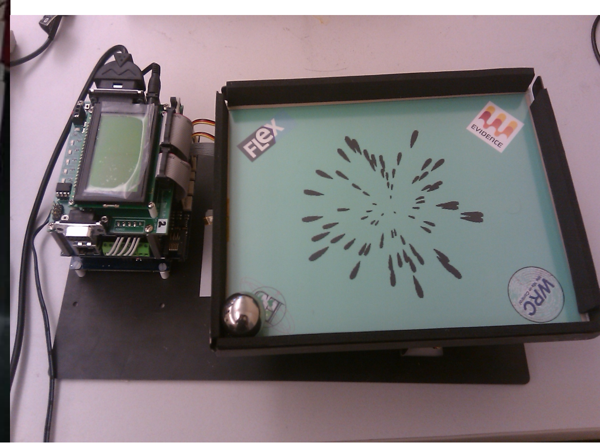
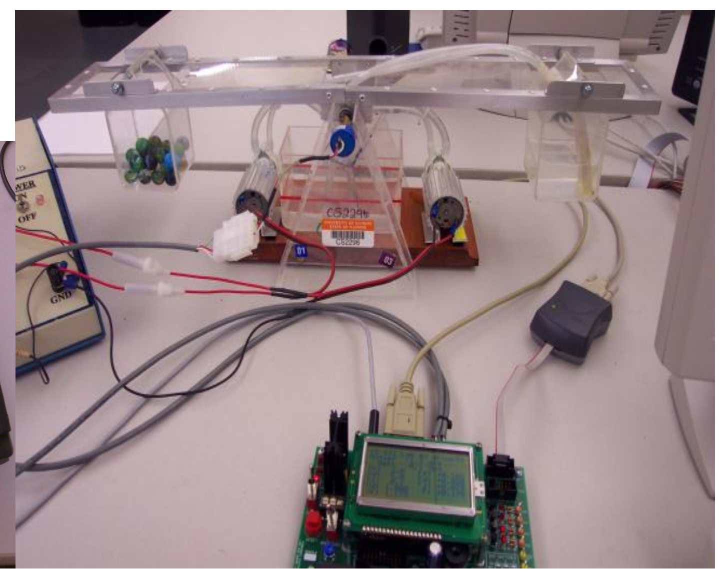
Overview

- **Today: basics of control theory: stability, proportional control and transfer function.**

- **A basic tutorial** on control theory is available on the web: see https://ctms.engin.umich.edu/CTMS/index.php?example=Introduction&section=ControlPID on PID controllers.

- Additional tutorials (more advanced and out of the scope of this course) on control theory and examples are available at https://ctms.engin.umich.edu/CTMS/index.php?aux=Home





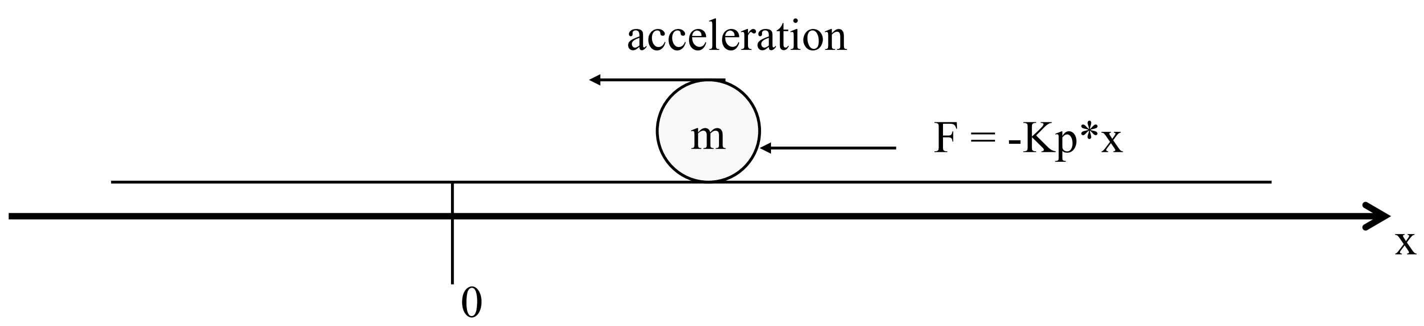
Automatic Control

There are many exciting automatic control applications. The challenging ones are those which are open loop (i.e. without control) unstable.

