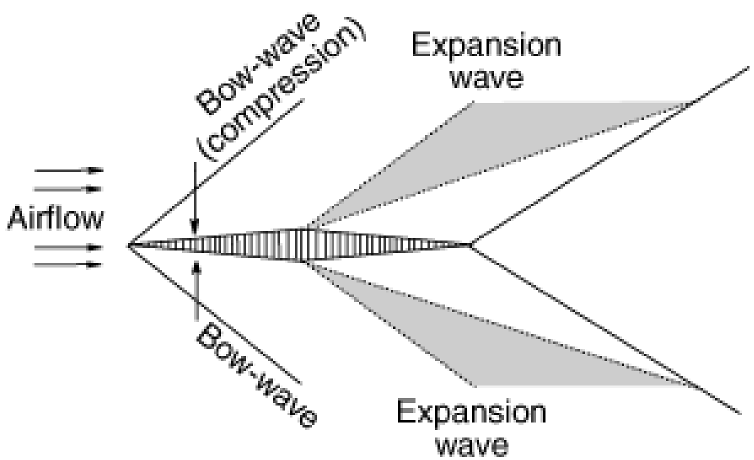
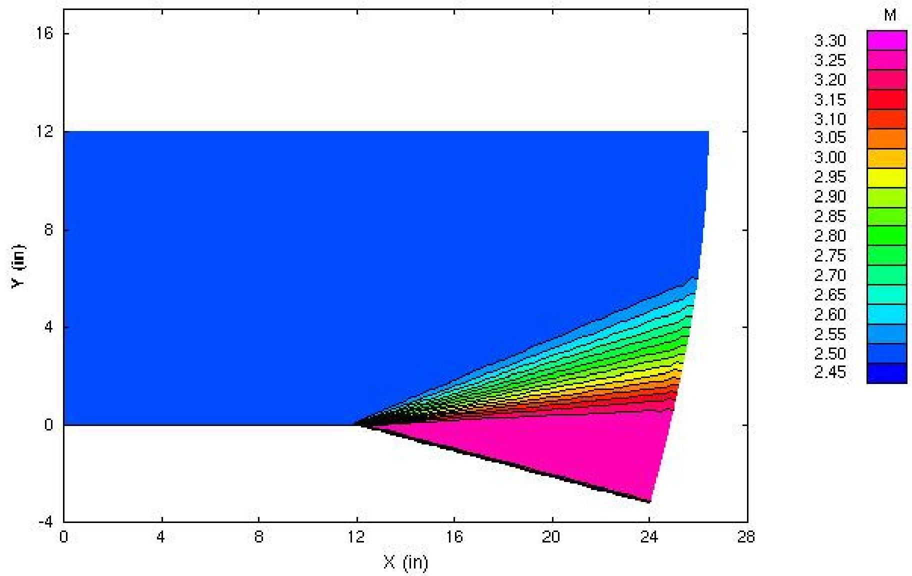
THE SOLUTION OF A PRANDTL-MEYER EXPANSION WAVE FLOW FIELD

