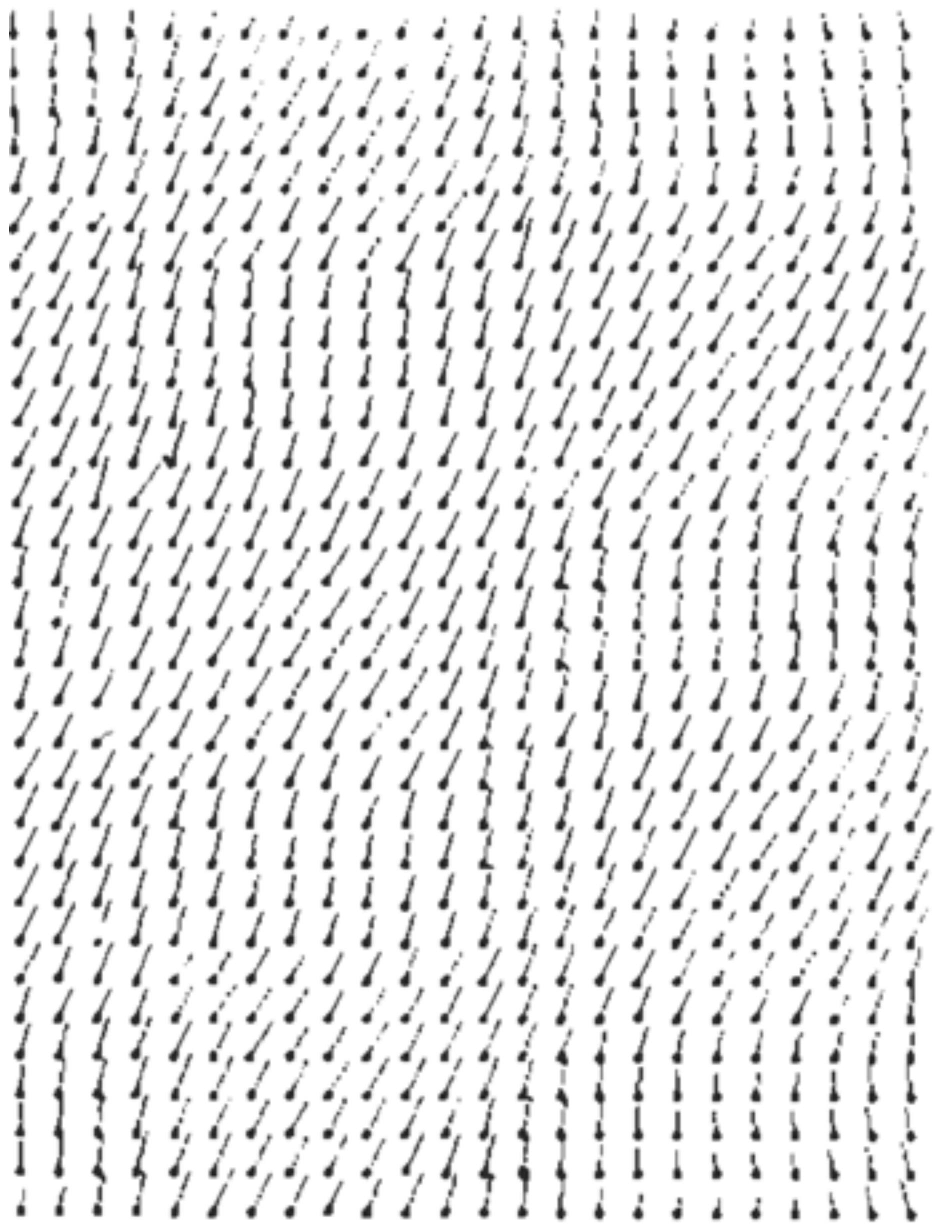
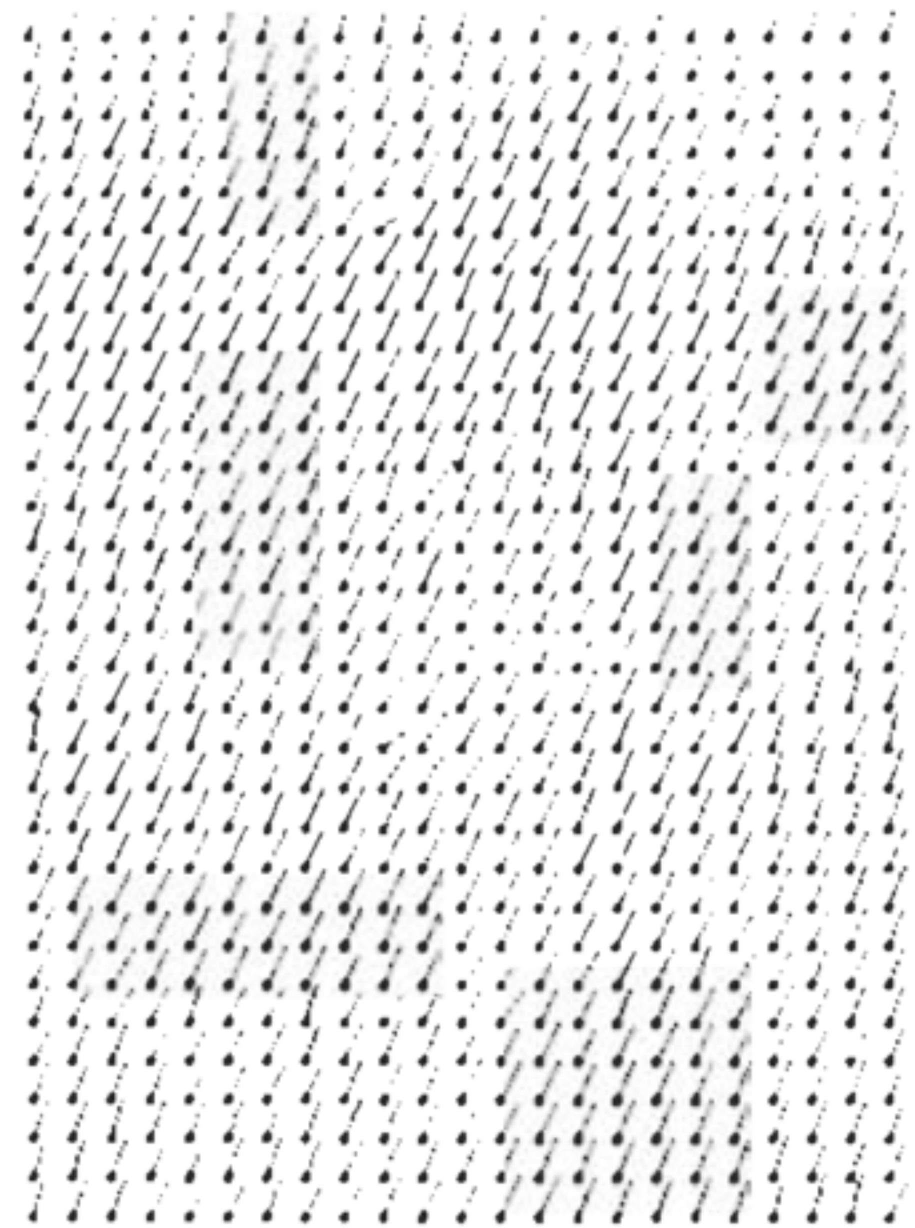



