^{-3t} \qquad t\ge 0$$

(c)

$$G(s)=\left(\frac{50}{s}-\frac{20}{s+1}-\frac{30s+20}{s^2+4}\right)e^{-s} \qquad g(t)=\left[50-20e^{-(t-1)}-30\cos 2(t-1)-5\sin 2(t-1)\right]u_s(t-1)$$

(d)

$$G(s)=\frac{1}{s}-\frac{s-1}{s^2+s+2}=\frac{1}{s}+\frac{1}{s^2+s+2}-\frac{s}{s^2+s+2}$$ Taking the inverse Laplace transform,

$$g(t)=1+1.069e^{-0.5t}\left[\sin 1.323t+\sin\left(1.323t-69.3^{o}\right)\right]=1+e^{-0.5t}\left(1.447\sin 1.323t-\cos 1.323t\right) \quad t\ge 0$$

(e) $g(t)=0.5t^2e^{-t} \qquad t\ge 0$

(f)

$$G(s)=\frac{0.4}{s}-\frac{0.9889}{s+3.6180}+\frac{2.5889}{s+1.3820}-\frac{2}{s+1}$$

$$g(t)=0.4-0.9889e^{-3.618t}+1.3820e^{-2.5889t}-2e^{-t}$$

(g) $G(s)=\frac{2}{(s+1)(s+2)}+\frac{2e^{-s}}{s+1}$

$$=\frac{2}{s+1}-\frac{2}{s+2}+\frac{2e^{-s}}{s+1}$$

⇨ $\mathcal{L}^{-1}\{\mathrm{G(s)}\}=2\mathrm{e}^{-\mathrm{t}}-2\mathrm{e}^{-2\mathrm{t}}+2\mathrm{e}^{-(\mathrm{t}-1)}u(t-1)$

3-17. Solve the following differential equations by means of the Laplace transform.

(a) $\frac{d^2 f(t)}{dt^2}+5\frac{df(t)}{dt}+4f(t)=e^{-2t}u_s(t)$ Assume zero initial conditions.

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] \quad = \quad sF(s)-f(0)$$

From: $\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] \quad = \quad s^n F(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(0)-\cdots-f^{(n-1)}(0)$

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\}=s^2F(s)-sf(0)-\dot{f}(0)$$

$$\mathcal{L}\left\{\frac{df(t)}{dt^2}\right\}=sF(s)-f(0)$$

$$\mathcal{L}\left\{e^{-2t}\right\}=\frac{1}{s+2}$$

$$\left(s^2+5s+4\right)F(s)=\frac{1}{s+2}$$

$$F(s)=\frac{1}{(s+2)\left(s^2+5s+4\right)}=\frac{1}{s^3+7s^2+14s+8}$$

$$\mathcal{L}^{-1}\left[F(s)\right]=-1/2\ e^{-2\,t}+1/6\ \mathrm{e}^{-4\,t}+1/3\ \mathrm{e}^{-t}$$

```
ilaplace(1/(s^3+7*s^2+14*s+8))
>>
ans = - 1/2 exp(-2 t) + 1/6 exp(-4 t) + 1/3 exp(-t)
```

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_2(t) \\ \dfrac{dx_2(t)}{dt} = -2x_1(t) - 3x_2(t) + u_s(t) \\ x_1(0) = 1, x_2(0) = 0 \end{cases}$$

(b) $x = x_1$

$$\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t) = u_s(t)$$

From

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$$

$$\mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} = ?s^2 X(s) - sx(0) - \dot{x}(0) = s^2 X(s) - 1$$

$$\mathcal{L}\left\{\frac{dx(t)}{dx}\right\} = ?sX(s) - x(0) = ?sX(s) - ?$$

$$\mathcal{L}\{u_s(t)\} = \frac{1}{s}$$

Or

$$\left(s^2 + 3s + 2\right) X(s) = \frac{1}{s}$$

$$X(s) = \frac{1}{s\left(s^2 + 3s + 2\right)}$$

ilaplace(1/(s^3+3*s^2+2*s))

ans = 1/2 + 1/2 exp(-2 t) - exp(-t)

