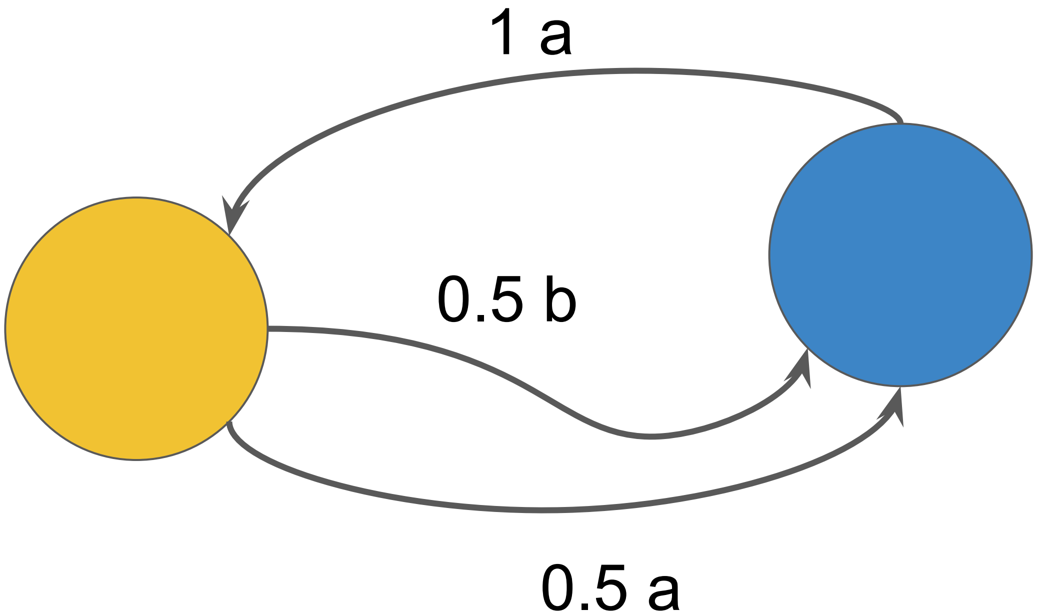
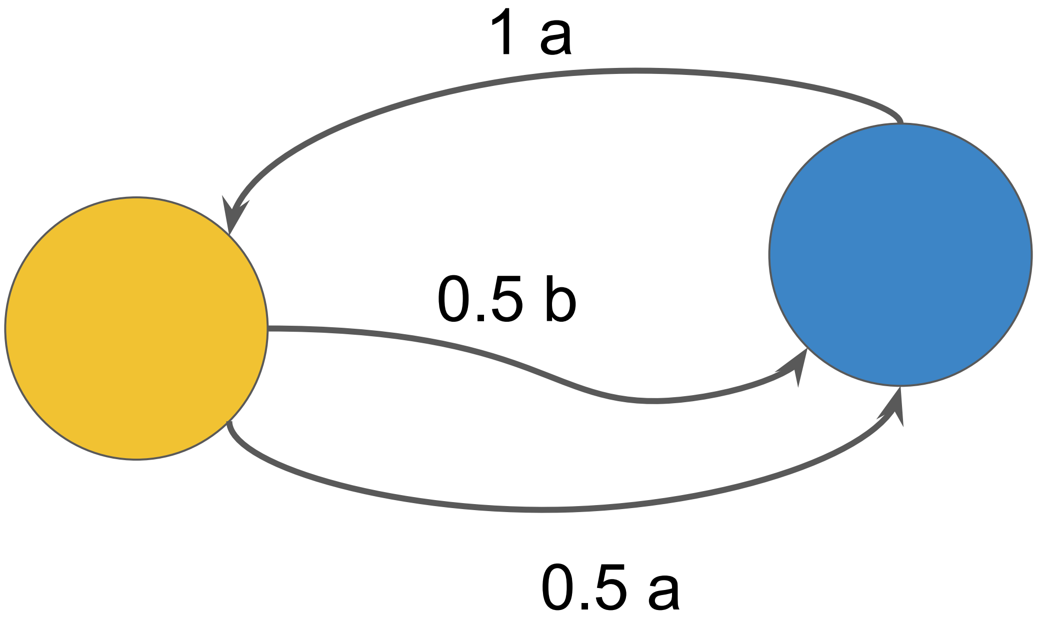
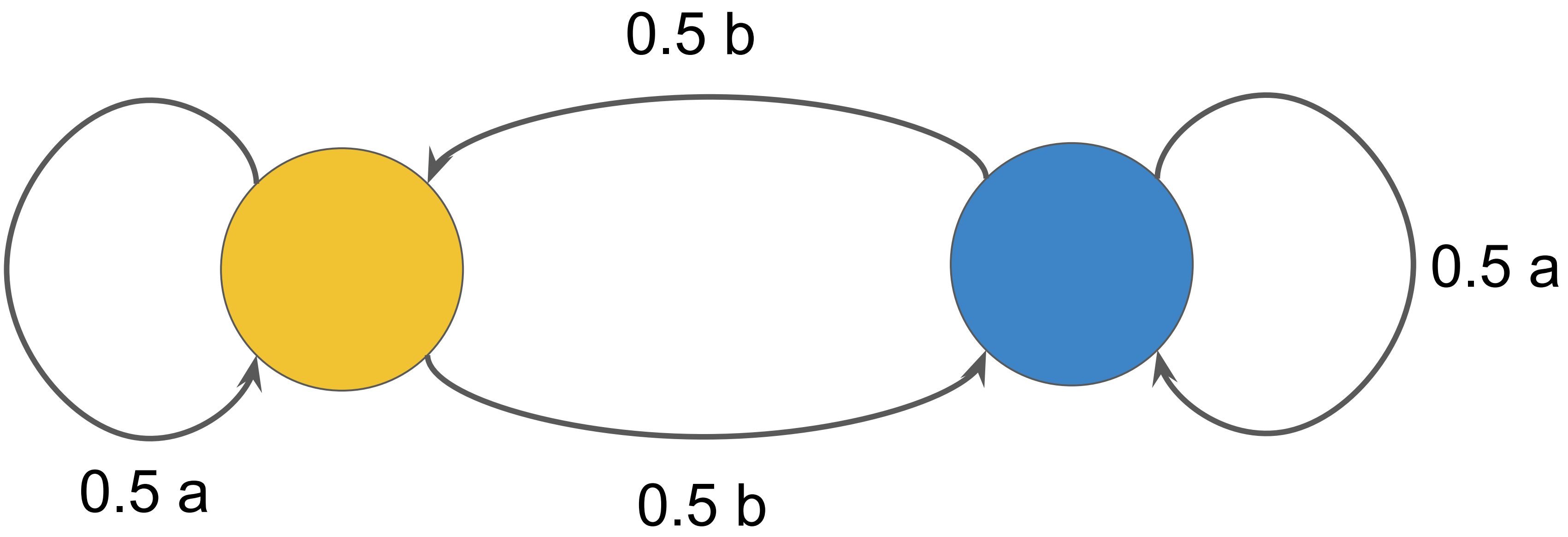
On the sequential probability ratio test in Hidden Markov Models

