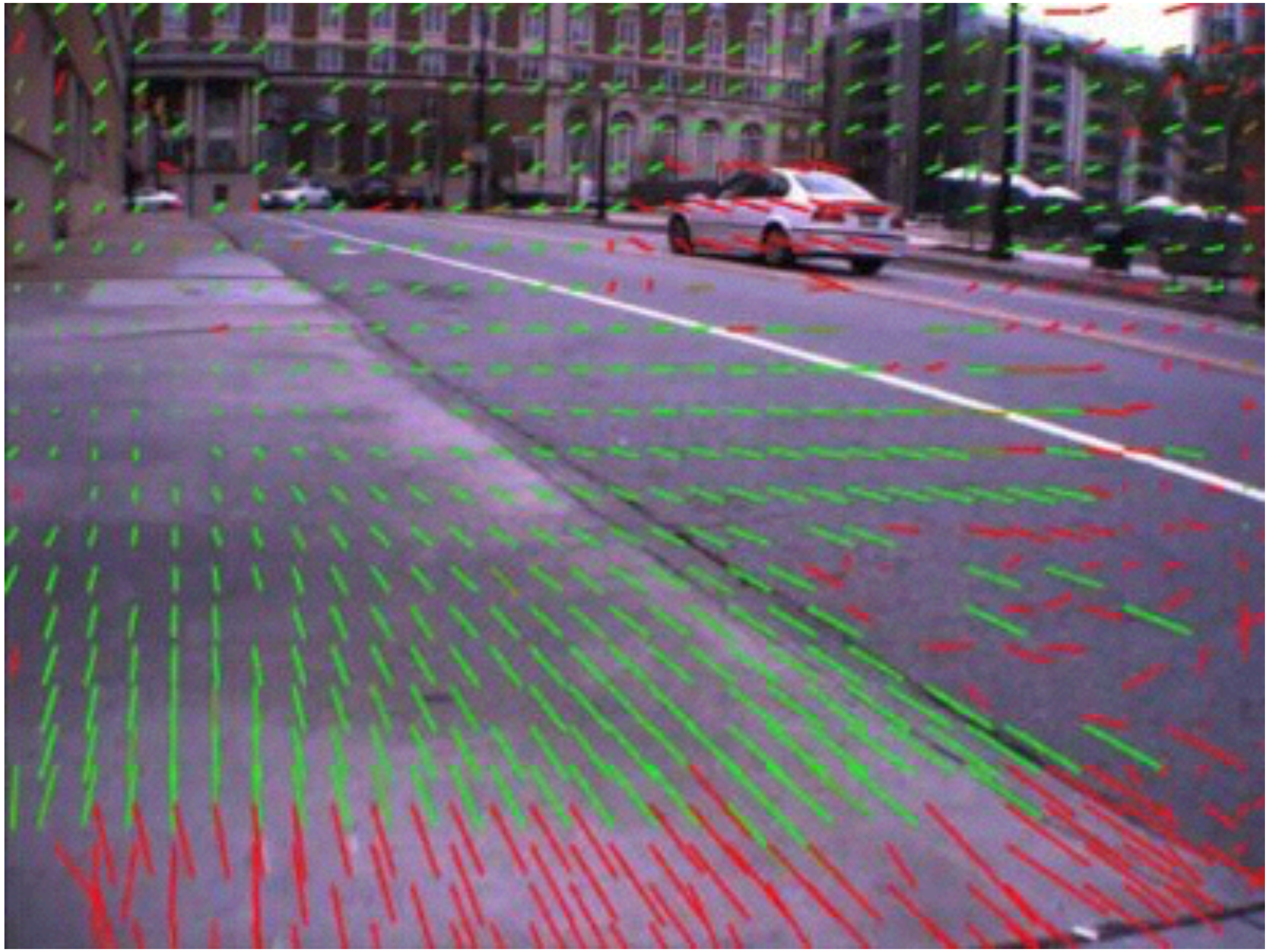


Optical Flow: Constant Flow

Computer Vision 16-385

Carnegie Mellon University (Kris Kitani)

Optical Flow

(a.k.a., Video Stabilization, Tracking, Stereo Matching, Registration)

Given a pair of images

$$\{I_t, I_{t+1}\}$$

Estimate the optical flow field

$$\{v(p_i), u(p_i)\}$$

$$I_x u + I_y v + I_t = 0$$

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

optical flow

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

How can we use the brightness constancy equation to estimate the optical flow?

unknown

$$I_x u + I_y v + I_t = 0$$

known

We need at least ____ equations to solve for 2 unknowns.

unknown

$$I_x u + I_y v + I_t = 0$$

known

Where do we get more equations (constraints)?