3-23. Consider the two-degree-of-freedom mechanical system shown in Fig. 3P-23, subjected to two applied forces, $f_1(t)$ and $f_2(t)$, and zero initial conditions. Determine system responses $x_1(t)$ and $x_2(t)$ when

(a) $f_1(t)=0,\ f_2(t)=u_s(t)$

(b) $f_1(t)=u_s(t),\ f_2(t)=u_s(t).$

Use the following parameter values:
$m_1=m_2=1\,kg,\ b_1=2Ns/m,\ b_2=1Ns/m,\ k_1=k_2=1N/m$.

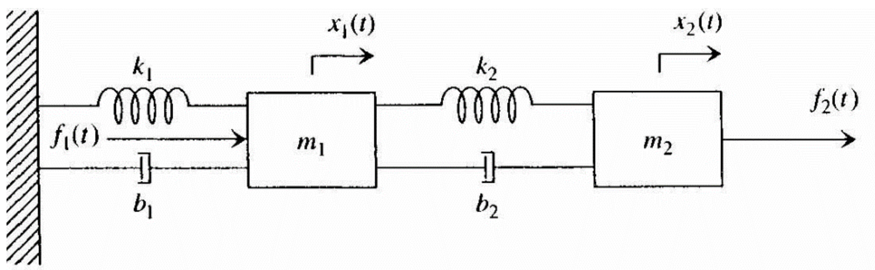
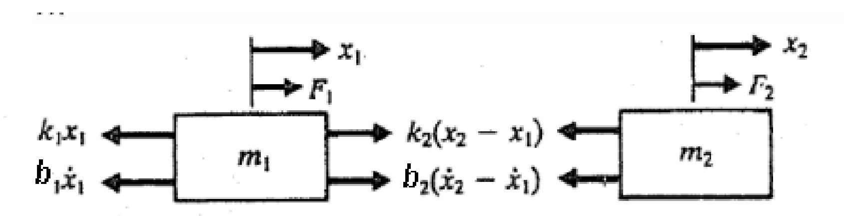


Figure 3P-23

Starting with the Free Body Diagram, we have



The equations of motion using the Newton's Law become:

$$m_1\ddot{x}_1 = f_1(t)-k_1x_1-b_1\dot{x}_1$$
$$m_2\ddot{x}_2 = f_2(t)-b_2(\dot{x}_2-\dot{x}_1)-k_2(x_2-x_1)$$

$$m_1\ddot{x}_1+(b_1+b_2)\dot{x}_1-b_2\dot{x}_2+(k_1+k_2)x_1-k_2x_2 = f_1(t)$$
$$m_2\ddot{x}_2-b_2\dot{x}_1+b_2\dot{x}_2-k_2x_1+k_2x_2 = f_2(t)$$

In Laplace domain with zero initial conditions we get:

$$\mathcal{L}\left\{\frac{d^2x_i(t)}{dt^2}\right\}=s^2X_i(s)\quad \mathcal{L}\left\{\frac{dx_i(t)}{dt}\right\}=sX_i(s)$$
$$\mathcal{L}\left\{\frac{df_i(t)}{dt^2}\right\}=sF_i(s)$$
$$i=1,2$$

$$m_1 s^2 X_1 + (b_1 + b_2) s X_1 - b_2 s X_2 + (k_1 + k_2) X_1 - k_2 X_2 = F_1(t)$$
$$m_2 s^2 X_2 - b_2 s X_1 + b_2 s X_2 - k_2 X_1 + k_2 X_2 = F_2(t)$$

Solve for *X1* and *X2* transfer functions. In matrix form:

$$\begin{bmatrix} m_1 s^2 + (b_1+b_2)s + (k_1+k_2) & -b_2 s - k_2 \\ -b_2 s - k_2 & m_2 s^2 + b_2 s + k_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$
$$\begin{bmatrix} s^2+4s+2 & -2s-1 \\ -2s-1 & s^2+2s+2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Pre-multiply by the inverse

$$A = \begin{bmatrix} s^2+4s+2 & -2s-1 \\ -2s-1 & s^2+2s+2 \end{bmatrix}$$
$$A^{-1} = \frac{1}{(s^2+4s+2)(s^2+2s+2)-(-2s-1)^2} \begin{bmatrix} s^2+2s+2 & 2s+1 \\ 2s+1 & s^2+4s+2 \end{bmatrix}$$