Optical Flow: Horn-Schunck

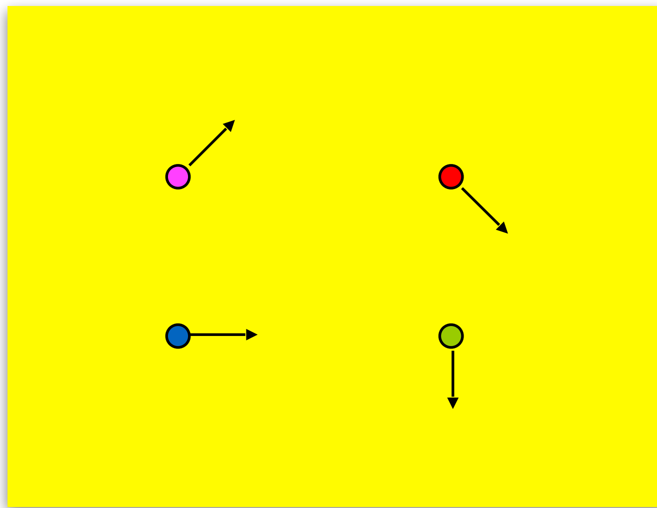
16-385 Computer Vision (Kris Kitani)

Carnegie Mellon University

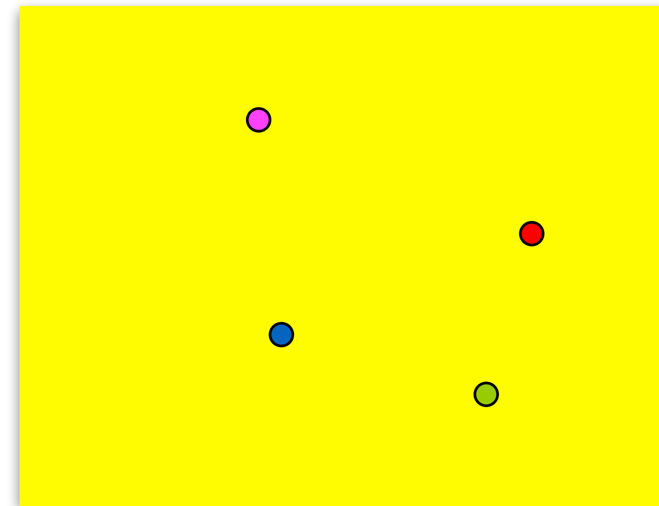
Horn-Schunck Optical Flow (1981)	**Lucas-Kanade Optical Flow (1981)**
brightness constancy	method of differences
small motion	
'smooth' flow (flow can vary from pixel to pixel)	**'constant' flow** (flow is constant for all pixels)
global method (dense)	local method (sparse)

Optical Flow

(Problem definition)



$$I(x, y, t)$$



$$I(x, y, t')$$

Estimate the motion (flow) between these two consecutive images

How is this different from estimating a 2D transform?

Key Assumptions

(unique to optical flow)

Color Constancy

(Brightness constancy for intensity images)

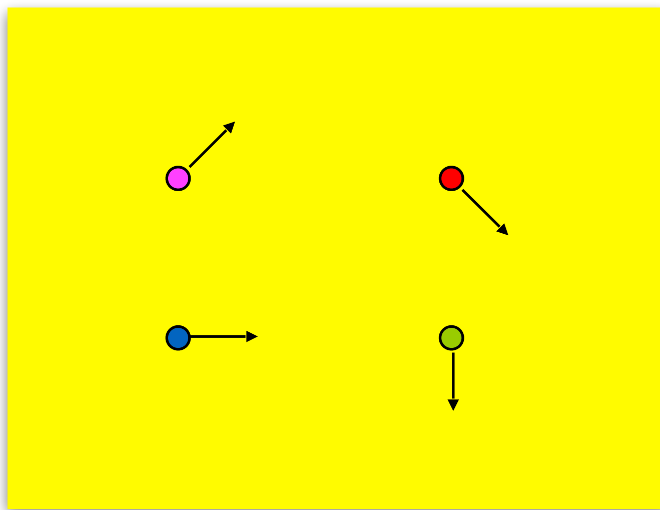
Implication: allows for pixel to pixel comparison
(not image features)

Small Motion

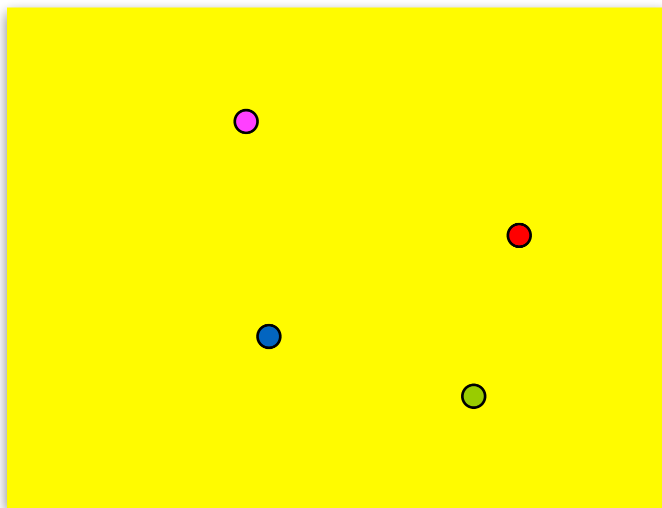
(pixels only move a little bit)

Implication: linearization of the brightness
constancy constraint

Approach



$I(x, y, t)$

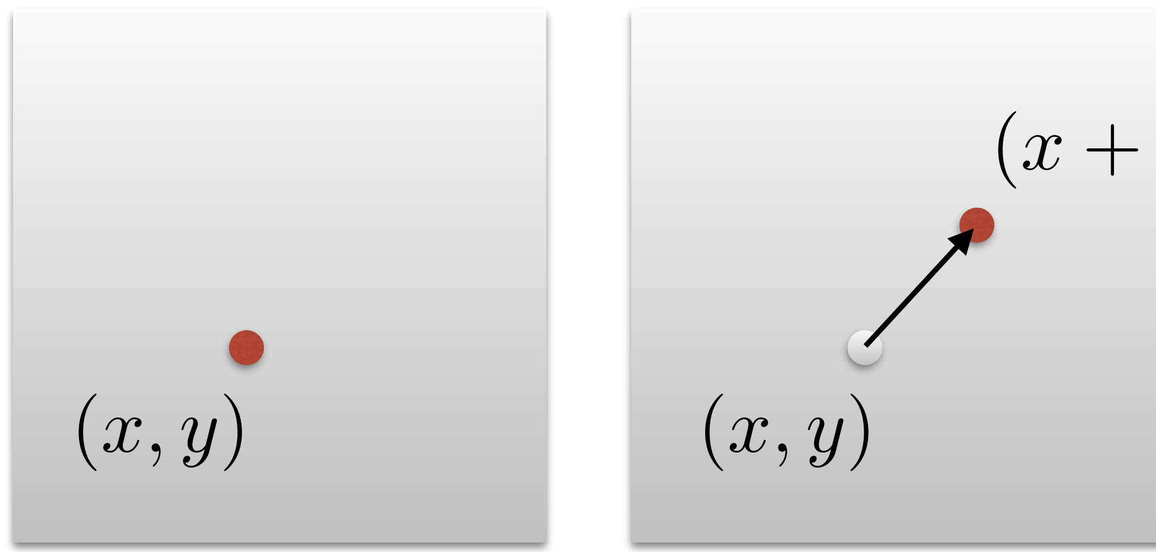


$I(x, y, t')$

Look for nearby pixels with the same color

(small motion) (color constancy)

