SF2822 Applied Nonlinear Optimization

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Project assignment 1C3

Authors:
Emil Backman
960304-3952
Oscar Bergqvist
950214-2855
Felix Mir
960710-6631

e-mail:
emilback@kth.se
obergq@kth.se
fmir@kth.se

1 Problem description

1.1 Basic Model description

This section will go through and describe the "blending problem" for petrochemicals that we received from the company Oljeblandaren. The problem is to find the optimal planning strategy, where we need to decide the amount of Crude oil 1 and 2 to be bought. For the Crude oil 2 we receive the exact amount ordered for the given price. This is not always the case for the crude oil 1, where the incoming supply is varying weekly and the pre-planned amount of crude oil 1 needs to be reached by buying extra, more expensive, crude oil if there is a deficit.

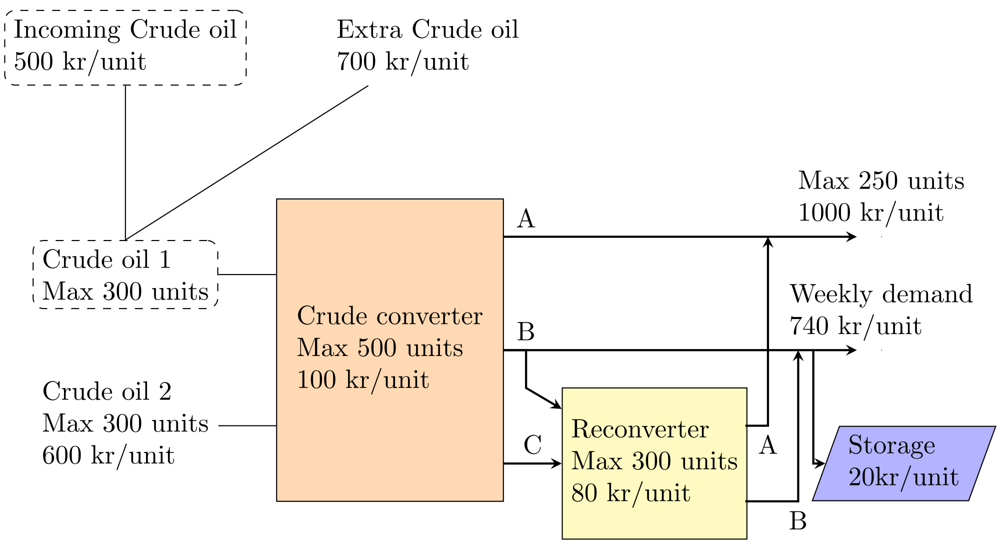

Figure 1: Blending map

The bought crude oil will go through a converter (Crude converter) and three different products will be obtained, which is here called A,B and C. The exchange (in percent) for the two crude oils is given in the following table:

	Prod. A	Prod. B	Prod. C
Crude oil 1	50	30	20
Crude oil 2	70	20	10

Table 1: Crude converter, exchange rate

Product C is always refined directly in a second step (the reconverter) whereas product A may be sold directly. Product B may either be sold directly or alternatively refined in a second step (the reconverter). The exchange (in percent) for products B and C in the reconverter is given by the following table:

	Prod. A	Prod. B
Prod. B	90	10
Prod. C	40	60

Table 2: Reconverter, exchange rate

The company wants to plan the production for the following three weeks, with the following conditions.

Each week one may buy at most 300 units of crude oil 1 to the price 500 kr/unit and at most 300 units of crude oil 2 to the price 600 kr/unit. The quantity of Crude oil 2 which will be available through the regular sources is expected to remain fixed during the time period in question. The situation for Crude oil 1 is less certain. After a more careful investigation, the analysts at Oljeblandarna believe that the supply of Crude oil 1 for each week is approximately normally distributed with mean 300 units and standard deviation 20 units. Because of technical reasons, the decision on purchased quality and production plan for all three weeks has to be determined before the time period in question. In case of possible shortage of Crude oil 1, Oljeblandarna needs to purchase additional quantities of Crude oil 1 to the price 700 kr/unit.