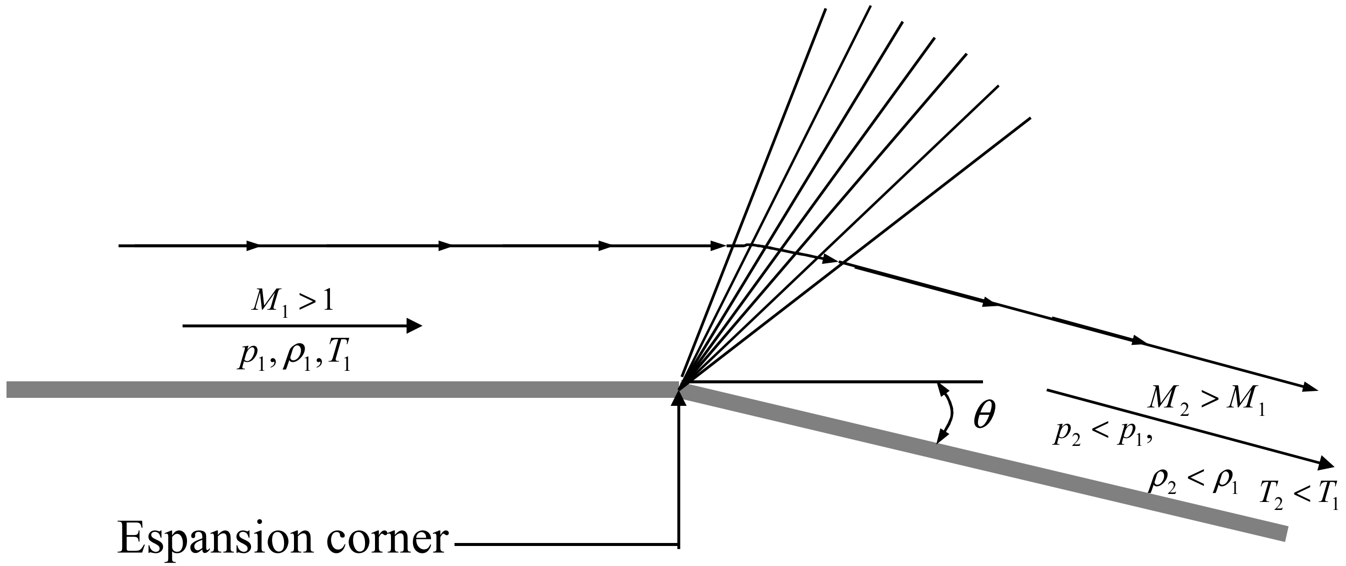
- Supersonic flows turn through convex corners of airfoils by means of expansion waves
- A Prandtl-Meyer expansion wave is an idealized expansion wave





THE SOLUTION OF A PRANDTL-MEYER EXPANSION WAVE FLOW FIELD

2D inviscid supersonic flow moving over a surface



MacCormack's space marching (or downstream marching) technique

Exact analytical solution of this problem exist, which helps to obtain a reasonable feeling for the accuracy of the numerical technique

THE GOVERNING EQS OF A TWO-DIMENSIONAL SUPERSONIC FLOW (I)

The governing equations:

Euler equations (inviscid flow) for a steady 2D flow in strong conservation form, for adiabatic flow and no body forces (*f=0*)

$$\frac{\partial F}{\partial x} = -\frac{\partial G}{\partial y} \qquad G = \begin{Bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho v\left(e + \dfrac{V^2}{2}\right) + pv \end{Bmatrix} \qquad F = \begin{Bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho u\left(e + \dfrac{V^2}{2}\right) + pu \end{Bmatrix}$$

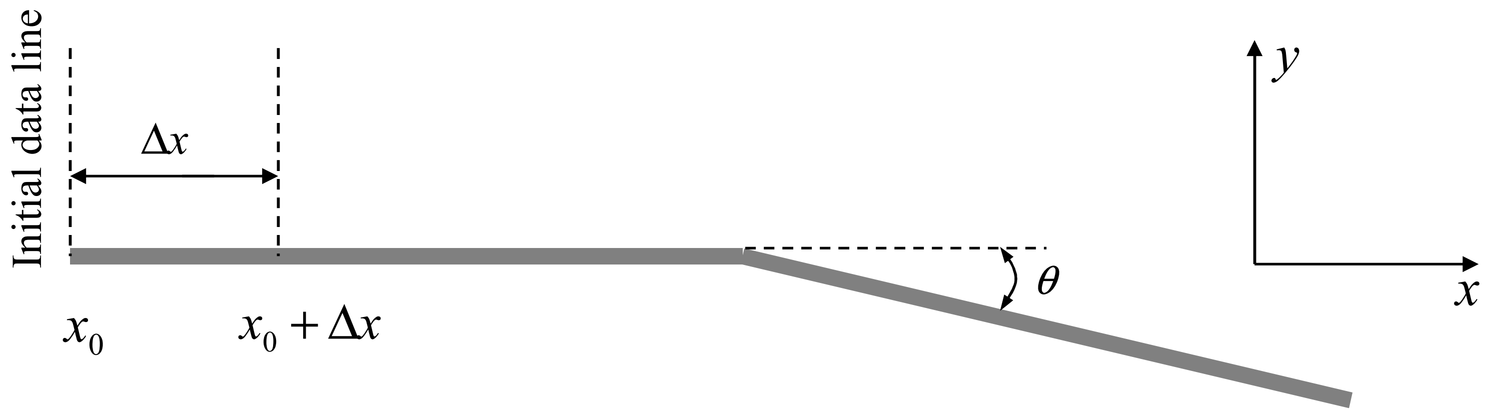
This strong conservation form allows us to apply a downstream marching solution