Brightness constancy



$I(x, y, t)$ $\quad$ $I(x, y, t + \delta t)$

Optical flow (velocities): (u, v) $\quad$ Displacement: $(\delta x, \delta y) = (u\delta t, v\delta t)$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For a really small time step…

Optical Flow Constraint equation

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

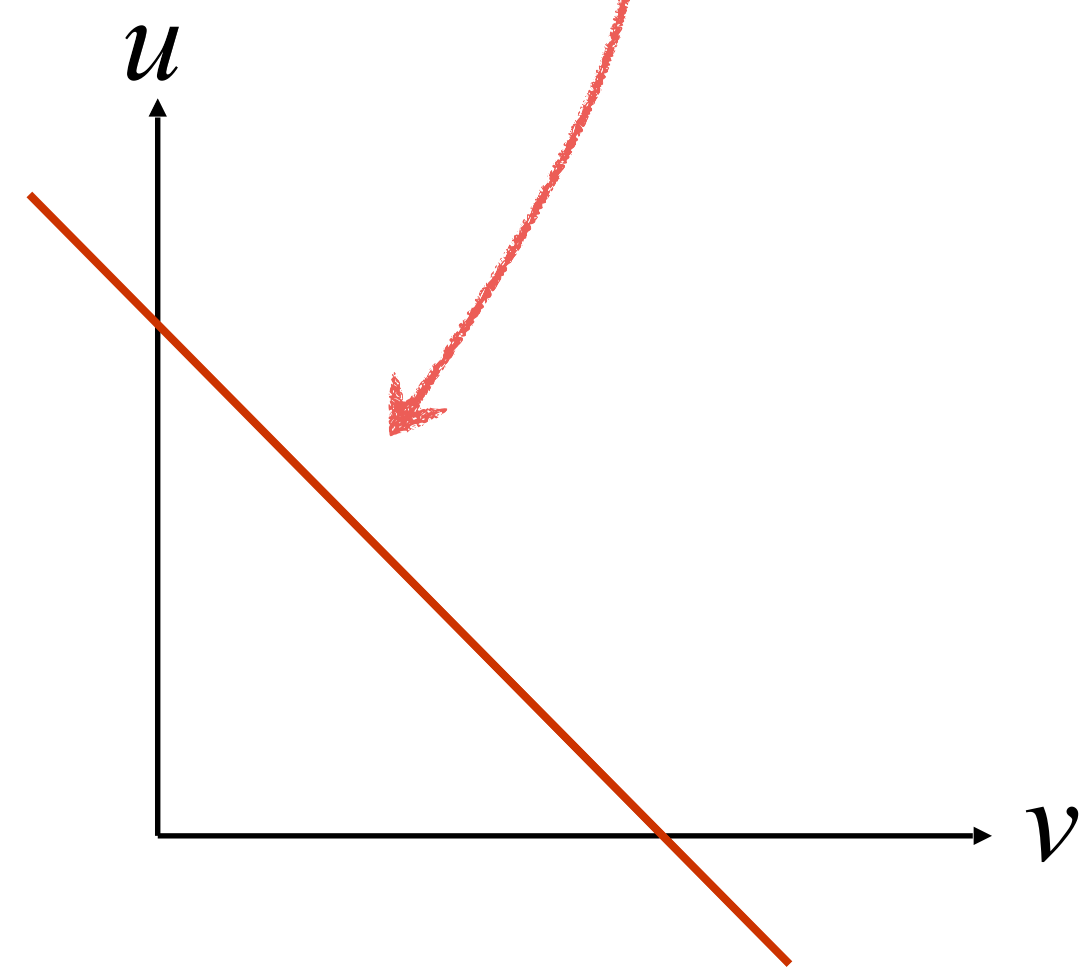
$$\cancel{I(x, y, t)} + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = \cancel{I(x, y, t)}$$

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

$$\boxed{I_x u + I_y v + I_t = 0}$$

Solution lies on a straight line

$$\boxed{I_x u + I_y v + I_t = 0}$$



The solution cannot be determined uniquely with a single constraint (a single pixel)

Where can we get an additional constraint?

Horn-Schunck Optical Flow (1981)

brightness constancy

small motion

'smooth' flow
(flow is smooth from pixel to pixel)

global method

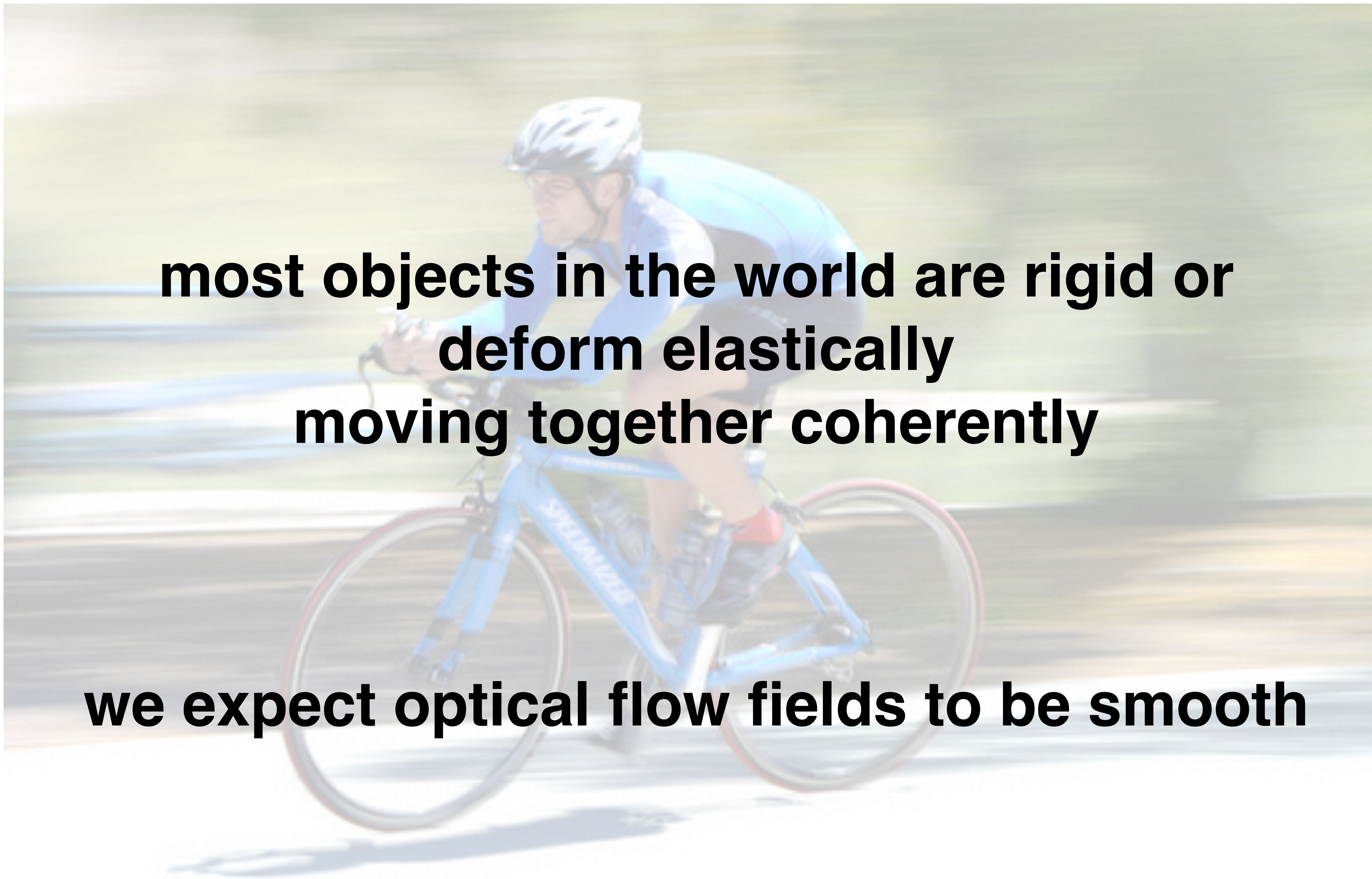
Lucas-Kanade Optical Flow (1981)