Furthermore, on may each week sell at most 250 units of product A to the price 1000 kr/unit. Product B may each week be wold to the price 740 kr/unit. The first week, only 30 units may be sold, second week and the third week, 130 units may be sold. The running cost for step 1 (the crude converter) is 100 kr/unit and the capacity is 500 units (for each of the weeks). The corresponding figures for step 2 (the reconverter) is 80 kr/unit and 300 units respectively. Produce A must be sold the same week that is is made, but product B may be stored during the weeks for which the planning is made. The storage cost for product B from one week to the next is 20 kr per unit. Product B that has been stored may be run through the reconverter.

The question is what production plan the company should hold during the three-week period in question to maximize its profit (sales profit minus purchase cost and running cost).

1.2 Advanced model description

The advanced model has one new feature compared to the basic model, the rest is the same. The linear relations that have been used above are approximations. To improve accuracy of the model, the model should include the storage non-linear dependency. The dependency is described as, if x units of product B are stored one week, one obtains $x - 0.01x^2$ units of product B the following week. This is due to the fact that some of the oil becomes waste during the storage (for $x > 100$, the above model is bad, but such large quantities are not expected in the storage).

2 Mathematical formulation

In this section the constraints and cost function of the basic and advanced problem are derived and presented. The summarized programs for the basic and advanced models are presented in Appendix A. For increased readability and ease of notation every parameter with sub-index i will concern $i \in \{1, 2, 3\}$ representing the 3 week that planning is due for.

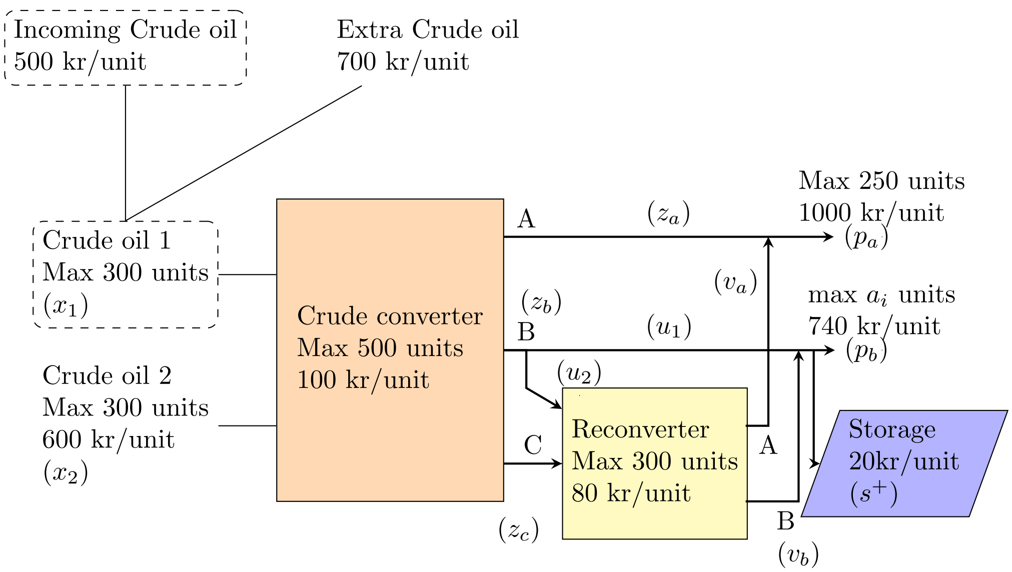

Figure 2: production flow, variable description

2.1 Basic model constraints

Let the variables x_{1i} and x_{2i} represent the number of units of crude oil 1 and crude oil 2 that enters the crude converter week i. The supply constraint of crude oil 2 is

$$x_{2i} \leq 300 \tag{1}$$

and the maximum capacity of the crude converter is

$$x_{1i} + x_{2i} \leq 500. \tag{2}$$

The variables z_{ai}, z_{bi}, and z_{ci} represents the yield of product A, B and C respectively from the crude converter for week i. The exchange from crude oil in the converter is given and thus the z variables are defined from the constraints

$$0.5x_{1i} + 0.7x_{2i} = z_{ai} \tag{3}$$

$$s^+_{i-1} + 0.3x_{1i} + 0.2x_{2i} = z_{bi} \tag{4}$$

$$0.2x_{1i} + 0.1x_{2i} = z_{ci} \tag{5}$$

Here the variable s^+ corresponds to units of stored product B available from the previous week. After the crude converter, product B is divided to u_{1i} and u_{2i}, where the first is the number of units to sell directly and the later the number of units to use in the reconverter. Thus