Where do we get more equations (constraints)?

$$I_x u + I_y v + I_t = 0$$

Assume that the surrounding patch (say 5x5) has
‘**constant flow**’

Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us equations

Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us 25 equations

$$I_x(\boldsymbol{p}_1)u + I_y(\boldsymbol{p}_1)v = -I_t(\boldsymbol{p}_1)$$
$$I_x(\boldsymbol{p}_2)u + I_y(\boldsymbol{p}_2)v = -I_t(\boldsymbol{p}_2)$$
$$\vdots$$
$$I_x(\boldsymbol{p}_{25})u + I_y(\boldsymbol{p}_{25})v = -I_t(\boldsymbol{p}_{25})$$

Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us 25 equations

$$\begin{bmatrix} I_x(\boldsymbol{p}_1) & I_y(\boldsymbol{p}_1) \\ I_x(\boldsymbol{p}_2) & I_y(\boldsymbol{p}_2) \\ \vdots & \vdots \\ I_x(\boldsymbol{p}_{25}) & I_y(\boldsymbol{p}_{25}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} I_t(\boldsymbol{p}_1) \\ I_t(\boldsymbol{p}_2) \\ \vdots \\ I_t(\boldsymbol{p}_{25}) \end{bmatrix}$$

Matrix form

Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us 25 equations

$$\begin{bmatrix} I_x(\boldsymbol{p}_1) & I_y(\boldsymbol{p}_1) \\ I_x(\boldsymbol{p}_2) & I_y(\boldsymbol{p}_2) \\ \vdots & \vdots \\ I_x(\boldsymbol{p}_{25}) & I_y(\boldsymbol{p}_{25}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} I_t(\boldsymbol{p}_1) \\ I_t(\boldsymbol{p}_2) \\ \vdots \\ I_t(\boldsymbol{p}_{25}) \end{bmatrix}$$

$$\underset{25 \times 2}{A} \qquad \underset{2 \times 1}{x} \qquad \underset{25 \times 1}{b}$$

How many equations? How many unknowns? How do we solve this?

Least squares approximation

$\hat{x} = \arg\min_{x} ||Ax - b||^2$ is equivalent to solving $A^\top A\hat{x} = A^\top b$

Least squares approximation

$$\hat{x} = \arg\min_x ||Ax - b||^2$$ is equivalent to solving $$A^\top A \hat{x} = A^\top b$$

To obtain the least squares solution solve:

$$\begin{array}{ccc} A^\top A & \hat{x} & A^\top b \\ \left[\begin{array}{cc} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{array} \right] & \left[\begin{array}{c} u \\ v \end{array} \right] = & - \left[\begin{array}{c} \sum_{p \in P} I_x I_t \\ \sum_{p \in P} I_y I_t \end{array} \right] \end{array}$$

where the summation is over each pixel $\boldsymbol{p}$ in patch $\boldsymbol{P}$

$$x = (A^\top A)^{-1} A^\top b$$

Least squares approximation

$$\hat{x} = \arg\min_x ||Ax - b||^2$$ is equivalent to solving $$A^\top A\hat{x} = A^\top b$$

To obtain the least squares solution solve:

$$\begin{array}{ccc} A^\top A & \hat{x} & A^\top b \end{array}$$

$$\begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} \sum_{p \in P} I_x I_t \\ \sum_{p \in P} I_y I_t \end{bmatrix}$$

where the summation is over each pixel $\boldsymbol{p}$ in patch $\boldsymbol{P}$

Sometimes called 'Lucas-Kanade Optical Flow'
(can be interpreted to be a special case of the LK method with a translational warp model)

When is this solvable?

$$A^\top A\hat{x} = A^\top b$$

$A^\top A$ should be invertible

$A^\top A$ should not be too small

λ_1 and λ_2 should not be too small

$A^\top A$ should be well conditioned

λ_1/λ_2 should not be too large (λ_1=larger eigenvalue)

Where have you seen this before?

$$A^\top A = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

Where have you seen this before?

$$\begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

Harris Corner Detector!