method of differences

'constant' flow
(flow is constant for all pixels)

local method

Smoothness



most objects in the world are rigid or deform elastically
moving together coherently

we expect optical flow fields to be smooth

Key idea

(of Horn-Schunck optical flow)

Enforce

brightness constancy

Enforce

smooth flow field

to compute optical flow

Key idea

(of Horn-Schunck optical flow)

Enforce

brightness constancy

Enforce

smooth flow field

to compute optical flow

Enforce **brightness constancy**

$$\boxed{I_x u + I_y v + I_t = 0}$$

For every pixel,

$$\min_{u,v} \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

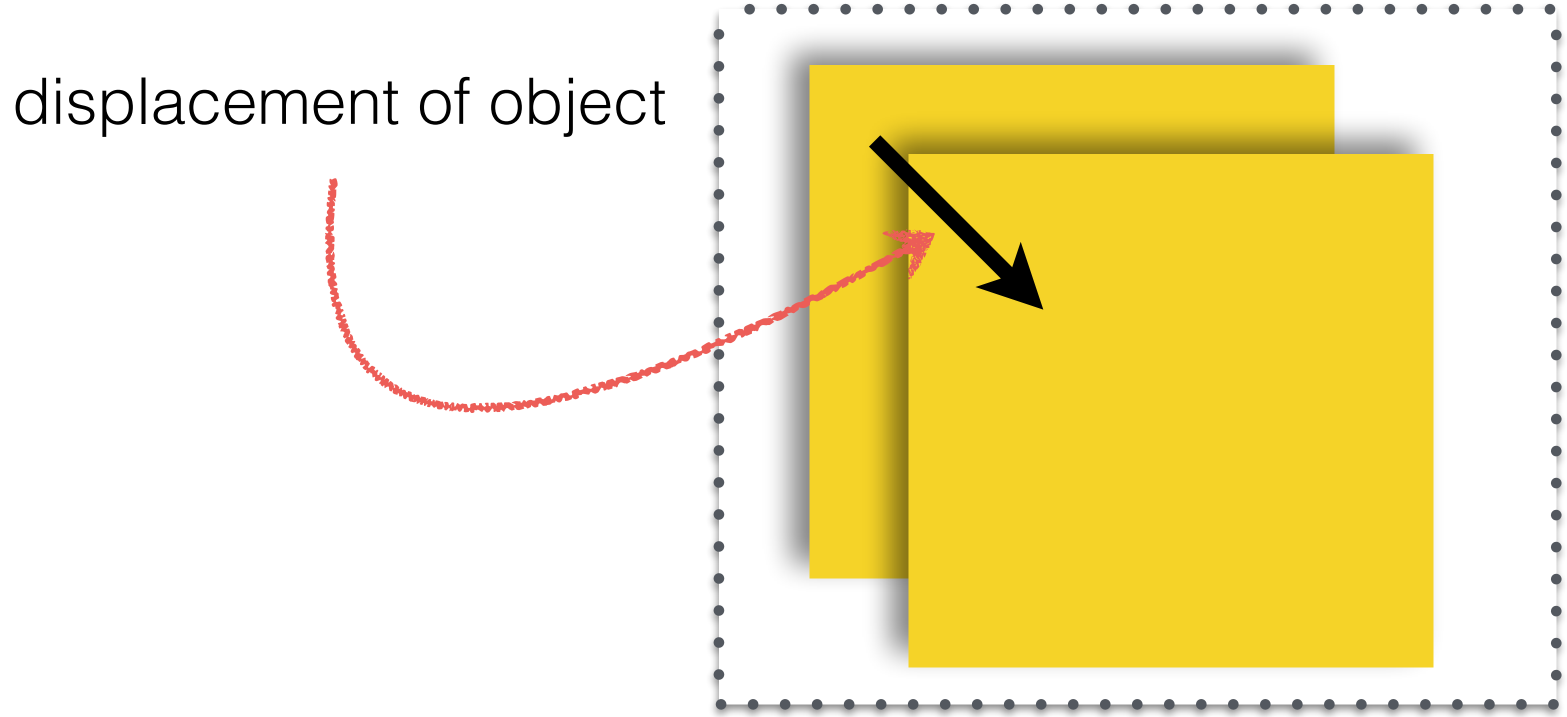
Enforce
brightness constancy

$$\boxed{I_x u + I_y v + I_t = 0}$$

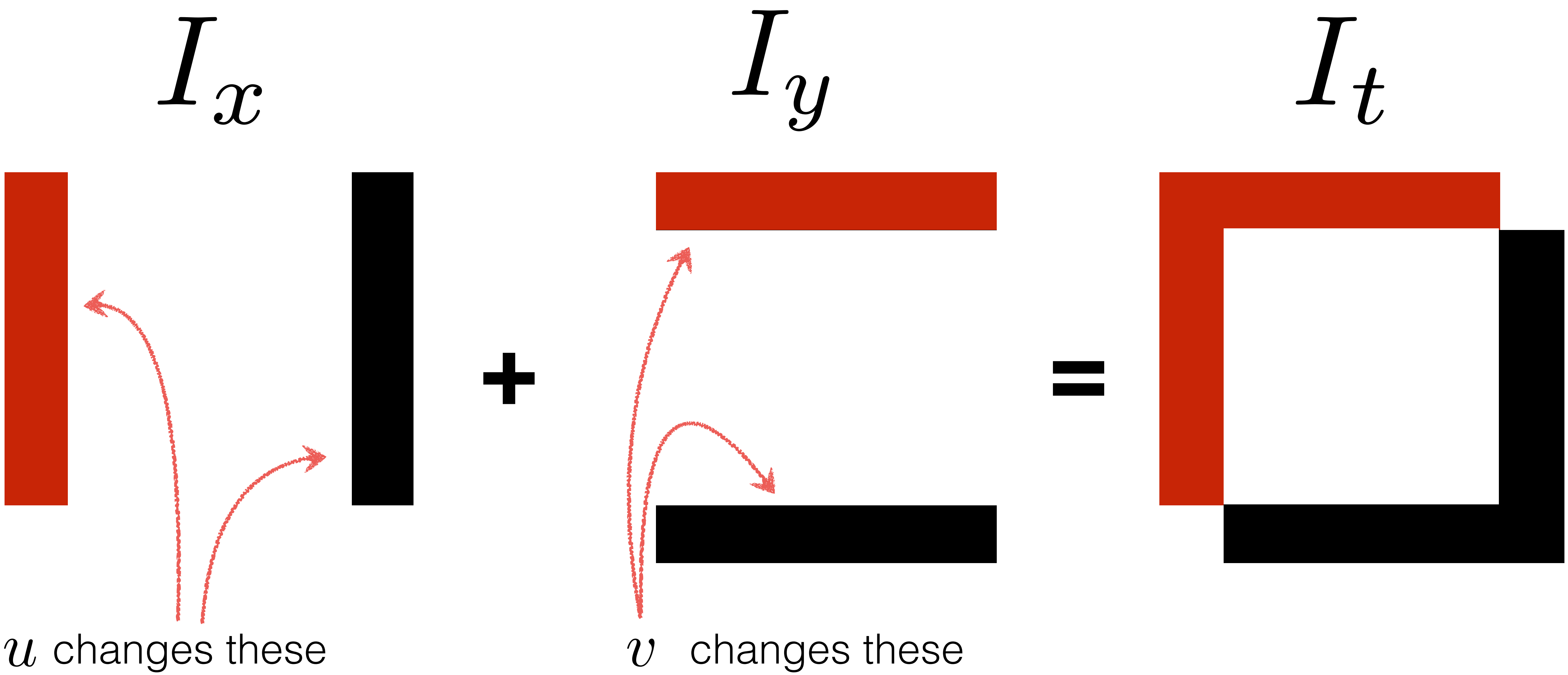
For every pixel,

$$\min_{u,v} \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

lazy notation for $I_x(i,j)$



Find the optical flow such that it satisfies:



Key idea

(of Horn-Schunck optical flow)