Solving for $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ we get

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = A^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{1}{(s^2+4s+2)(s^2+2s+2)-(2s+1)^2} \begin{bmatrix} s^2+2s+2 & 2s+1 \\ 2s+1 & s^2+4s+2 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$
$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} (s^2+2s+2)F_1 - (2s+1)F_2 \\ (2s+1)F_1 + (s^2+4s+2)F_2 \end{bmatrix}}{(s^2+4s+2)(s^2+2s+2)-(2s+1)^2}$$

(a)

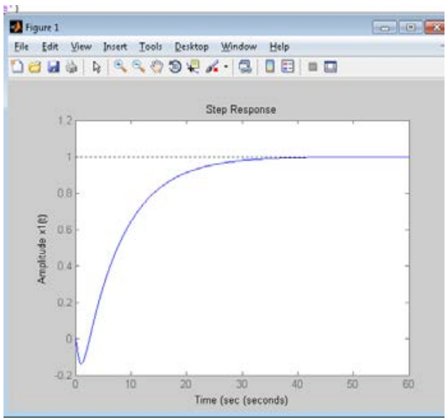
$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} \dfrac{-(2s+1)}{s} \\ \dfrac{+(s^2+4s+2)}{s} \end{bmatrix}}{(s^2+4s+2)(s^2+2s+2)-(2s+1)^2} = \frac{\begin{bmatrix} \dfrac{-(2s+1)}{s} \\ \dfrac{+(s^2+4s+2)}{s} \end{bmatrix}}{s^4+6s^3+8s^2+8s+1} = \frac{1}{s} \frac{\begin{bmatrix} -(2s+1) \\ (s^2+4s+2) \end{bmatrix}}{s^4+6s^3+8s^2+8s+1}$$

We can use MATLAB to find the time responses.

x1:

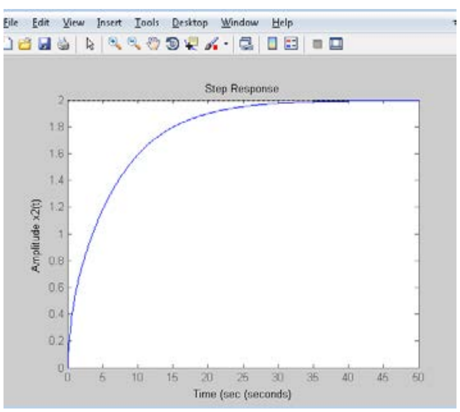
```
num = [-2 1];
den = [1 6 8 1];
G = tf (num,den);
```

```
step(G);
title ('Step Response')
xlabel ('Time (sec')
ylabel ('Amplitude x1(t)')
```



x_2:

```
num = [1 4 2];
den = [1 6 8 1];
G = tf (num,den);
step(G);
title ('Step Response')
xlabel ('Time (sec')
ylabel ('Amplitude x2(t)')
```



3-24. Express the following set of first-order differential equations in the vector-matrix form of $\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$.

(a)
$$\frac{dx_1(t)}{dt} = -x_1(t) + 2x_2(t)$$
$$\frac{dx_2(t)}{dt} = -2x_2(t) + 3x_3(t) + u_1(t)$$
$$\frac{dx_3(t)}{dt} = -x_1(t) - 3x_2(t) - x_3(t) + u_2(t)$$

(b)
$$\frac{dx_1(t)}{dt} = -x_1(t) + 2x_2(t) + 2u_1(t)$$
$$\frac{dx_2(t)}{dt} = 2x_1(t) - x_3(t) + u_2(t)$$
$$\frac{dx_3(t)}{dt} = 3x_1(t) - 4x_2(t) - x_3(t)$$

Solution:

a)

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -2 & 3 \\ -1 & -3 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

b)

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & -1 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

3-36. Consider the electrical circuits shown in Figs. 3P-36(a) and (b).