$$u_{1i} + u_{2i} = z_{bi} \tag{6}$$

The capacity of the reconverter is given by

$$z_{ci} + u_{2i} \leq 300 \tag{7}$$

and the yield of product A and B, represented by the variables v_{ai} and v_{bi} is given as

$$0.9u_{2i} + 0.4z_{ci} = v_{ai} \tag{8}$$

$$0.1u_{2i} + 0.6z_{ci} = v_{bi}. \tag{9}$$

The variables p_{ai} and p_{bi} corresponds to the number of units of product A and B that are sold. Since we can not sell more than we produce

$$z_{ai} + v_{ai} \geq p_{ai} \tag{10}$$

$$u_{1i} + v_{bi} \geq p_{bi}. \tag{11}$$

The sales are limited by the demand in the market which is specified by

$$p_{ai} \leq 250 \tag{12}$$

$$p_{bi} \leq a_i \tag{13}$$

where $a = (30, 130, 130)$. The demand of product B is 30 units the first week and 130 units the other two weeks. Moreover, there is an option to store product B until the following weeks, this stored excess is given by s^+_i. We also assume that there is no stored product before week 1. Thus we have

$$u_{1i} + v_{bi} - p_{bi} = s^+_i \tag{14}$$

$$s^+_0 = 0. \tag{15}$$

2.2 Advanced model constraints

In the advanced model it is assumed that if s units is stored one week, $s - 0.01s^2$ units of useful product B is obtained the next week. Thus the constraint in equation (4) needs to be changed to

$$s^+_{i-1} - 0.01(s^+_{i-1})^2 + 0.3x_{1i} + 0.2x_{2i} = z_{bi}. \tag{16}$$

No other changes from the basic model are made in the advanced model.

2.3 Cost function

The cost function of the basic and the advanced model is the same and can be decomposed into a stochastic part C_1 and a deterministic part C_2 as

$$C(Y, x_1, x_2, z_c, u_2, p_a, p_b, s^+) = C_1(Y, x_1) + C_2(x_1, x_2, z_c, u_2, p_a, p_b, s^+). \quad (17)$$

The stochastic part depends on the supply Y of cheap crude oil 1 each week, which is a normally distributed random variable with mean 300 units and standard deviation 20 units. If $x_{1i} > Y_i$ additional crude oil 1 has to be bought from another provider at a higher cost. Thus

$$C_1(Y, x_1) = \sum_{i=1}^{3} C_{1i}(Y_i, x_{1i}) \quad (18)$$

$$C_{1i}(Y_i, x_{1i}) = \begin{cases} 500x_{1i} & x_{1i} \leq Y_i \\ 500Y_i + 700(x_{1i} - Y_i) & x_{1i} > Y_i. \end{cases} \quad (19)$$

The deterministic part of the cost is given by the cost of buying crude oil 2, the running cost of the crude converter, the storage cost, the cost of running the reconverter and finally the income from sales of product A and B. Thus C_2 is given as

$$C_2 = \sum_{i=1}^{3} 600x_{2i} + 100(x_{1i} + x_{2i}) + 80(z_{ci} + u_{2i}) + 20s_i^+ - 1000p_{ai} - 740p_{bi}. \quad (20)$$

The expected cost is given by

$$E[C] = E[C_1 + C_2] = E[C_1] + C_2 = \sum_{i=1}^{3} E[C_{1i}] + C_2. \quad (21)$$

Letting

$$\mathrm{pdf}(y, 300, 20) = \frac{1}{20\sqrt{2\pi}} e^{-\frac{(y-300)^2}{2\cdot 20^2}} \quad (22)$$

denote the probability density function of Y (evaluated at y) we have that

$$
\begin{aligned}
E[C_{1i}] &= \int_{-\infty}^{\infty} C_{1i}(y, x_{1i})\ \mathrm{pdf}(y, 300, 20)dy \\
&= \int_{-\infty}^{x_{1i}} (500y + 700(x_{1i} - y))\ \mathrm{pdf}(y, 300, 20)dy \\
&\quad + \int_{x_{1i}}^{\infty} 500x_{1i}\ \mathrm{pdf}(y, 300, 20)dy \\
&= -200 \int_{-\infty}^{x_{1i}} y\ \mathrm{pdf}(y, 300, 20)dy \\
&\quad + 200x_{1i} \int_{-\infty}^{x_{1i}} \ \mathrm{pdf}(y, 300, 20)dy \\
&\quad + 500x_{1i} \int_{-\infty}^{\infty} \mathrm{pdf}(y, 300, 20)dy \\
&= -200 \int_{-\infty}^{x_{1i}} y\mathrm{pdf}(y, 300, 20)dy \\
&\quad + 200x_{1i}\ \Phi(\frac{x_{1i} - 300}{20}) \\
&\quad + 500x_{1i}
\end{aligned} \tag{23}
$$