Enforce

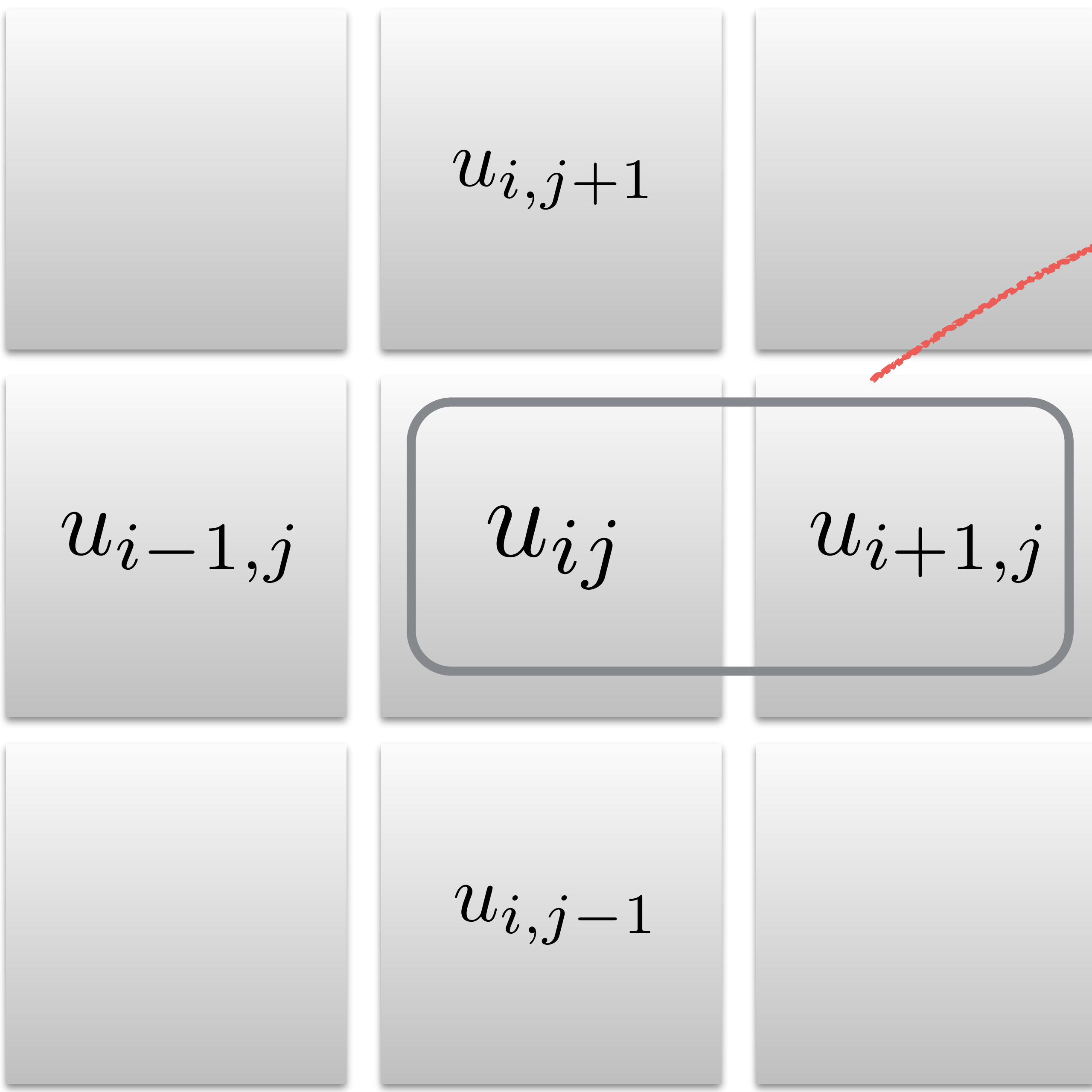
brightness constancy

Enforce

smooth flow field

to compute optical flow

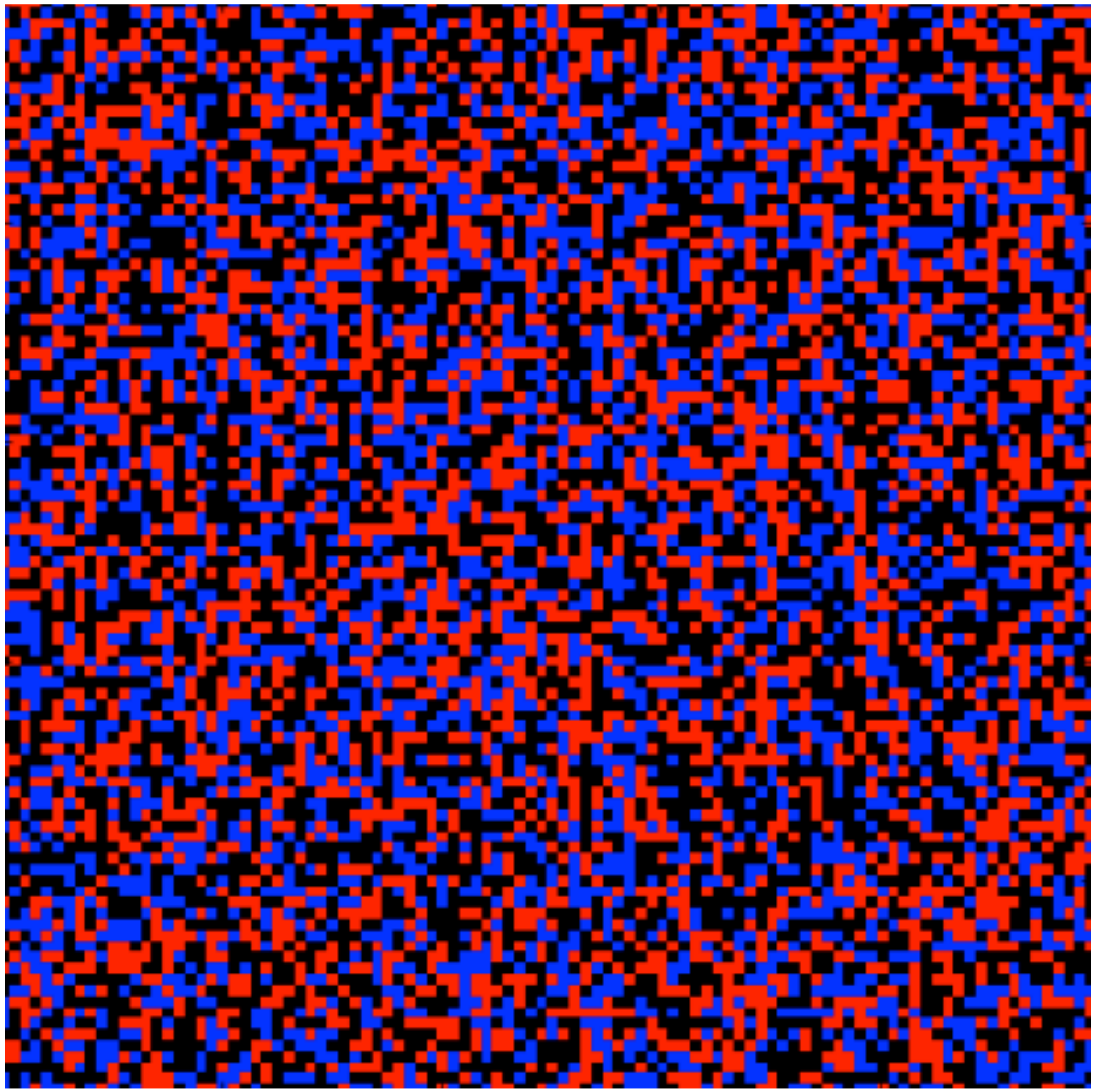