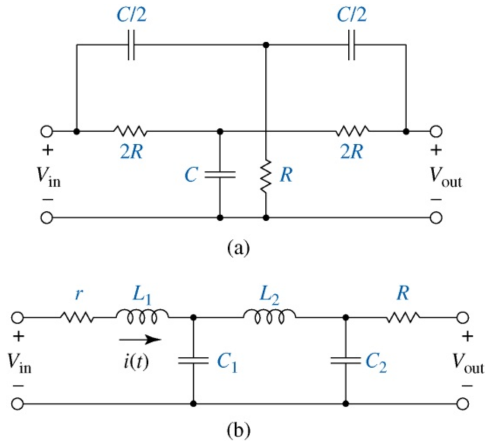


Figure 3P-36

For each circuit:

(a) Find the dynamic equations and state variables.

(b) Determine the transfer function.

(c) Use MATALB to plot the step response of the system.

a) According to the circuit:

$$\begin{cases} \dfrac{v_{in} - v_1}{2R} + C\dfrac{d}{dt}v_1 + \dfrac{v_{out} - v_1}{2R} = 0 \\ \dfrac{C}{2}\dfrac{d}{dt}(v_{in} - v_2) - \dfrac{v_2}{R} + \dfrac{C}{2}\dfrac{d}{dt}(v_{out} - v_2) = 0 \\ \dfrac{C}{2}\dfrac{d}{dt}(v_2 - v_{out}) + \dfrac{v_1 - v_{out}}{2R} = 0 \end{cases}$$

By using Laplace transform we have:

$$\begin{cases} \dfrac{V_{in}(s) - V_1(s)}{2R} + CsV_1(s) + \dfrac{V_{out}(s) - V_1(s)}{2R} = 0 \\ \dfrac{Cs}{2}\left(V_{in}(s) - V_2(s)\right) - \dfrac{V_2(s)}{R} + \dfrac{Cs}{2}\left(V_{out}(s) - V_2(s)\right) = 0 \\ \dfrac{Cs}{2}\left(V_2(s) - V_{out}(s)\right) + \dfrac{V_1(s) - V_{out}(s)}{2R} = 0 \end{cases}$$

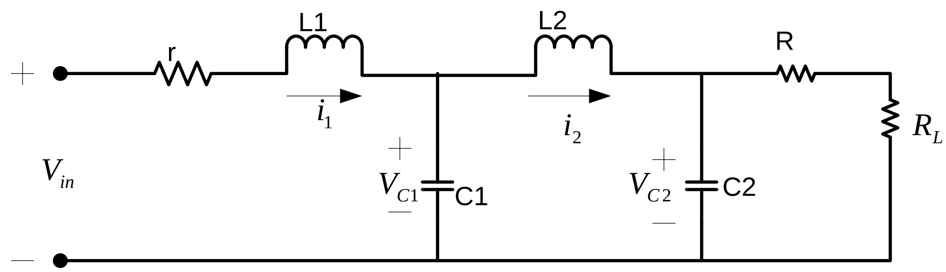
From above equations:

$$\begin{cases} V_1(s) = \dfrac{1}{2(RCS + 1)}\left(V_{in}(s) + V_{out}(s)\right) \\ V_2(s) = \dfrac{RCS}{2(RCS + 1)}\left(V_{in}(s) + V_{out}(s)\right) \end{cases}$$

Substituting $V_1(s)$ and $V_2(s)$ into preceding equations, we obtain:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{R^2C^2s^2 + 1}{R^2C^2s^2 + 4RCs + 1}$$

b) Measuring V_{out} requires a load resistor, which means:



Then we have:

$$\begin{cases} L_1 \dfrac{d}{dt} i_1 = v_{in} - r i_1 - v_{C1} \\ C_1 \dfrac{d}{dt} v_{C1} = i_1 - i_2 \\ L_2 \dfrac{d}{dt} i_2 = v_{C1} - v_{C2} \\ C_2 \dfrac{d}{dt} v_{C2} = i_2 - \dfrac{v_{C2}}{R + R_L} \end{cases}$$