where $\Phi(x)$ is the cumulative density function of the standard normal distribution. The first integral is evaluated as.

$$
\begin{aligned}
&\int_{-\infty}^{x_{1i}} y\mathrm{pdf}(y, 300, 20)dy = \int_{-\infty}^{x_{1i}} \frac{y}{20\sqrt{2\pi}} e^{-\frac{(y-300)^2}{2\cdot 20^2}} dy = \left[\frac{(y-300)^2}{2\cdot 20^2} = t^2 \right] \\
&\quad = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{x_{1i}-300}{2\cdot 20^2}} (\sqrt{2}\cdot 20t + 300) e^{-t^2} dy \\
&\quad = -400\mathrm{pdf}(x_{1i}, 300, 20) + \frac{300}{\sqrt{\pi}} \int_{-\inf}^{\frac{x_{1i}-300}{\sqrt{2}\cdot 20}} e^{-t^2} dt = \left[t^2 = \frac{s^2}{2} \right] \\
&\quad = -400\mathrm{pdf}(x_{1i}, 300, 20) + 300\Phi(\frac{x_{1i} - 300}{20}).
\end{aligned} \tag{24}
$$

Inserting this result in (23) we get

$$
\begin{aligned}
E[C_{1i}] = {} & 80000\mathrm{pdf}(x_{1i}, 300, 20) + 500x_{1i} \\
& + (200x_{1i} - 60000)\Phi(\frac{x_{1i} - 300}{20}).
\end{aligned} \tag{25}
$$

Inserting this in (21) we get the expected cost

$$
\begin{aligned}
&E[C] = \\
&\sum_{i=1}^{3} \big[80000\mathrm{pdf}(x_{1i}, 300, 20) + 500x_{1i} + (200x_{1i} - 60000)\Phi(\frac{x_{1i} - 300}{20}) \\
&\quad + 600x_{2i} + 100(x_{1i} + x_{2i}) + 80(z_{ci} + u_{2i}) + 20s_i^+ - 1000p_{ai} - 740p_{bi} \big]
\end{aligned} \tag{26}
$$

3 Results and Discussion

3.1 Basic model

The optimal value of the basic model, found by GAMS, is -295139.4 which corresponds to a profit of 295139.4 kr. This is a global optimum to the problem as explained in the following discussion.

3.1.1 Production flows

The optimal production flow as found by GAMS is presented in the following table

Variables	Week 1	Week 2	Week 3
x_1	293.958	294.356	294.356
x_2	0.148	71.539	85.644
z_a	147.082	197.255	207.129
z_b	88.217	116.720	105.436
z_c	58.806	66.025	67.436
u_1	0	87.459	87.772
u_2	88.217	29.261	17.663
v_a	102.918	52.745	42.871
v_b	44.105	42.541	42.228
p_a	250.000	250.000	250.000
p_b	30.000	130.000	130.000
s^+	14.105	0	0

Table 3: Production flows of the basic model

The data from the above table will be demonstrated in three different figures, one of every week. The first weeks production flow will look like:

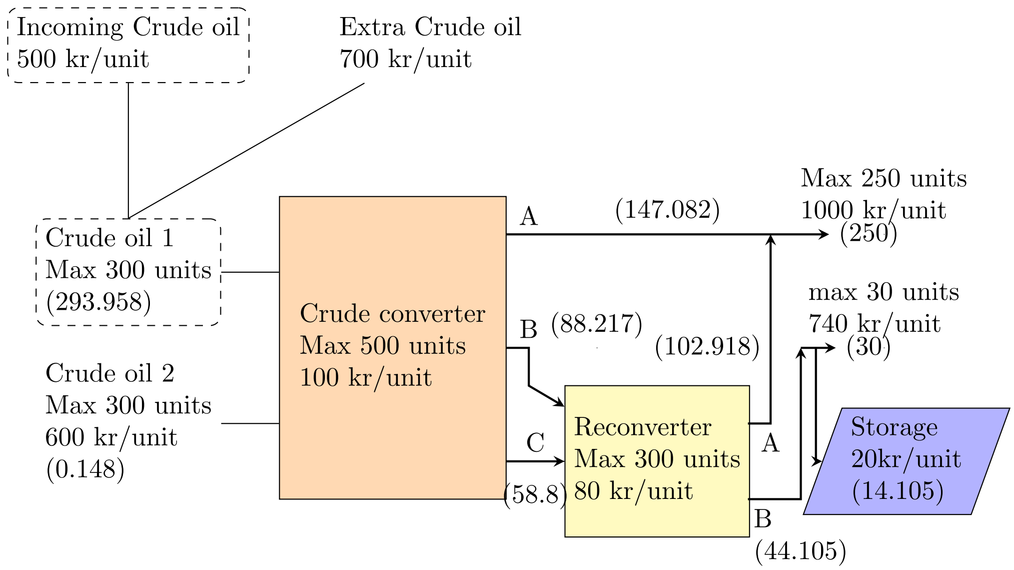

Figure 3: Production flow,week 1, Basic model

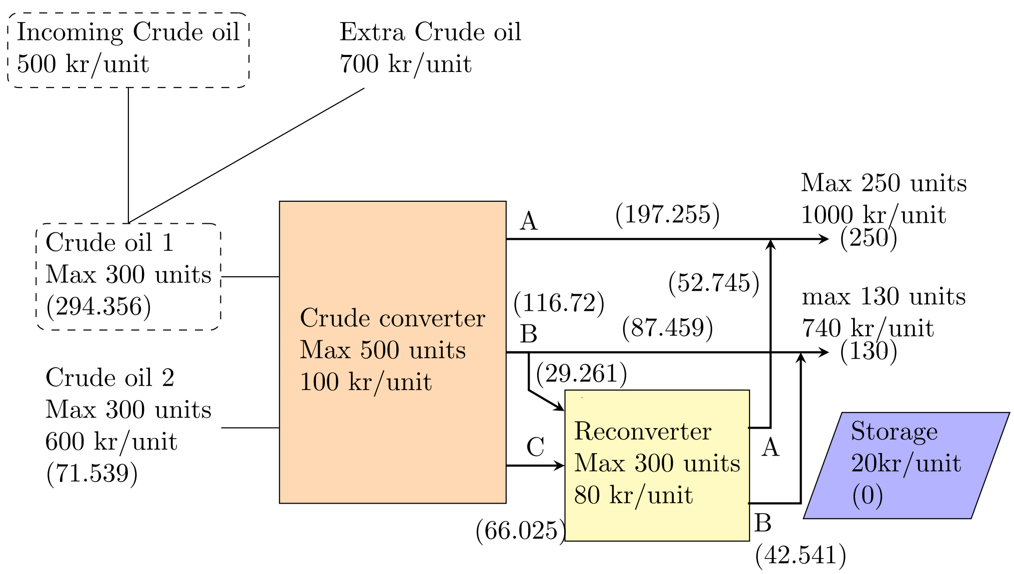

Figure 4: Production flow,week 2, Basic model

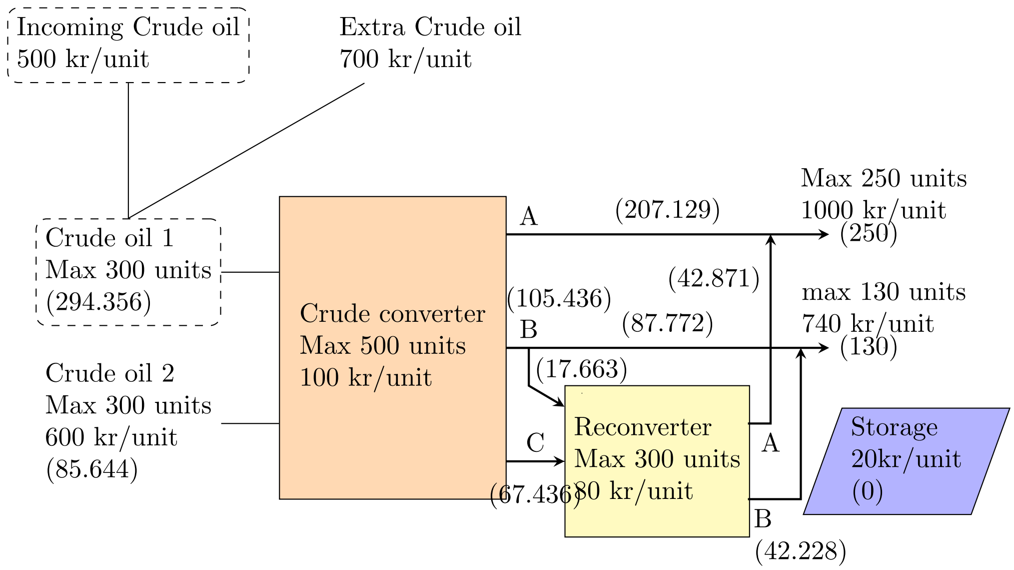

Figure 5: Production flow,week 3, Basic model

3.1.2 Discussion of optimality

This section aim is to examine if the solution found by GAMS is globally optimal. Consider the program

$$
\textbf{(P)} \qquad \begin{array}{lll} \min\limits_{x} & f(x) & \\ \text{s.t.} & g_i(x) \geq 0 & i \in \mathcal{I} \\ & g_i(x) = 0 & i \in \mathcal{E} \end{array} \tag{27}
$$

From the lecture notes [1] it is known that if f is a convex function, if $g_i, i \in \mathcal{I}$ are concave functions and $g_i, i \in \mathcal{E}$ are affine functions and the local optimally conditions are satisfied in a point, then the point is a global optimum to (P).

All the constraints in the basic program are linear and thus satisfies these conditions. GAMS converges to a point which satisfies the local optimality conditions. It is thus sufficient to check if the cost function is convex to confirm global optimality. From the lecture notes, we know that the sum of convex functions is a convex function. The cost function is a sum of a linear and a non linear part. Linear functions are convex and the cost function will thus be convex if the nonlinear part can be proved to be convex. The nonlinear part of the cost function is:

$$
\begin{aligned}
&E[C_{nonlinear}] = \\