THE GOVERNING EQS OF A TWO-DIMENSIONAL SUPERSONIC FLOW (II)

Downstream marching solution: $$\frac{\partial F}{\partial x} = -\frac{\partial G}{\partial y} \qquad \frac{\partial G}{\partial y} \to \frac{\partial F}{\partial x} \to F$$

If the flow field variables are given at x_o as a function of y (initial data line), then the y derivative of G is known along this line → the x derivative of F can be calculated → we can advance the flow field variables to the next vertical line located at $x_o+\Delta x$

Solution can be carried out by marching in steps of Δx along the x direction



THE GOVERNING EQS OF A TWO-DIMENSIONAL SUPERSONIC FLOW (III)

We denote:

$$F_1 = \rho u$$

$$F_2 = \rho u^2 + p$$

$$F_3 = \rho u v$$

$$F_4 = \rho u\left(e + \frac{u^2 + v^2}{2}\right) + pu = \rho u\left(\frac{1}{\gamma - 1}\frac{p}{\rho} + \frac{u^2 + v^2}{2}\right) + pu = \frac{1}{\gamma - 1} pu + \rho u \frac{u^2 + v^2}{2} + pu$$

$$e = c_v T = \frac{RT}{\gamma - 1} = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

$$F_4 = \frac{\gamma}{\gamma - 1} pu + \rho u \frac{u^2 + v^2}{2}$$

$$G_1 = \rho v$$

$$G_2 = \rho u v$$

$$G_3 = \rho v^2 + p$$

$$G_4 = \rho v\left(e + \frac{u^2 + v^2}{2}\right) + pv$$

$$G_4 = \frac{\gamma}{\gamma - 1} pv + \rho v \frac{u^2 + v^2}{2}$$

THE GOVERNING EQS OF A TWO-DIMENSIONAL SUPERSONIC FLOW (IV)

Extra work to do:

solving the eq. $\frac{\partial F}{\partial x} = -\frac{\partial G}{\partial y}$ we get F_1, F_2, F_3, F_4 and we need G_1, G_2, G_3, G_4.

(1) Decode the primitive variables (u,v,p,T) from the flux variables F_1, F_2, F_3, F_4

$$\rho = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad \text{where} \quad A = \frac{F_3^2}{2F_1} - F_4$$

$$u = \frac{F_1}{\rho} \qquad B = \frac{\gamma}{\gamma - 1} F_1 F_2$$

$$v = \frac{F_3}{F_1} \qquad C = -\frac{\gamma + 1}{2(\gamma - 1)} F_1^3$$

$$p = F_2 - F_1 u$$

$$T = \frac{p}{\rho R}$$

THE GOVERNING EQS OF A TWO-DIMENSIONAL SUPERSONIC FLOW (V)

(2) Elements G_1, G_2, G_3, G_4 are more desirably expressed in terms of F_1, F_2, F_3, F_4 than in terms of the primitive variables.

We get F_1, F_2, F_3, F_4 at a point and we need G_1, G_2, G_3, G_4 to calculate in the next point. $\frac{\partial G}{\partial y} \to \frac{\partial F}{\partial x} \to F$

$$G_1 = \rho v = \rho \frac{F_3}{F_1}$$

$$G_2 = F_3$$

$$G_3 = \rho v^2 + p = \rho \left(\frac{F_3}{F_1}\right)^2 + p = \rho \left(\frac{F_3}{F_1}\right)^2 + F_2 - \frac{F_1^2}{\rho}$$