When

$$v_{out} = \frac{R_L}{R + R_L} v_{C2}$$

If $R_L >> R$, then $v_{out} = v_{C2}$

By using Laplace transform we have:

$$\begin{cases} L_1 s I_1(s) = V_{in}(s) - r I_1(s) - V_{C1}(s) \\ C_1 s V_{C1}(s) = I_1(s) - I_2(s) \\ L_2 s I_2(s) = V_{C1}(s) - V_{C2}(s) \\ C_2 s V_{C2}(s) = I_2(s) - \dfrac{V_{C2}(s)}{R + R_L} \end{cases}$$

Therefore:

$$I_2(s) = \frac{C_2(R + R_L) + 1}{R + R_L} V_{C2}(s)$$

$$V_{C1}(s) = \frac{L_2C_2s(R + R_L) + s + (R + R_L)}{R + R_L} V_{C2}(s)$$

$$I_1(s) = \frac{L_2C_2C_1s^2(R + R_L) + C_1s^2 + C_1s(R + R_L) + C_2(R + R_L) + 1}{R + R_L} V_{C2}$$

$\frac{V_{C2}(s)}{V_{in}(s)}$ can be obtained by substituting above expressions into the first equation of the state variables of the system.

3-37. The following differential equations represent linear time-invariant systems. Write the dynamic equations (state equations and output equations) in vector-matrix form.

(a) $$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+y(t)=5r(t)$$

(b) $$2\frac{d^3y(t)}{dt^3}+3\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+2y(t)=r(t)$$

(c) $$\frac{d^3y(t)}{dt^3}+5\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+y(t)+\int_0^t y(\tau)d\tau=r(\tau)$$

(d) $$\frac{d^4y(t)}{dt^4}+1.5\frac{d^3y(t)}{dt^3}+2.5\frac{dy(t)}{dt}+y(t)=2r(t)$$

(a) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1$$

(b) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2y}{dt^2}$

State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2.5 & -1.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1$$

(c) State variables: $x_1 = \int_0^t y(\tau)d\tau, \quad x_2 = \frac{dx_1}{dt}, \quad x_3 = \frac{dy}{dt}, \quad x_4 = \frac{d^2y}{dt^2}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

(d) **State variables:** $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2y}{dt^2}, \quad x_4 = \frac{d^3y}{dt^3}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2.5 & 0 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

3-41. Given a system described by the dynamic equations:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \qquad y(t) = \mathbf{C}\mathbf{x}(t)$$

(a) $$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

(b) $$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

(c) $$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

(1) Find the eigenvalues of **A**.

(2) Find the transfer-function relation between $\mathbf{X}(s)$ and $U(s)$.

(3) Find the transfer function $Y(s)/U(s)$.

(a) **(1)** **Eigenvalues of A:** $2.325, \quad -0.3376 + j0.5623, \quad -0.3376 - j0.5623$

(2) **Transfer function relation:**

$$\mathbf{X}(s)=\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}U(s)=\frac{1}{\Delta(s)}\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s)=\frac{1}{\Delta(s)}\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s\left(s+3\right) & s \\ -s & -2s-1 & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s)=\frac{1}{\Delta(s)}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}U(s)$$