&\sum_{i=1}^{3} \left[80000\text{pdf}(x_{1i}, 300, 20) + (200x_{1i} - 60000)\Phi(\frac{x_{1i} - 300}{20})\right] \\
&= 200 \sum_{i=1}^{3} \left[400 \frac{1}{20\sqrt{2\pi}} e^{-\frac{(y-300)^2}{2 \cdot 20^2}} + (x_{1i} - 300) \int_{-\infty}^{x_{1i}} \phi(y)dy\right]
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
&\frac{dE[C_{nonlinear}]}{dx} = E[\frac{dC_{nonlinear}}{dx}] = \\
&200 \sum_{i=1}^{3} \left[400 \frac{1}{20\sqrt{2\pi}} (\frac{3}{4} - \frac{x}{400}) e^{-\frac{(y-300)^2}{2 \cdot 20^2}} + \int_{-\infty}^{x_{1i}} \phi(y)dy \right. \\
&\left. + (x_{1i} - 300) \frac{1}{20\sqrt{2\pi}} e^{-\frac{(y-300)^2}{2 \cdot 20^2}}\right] \\
&200 \sum_{i=1}^{3} \left[\int_{-\infty}^{x_{1i}} \phi(y)dy\right]
\end{aligned} \tag{29}
$$

$$
\frac{d^2}{dx^2} E[C_{nonlinear}] = E[\frac{d^2 C_{nonlinear}}{dx^2}] = 200 \sum_{i=1}^{3} \phi(x_{1i}) \tag{30}
$$

A twice differentiable function is strictly convex if its second derivative is positive at all points. The normal pdf is always positive, and therefore the expected nonlinear cost is convex. Moreover, there is no stationary point, as $\frac{dC_{nonlinear}}{dx}$ is strictly positive. Thus, the local minimizer found by GAMS in the previous section is a global minimizer.

3.2 Advanced model

The optimal value found by GAMS, of the advanced model was -293886.6 which corresponds to a profit of 293886.6 kr. As explained in the following discussion this is not necessarily a global optimum, but an upper bond for the deviation is proposed.