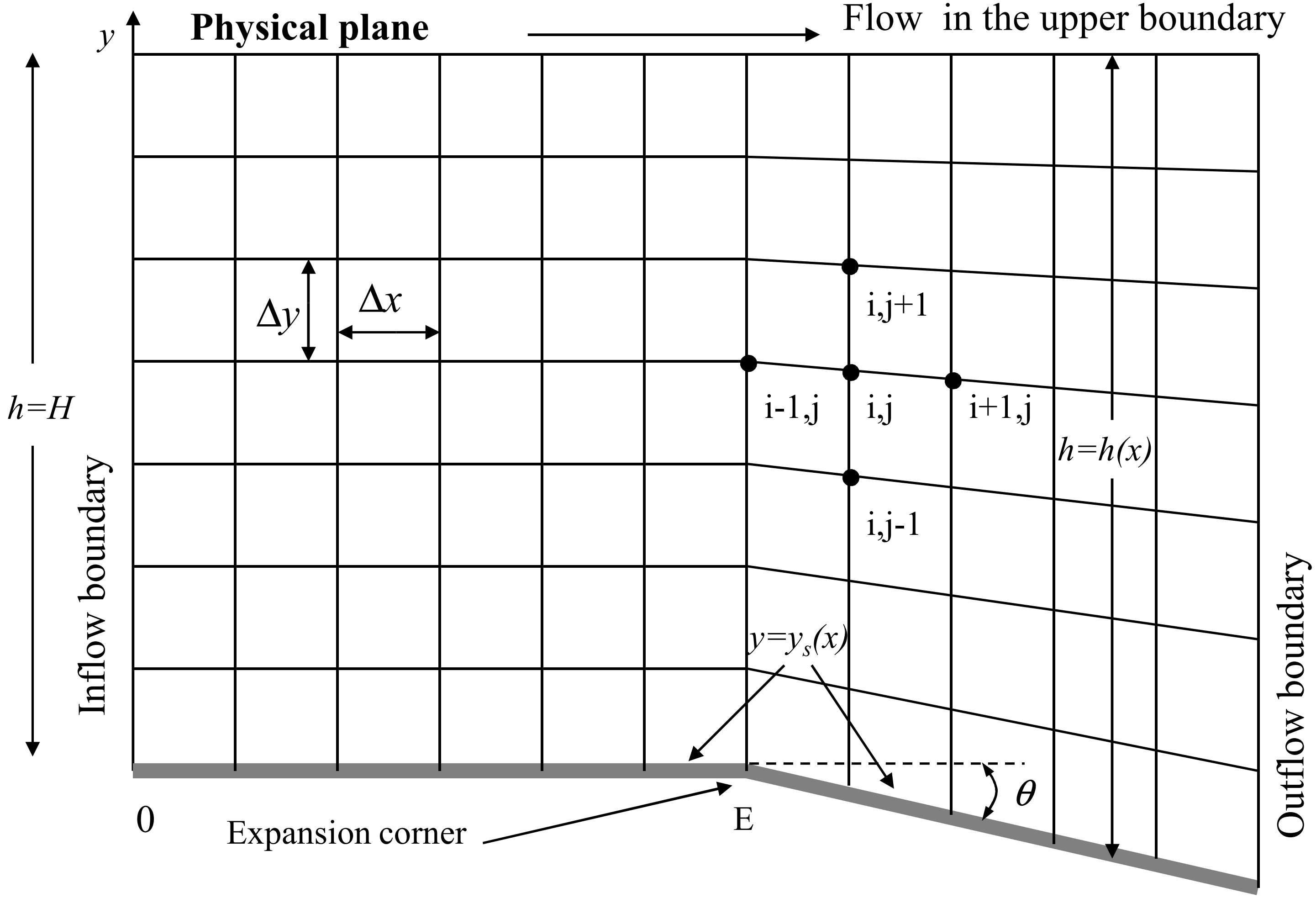
←Obtention of G's as a function of the F's

$$p = F_2 - \rho u^2 = F_2 - \frac{F_1^2}{\rho}$$

$$G_4 = \frac{\gamma}{\gamma - 1} p v + \rho v \frac{u^2 + v^2}{2} = \frac{\gamma}{\gamma - 1}\left(F_2 - \frac{F_1^2}{\rho}\right)\frac{F_3}{F_1} + \frac{\rho}{2}\frac{F_3}{F_1}\left[\left(\frac{F_1}{\rho}\right)^2 + \left(\frac{F_3}{F_1}\right)^2\right]$$

THE GOVERNING EQS OF A TWO-DIMENSIONAL SUPERSONIC FLOW (VI)

