Homework Assignment 12: Applied Probabilistic Models

Generating Functions

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1 Exercises

Exercises solved in this work are provided in the book Grinstead and Snell [1].

1.1 Generating Functions for Discrete Densities

For these exercises generating functions for discrete densities are discussed. These exercises are related to branching processes

1.1.1 Exercise 1 page 392

The exercise describes a branching process. Let $h(z)$ be the ordinary generating function for the p_i:

$$h(z) = p_0 + p_1 z + p_2 z^2 + \cdots . \tag{1}$$

By Theorem 10.2 in Grinstead and Snell [1], if the mean number m of offspring produced by a single parent is ≤ 1, then $d = 1$ and the process dies out with probability 1. If $m > 1$ then $d < 1$ and the process dies out with probability d.

(a) For this case $h'(z)|_{z=1} = m = \frac{1}{4} + \frac{1}{2}(1) = \frac{3}{4}$. Since $m < 1$, then $d = 1$.

(b) For this case $h'(z)|_{z=1} = m = \frac{1}{3} + \frac{2}{3}(1) = 1$. Since $m = 1$, then $d = 1$.

(c) For this case $h'(z)|_{z=1} = m = \frac{4}{3}$. Since $m > 1$, then the process dies out with probability d. To find this value, the exercise states that at most two offspring can be produced, therefore the condition $z = h(z)$ yields the equation

$$d = p_0 + p_1 d + p_2 d^2, \tag{2}$$

which is satified by $d = 1$ and $d = p_0/p_2$. Thus, in addition to the root $d = 1$, this second root $d = \frac{1}{2}$, and represents the probability that the process will die out.

(d) For this case $h'(z)|_{z=1} = m = \sum_{j=0}^{\infty} \left(\frac{n}{2^{n+1}}\right)$. In looks like $m \to 1$ as it is added up to ∞ offspring, then $d = 1$.

(e) For this case $h'(z)|_{z=1} = m = \sum_{j=0}^{\infty} \frac{j}{3}\left(\frac{2}{3}\right)^j$. This sumation is greater than 1 then, $m > 1$. To calculate d if notice this geometric series has the form $\frac{1}{3-2z}$, then the condition $z = h(z)$ yields the equation

$$(3-2z)z = 1,$$
$$2z^2 - 3z + 1 = 0.$$

which is satified by $d = 1$ and $d = \frac{1}{2}$.

(f) To estimate d numerically it is used R software [2], with the following code:

```
pj <- function (j){
  return (exp(-2)*2^j/factorial(j)) #Probability pj of j offspring
}

d <- pj(0)

for (i in 0:100){
  d_new <- 0
  for (j in 0:100){  #Sumation of d for j offspring
  d_new <- d_new + pj(j)*(d^j)
  }
  d <- d_new
}
```

branching.R

This experiment estimates a value of $d \approx 0.2032$.

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1.1.2 Exercise 3 page 392

In the chain letter problem the expected number of letters we send is $m = p_1 + 2p_2$ and the expected payoff is equal to $-100 + 50(m + m^{12})$. The expected profit is asked to be calculated.

Case a:

If $p_0 = \frac{1}{2}, p_1 = 0,$ and $p_2 = \frac{1}{2}$; then $m = 0 + 2\left(\frac{1}{2}\right) = 1$. Therefore:

$$\mathbb{E}(\text{Profit}) = -100 + 50(1 + 1^{12}) = 0.$$

Case b:

If $p_0 = \frac{1}{6}, p_1 = \frac{1}{2},$ and $p_2 = \frac{1}{3}$; then $m = \frac{1}{2} + 2\left(\frac{1}{3}\right) = \frac{7}{6}$. Therefore:

$$\mathbb{E}(\text{Profit}) = -100 + 50\left[\frac{7}{6} + \left(\frac{7}{6}\right)^{12}\right] = 276.26.$$

If $p_0 > \frac{1}{2}$ we canot expect to make a profit because let say $p_0 = 0.51$, then p_1 and p_2 are force to take other two values, assume $p_1 = 0.16$, and $p_2 = 0.33$. Therefore:

$$\mathbb{E}(\text{Profit}) = -100 + 50\left[0.49 + (0.49)^{12}\right] = -75.49.$$

$\blacksquare$

1.2 Generating Functions for Continuous Densities

For the continuous case, the moment generating function $g(t)$ for X is defined as:

$$g(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx. \tag{3}$$

1.2.1 Exercise 1 page 401

For this exercise let X be a continuous random variable with values in $[0, 2]$ and for each case there is a given density function f_X.

Case a:

$$\begin{aligned} g(t) &= \int_0^2 \frac{1}{2} e^{tx} dx \\ &= \frac{1}{2} \left[\frac{e^{tx}}{t} \right]_0^2 = \frac{1}{2} \left(\frac{e^{2t-1}}{t} \right) = \frac{e^{2t} - 1}{2t}. \end{aligned}$$

Case b:

$$\begin{aligned} g(t) &= \int_0^2 \frac{1}{2} x e^{tx} dx \\ &= \frac{1}{2} \left[\frac{(tx-1)e^{tx}}{t^2} \right]_0^2 = \frac{1}{2} \left[\frac{(2t-1)e^{2t}+1}{t^2} \right] = \frac{(2t-1)e^{2t}+1}{2t^2}. \end{aligned}$$