3.2.1 Production flows

The optimal production flow, for the advanced program, as found by GAMS is presented in the following table

Variables	Week 1	Week 2	Week 3
x_1	290.293	294.356	294.356
x_2	3.533	73.729	85.644
z_a	147.620	198.788	207.129
z_b	87.795	114.967	105.436
z_c	58.412	66.244	67.436
u_1	0	87.508	87.772
u_2	87.795	27.460	17.663
v_a	102.380	51.212	42.871
v_b	43.827	42.492	42.228
p_a	250.000	250.000	250.000
p_b	30.000	130.000	130.000
s^+	13.827	0	0

Table 4: Production flows of the advanced model

3.2.2 Discussion of optimality

From the discussion in section 3.1.2 we know that the cost function is convex. The only change in the advanced model is in the constraint

$$s_{i-1}^+ - 0.01(s_{i-1}^+)^2 + 0.3x_{1i} + 0.2x_{2i} = z_{bi} \tag{31}$$

which can be written as

$$z_{bi} - s_{i-1}^+ + 0.01(s_{i-1}^+)^2 - 0.3x_{1i} - 0.2x_{2i} = 0 \tag{32}$$

In order to see if a solution is a global optimum, it needs to be proved that the function

$$g = z_{bi} - s_{i-1}^+ + 0.01(s_{i-1}^+)^2 - 0.3x_{1i} - 0.2x_{2i} \tag{33}$$

is affine. Since $(s_{i-1}^+)^2$ is a nonlinear function, g is not affine. Thus, it is not certain that any solution to the advanced problem is a global optimum.

In order to get a bound for the deviation from the global optimality, we consider the relaxation of the advanced constraint obtained by changing the constraint from an equality constraint into an inequality constraint. This makes the feasible region bigger without changing the cost function, which is a relaxation according to the definition in the lecture slides [1]. The relaxation has the constraint:

$$s_{i-1}^+ - 0.01(s_{i-1}^+)^2 + 0.3x_{1i} + 0.2x_{2i} \geq z_{bi} \tag{34}$$

$$s_{i-1}^+ - 0.01(s_{i-1}^+)^2 + 0.3x_{1i} + 0.2x_{2i} - z_{bi} \geq 0 \tag{35}$$

The relaxation has a convex cost function, and its only nonlinear constraint is on the form

$$g \geq 0 \tag{36}$$

where g is a concave function (since $-0.01(s_{i-1}^+)^2$ is concave). Thus the relaxed problem is a convex problem, and any solution to it is a global optimum. From the lecture slides we know that if P_R is a relaxation of P then it is true for the optimal objective values that

$$\text{optval}(P_R) \leq \text{optval}(P) \tag{37}$$

for a minimisation problem. Since any solution is globally optimal for our relaxation, a lower bound to the global optimal value of the advanced problem is any solution of the relaxed problem.

By inspecting the dual value of the advanced constraint in the advanced problem we can avoid solving the relaxed problem, even though it is simple in this case. If increasing the right hand side of the constraint on standard form $g = 0$ (that is, increasing 0 to 1) will change the objective by *lambda* units, where *lambda* is the dual of the constraint. We find that *lambda* is positive for all weeks, meaning that entering the zone where $g > 0$ results in higher objective values. Thus, there is no improvement in the solution for the relaxation over the original solution. This indicates that our solution is globally optimal, which is also confirmed by solving the relaxed problem in GAMS, which yields the same solution.

4 Possible improvements

4.1 Integer formulation

In the straight forward interpretation of the given problem, one might just assume this to be a ordinary (**NLP**) and solve it as such. But as one could suspect this solution is not feasible in reality as barrels of oil are not available in continuous values, but rather as integer values. Thus one would need instead to solve (**INLP**), which is not within the scope of the material covered in this course.

4.2 Stochatisc formulation

The model is simplified in the manner that it does not consider all the risks. One big risks in real life is to predict the demand for the companies product. In most cases it cannot be predicted and thus it should be stochastic. Moreover the model can be improved by taking into account of risk that can disrupt production flows, e.g. power failure or other failures in production caused by internal or external factors. One big risk for an oil company is the legislation. There could be an risk for increased taxes against fossil fuels or prohibit it as fuel, as mega trends go towards sustainability. This would have a huge impact on the demand for oil. Thus a stochastic formulation is more realistic.