Oscar Darwin & Stefan Kiefer


1 a
0.5 b
0.5 a



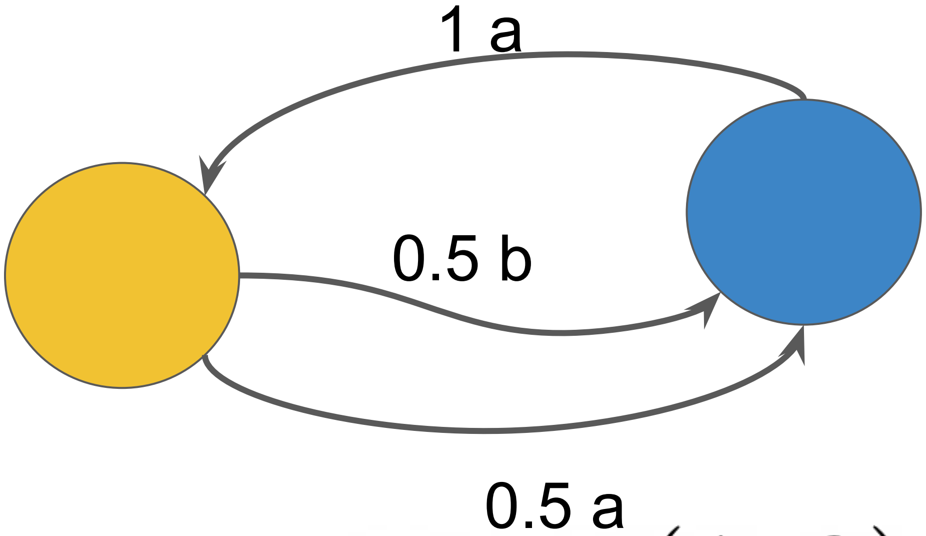
EVENTS: {The first 3 emitted letters are a, a, b}
{After the first b, the letters a and then b are emitted}



Both States are
Equivalent

$$\Psi(a) = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$\Psi(b) = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$$



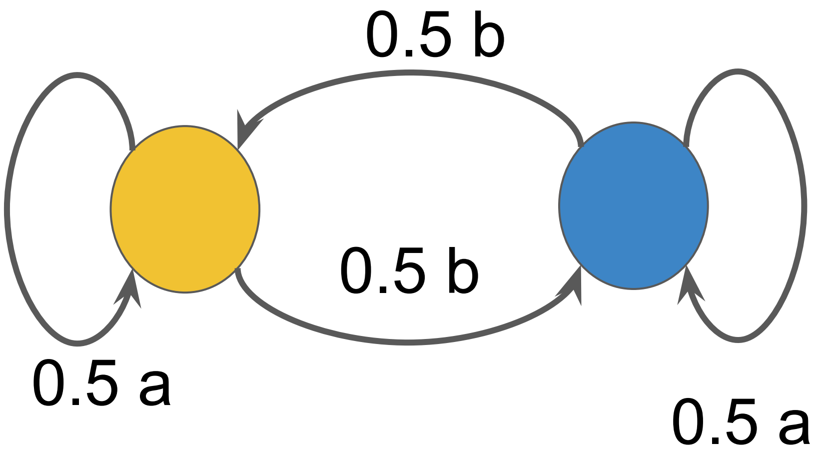
$$\pi_1 = (1, 0)$$

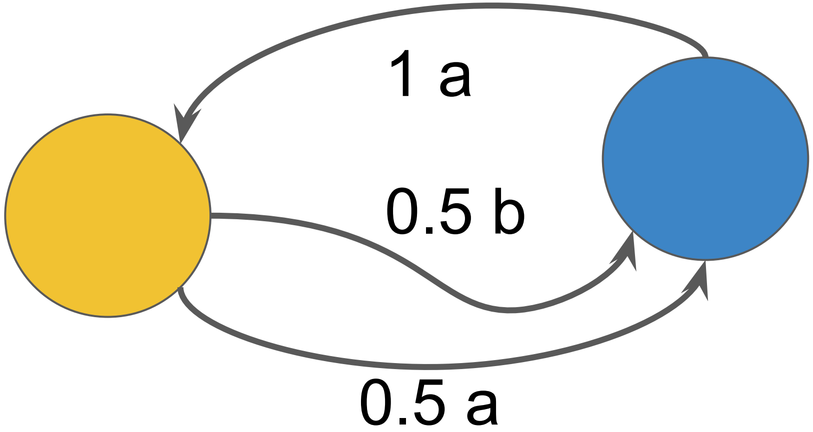
$$\pi_2 = (0, 1)$$

$$\mathbb{P}_{\pi_1}(aab\Sigma^\omega) = \pi_1 \Psi(a)\Psi(a)\Psi(b)1^T$$

Equivalence

- Two initial states are equivalent if their respective probability distributions on the set of infinite words agree on all events.
- equivalence is decidable in $O(|\Sigma||Q|^3)$ time. [Tzeng '92, KMOWW '11]