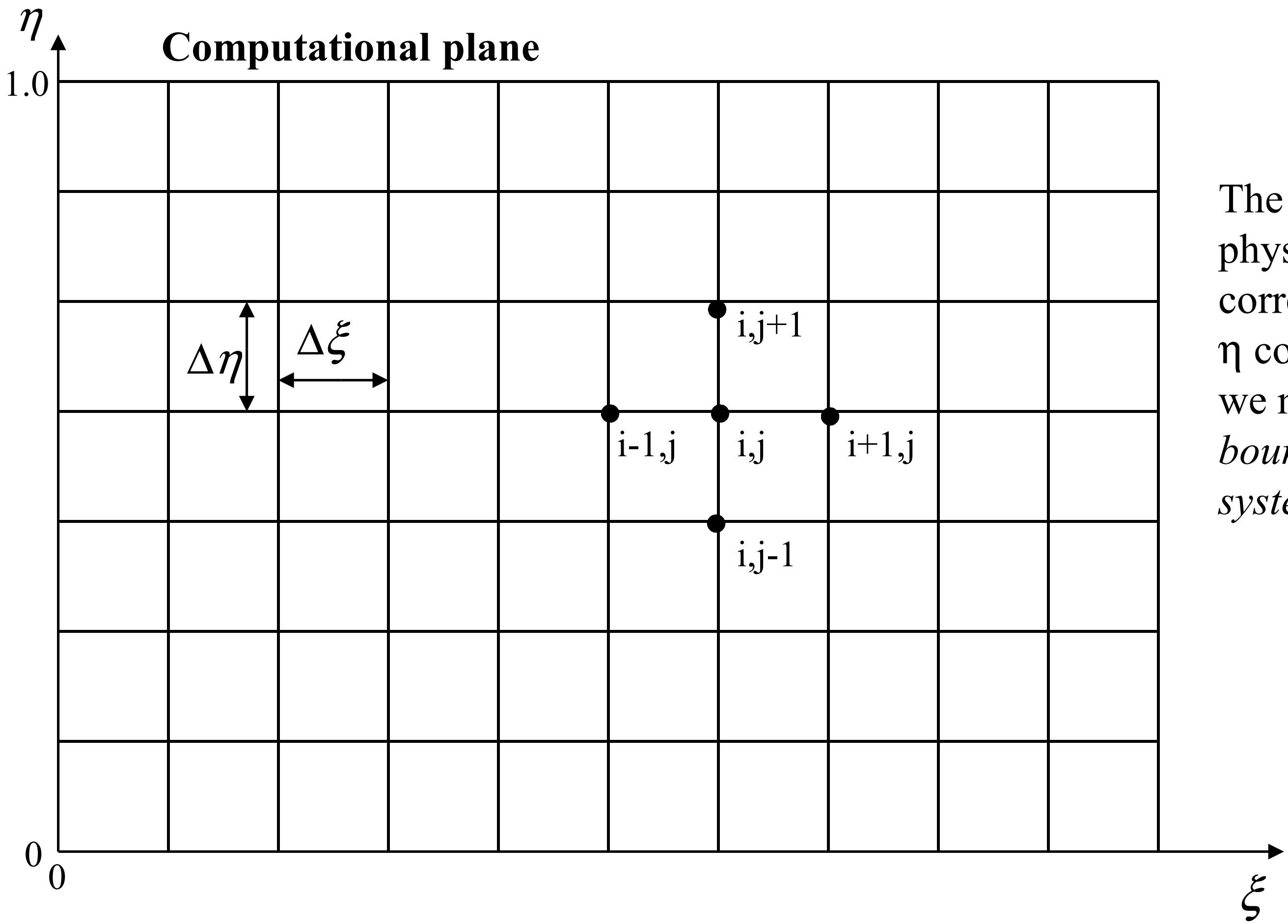
The transformation: grid generation, equation transformation



Not a completely rectangular grid.

Physical plane must be transformed to a computational plane where the finite-difference grid is rectangular

THE GOVERNING EQS OF A TWO-DIMENSIONAL SUPERSONIC FLOW (VII)



The bottom surface in the physical plane should correspond to a constant η coordinate curve; i.e. we need to establish a *boundary-fitted coordinate system.*

THE GOVERNING EQS OF A TWO-DIMENSIONAL SUPERSONIC FLOW (VIII)

The transformation:

$y_s(x)$ - y location of the lower surface

h(x) - local height from the lower to the upper boundary in the physical plane

$$\xi = x \qquad \eta = \frac{y - y_s(x)}{h(x)}$$

We can carry out the finite-difference calculations on the rectangular grid in the $\xi - \eta$ plane.

