Exercise 1

Show that $\dfrac{\partial(\vec{a}^T\vec{x})}{\partial\vec{x}} = \vec{a}^T$

$\vec{a}^T\vec{x}$ is the linear function $\sum_{i=0}^{n-1} a_i x_i$

$$\frac{\partial(\vec{a}^T\vec{x})}{\partial\vec{x}} = \left(\frac{\partial(\vec{a}^T\vec{x})}{\partial x_0} \quad \frac{\partial(\vec{a}^T\vec{x})}{\partial x_1} \quad \cdots \quad \frac{\partial(\vec{a}^T\vec{x})}{\partial x_{n-1}} \right)$$

$$= (a_0 \quad a_1 \quad \cdots \quad a_{n-1}) = \vec{a}^T. \quad \square$$

Show that $\dfrac{\partial(\vec{a}^T A\vec{a})}{\partial\vec{a}} = \vec{a}^T(A + A^T)$

$\dfrac{\partial(\vec{a}^T A\vec{a})}{\partial\vec{a}}$, $\quad f(\vec{a}) = A\vec{a}$

product rule for dot product

$$= \frac{\partial(\vec{a}^T f(\vec{a}))}{\partial\vec{a}} = \vec{a}^T\frac{\partial f(\vec{a})}{\partial\vec{a}} + f(\vec{a})^T\frac{\partial\vec{a}}{\partial\vec{a}}$$

$$= \vec{a}^T A + (A\vec{a})^T = \vec{a}^T A + \vec{a}^T A^T = \vec{a}^T(A + A^T). \quad \square$$

Show that $\frac{\partial(\vec{x}-A\vec{s})^T(\vec{x}-A\vec{s})}{\partial\vec{s}} = -2(\vec{x}-A\vec{s})^T A$

$$\frac{\partial}{\partial\vec{s}}(\vec{x}^T-\vec{s}^TA^T)(\vec{x}-A\vec{s})$$

$$= \frac{\partial}{\partial\vec{s}}\vec{x}^T\vec{x} - \frac{\partial}{\partial\vec{s}}(\vec{x}^TA\vec{s} - \vec{s}^TA^T\vec{x}) + \frac{\partial}{\partial\vec{s}}\vec{s}^TA^TA\vec{s}$$

term 1:

$$\frac{\partial}{\partial\vec{s}}\vec{x}^T\vec{x} = 0$$

term 2:

$(\vec{x}^TA\vec{s})^T = \vec{s}^TA^T\vec{x}$, This is a scalar so

$\vec{x}^TA\vec{s} = \vec{s}^TA^T\vec{x}$

I can therefore combine term 2 to

$2\vec{s}^TA^T\vec{x}$

so $\frac{\partial}{\partial\vec{s}}2\vec{s}^TA^T\vec{x} = 2\frac{\partial}{\partial\vec{s}}\vec{s}^TA^T\vec{x}$

set $y(\vec{x}) = A^T\vec{x}$

$$2\frac{\partial}{\partial\vec{s}}\vec{s}^Ty(\vec{x}) = 2\cdot y(\vec{x})^T = 2\cdot\vec{x}^TA$$

term 3:

$$\frac{\partial}{\partial \vec{s}} \vec{s}^{\,T} A^T A \vec{s} = \vec{s}^{\,T}(A^T A + A^T A)$$
$$= 2\vec{s}^{\,T} A^T A$$

So
$$\frac{\partial (\vec{x} - A\vec{s})^T (\vec{x} - A\vec{s})}{\partial \vec{s}} = -2\vec{x}^{\,T} A + 2\vec{s}^{\,T} A^T A$$
$$= -2(\vec{x}^{\,T} A - \vec{s}^{\,T} A^T A)$$
$$= -2(\vec{x}^{\,T} - \vec{s}^{\,T} A^T) A$$
$$= -2(\vec{x} - A\vec{s})^T A. \quad \square$$

Find the second derivative of $(\vec{x} - A\vec{s})^T(\vec{x} - A\vec{s})$

This is $\frac{\partial}{\partial \vec{s}} - 2(\vec{x} - A\vec{s})^T A$

$$\frac{\partial}{\partial \vec{s}} -2(\vec{x} - A\vec{s})^T A = -2 \frac{\partial}{\partial \vec{s}} (\vec{x}^{\,T} A - \vec{s}^{\,T} A^T A)$$

$$= 2 \frac{\partial}{\partial \vec{s}} \vec{s}^{\,T} A^T A = 2A^T A. \quad \square$$