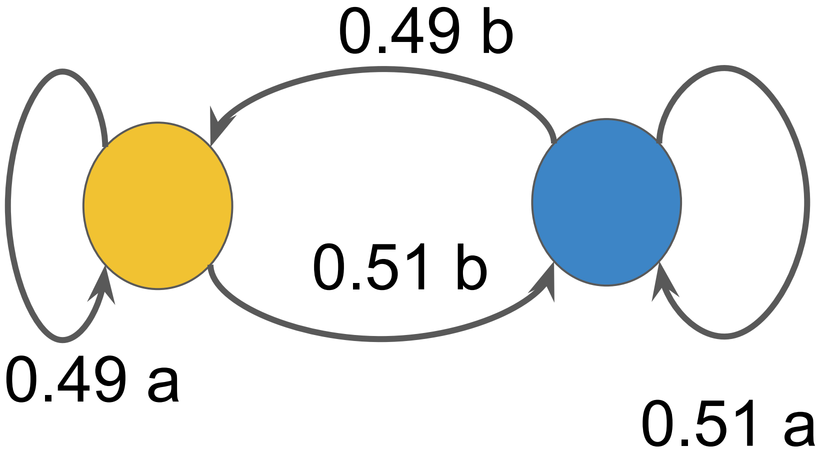
The event {substring “bb” appears infinitely often}

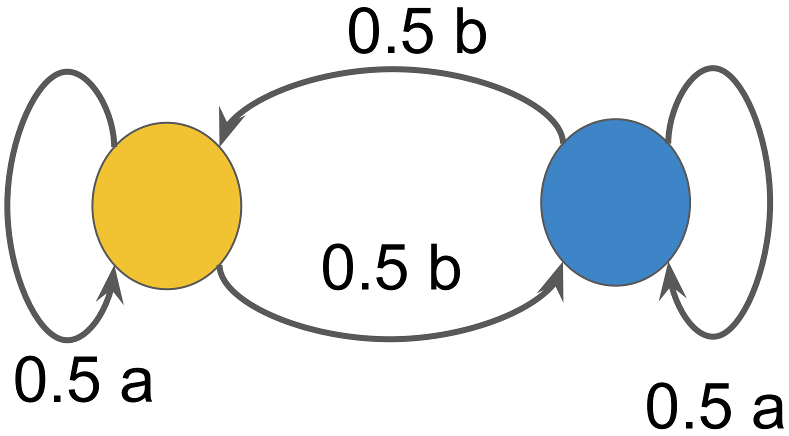
Distinguishability

- Two initial states are distinguishable if there is some event E such that

$$\mathbb{P}_1(E) = 1 \, \mathbb{P}_2(E) = 0$$

- Distinguishability is decidable in polynomial time [Kiefer,Chen ‘14]


0.49 b
0.51 b
0.49 a
0.51 a


0.5 b
0.5 b
0.5 a
0.5 a

Likelihood Ratios and Exponents

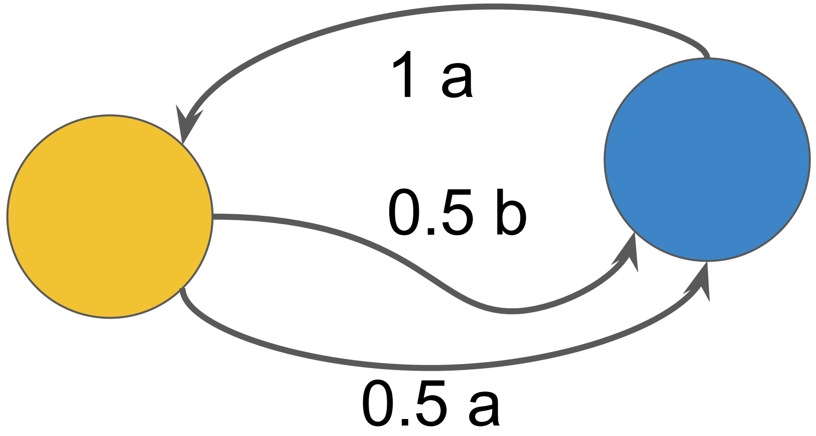
$$\frac{\mathbb{P}_{\pi_1}(aab\Sigma^\omega)}{\mathbb{P}_{\pi_2}(aab\Sigma^\omega)} \qquad L_3 = \frac{\mathbb{P}_{\pi_1}(aab\Sigma^\omega)}{\mathbb{P}_{\pi_2}(aab\Sigma^\omega)}$$

$$\frac{\mathbb{P}_{\pi_1}(aab\Sigma^\omega)}{\mathbb{P}_{\pi_2}(aab\Sigma^\omega)} \qquad L_3 = \frac{\mathbb{P}_{\pi_1}(aab\Sigma^\omega)}{\mathbb{P}_{\pi_2}(aab\Sigma^\omega)}$$