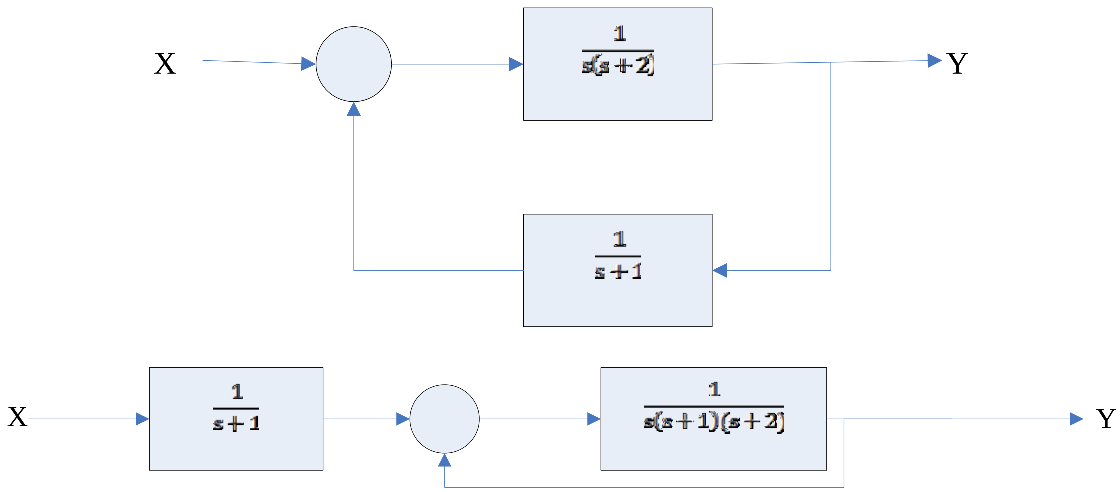
$$\Delta(s)=s^3+3s^2+2s+1$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)}=\mathbf{C}(s)\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\frac{1}{\Delta(s)}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}=\frac{1}{s^3+3s^2+2s+1}$$

Chapter 4

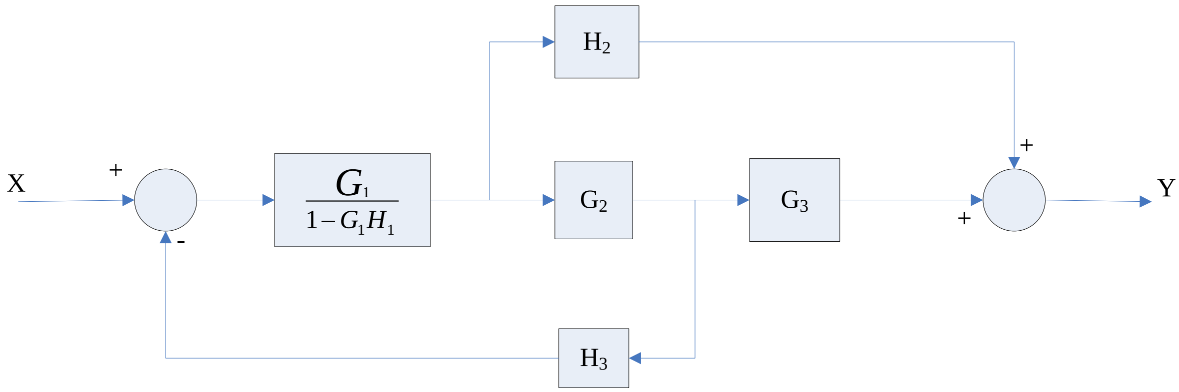
4-2)