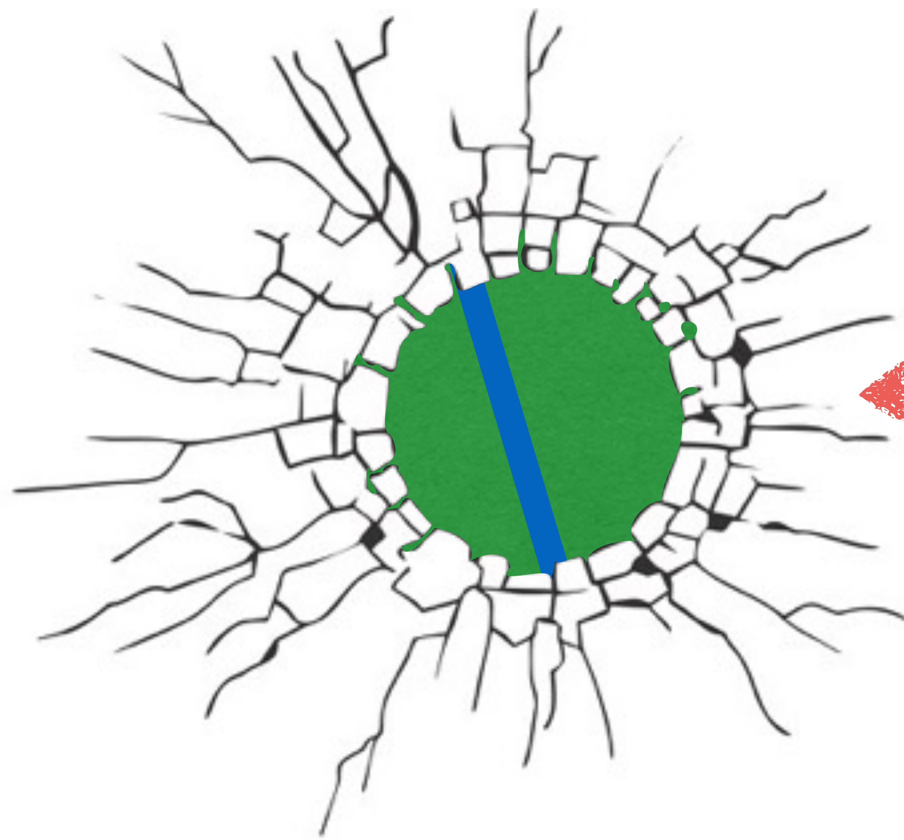
Implications

- Corners are when $\lambda 1$, $\lambda 2$ are big; this is also when Lucas-Kanade optical flow works best
- Corners are regions with two different directions of gradient (at least)
- Corners are good places to compute flow!

What happens when you have no 'corners'?

You want to compute optical flow.
What happens if the image patch contains only a line?