You will learn the key concepts in 2 lectures and will be able to control the "amazing ball" system, at the lab by yourselves.

A Simple System with Proportional Control

- The control is proportional to the size of the error
- Consider a marble on a flat and perfectly leveled table.
  - Any point can be an equilibrium point (just pick one)
  - Its motion can be described by Newton's law $F = ma$, or $x'' = F/m$
  - suppose that we want to keep the marble at $x = 0$, by applying proportional control: $F = -k_p x$.
- The feedback is negative since if the marble *position error* is negative, it pushes with a positive force and vice versa.
- K_p is a positive number known as proportional control constant.



- Quiz: Can you describe the motion of this marble under proportional control?

Concept of Stability

- Error is the difference between the set point (where you want the state to be: x=0 in the marble example!) and the actual state (current x position of the marble!).

- Set point can follow a prescribed trajectory, e.g., an orbit. Or it can be a constant, e.g., keep room temperate at 75 F.

- The objective of controlling the marble is to put it at rest at the origin. I.e. $x = 0$, and $x' = 0$, from any initial condition. Or for that matter, after the marble was disturbed by some force.
  – This intuition is formalized as the notion of stability. Or more precisely, asymptotic stability, I.e. the ***<u>error</u>*** will converge to zero over time.

  – The opposite of stability is instability, meaning that the ***<u>error</u>*** will grow larger and larger without bound. That is, the marble will leave the origin for good.

  – In between, there is marginal stability, the ***<u>error</u>*** will stay within some bound.

Stability

- Quiz: according to your intuition, is the marble under proportional control
  - **Stable**
  - **Unstable**
  - **Marginally stable**?

Stability and Eigenvalues

- What is the mathematical tool that would allow us to reason about the stability of a controlled motion?

- Most systems are modeled by linear systems, or by a collection of linear systems via piecewise linearization. For example, a fighter jet F16, is controlled by six linear models for six different flight conditions.

- From the calculus courses, we know that the solution of Linear Differential equations. e.g., 3*y'' + 4*y' + 5*y = 0, is sum of exponentials:
  - $y(t) = c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t) + \ldots.$
    - where the exponents λ_l, λ_2 ... are known as eigenvalues of the system.
    - eigenvalues can be complex, e.g. $\exp(RE(\lambda)\ t)*\exp(IM(\lambda)\ jt)$

Stability Analysis

- What is the relationship between the magnitude of eigenvalues and the notion of (asymptotic) stability, marginal stability and instability?

- Study of Stability (does the error grow, shrink to zero or is it bounded?)
  - **Stability**: all eigenvalues' real parts are negative
    - Stable with oscillation that dies down: imaginary part non zero
    - Stable with no oscillation: imaginary part = 0
  - **Instability**: at least one eigenvalue's real part is greater than zero.
  - **Marginal stability**: all the real parts not greater than zero but some equal to zero.

- For example, suppose that the eigenvalue is s = 5 + 6j
- Is the system unstable? The error y(t):

y(t) =exp((5+6j)t)
=exp(5 t)* exp(6 j t)

- exp(5 t) grows larger and larger
- exp(6j t) = cos(6 t) + j sin(6 t) oscillates with a constant magnitude