The partial differential equations for the flow are numerically solved in the transformed space and therefore must be appropriately transformed for use in the transformed, computational plane: $\frac{\partial F}{\partial x} = -\frac{\partial G}{\partial y}$ must be transformed into terms dealing with η and ξ.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\left(\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial \eta}{\partial x}\right)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}\left(\frac{\partial \xi}{\partial y}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial \eta}{\partial y}\right)$$

$$\frac{\partial \xi}{\partial x} = 1 \qquad \frac{\partial \xi}{\partial y} = 0 \qquad \frac{\partial \eta}{\partial x} = -\frac{1}{h}\frac{dy_s}{dx} - \frac{\eta}{h}\frac{dh}{dx} \qquad \frac{\partial \eta}{\partial y} = \frac{1}{h}$$

THE GOVERNING EQS OF A TWO-DIMENSIONAL SUPERSONIC FLOW (IX)

A faster calculation of $\dfrac{\partial \eta}{\partial x}$

From the physical plane and denoting the x location of the expansion corner by $x=E$:

For $x \leq E:$ $\quad y_s = 0$, $\quad h = H$

For $x \geq E:$ $\quad y_s = -(x-E)\tan\theta$, $\quad h = H + (x-E)\tan\theta$

Differentiating these expressions:

For $x \leq E$ $\quad \dfrac{dy_s}{dx} = 0$, $\quad \dfrac{dh}{dx} = 0$

For $x \geq E$ $\quad \dfrac{dy_s}{dx} = -\tan\theta$, $\quad \dfrac{dh}{dx} = \tan\theta$

Result:

$$\frac{\partial \eta}{\partial x} = \begin{cases} 0 & for \quad x \leq E \quad (8.25a) \\ (1-\eta)\dfrac{\tan\theta}{h} & for \quad x \geq E \quad (8.25b) \end{cases}$$

THE GOVERNING EQS OF A TWO-DIMENSIONAL SUPERSONIC FLOW (X)

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}+\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y}=\frac{1}{h}\frac{\partial}{\partial \eta}$$

Substituting $\frac{\partial \xi}{\partial x},\frac{\partial \xi}{\partial y},\frac{\partial \eta}{\partial x},\frac{\partial \eta}{\partial y}$ into $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ where $\frac{\partial \eta}{\partial x}$ is given by equations in previous page.

$$\frac{\partial F}{\partial x}=-\frac{\partial G}{\partial y}$$

$$\frac{\partial F}{\partial \xi}+\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial F}{\partial \eta}=-\frac{1}{h}\frac{\partial G}{\partial \eta}$$

$$\frac{\partial F}{\partial \xi}=-\left[\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial F}{\partial \eta}+\frac{1}{h}\frac{\partial G}{\partial \eta}\right]$$

$$Continuity: \quad \frac{\partial F_1}{\partial \xi}=-\left[\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial F_1}{\partial \eta}+\frac{1}{h}\frac{\partial G_1}{\partial \eta}\right]$$

$$x \quad momentum: \quad \frac{\partial F_2}{\partial \xi}=-\left[\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial F_2}{\partial \eta}+\frac{1}{h}\frac{\partial G_2}{\partial \eta}\right]$$

$$y \quad momentum: \quad \frac{\partial F_3}{\partial \xi}=-\left[\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial F_3}{\partial \eta}+\frac{1}{h}\frac{\partial G_3}{\partial \eta}\right]$$

$$Energy: \quad \frac{\partial F_4}{\partial \xi}=-\left[\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial F_4}{\partial \eta}+\frac{1}{h}\frac{\partial G_4}{\partial \eta}\right]$$