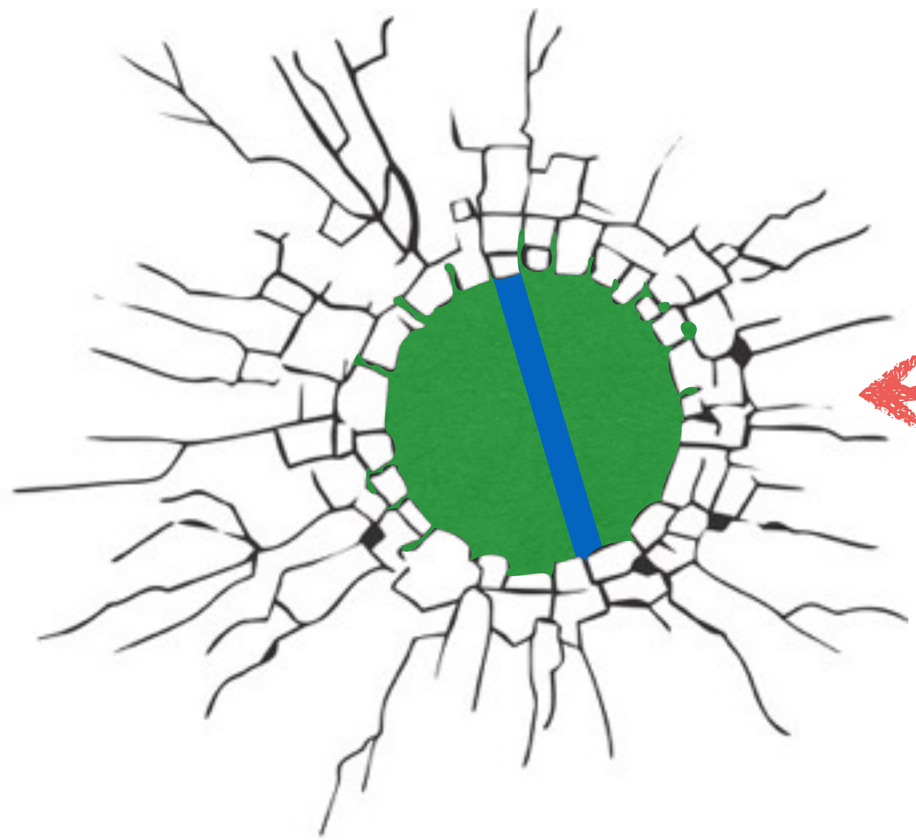
Aperture Problem



small visible
image patch

In which direction is the line moving?

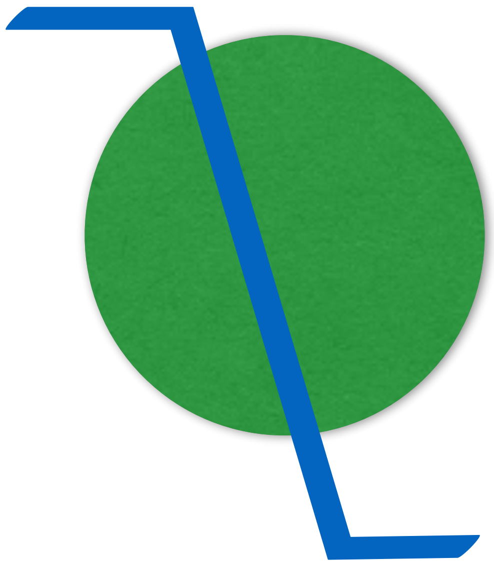
Aperture Problem



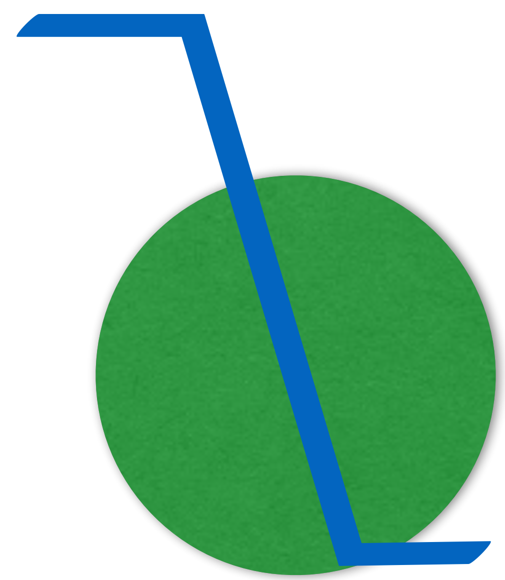
small visible
image patch

In which direction is the line moving?