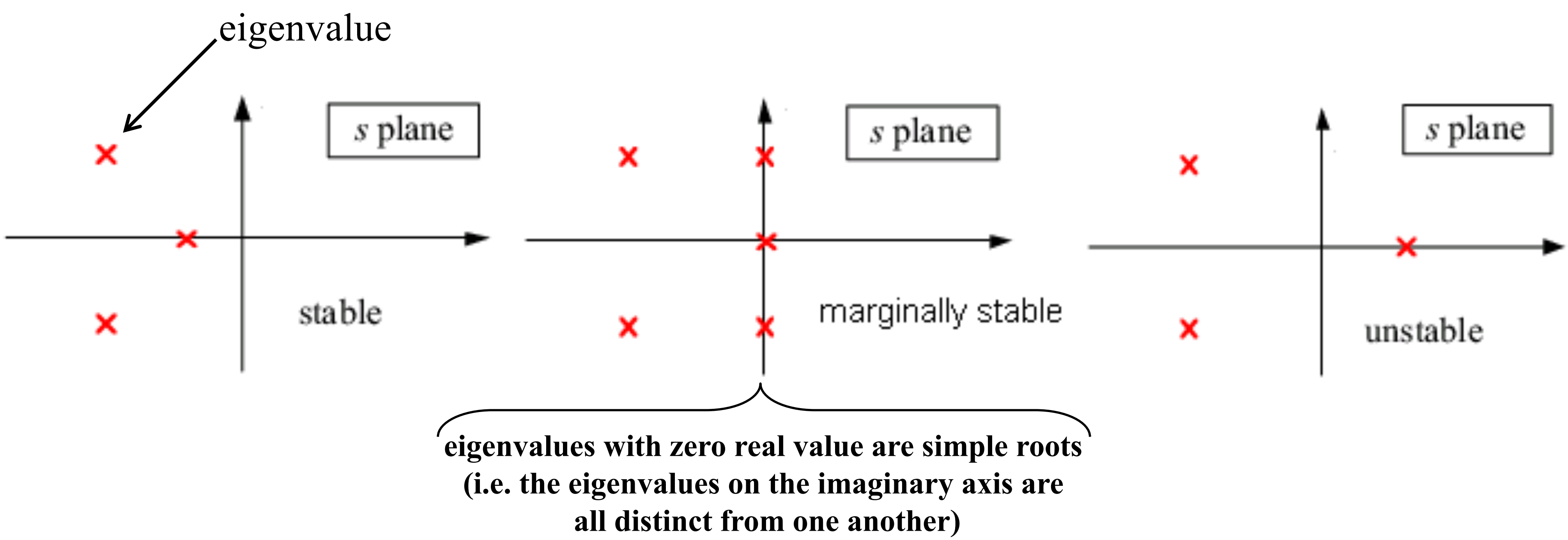
- The combined effect is that the oscillation grows larger and larger ➔ **system is unstable!!!**

Solving Linear Differential Equations Using Laplace Transform

- Linear Differential Equations can be solved by using Laplace transform:

- $3*y'' + 4*y' + 5*y = 0$ (**replace nth derivative with s^{n}**)
- $3s^2 + 4s + 5 = 0$

- Solve for s and you will get the eigenvalues to study the stability of the given system.



Basic Laplace transform rules

- **Basic Laplace transform**: it allows us to easily solve linear differential equations that describe many physical systems and it allows us to reason about a physical system stability!

$$\frac{d}{dt}x(t) = x^{'}(t) \leftrightarrow s \cdot X(s) - \cancel{x(0)}$$

$$\frac{d^2}{dt^2}x(t) = x^{''}(t) \leftrightarrow s^2 \cdot X(s) - s \cdot \cancel{x(0)} - \cancel{x'(0)}$$

$$\int_0^t x(\tau)d\tau \leftrightarrow \frac{X(s)}{s}$$

Notice that x(0) and x’ (0) are the initial condition. In the study of control system behaviors, we are interested in the steady state responses which is independent of initial conditions. Thus, we normally set x(0) = 0.

Basic Laplace transform rules

- **Basic Laplace transform**: for our practical purpose, we can just ignore initial conditions and use the following rules

$$\frac{d}{dt}x(t) = x^{'}(t) \leftrightarrow s \cdot X(s)$$

$$\frac{d^2}{dt^2}x(t) = x^{''}(t) \leftrightarrow s^2 \cdot X(s)$$

$$\int_0^t x(\tau)d\tau \leftrightarrow \frac{X(s)}{s}$$

Solving Linear Differential Equations Using Laplace Transform

- Now, let us analyze the stability of our marble under proportional control.

- From Newton's law, we have $F = m x''$
- From proportional control we have $F = -k_p * x$

- Hence, we have $mx'' = -k_p * x$, with $m > 0$ and $k_p > 0$,

- $- k_p x = mx''$ ➔ $x'' = - k_p x/m$ ➔ $s^2 = - k_p /m$

- $s = \mathrm{SQRT}(- k_p /m) = +\mathrm{SQRT}(k_p /m)j = -\mathrm{SQRT}(k_p /m)j$

- ➔ **two complex conjugates solutions with real part equal to zero**

- **<u>The marble under proportional control is therefore marginally stable!</u>**

Marble Under P control and its transfer function representation

- $F = m\,y''$, //where m is the mass of the marble
- $F = -\,k_p * y$ // proportional control

