KTH Engineering Sciences

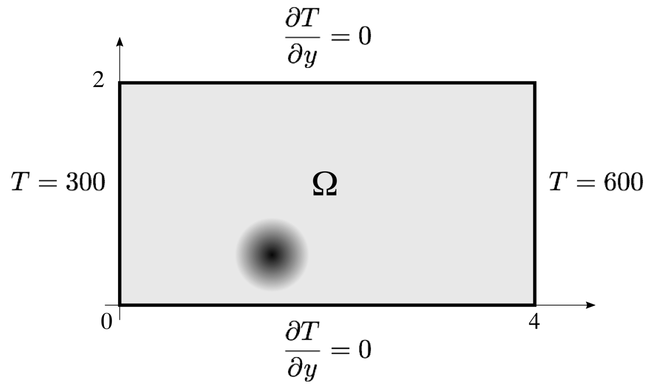
Computer Exercise 4
Numerical solution of elliptic PDE problems

In this exercise you will solve elliptic PDEs in two dimensions. The physical background is as follows: Heat is conducted through a rectangular metal block, being the region $\Omega = [0 \leq x \leq 4, 0 \leq y \leq 2]$ when placed in a xy-coordinate system. The block is kept at constant temperature at the two sides $x = 0$ and $x = 4$. It is isolated at the other two sides. An external source modeled by the function $f(x, y)$ heats the block. The following elliptic problem for the temperature distribution $T(x, y)$ can then be formulated:

$$-\Delta T = f, \quad (x, y) \in \Omega,$$

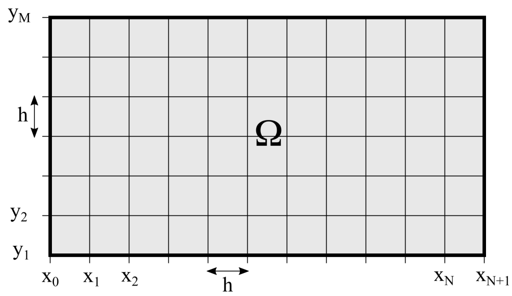
with boundary conditions

$$\begin{aligned} T(0, y) &= 300, \qquad & 0 \leq y \leq 2, \\ T(4, y) &= 600, \qquad & 0 \leq y \leq 2, \\ \frac{\partial T}{\partial y}(x, 0) &= 0, \qquad & 0 < x < 4, \\ \frac{\partial T}{\partial y}(x, 2) &= 0, \qquad & 0 < x < 4. \end{aligned}$$



Part 1: Finite difference approximation

Solve the elliptic PDE problem. Use the finite difference method as described in Edsberg, Chapter 7. The method should be of at least order two. Discretize the rectangular domain into a quadratic mesh with the same, uniform, stepsize h in the x- and y-directions. A suggested numbering of the grid points is given in the figure on the right, where for instance, $N = 39$ and $M = 21$ if $h = 0.1$.



a) Solve the problem with $f \equiv 0$ and $h = 0.1$. Visualize the solution $T(x, y)$ with colors, using the Matlab function `imagesc`. What is the T-value at $(x, y) = (2, 1)$? Derive the analytic solution. Why is the numerical solution almost exact?

b) Solve the problem with

$$f(x, y) = 2000 \exp\left(-5(x - 1.5)^2 - 10(y - 0.5)^2\right).$$

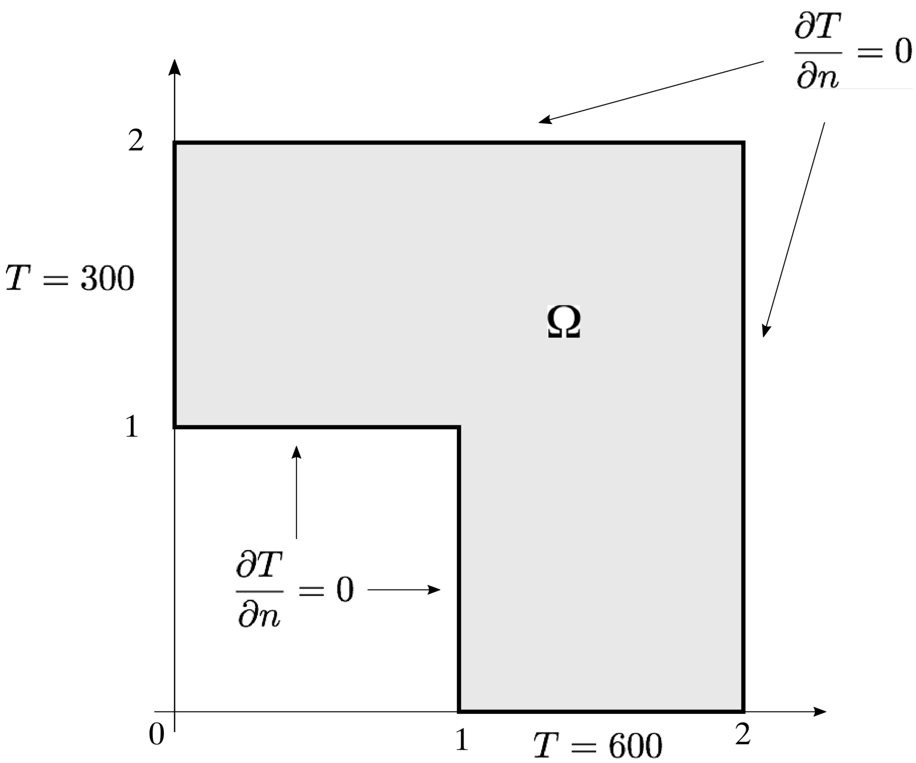
Report the T-values obtained at $(2, 1)$ when you use $h = 0.1$, $h = 0.05$ and $h = 0.025$ (three T-values). Here it is important to use MATLAB's `sparse` format for the matrices. Otherwise the problems with small h take very long time. Visualize the solution as above for $h = 0.05$. This time also plot it with the `mesh` and `contour` commands. (You do not need to find the analytic solution for this case!) Comment on your results. Is the result as expected?

OBS! For full credit your implementation should exhibit second order accuracy when the grid is refined.

Part 2: Comsol Multiphysics

a) Solve the problem in Part 1b with Comsol Multiphysics. Draw the geometry, set the PDE coefficients, specify boundary conditions, generate the mesh and compute the solution. Check your numerical result: what is the T-value at the point $(2, 1)$? How many elements (triangles) have been generated in the mesh? Make a refinement of the grid and find again $T(2, 1)$. How many elements are there now? Plot a 2D-graph of the solution.

b) Solve the problem using Comsol Multiphysics in the following domain.



In this problem the left and bottom boundaries have prescribed temperatures (300 and 600). The remaining boundaries are heat insulated, i.e. the normal derivative is equal to zero along these boundaries. There is no external source, $f \equiv 0$. Create the domain with the Geometry builder. Specify the boundary conditions. Set the PDE-parameters and generate the mesh. How many elements are generated? Solve the problem and plot the solution. What is the value of T in the points $(1, 1)$ and $(2, 2)$? Refine the mesh and solve again. How many elements are there now? What is $T(1, 1)$ and $T(2, 2)$?