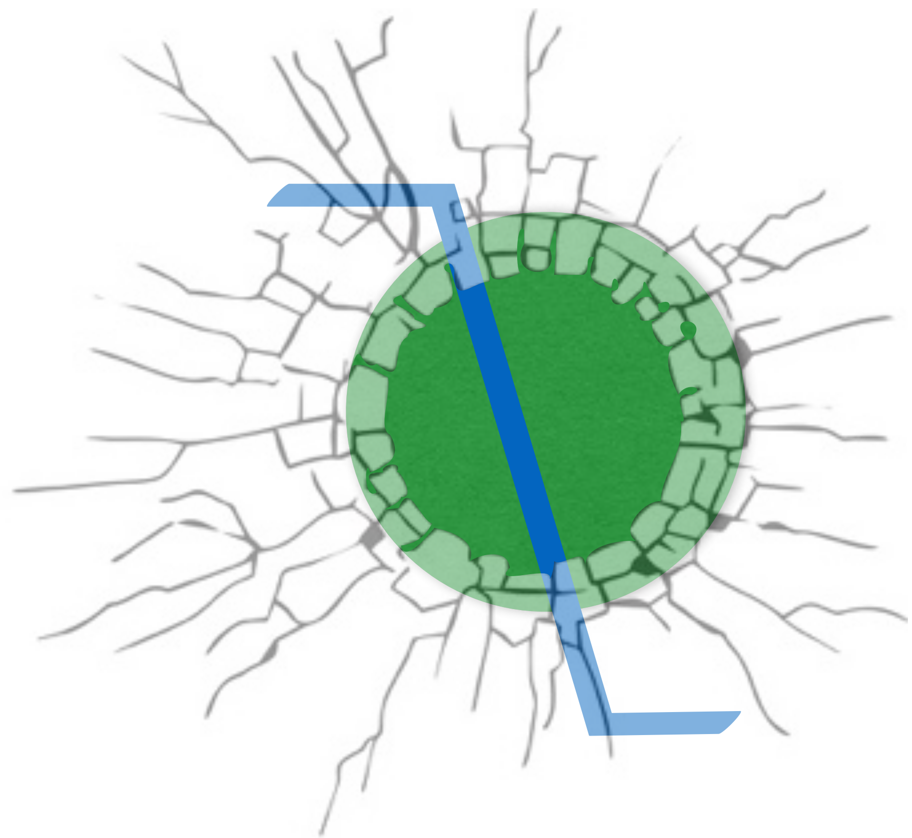
Aperture Problem



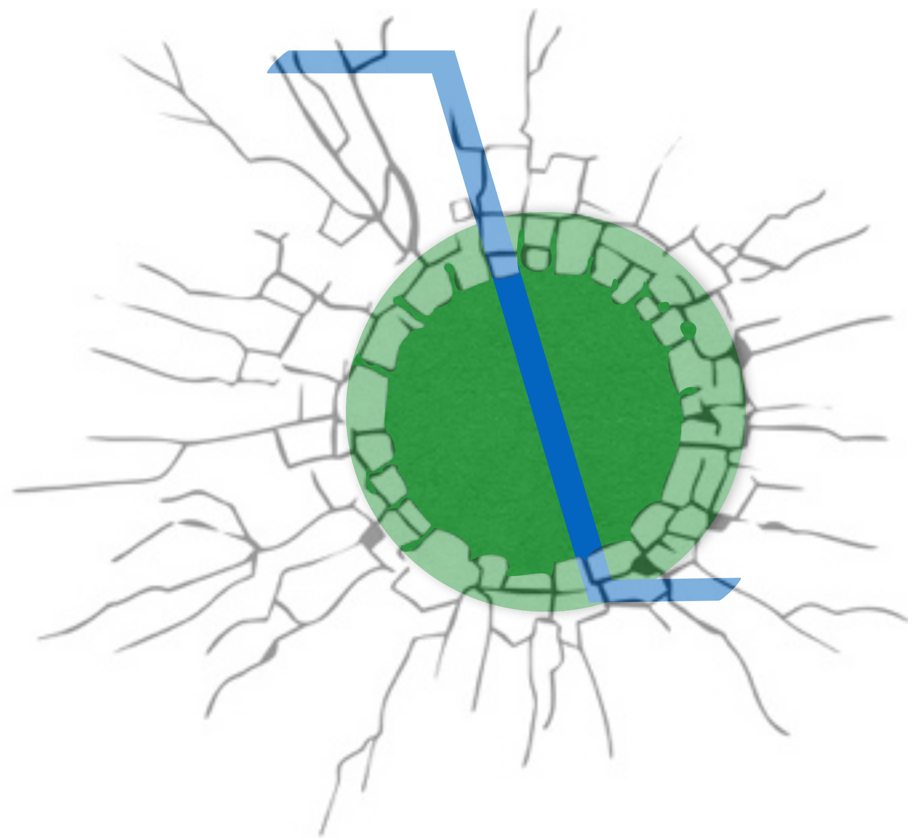
Aperture Problem

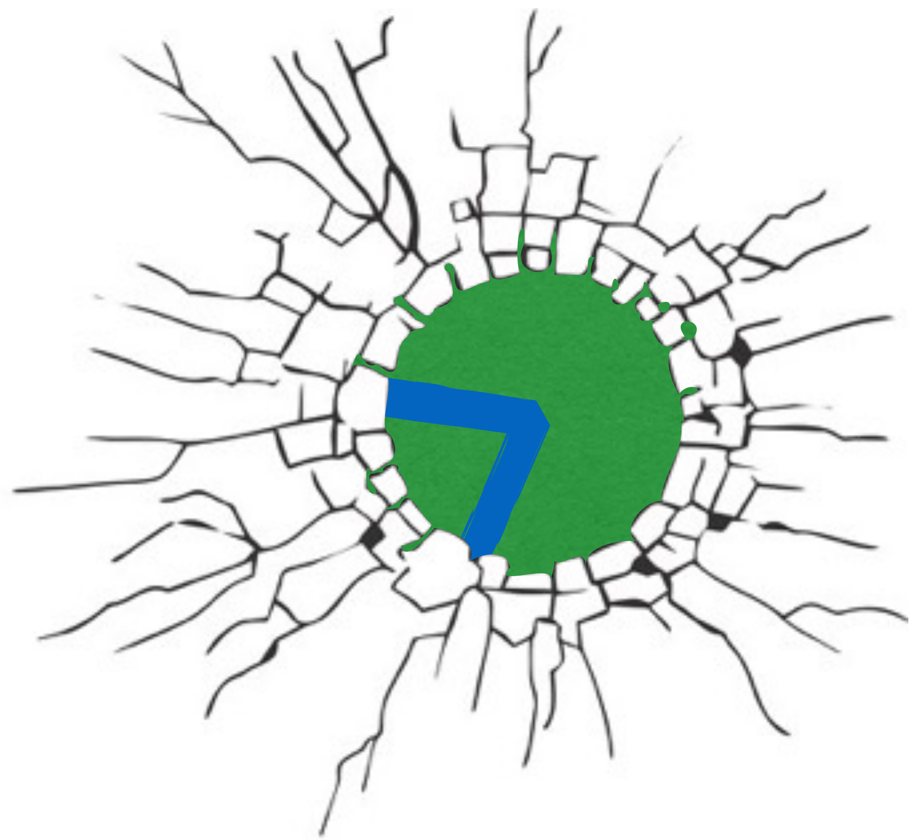


Aperture Problem

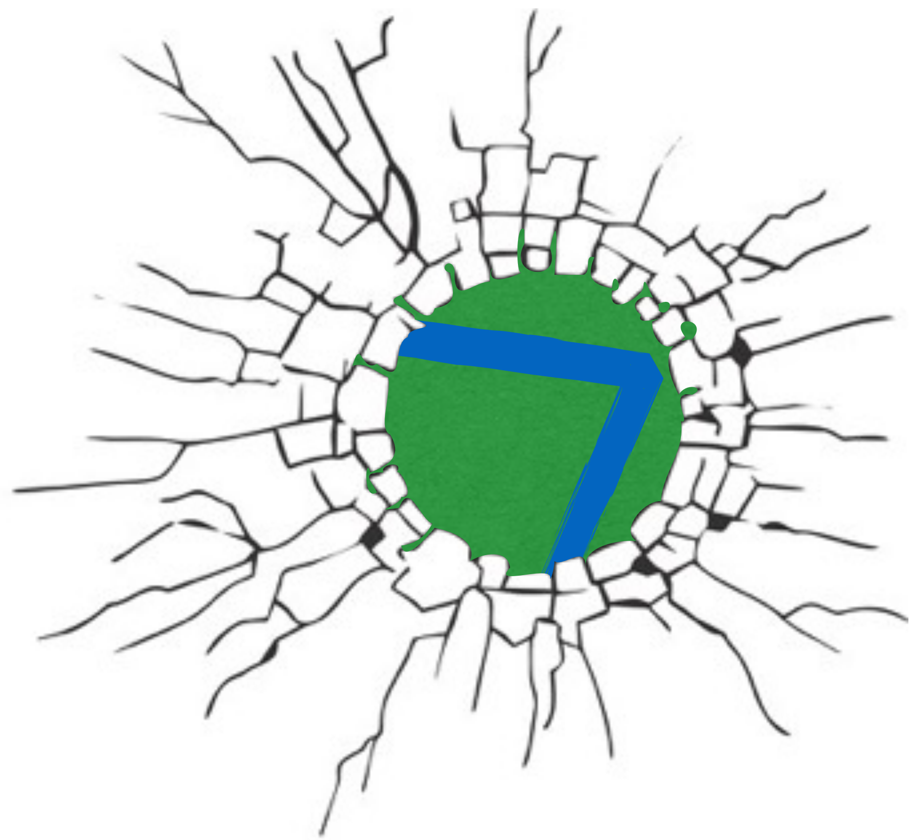


Aperture Problem





Want patches with different gradients to
the avoid aperture problem



Want patches with different gradients to the avoid aperture problem

H(x,y) = y

1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5

optical flow: (1,1)

I(x,y)

-	-	-	-
-	1	1	1
-	2	2	2
-	3	3	3
-	4	4	4

$$I_x u + I_y v + I_t = 0$$

Compute gradients

$$I_x(3,3) = 0$$

$$I_y(3,3) = 1$$

$$I_t(3,3) = I(3,3) - H(3,3) = -1$$

Solution:

$$v = 1$$

We recover the v of the optical flow but not the u.

This is the aperture problem.