KTH Engineering Sciences

Computer Exercise 5

Numerical experiments with hyperbolic PDE problems

In this exercise you shall make some numerical experiments with finite difference methods applied to two different hyperbolic PDE-problems.

Part 1: Model problem

Consider the model problem for a hyperbolic PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad a > 0, \qquad 0 \le x \le 2, \qquad t > 0,$$

with initial condition $u(x,0) \equiv 0$. The boundary condition at $x = 0$ models a a wave with period time τ entering from the left into the domain. The wave can be either a sine wave,

$$u(0,t) = g_{\sin}(t) := \sin(2\pi t/\tau),$$

or a square wave,

$$u(0,t) = g_{\mathrm{sq}}(t) := \begin{cases} 1, & n\tau < t \le (n+1/2)\tau, \\ -1, & (n+1/2)\tau < t \le (n+1)\tau, \end{cases} \qquad n = 0, 1, 2, \ldots$$

Discretize the x-interval into N equidistant subintervals and define gridpoints $x_i = i\Delta x, i = 0, 1, 2, \ldots, N$, where $\Delta x = 2/N$. The Courant number is $\sigma = a\Delta t/\Delta x$. Let $N = 100$, $\tau = 1$ and $a = 1$. Implement the upwind, Lax–Friedrich and Lax–Wendroff methods. For the last two, use extrapolation as numerical boundary condition at $x = 2$ (see Section 8.2.6 in Edsberg.) Run the three methods on the time interval $[0, 2\tau]$. Present the results in a 2D graph with $u(x, 2\tau)$ as a function of x for *all three methods plotted on top of each other in the same figure*, for easy comparison.

Experiment with different σ-values for both the sine and the square wave case. Draw conclusions and try to explain the observed behaviour.

- When are the methods stable/unstable? Are your experimental results in agreement with the theoretical stability results? How does the stability depend on σ and on the boundary condition function?
- Which method(s) give overly smeared numerical solutions? Which method(s) introduce spurious oscillations? When does this happen?
- Which method is best for the sine and the square wave case, respectively?

At the examination you will be asked to run the code for various Courant numbers σ with either g_{sq} or $g_{\sin}$ as boundary condition. Therefore, make sure to write your program so that it is simple to change σ and to select the boundary condition function.

Part 2: Heat exchanger application

In this part the exchanger in Edsberg, Example 8.1 is studied. A fluid of temperatute $T(x,t)$ is flowing with constant speed v in a pipe. Outside the pipe there is a cooling medium that keeps a constant low temperature T_{cool}. The temperature of the fluid in the pipe is initially cool, i.e. $T = T_{\text{cool}}$ but within a short time period hot fluid with fluctuating temperature enters the pipe. The task is to study how the temperature $T(x,t)$ of the fluid in the pipe depends on x and t.

The following PDE-model is given:

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} + k(T - T_{\text{cool}}) = 0, \qquad 0 \le x \le L, \qquad t > 0, \tag{1}$$

with the initial condition

$$T(x,0) \equiv T_{\text{cool}},$$

and the boundary condition

$$T(0,t) = \begin{cases} T_{\text{cool}} + (T_{\text{hot}} - T_{\text{cool}})\sin(\pi t), & 0 \le t \le 0.5, \\ T_{\text{hot}}, & 0.5 \le t \le 2, \\ T_{\text{hot}} + T_{\text{cool}}\sin(4\pi(t-2)), & t > 2. \end{cases}$$

The length of the heat exchanger is $L = 3$. The heat exchange parameter $k = 0.2$, the velocity of the fluid is $v = 1$, the cooling temperature $T_{\text{cool}} = 10$ and the hot temperature is $T_{\text{hot}} = 100$.

(a) Use the upwind and the Lax–Wendroff methods to simulate the temperature in the pipe for $0 \le t \le 5$. Choose suitable stepsizes Δx and Δt. Present T as a function of both t and x in two 3D graphs next to each other in two subplots.

Hint: For advection equations with lower order terms and a source,

$$u_t + au_x + bu = c,$$

the upwind method is easy to extend by adding a term $-\Delta t(bu_{i,k} - c)$. The extension of Lax–Wendroff is more involved. In the Exercise lecture it will be shown to be

$$\begin{aligned} u_{i,k+1} = u_{i,k} &- \frac{\sigma(1 - b\Delta t)}{2}(u_{i+1,k} - u_{i-1,k}) + \frac{\sigma^2}{2}(u_{i+1,k} - 2u_{i,k} + u_{i-1,k}) \\ &- \Delta t\left(1 - \frac{b\Delta t}{2}\right)(bu_{i,k} - c), \end{aligned}$$

where $\sigma = a\Delta t/\Delta x$.

(b) Make 2D plots of $T(t,x)$ for $0 \le x \le L$ at the time points $t \approx 2.5$ and $t \approx 5$. Plot the solutions from the two methods on top of each other in the same figure/subplot.

Experiment with different σ and Δx. Compare the two methods. Try to explain your observations using the theoretical results about the methods. Which method is most accurate? How much larger Δx can you use in the most accurate method? How does this depend on σ?

Write your program so that it is simple to change σ and Δx (or N) so that you can easily show different results at the examination.