4.3 Dependency of costs

The model could be improved by taking dependencies and extra costs into account. E.g there could be an maintenance cost that depend on usage of the converters instead of being constant.

A Programs

The program for the basic exercise:

$$\begin{aligned}
\min_{x} \quad & \sum_{i=1}^{3} \Big[80000\mathrm{pdf}(x_{1i}, 300, 20) + 500x_{1i} \\
& \quad + (200x_{1i} - 60000)\Phi(\frac{x_{1i} - 300}{20}) \\
& \quad + 600x_{2i} + 100(x_{1i} + x_{2i}) + 80(z_{ci} + u_{2i}) \\
& \quad + 20s_i^+ - 1000p_{ai} - 740p_{bi}\Big]
\end{aligned}$$

$$\textbf{(P)} \quad \begin{aligned}
\text{s.t.} \quad & x_{2i} \le 300 && \forall i \in \{1, 2, 3\} \\
& x_{1i} + x_{2i} \le 500. \\
& 0.5x_{1i} + 0.7x_{2i} = z_{ai} \\
& s_{i-1}^+ + 0.3x_{1i} + 0.2x_{2i} = z_{bi} \\
& 0.2x_{1i} + 0.1x_{2i} = z_{ci} \\
& u_{1i} + u_{2i} = z_{bi} \\
& z_{ci} + u_{2i} \le 300 \\
& 0.9u_{2i} + 0.4z_{ci} = v_{ai} \\
& 0.1u_{2i} + 0.6z_{ci} = v_{bi} \\
& z_{ai} + v_{ai} \ge p_{ai} \\
& u_{1i} + v_{bi} \ge p_{bi} \\
& p_{ai} \le 250 \\
& p_{bi} \le a_i \\
& u_{1i} + v_{bi} - p_{bi} = s_i^+ \\
& s_0^+ = 0 \\
& x_{1i}, x_{2i}, z_{ai}, z_{bi}, z_{ci}, u_{1i}, u_{2i} \ge 0 \\
& v_{ai}, v_{bi}, p_{ai}, p_{bi}, s_0^+ \ge 0
\end{aligned} \tag{38}$$

The program for the advanced exercise:

$$
\textbf{(P)} \quad
\begin{aligned}
\min_x \quad & \sum_{i=1}^{3} \Big[80000\text{pdf}(x_{1i}, 300, 20) + 500x_{1i} \\
& \qquad + (200x_{1i} - 60000)\Phi(\frac{x_{1i} - 300}{20}) \\
& \qquad + 600x_{2i} + 100(x_{1i} + x_{2i}) + 80(z_{ci} + u_{2i}) \\
& \qquad + 20s_i^+ - 1000p_{ai} - 740p_{bi}\Big] \\
\text{s.t.} \quad & x_{2i} \leq 300 \qquad\qquad \forall i \in \{1, 2, 3\} \\
& x_{1i} + x_{2i} \leq 500. \\
& 0.5x_{1i} + 0.7x_{2i} = z_{ai} \\
& s_{i-1}^+ - 0.01(s_{i-1}^+)^2 + 0.3x_{1i} + 0.2x_{2i} = z_{bi} \\
& 0.2x_{1i} + 0.1x_{2i} = z_{ci} \\
& u_{1i} + u_{2i} = z_{bi} \\
& z_{ci} + u_{2i} \leq 300 \\
& 0.9u_{2i} + 0.4z_{ci} = v_{ai} \\
& 0.1u_{2i} + 0.6z_{ci} = v_{bi} \\
& z_{ai} + v_{ai} \geq p_{ai} \\
& u_{1i} + v_{bi} \geq p_{bi} \\
& p_{ai} \leq 250 \\
& p_{bi} \leq a_i \\
& u_{1i} + v_{bi} - p_{bi} = s_i^+ \\
& s_0^+ = 0 \\
& x_{1i}, x_{2i}, z_{ai}, z_{bi}, z_{ci}, u_{1i}, u_{2i} \geq 0 \\
& v_{ai}, v_{bi}, p_{ai}, p_{bi}, s_0^+ \geq 0
\end{aligned}
\tag{39}
$$

References

[1] Forsgren, Anders. (2019, Mars) Lecture notes, lecture 3. KTH, Stockholm, Sweden. . [Online]. Available: https://kth.instructure.com/files/1833275/