▶ **Lemma 2.** *Let π_1, π_2 be initial states.*

1. $\lim_{n\to\infty} L_n$ *exists* $\mathbb{P}_{\pi_2}$*-almost surely and lies in* $[0, \infty)$.
2. $\lim_{n\to\infty} L_n = 0$ $\mathbb{P}_{\pi_2}$*-almost surely if and only if* π_1 *and* π_2 *are distinguishable.*

*[Kiefer,Chen '14]


1 a
0.5 b
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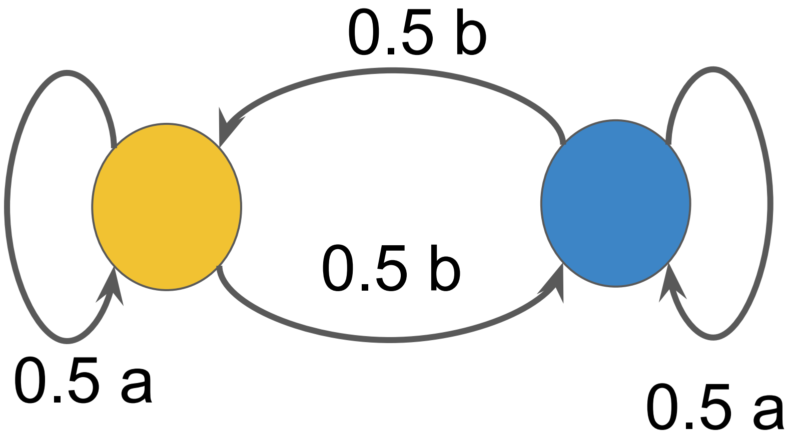
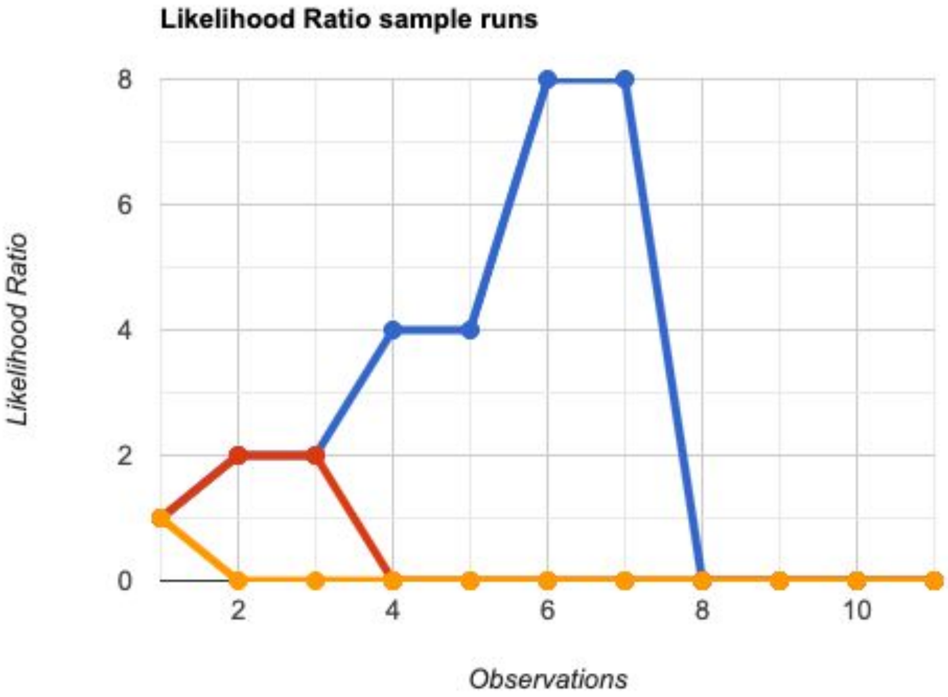
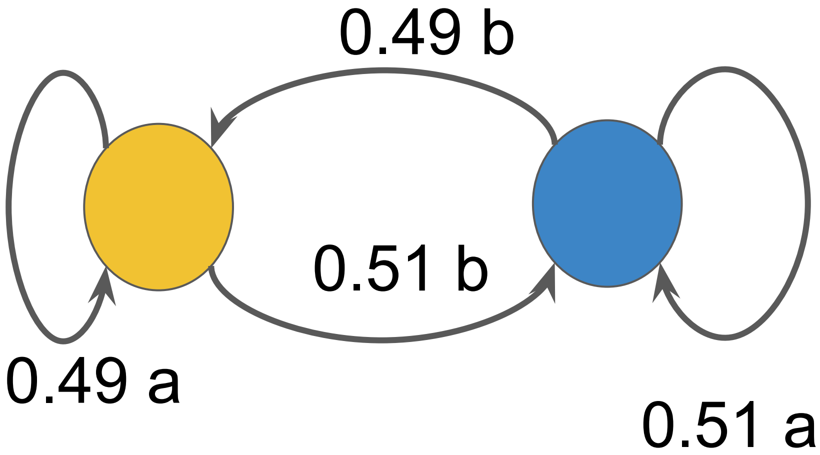
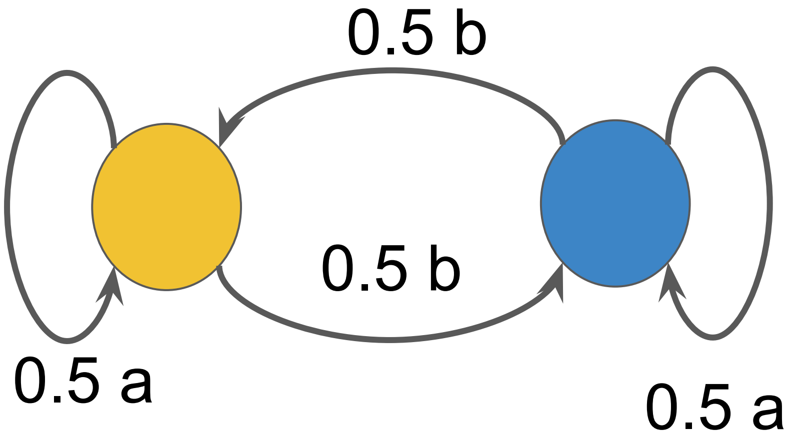
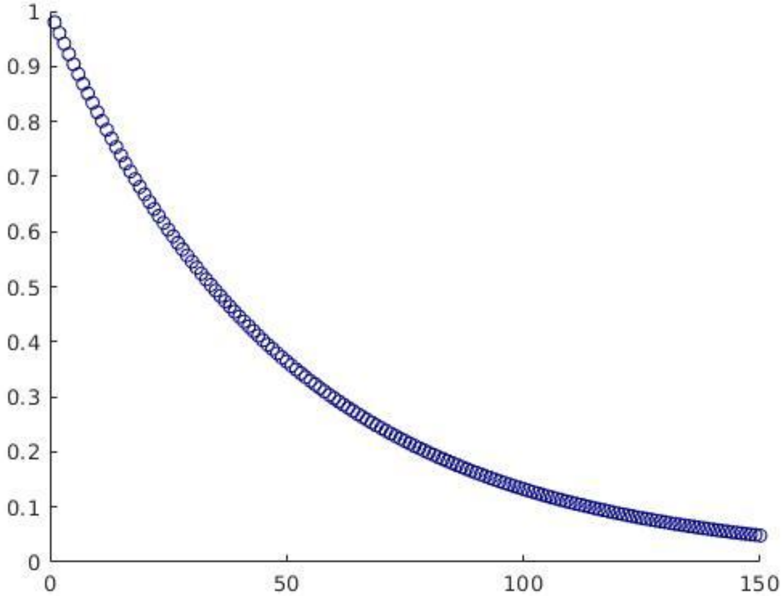