Enforce **smooth flow field**



u-component of flow

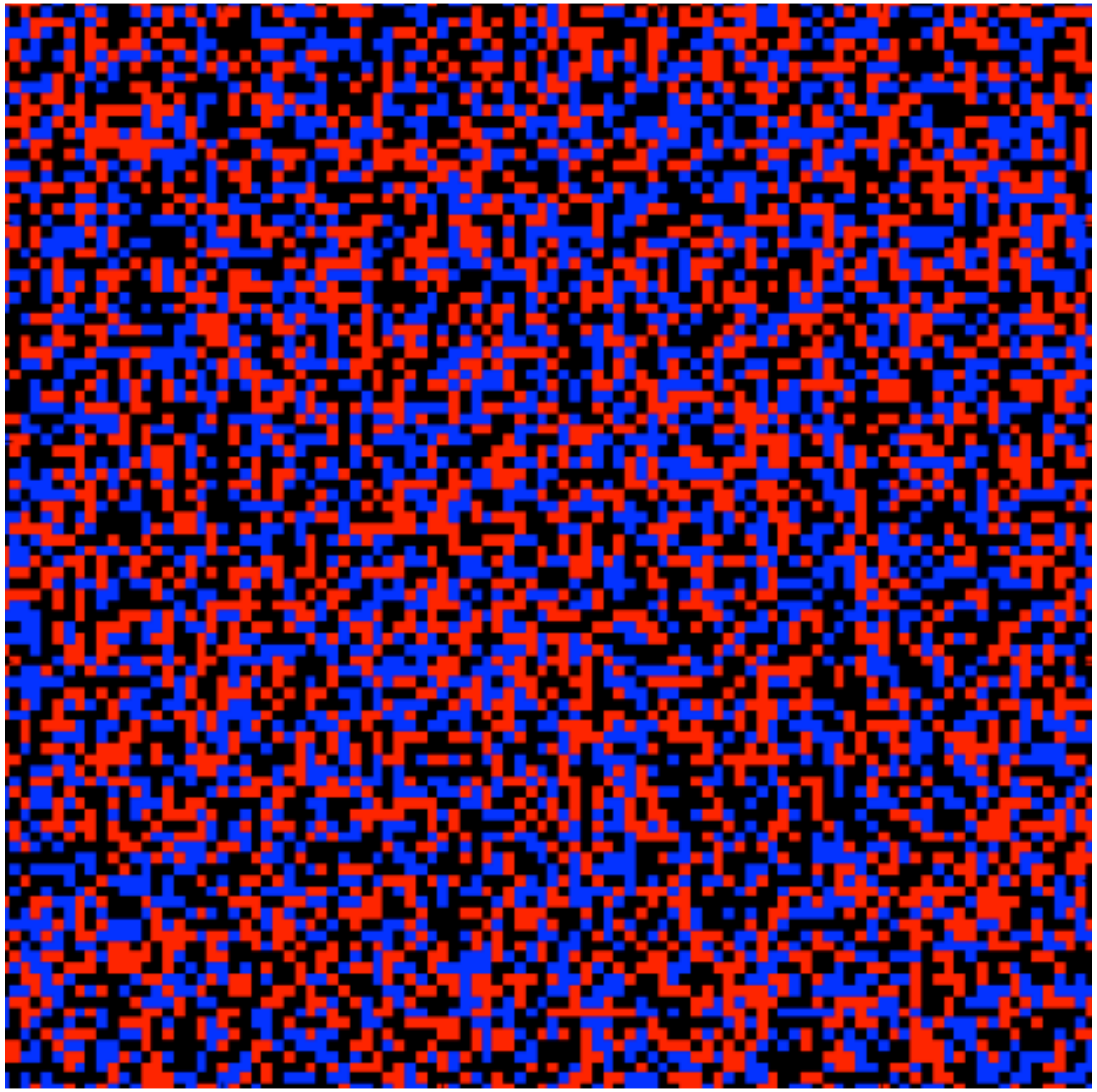
$$\min_{\boldsymbol{u}} (u_{i,j} - u_{i+1,j})^2$$