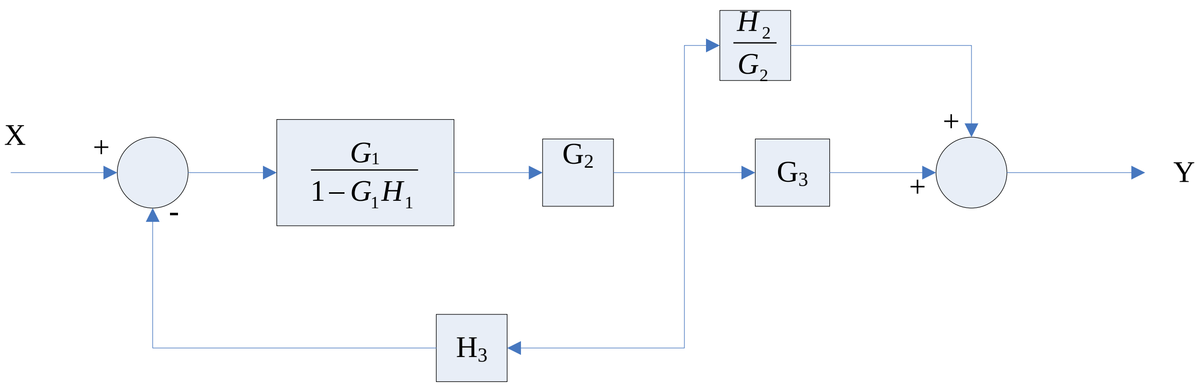


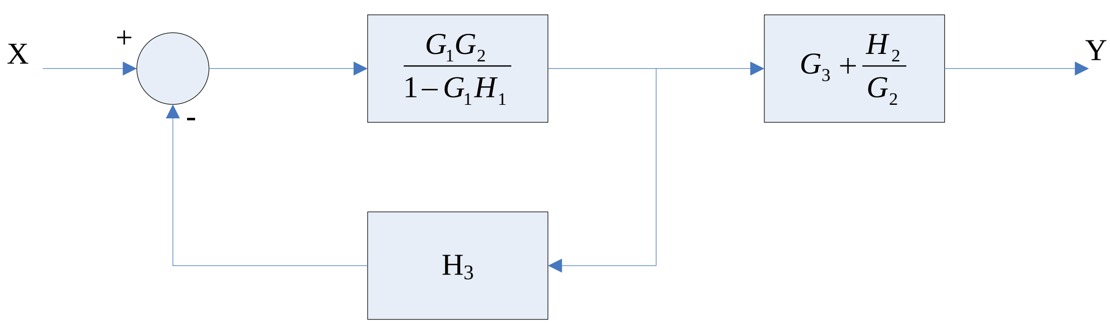
Characteristic equation: $s(s+1)(s+2)+\ 1=0$

⇨ $s^3+3s^2+2s+1=0$

4-3)







$$\frac{Y(s)}{X(s)} = \frac{\dfrac{G_1G_2}{1-G_1H_1}}{1+\dfrac{G_1G_2H_3}{1-G_1H_1}}\left(G_3+\frac{H_2}{G_2}\right) = \frac{G_1G_2G_3 + G_1H_2}{1-G_1H_1+G_1G_2H_3}$$

4-8)

(a)

$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{\dfrac{10(s+4)}{s(s+1)}}{1+\dfrac{5s}{s(s+1)}+\dfrac{10(s+2)}{s(s+1)}} = \frac{10(s+4)}{s^2+16s+20}$$

(b)

$$\left.\frac{Y(s)}{E(s)}\right|_{N=0} = \left.\frac{Y(s)/R(s)}{E(s)/R(s)}\right|_{N=0} = \frac{\dfrac{10(s+4)}{s(s+1)}}{1+\dfrac{5s}{s(s+1)}-\dfrac{20}{s(s+1)}} = \frac{10(s+4)}{s^2+6s-20}$$

(c)

$$\left.\frac{Y(s)}{N(s)}\right|_{R=0} = \frac{1}{1+\dfrac{5s}{s(s+1)}+\dfrac{10(s+2)}{s(s+1)}} = \frac{s(s+1)}{s^2+16s+20}$$

(d)

$$Y(s) = \left.\frac{Y(s)}{R(s)}\right|_{N=0} R(s) + \left.\frac{Y(s)}{N(s)}\right|_{R=0} N(s)$$

4-22)

(a)

$$\frac{Y_5}{Y_1}=\frac{G_1G_2G_3+G_3G_4}{\Delta} \qquad \frac{Y_2}{Y_1}=\frac{1+G_3H_2}{\Delta} \qquad \frac{Y_5}{Y_2}=\frac{Y_5/Y_1}{Y_2/Y_1}=\frac{G_1G_2G_3+G_3G_4}{1+G_3H_2}$$

$$\Delta=1+G_1H_1+G_3H_2+G_3G_4H_3+G_1G_2G_3H_3+G_1G_3H_1H_2$$

(b)

$$\frac{Y_5}{Y_1}=\frac{G_1G_2G_3+G_3G_4}{\Delta} \qquad \frac{Y_2}{Y_1}=\frac{1+G_3H_2+H_4}{\Delta} \qquad \frac{Y_5}{Y_2}=\frac{Y_5/Y_1}{Y_2/Y_1}=\frac{G_1G_2G_3+G_3G_4}{1+G_3H_2+H_4}$$