0.5 b
0.5 b
0.5 a
0.5 a

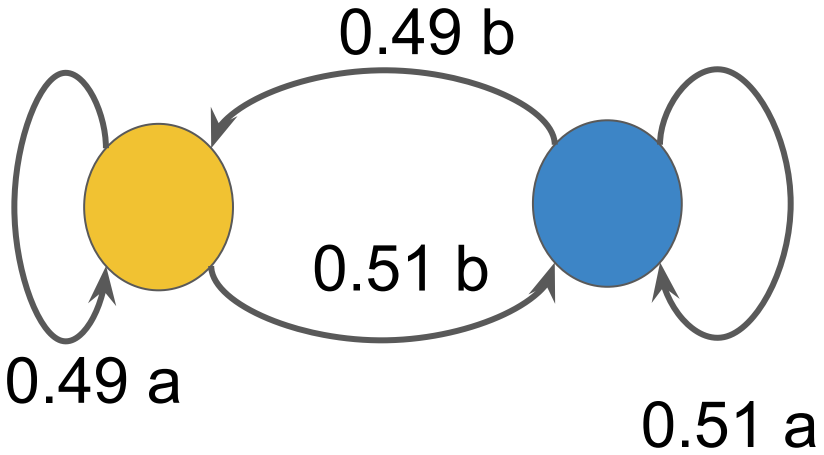

Likelihood Ratio sample runs
Likelihood Ratio
8
6
4
2
0
2
4
6
8
10
Observations

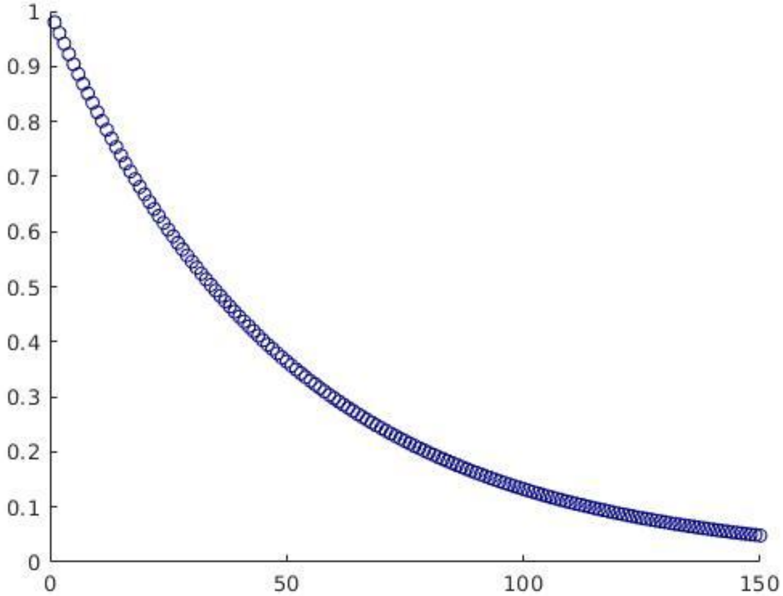






- What's the value of $\lim_{n\to\infty} \frac{1}{n} \ln L_n$





▶ **Theorem 8.** *For any initial states* π_1, π_2 *the limit* $\lim_{n\to\infty} \frac{1}{n} \ln L_n$ *exists* $\mathbb{P}_{\pi_2}$*-almost surely. Furthermore, we have* $\lim_{n\to\infty} \frac{1}{n} \ln L_n$ *takes at most* $|Q|^2 + 1$.