Which flow field optimizes the objective? $\min_{\boldsymbol{u}} (u_{i,j} - u_{i+1,j})^2$

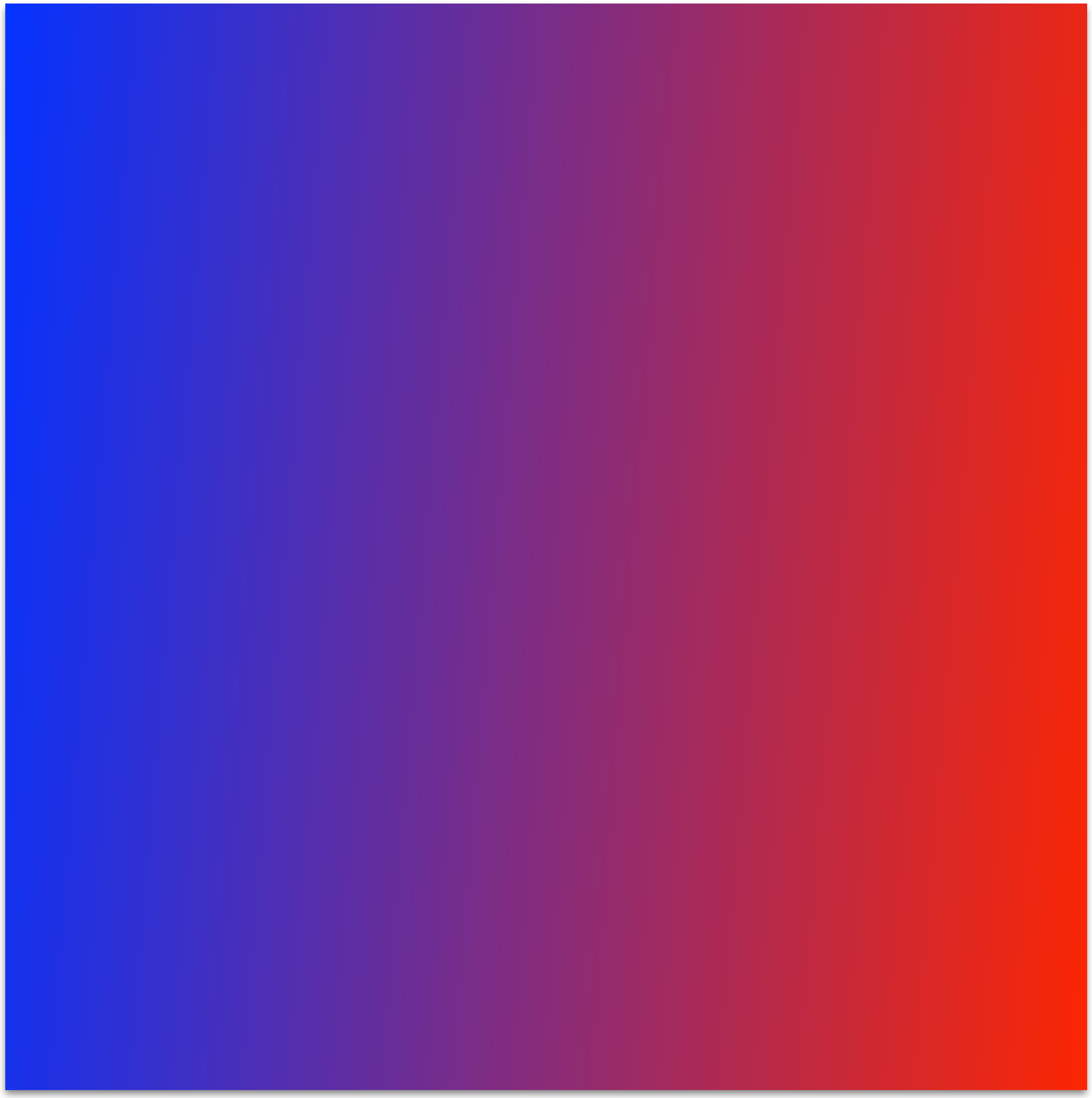


$$\sum_{ij} (u_{ij} - u_{i+1,j})^2 \quad ? \quad \sum_{ij} (u_{ij} - u_{i+1,j})^2$$

Which flow field optimizes the objective? $\min_{\boldsymbol{u}} (u_{i,j} - u_{i+1,j})^2$



big



small

Key idea

(of Horn-Schunck optical flow)

Enforce

brightness constancy

Enforce

smooth flow field

to compute optical flow

bringing it all together…

Horn-Schunck optical flow

$$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ \underbrace{E_s(i,j)}_{\text{smoothness}} + \underbrace{\lambda}_{\text{weight}} \underbrace{E_d(i,j)}_{\text{brightness constancy}} \right\}$$

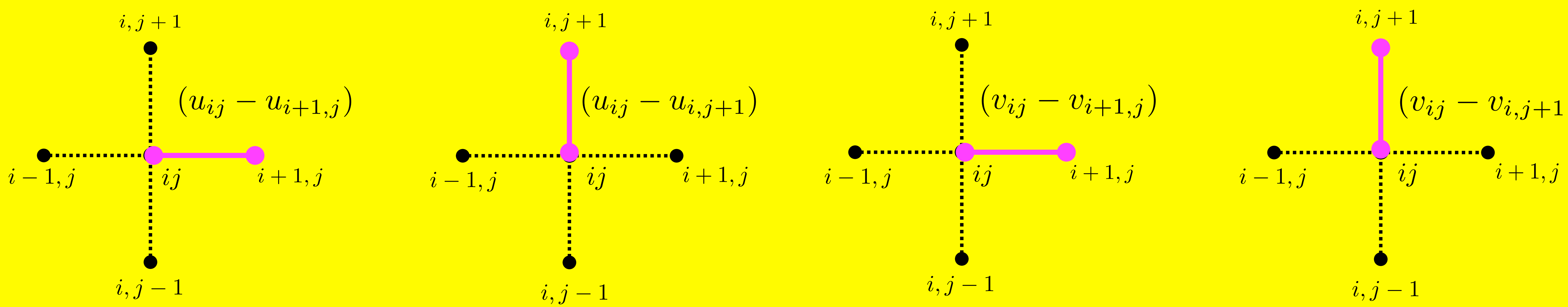
HS optical flow objective function

Brightness constancy

$$E_d(i,j) = \Big[I_x u_{ij} + I_y v_{ij} + I_t \Big]^2$$

Smoothness

$$E_s(i,j) = \frac{1}{4}\Big[(u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \Big]$$



why not all four neighbors?

How do we solve this minimization problem?

$$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

How do we solve this minimization problem?

$$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

Compute partial derivative, derive update equations

(gradient decent!)

Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \underbrace{\frac{1}{4}\left[(u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right]}_{\text{smoothness term}} + \lambda \underbrace{\left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2}_{\text{brightness constancy}} \right\}$$

Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \frac{1}{4} \left[(u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

it's not so bad…

$$\frac{\partial E}{\partial u_{kl}} =$$

how many u terms depend on k and l?

Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \frac{1}{4} \left[(u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

it's not so bad…

$$\frac{\partial E}{\partial u_{kl}} =$$

how many u terms depend on k and l?

FOUR from smoothness

ONE from brightness constancy

Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \frac{1}{4} \left[(u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

it's not so bad…

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

how many u terms depend on k and l?

FOUR from smoothness

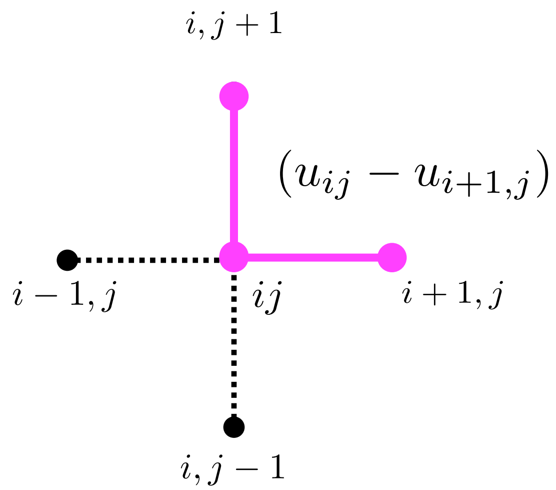
ONE from brightness constancy

Compute the partial derivatives of this huge sum!

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_x u_{ij}+I_y v_{ij}+I_t\right]^2\right\}$$

$$(u_{ij}^2-2u_{ij}u_{i+1,j}+u_{i+1,j}^2) \qquad (u_{ij}^2-2u_{ij}u_{i,j+1}+u_{i,j+1}^2)$$

(variable will appear four times in sum)

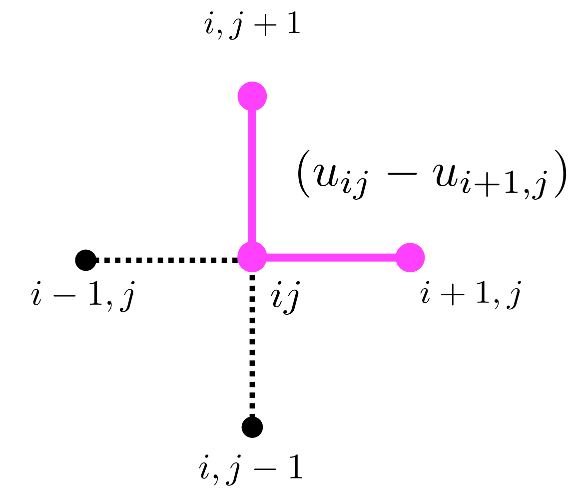


Compute the partial derivatives of this huge sum!

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_x u_{ij}+I_y v_{ij}+I_t\right]^2\right\}$$

$(u_{ij}^2-2u_{ij}u_{i+1,j}+u_{i+1,j}^2)$ $\qquad$ $(u_{ij}^2-2u_{ij}u_{i,j+1}+u_{i,j+1}^2)$

(variable will appear four times in sum)



$$\frac{\partial E}{\partial u_{kl}}=2(u_{kl}-\bar{u}_{kl})+2\lambda(I_x u_{kl}+I_y v_{kl}+I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}}=2(v_{kl}-\bar{v}_{kl})+2\lambda(I_x u_{kl}+I_y v_{kl}+I_t)I_y$$

short hand for
local average

$$\bar{u}_{ij}=\frac{1}{4}\left\{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}\right\}$$

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

Where are the extrema of E?

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

Where are the extrema of E?

(set derivatives to zero and solve for unknowns u and v)

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

Where are the extrema of E?

(set derivatives to zero and solve for unknowns u and v)

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

this is a linear system $$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$ *how do you solve this?*

ok, take a step back, why are we doing all this math?

We are solving for the optical flow (u,v) given two constraints

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_x u_{ij}+I_y v_{ij}+I_t\right]^2\right\}$$

smoothness

brightness constancy

We need the math to minimize this

(back to the math)

Partial derivatives of Horn-Schunck objective function E:

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

Where are the extrema of E?

(set derivatives to zero and solve for unknowns u and v)

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

how do you solve this?

$$(1 + \lambda I_x^2) u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2) v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

Recall $\boldsymbol{x} = \mathbf{A}^{-1} \boldsymbol{b} = \frac{\mathrm{adj}\mathbf{A}}{\det \mathbf{A}} \boldsymbol{b}$

$$(1+\lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1+\lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

Recall $\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{b} = \dfrac{\mathrm{adj}\mathbf{A}}{\det \mathbf{A}}\boldsymbol{b}$

Same as the linear system:

$$\underbrace{\{1+\lambda(I_x^2+I_y^2)\}}_{\text{(det A)}}u_{kl} = (1+\lambda I_x^2)\bar{u}_{kl} - \lambda I_x I_y \bar{v}_{kl} - \lambda I_x I_t$$

$$\underbrace{\{1+\lambda(I_x^2+I_y^2)\}}_{\text{(det A)}}v_{kl} = (1+\lambda I_y^2)\bar{v}_{kl} - \lambda I_x I_y \bar{u}_{kl} - \lambda I_y I_t$$

$$\{1 + \lambda(I_x^2 + I_y^2)\}u_{kl} = (1 + \lambda I_x^2)\bar{u}_{kl} - \lambda I_x I_y \bar{v}_{kl} - \lambda I_x I_t$$

$$\{1 + \lambda(I_x^2 + I_y^2)\}v_{kl} = (1 + \lambda I_y^2)\bar{v}_{kl} - \lambda I_x I_y \bar{u}_{kl} - \lambda I_y I_t$$

Rearrange to get update equations:

$$\boxed{\begin{aligned} \underbrace{\hat{u}_{kl}}_{\text{new value}} &= \underbrace{\bar{u}_{kl}}_{\text{old average}} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x \\ \hat{v}_{kl} &= \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y \end{aligned}}$$

Recall: $$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

When lambda is small (lambda inverse is big)...

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x$$

new value

old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

Recall: $$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

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goes to zero

new value

old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \cancel{\frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2}} I_y$$

goes to zero

Recall: $$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

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$$\hat{u}_{kl} = \bar{u}_{kl} - \cancel{\frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x}$$

goes to zero

new value

old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \cancel{\frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y}$$

goes to zero

...we only care about smoothness.

ok, take a step back, why did we do all this math?

We are solving for the optical flow (u,v) given two constraints

$$\sum_{ij} \left\{ \frac{1}{4} \left[(u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

smoothness

brightness constancy

We needed the math to minimize this (now to the algorithm)

Horn-Schunck Optical Flow Algorithm

```
1. Precompute image gradients
```
$I_y \quad I_x$

```
2. Precompute temporal gradients
```
I_t

```
3. Initialize flow field
```
$\boldsymbol{u} = \mathbf{0}$

$\boldsymbol{v} = \mathbf{0}$

```
4. While not converged

   Compute flow field updates for each pixel:
```

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x \qquad \hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

Just 8 lines of code!