Case c:

$$\begin{aligned} g(t) &= \int_0^2 \left(1 - \frac{x}{2}\right) e^{tx} dx \\ &= \left[\frac{e}{2t^2} - \frac{(x-2)e^{tx}}{2t} \right]_0^2 = \left[-\frac{(t(x-2)-1)e^{tx}}{2t^2} \right]_0^2 = \frac{e^{2t}}{2t^2} - \frac{2t+1}{2t^2} = \frac{e^{2t}-2t-1}{2t^2}. \end{aligned}$$

Case d:

$$\begin{aligned} g(t) &= \int_0^2 |1-x|\, e^{tx} dx \\ &= \left[\frac{(x-1)(tx-t-1)e^{tx}}{t^2|x-1|} \right]_0^2 = \left[\frac{(t-1)e^{2t}}{t^2} + \frac{2e^t}{t^2} - \frac{t+1}{t^2} \right] = \frac{(t-1)e^{2t}+2e^t-t-1}{t^2}. \end{aligned}$$

Case e:

$$\begin{aligned} g(t) &= \int_0^2 \frac{3}{8} x^2 e^{tx} dx \\ &= \left[\frac{3(t^2x^2-2tx+2)e^{tx}}{8t^3} \right]_0^2 = \frac{(6t^2-6t+3)e^{2t}-3}{4t^3}. \end{aligned}$$

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1.2.2 Exercise 6 page 402

According to Grinstead and Snell [1], $k_X(\tau)$, called the characteristicfunction of X is the Fourier transform of f_X, and has an inverse given by

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau x} k_X(\tau) d\tau. \tag{4}$$

Therefore:

$$
\begin{aligned}
f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\tau x} e^{-|\tau|} d\tau, \\
&= \frac{\frac{ix}{x^2+1} + \frac{1}{x^2+1}}{2\pi} + \frac{\frac{1}{x^2+1} - \frac{ix}{x^2+1}}{2\pi} \\
&= \frac{1}{\pi(x^2+1)}
\end{aligned}
$$

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1.2.3 Exercise 10 page 403

(a)

$$
\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} x \frac{e^{-|x|}}{2} dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x}{|x|} e^{-|x|}|x| dx \\
&= -\frac{e^{-|x|}|x|}{2} - \frac{e^{-|x|}}{2} \\
&= \left[\frac{e^{-|x|}(-|x|-1)}{2}\right]_{-\infty}^{\infty} \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{V}(X) &= \int_{-\infty}^{\infty} x^2 \frac{e^{-|x|}}{2} dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|} dx \\
&= \frac{1}{2}\left(\frac{x}{|x|}\right)\int_{-\infty}^{\infty} x^2 e^{-|x|}\frac{|x|}{x} dx \\
&= \frac{1}{2}\left(\frac{x}{|x|}\right)\int_{-\infty}^{\infty} x e^{-|x|}|x| \\
&= \frac{1}{2}\left[\frac{x(-2e^{-|x|}|x| - x^2 e^{-|x|} - 2e^{-|x|})}{|x|}\right] \\
&= -\left[\frac{xe^{-|x|}|x|(2|x| - x^2 + 2)}{2|x|}\right]_{-\infty}^{\infty} \\
&= 2.
\end{aligned}
$$

(b) Let X_1 be a trial process. The moment generating function would be

$$\begin{aligned}
g(t) &= \mathbb{E}(e^{tx}) \\
&= \int_{-\infty}^{\infty} e^{tx}\left[\frac{e^{-|x|}}{2}\right] dx \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} e^{xt+x} dx + \int_{0}^{\infty} e^{xt-x} dx\right] \\
&= \frac{1}{2}\left[\left[\frac{e^{x(t+1)}}{t+1}\right]_{-\infty}^{0} - \left[\frac{e^{x(1-t)}}{1-t}\right]_{0}^{\infty}\right] \\
&= \frac{1}{2}\left[\frac{1}{t+1} + \frac{1}{1-t}\right] \\
&= \frac{1}{2}\left[\frac{2}{1-t^2}\right] \\
&= \frac{1}{1-t^2}.
\end{aligned}$$

If S_n is $X_1 + X_2 + ... + X_n$, then the moment generating function is given by

$$\begin{aligned}
[g(t)]^n &= \left(\frac{1}{1-t^2}\right)^n \\
&= \frac{1}{(1-t^2)^n}.
\end{aligned}$$

Then, if $S_n^* = \dfrac{(S_n - n\mu)}{\sqrt{n\sigma^2}}$, the moment generating function is given by

$$\begin{aligned}
\left[g\left(\frac{t}{\sqrt{n}}\right)\right]^n &= \left(\frac{1}{1-t^2}\right)^n \\
&= \left[\frac{1}{\left[1-\left(\frac{t}{\sqrt{n}}\right)^2\right]}\right]^n \\
&= \frac{1}{\left[1-\left(\frac{t}{\sqrt{n}}\right)^2\right]^n}
\end{aligned}$$

(c) According to the obtained expression of S_n^* as $n \to \infty$ then the expression may reduce to $\dfrac{1}{1^n}$. When the limit is calculated it is equal to 1.

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References

[1] Charles Miller Grinstead and James Laurie Snell. *Introduction to probability*. American Mathematical Soc., 2012.

[2] The R Foundation. The R Project for Statistical Computing. `https://www.r-project.org/`, 2020.