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Combining hidden Markov models for comparing the dynamics of multiple sleep electroencephalograms

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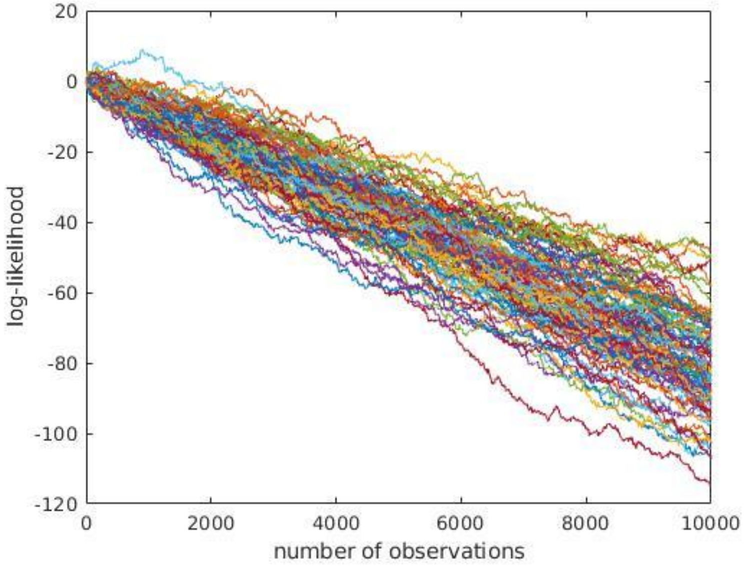
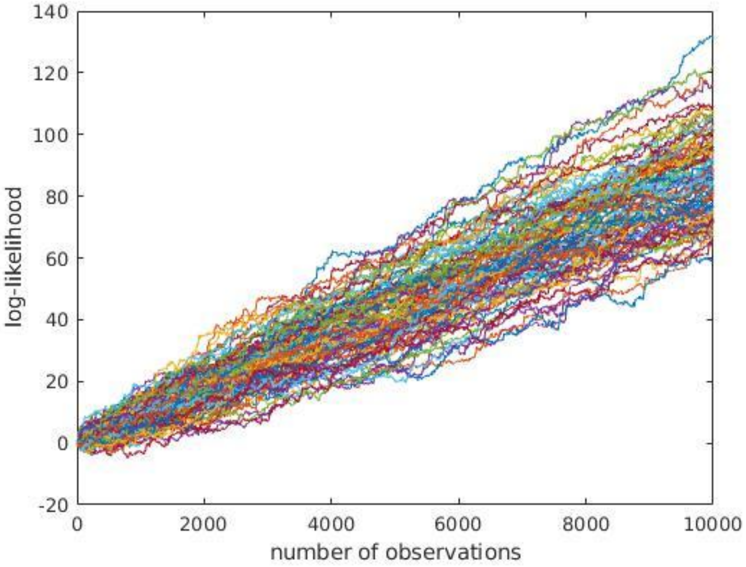
In this manuscript, we consider methods for the analysis of populations of electroencephalogram signals during sleep for the study of sleep disorders using hidden Markov models (HMMs). Notably, we propose an easily implemented method for simultaneously modeling multiple time series that involve large amounts of data. We apply these methods to study sleep-disordered breathing (SDB) in the Sleep Heart Health Study (SHHS), a landmark study of SDB and cardiovascular consequences. We use the entire, longitudinally collected, SHHS cohort to develop HMM population parameters, which we then apply to obtain subject-specific Markovian predictions. From these predictions, we create several indices of interest, such as transition frequencies between latent states. Our HMM analysis of electroencephalogram signals uncovers interesting findings regarding differences in brain activity during sleep between those with and without SDB. These findings include stability of the percent time spent in HMM latent states across matched diseased and non-diseased groups and differences in the rate of transitioning.

Table V. Averaged expected numbers of transitions per hour for healthy and diseased individuals.

	Healthy subgroup To state				
From state	1	2	3	4	5
1	13.30	1.66	0.59	1.08	0.15
2	2.01	19.94	0.16	3.74	0.07
3	0.38	0.08	17.63	2.83	0.25
4	0.98	4.19	2.42	36.74	0.10
5	0.12	0.05	0.37	0.05	10.12
	Diseased subgroup To state				
From state	1	2	3	4	5
1	9.68	2.71	0.65	1.90	0.15
2	3.05	17.32	0.18	4.21	0.03
3	0.39	0.05	17.48	2.96	0.36
4	1.86	4.69	2.43	38.41	0.17
5	0.11	0.02	0.50	0.07	9.61

Sequential Probability Ratio Test (SPRT)





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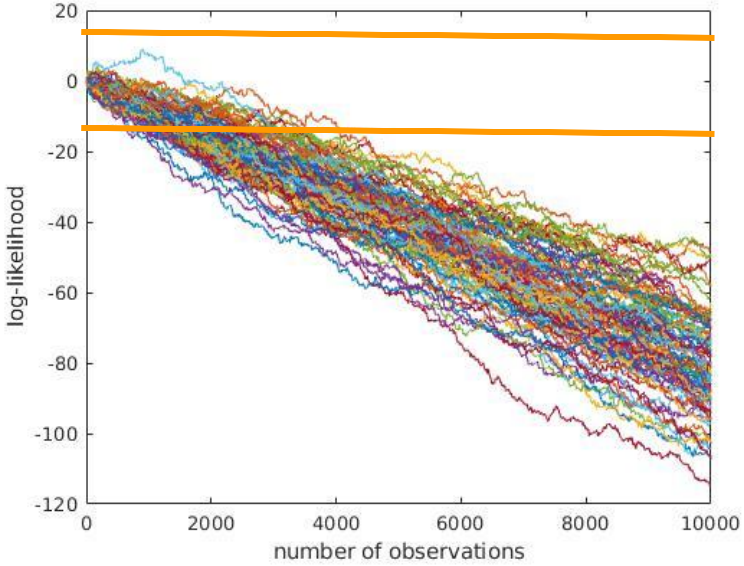