$$\Delta=1+G_1H_1+G_3H_2+G_3G_4H_3+G_1G_2G_3H_3+H_4+G_1G_3H_1H_2+G_1H_1H_4$$

(c)

$$\frac{Y_5}{Y_1}=\frac{G_1G_2G_3+G_4}{\Delta} \qquad \frac{Y_2}{Y_1}=\frac{1+G_2G_3H_3}{\Delta} \qquad \frac{Y_5}{Y_2}=\frac{Y_5/Y_1}{Y_2/Y_1}=\frac{G_1G_2G_3+G_3G_4}{1+G_2G_3H_3}$$

$$\Delta=1+G_1H_1+G_2G_3H_3+G_1G_2H_2-G_2G_4H_2H_3$$

(d)

$$\frac{Y_5}{Y_1}=\frac{G_3G_4+G_1G_2G_3}{\Delta} \qquad \frac{Y_2}{Y_1}=\frac{1+G_2H_2}{\Delta} \qquad \frac{Y_5}{Y_2}=\frac{Y_5/Y_1}{Y_2/Y_1}=\frac{G_3G_4+G_1G_2G_3}{1+G_2H_2}$$

$$\Delta=1+G_1H_1+G_2H_2+G_3G_4H_3+G_1G_2G_3H_3-G_4H_1H_2$$

(e)

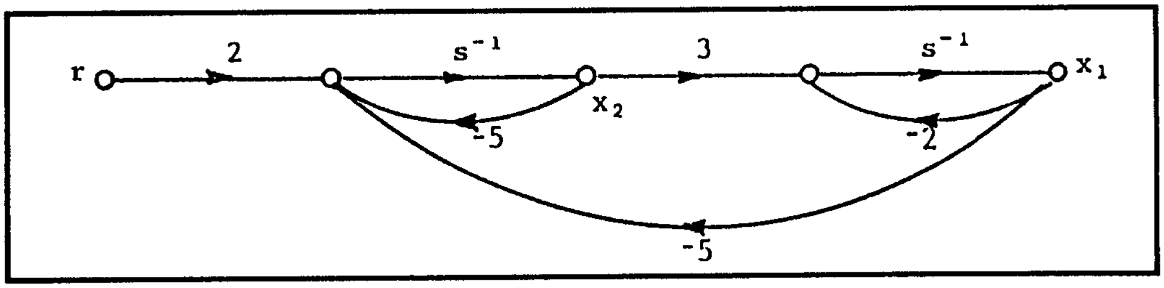
$$\frac{Y_5}{Y_1}=\frac{G_1G_2G_3(1+H_4)+G_4G_5(1+G_2H_1)}{\Delta} \qquad \frac{Y_2}{Y_1}=\frac{1+G_2H_1+G_3H_2+H_4+G_2H_1H_4+G_3H_2H_4}{\Delta}$$

$$\frac{Y_5}{Y^2}=\frac{Y_5/Y_1}{Y_2/Y_1}=\frac{G_1G_2G_3(1+H_4)+G_4G_5(1+G_2H_{1)}}{1+G_2H_1+G_3H_2+H_4+G_2H_1H_4+G_3H_2H_4}$$

$$\Delta=1+G_2H_1+G_3H_2+H_4+G_4G_5H_3+G_1G_2G_3H_3+G_2H_1H_4+G_3H_2H_4+G_1G_2G_3H_3H_4+G_2G_4H_1H_3$$

4-32)

(a) State diagram:



(b) Characteristic equation: $\Delta = 1 + 2s^{-1} + 5s^{-1} + 15s^{-1} + 10s^{-2} = 0 \qquad s^2 + 7s + 25 = 0$

(c) Transfer functions:

$$\frac{X_1(s)}{R(s)} = \frac{6s^{-2}}{\Delta} = \frac{6}{s^2 + 7s + 25} \qquad \frac{X_2(s)}{R(s)} = \frac{2s^{-1}(1 + 2s^{-1})}{\Delta} = \frac{2(s+2)}{s^2 + 7s + 25}$$