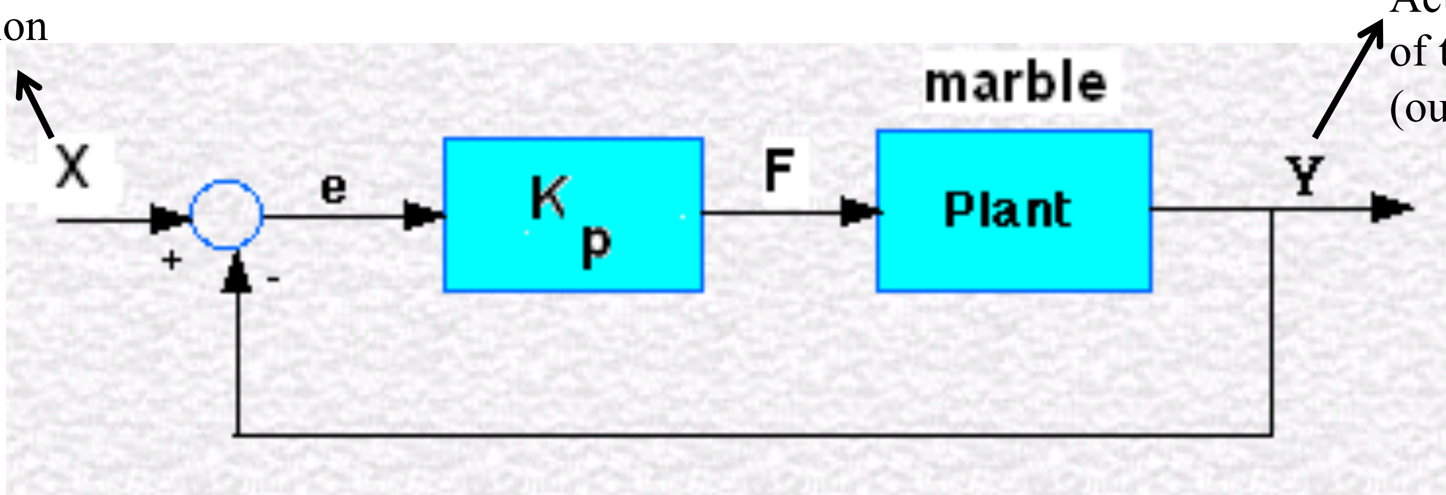
- $m*y'' + k_p * y = 0$ //putting it together
- $m*s^2 + k_p = 0$ //this describes the motion of the system initially at rest in s domain
  //the solutions of s are the exponents of the solution of the diff. equation

$x(t) = 0$

input: position set point x(t) → **Marble under P control** → output: current system position y(t)

Desired position of the marble (input)

Actual position of the marble (output)



Marble Under P control and its transfer function representation

- F = m y’ ’ , //where m is the mass of the marble
- F = -kp*y // proportional control

- m*y’ ’ + kp*y = 0 //putting it together
- m*s^2 + kp = 0 //this describes the motion of the system initially at rest in s domain
  //the solutions of s are the exponents of the solution of the diff. equation

input: position set point x(t) → Marble under P control → output: current system position y(t)