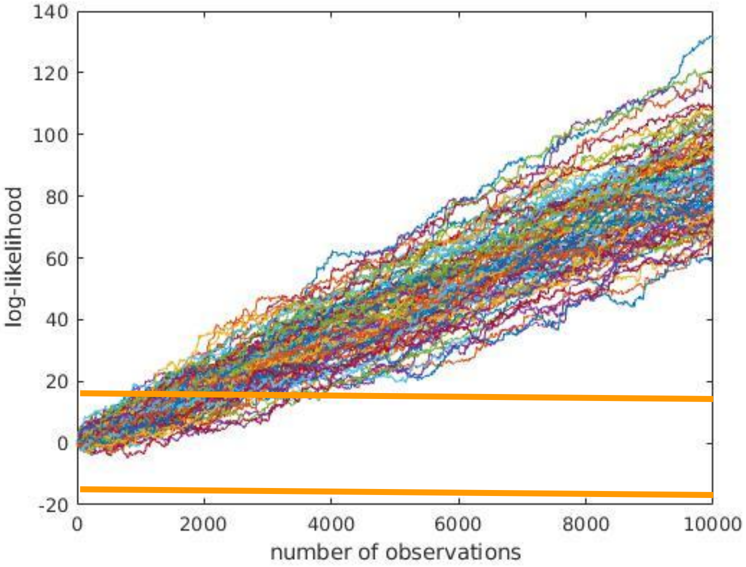

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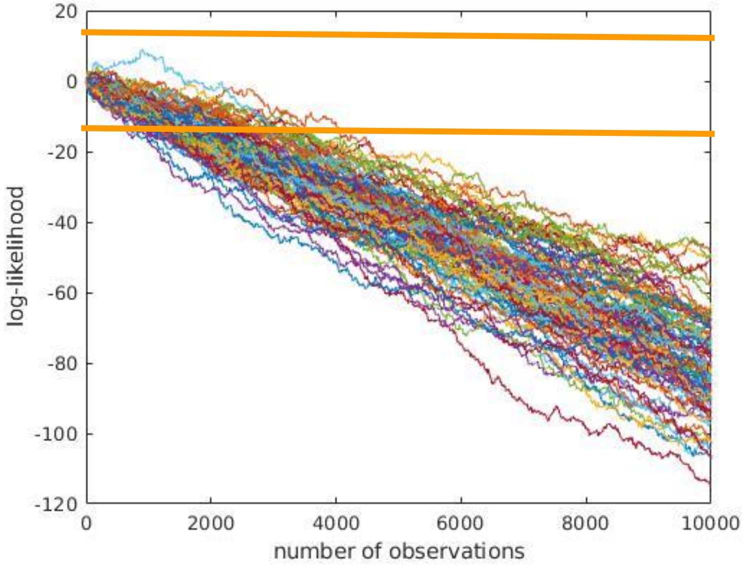
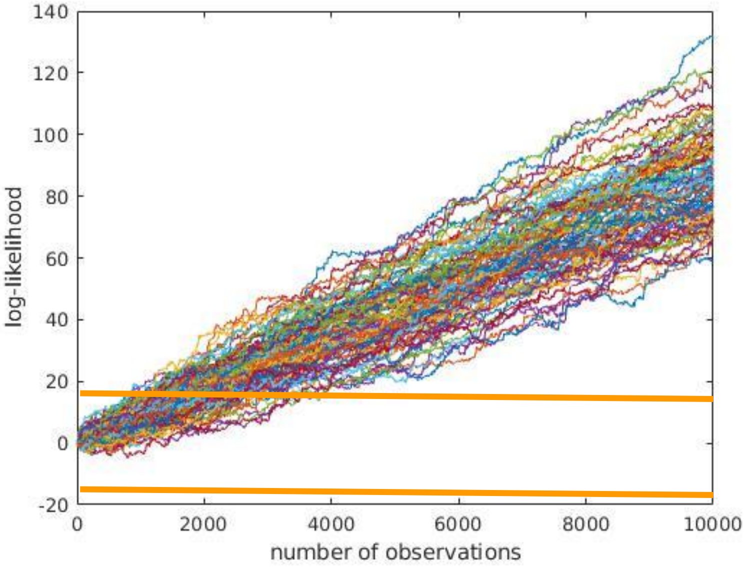
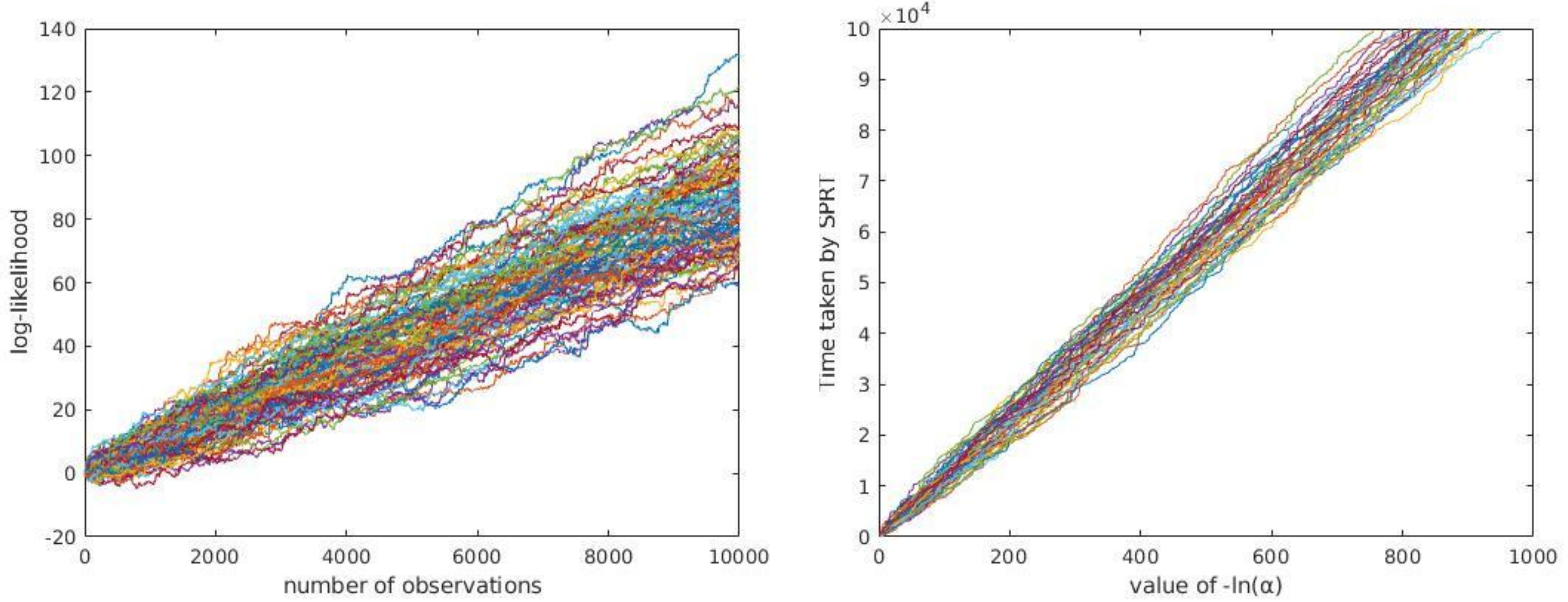