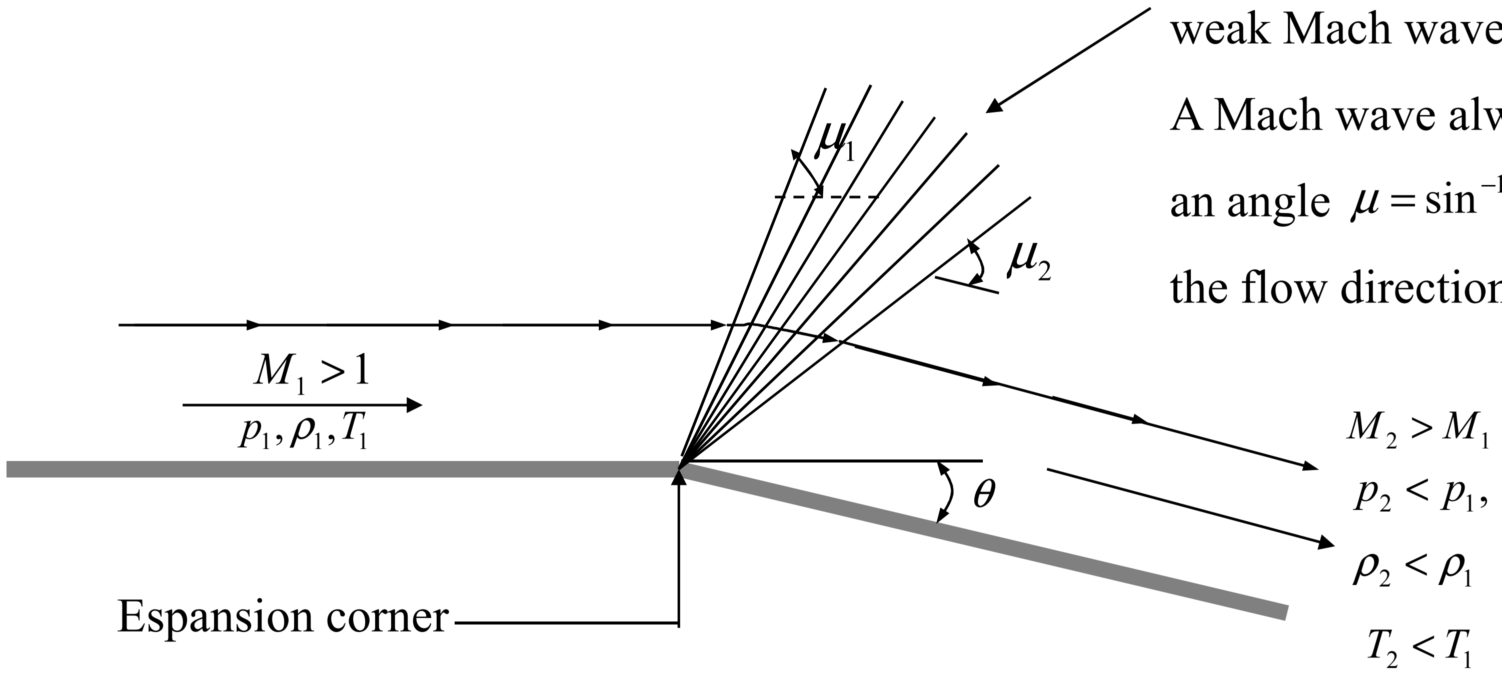
Governing equations in dimensional form to be solved numerically in the computational plane.

PRANDTL-MEYER EXPANSION WAVE SOME PHYSICAL CHARACTERISTICS (I)

2D inviscid supersonic flow moving over a surface

Infinite number of infinitely weak Mach waves.

A Mach wave always shows an angle $\mu = \sin^{-1}\frac{1}{M}$ with the flow direction.



We will now obtain the analytical solution of this problem:
M_2, p_2, ρ_2 and T_2 will be calculated analitically from M_1, p_1, ρ_1 and T_1 .