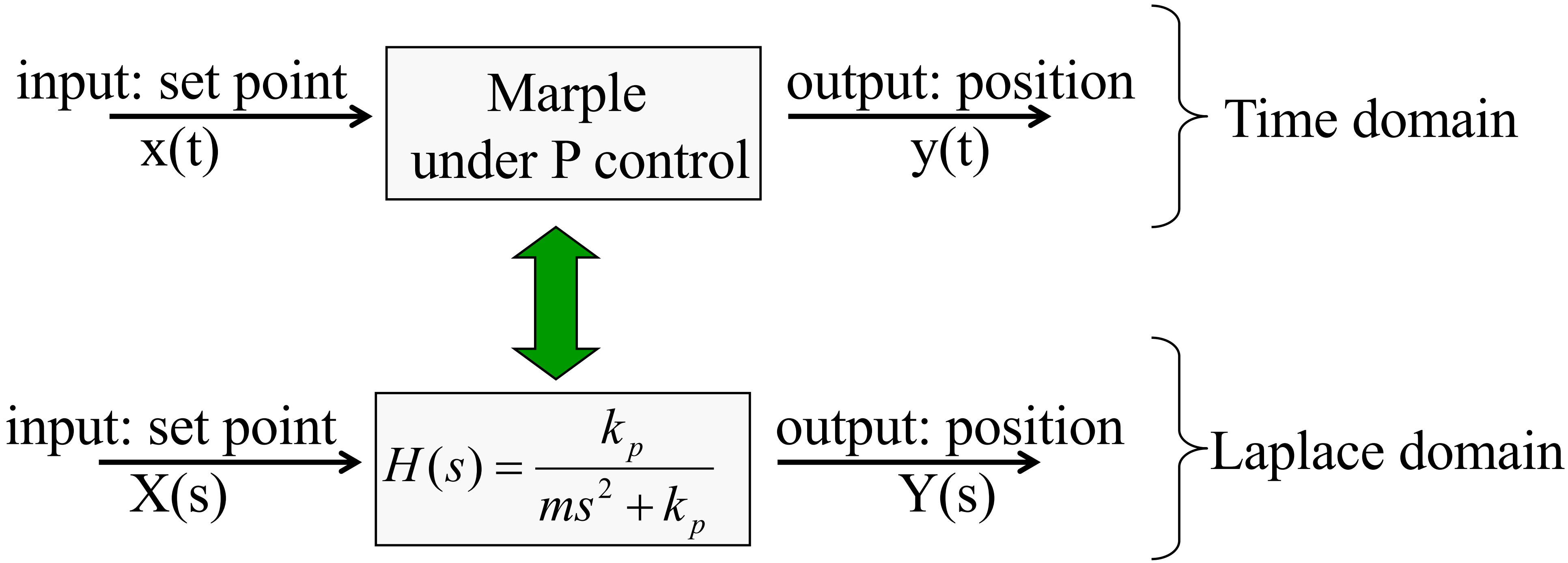
$$my''(t) = k_p(x(t) - y(t)) \quad \Rightarrow \quad ms^2Y(s) = k_p(X(s) - Y(s))$$

Transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{k_p}{ms^2 + k_p}$$

roots of denominator are the eigenvalues of the system

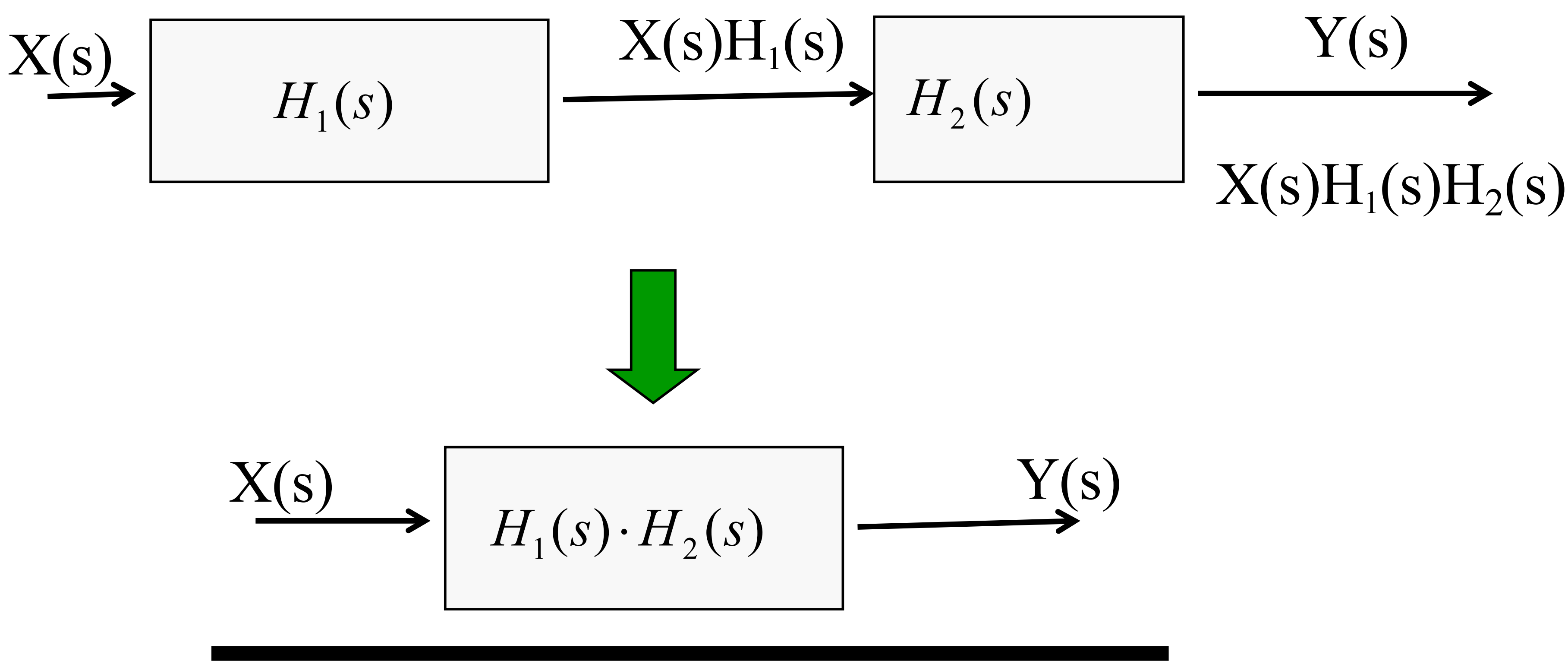
Marble Under P control and its Transfer Function representation