▶ **Proposition 4.** *Suppose* π_1 *and* π_2 *are distinguishable. Let* $\alpha, \beta \in (0,1)$. *By choosing* $A = \ln \frac{\alpha}{1-\beta}$ *and* $B = \ln \frac{1-\alpha}{\beta}$, *we have* $\mathbb{P}_{\pi_1}(\mathrm{SPRT}_{\alpha,\beta} = \pi_2) \leq \alpha$ *and* $\mathbb{P}_{\pi_2}(\mathrm{SPRT}_{\alpha,\beta} = \pi_1) \leq \beta$.



140
120
100
80
60
40
20
0
-20
log-likelihood
0
2000
4000
6000
8000
10000
number of observations
×10^4
10
9
8
7
6
5
4
3
2
1
0
Time taken by SPRT
0
200
400
600
800
1000
value of -ln(α)

Expected Time of SPRT

- In the independence case, we have the Wald formula

$$\mathbb{E}_{s_2}[N_{\alpha,\beta}] = \frac{\beta \ln \frac{1-\alpha}{\beta} + (1-\beta) \ln \frac{\alpha}{1-\beta}}{\ell}.$$

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- In the case of one likelihood exponent

$$\mathbb{P}_{\pi_2}\left(N_{\alpha,\beta} \sim \frac{\ln \alpha}{\ell} \quad (\text{as } \alpha, \beta \to 0)\right) = 1.$$

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- In the case of two or more likelihood exponents, we condition on hitting a particular likelihood exponent

$$\mathbb{P}_{\pi_2}\left(N_{\alpha,\beta} \sim \frac{\ln \alpha}{\ell} \quad (\text{as } \alpha, \beta \to 0) \;\middle|\; \lim_{n\to\infty} \frac{1}{n} \ln L_n = \ell\right) = 1.$$

Computability

▶ **Theorem 17.** *Given an HMM and initial states* π_1, π_2,

1. *one can compute* $\mathbb{P}_{\pi_2}(\lim_{n\to\infty} \frac{1}{n} \ln L_n = -\infty)$ *and* $\mathbb{P}_{\pi_2}(\lim_{n\to\infty} \frac{1}{n} \ln L_n = 0)$ *in PSPACE;*

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2. *one can decide whether* $\mathbb{P}_{\pi_2}(\lim_{n\to\infty} \frac{1}{n} \ln L_n = 0) = 0$ *(i.e.,* $0 \notin \Lambda_{\pi_1,\pi_2}$*) in polynomial time;*

Computability

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1. *one can compute* $\mathbb{P}_{\pi_2}(\lim_{n\to\infty} \frac{1}{n} \ln L_n = -\infty)$ *and* $\mathbb{P}_{\pi_2}(\lim_{n\to\infty} \frac{1}{n} \ln L_n = 0)$ *in PSPACE;*
2. *one can decide whether* $\mathbb{P}_{\pi_2}(\lim_{n\to\infty} \frac{1}{n} \ln L_n = 0) = 0$ *(i.e.,* $0 \notin \Lambda_{\pi_1,\pi_2}$*) in polynomial time;*
3. *deciding whether* $\mathbb{P}_{\pi_2}(\lim_{n\to\infty} \frac{1}{n} \ln L_n = 0) = 1$*, whether* $\mathbb{P}_{\pi_2}(\lim_{n\to\infty} \frac{1}{n} \ln L_n = -\infty) = 0$*, and whether* $\mathbb{P}_{\pi_2}(\lim_{n\to\infty} \frac{1}{n} \ln L_n = -\infty) = 1$ *are all PSPACE-complete problems.*