PRANDTL-MEYER EXPANSION WAVE SOME PHYSICAL CHARACTERISTICS (II)

μ_1 Angle between the leading Mach wave and the upstream flow direction

μ_2 Angle between the trailing Mach wave and the downstream flow direction

$$\mu_1 = \sin^{-1}\frac{1}{M_1} \quad \text{and} \quad \mu_2 = \sin^{-1}\frac{1}{M_2}$$

Flow through a expansion wave is isentropic. As the flow passes through the expansion wave: M increases, p, T and ρ decrease. Inside the wave, the flow is 2D

The analytical solution of the flow across a centered expansion waves depends on the simple relation:

$$f_2 = f_1 + \theta$$

$$f = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1(M^2-1)}{\gamma+1}} - \tan^{-1}\sqrt{M^2-1}$$

Prandtl-Meyer function for a calorically perfect gas (FoA, p. 532-537)

PRANDTL-MEYER EXPANSION WAVE SOME PHYSICAL CHARACTERISTICS (III)

Analytical solution: $M_1 \rightarrow f_1 \xrightarrow{\theta} f_2 \rightarrow M_2 \rightarrow p_2, T_2, \rho_2$

| isentropic flow relations

$$p_2 = p_1 \left\{ \frac{1 + [(\gamma - 1)/2] M_1^2}{1 + [(\gamma - 1)/2] M_2^2} \right\}^{\gamma/(y-1)}$$

$$T_2 = T_1 \frac{1 + [(\gamma - 1)/2] M_1^2}{1 + [(\gamma - 1)/2] M_2^2}$$

$$\rho_2 = \frac{p_2}{RT_2}$$

With all these equations, flow in 2 is determined

At the corner itself, there is a singular point at which the streamline at the wall experiences a discontinuous change in direction and where the flow properties are discontinuous. This will have an impact in the numerical method.

PROJECT 3: PRANDTL-MEYER EXPANSION WAVE

MacCormack's predictor-corrector explicit finite difference method.

Courant condition: $\Delta\xi = C\Delta\eta \,/\, |\tan(\theta \pm \mu)|_{\max}$ (the maximum obtained for each x-step)

Lower boundary condition: velocity tangent to the wall

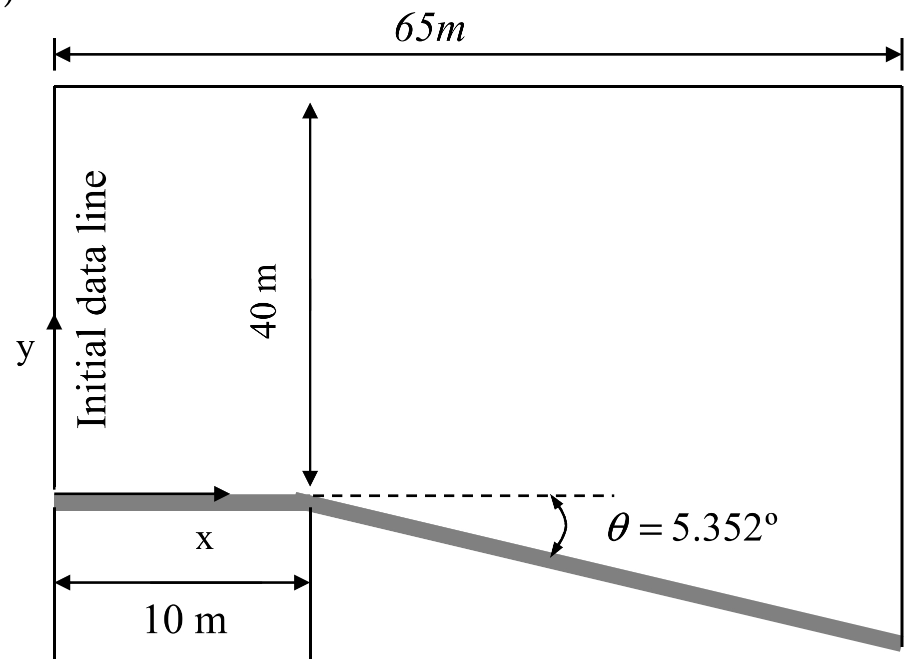
$$h = \begin{cases} 40 \quad m & 0 \le x \le 10m \\ 40 + (x-10)\tan\theta & 10 \le x \le 65m \end{cases}$$

$M_1 = 2$

$p_1 = 1.10 \times 10^5 \dfrac{N}{m^2}$

$\rho_1 = 1.23 \dfrac{kg}{m^3}$

$T_1 = 286.1 \quad K$



Physical plane, drawn to scale

Suggested values:

C=0.5   Δη=0.1

Tricks:

- Each x-step, numerical viscosity must be added due to the discontinuity in the sharp corner (CFD, p.391-392).
- Read carefully how to apply the lower boundary condition (CFD, p.392-395).