$$Y(s)=H(s)X(s)=\frac{k_p}{ms^2+k_p}X(s)$$

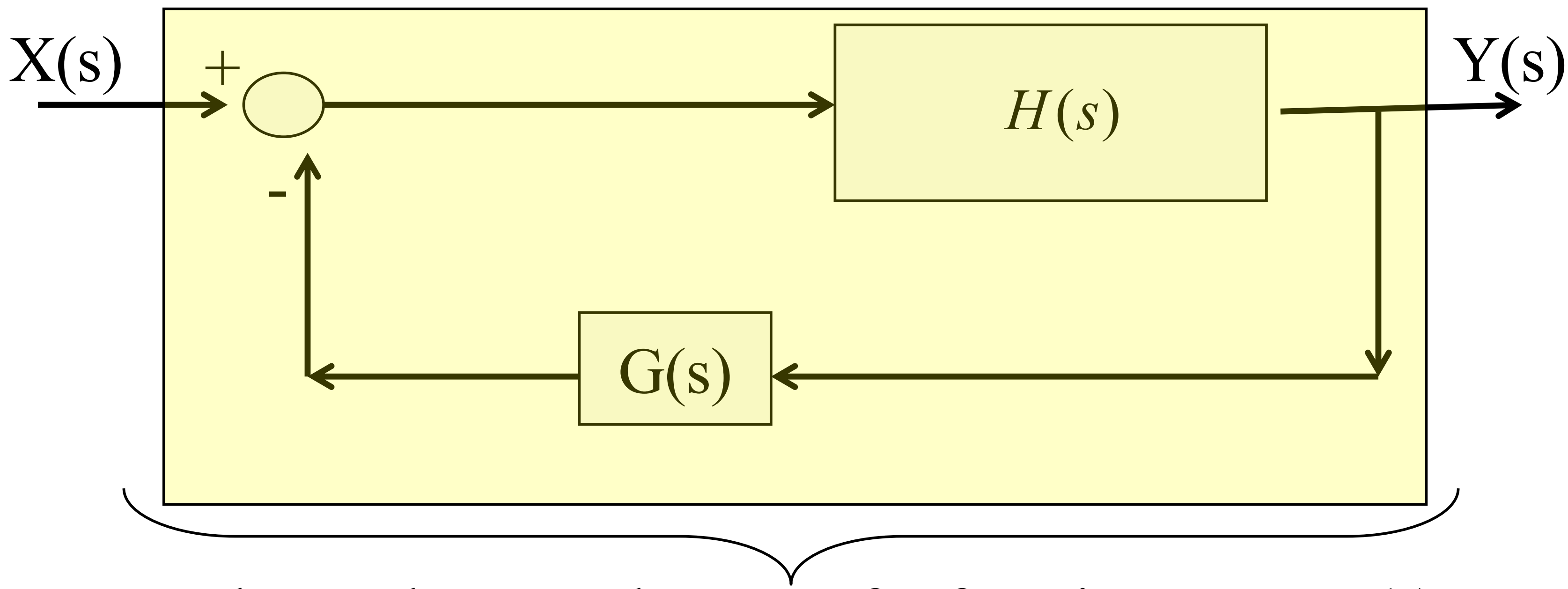
- Given the <u>transfer function</u> H(s) of a system and the Laplace transform X(s) of the input, the Laplace transform of the output is Y(s)=H(s)X(s)
- Roots of denominator of H(s) carry information about STABILITY of the system (see slide on stability analysis!)

Cascade of linear systems



Transfer functions allow us to “plug and play” with different components modularly, instead of creating large systems of differential equations each time.

Transfer function of a feedback system



The total system has transfer function $H_{feedback}(s)$

•What is the value of $H_{feedback}(s)$?

Appendix: Input-output Stability

- To avoid confusion with other stability definitions you may have learned in other classes, we study *bounded input/bounded output stability* (**BIBO stability**).

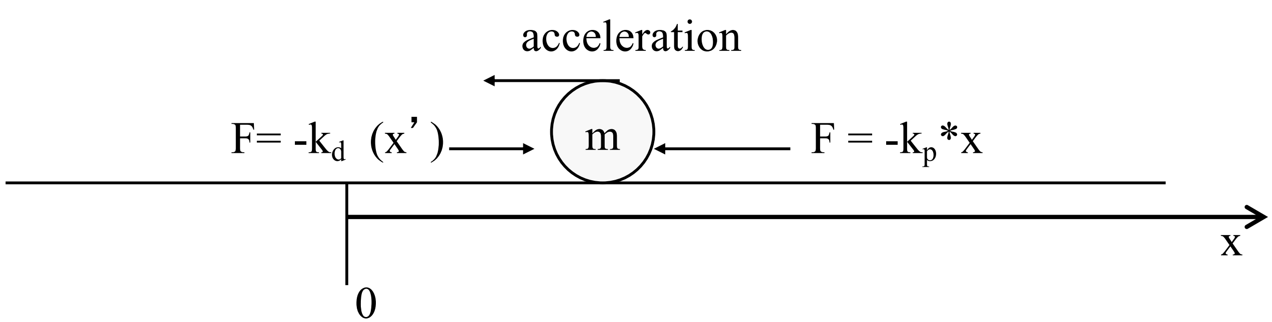
- A system is BIBO stable if for any time t_0, and any bounded input $u(t)$, $t \geq t_0$, the output is bounded on $t_0 \leq t < \infty$ assuming initial system state $x(t_0) = 0$.

- BIBO stability is a weaker form of system stability when compared to asymptotic stability. In fact given a Linear Time Invariant (LTI) system, asymptotic stability implies BIBO stability but the converse is false.

- BIBO stability can be checked by looking at roots of denominator of the system transfer function after performing pole-zero cancellation.

Intuition of Proportional + Derivative Control

- When the position error, x, is positive, proportional control force is negative. It pushes the marble back to the origin (setpoint) at 0.
- When the marble is moving towards setpoint, the velocity is negative. So the force due to derivative control $-k_d*x'$ is positive. This counters the proportional force and slows down the marble's motion towards the origin.
- In summary, proportional control is like your car's gas pedal that moves the car towards the setpoint (where you want it to be) while derivation control is like your car's brake.



Marble Under P + D control

- $F = m x''$, //where m is the mass of the marble
- $F = -k_p * x$ // proportional control
- $F = - k_p * x - k_d * x'$ //proportional + derivative control

- $m*x'' + k_d * x' + k_p * x = 0$ //putting it together
- $m*s^2 + k_d * s + k_p = 0$
- $s = (- k_d \pm \mathrm{SQRT}(k_d^2 - 4m* k_p))/2m$

- $m > 0$, $k_p > 0$ and $k_d > 0$

- Is this system stable (try some values for $k_p > 0$ and $k_d > 0$)?
- For example, suppose $m=1$, $k_p = 2$ and $k_d = 1$
- ➔ $s_{1,2} = (-1 \pm \mathrm{SQRT}(1-8))/2= -1/2 \pm 1.32\, j$
- The system is now stable!

How to Tune a Simple PD Controller

- Experimental tuning procedure
  - first set derivative gain to zero.
  - Slowly increase the proportional gain until the device oscillates around the set point with a speed that is satisfactory. At this point,
    - the real part of the eigenvalue is _________
    - the imaginary part of the eigenvalue determines the __________ of oscillation
  - Slowly increase the derivative gain until the device settles down at the setpoint. At this point, the real part of the eigenvalue is ____________

- Fine tuning
  - If the motion movement towards setpoint is too slow, we can __________ proportional gain or ___________ derivative gain

  - If the motion overshoots the setpoint too much, we can __________ the derivative gains or __________ proportional gain

- Avoid to use too large a proportional gain and too large a derivative gain at the same time. This will saturate the actuator. (Like slam gas and brake).