Puzzles of Cardinality

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Recall that the *cardinality* of a finite set A, denoted $|A|$, is simply the number of elements in the set. Finding cardinalities is one of the main goals of combinatorics and can be quite challenging in general, but for small sets, counting the number of elements should be easy, right? For example, what is the cardinality of the following?

$$A = \{2, |A|\}.$$

Something strange is going on here. If $|A| = 2$, then we have $A = \{2\}$ which only contains one element. But if $|A| = 1$, then $A = \{1, 2\}$ and thus contains two elements. What are we to make of this?

Perhaps you want to say that sets are not allowed to contain their own cardinality. But we surely agree that $\{1, 2, 3\}$ is a set, and it definitely contains its cardinality, 3. In fact, the following very nice problem appeared on the 1996 William Lowell Putnam Mathematical Competition [1]:

> Define a "selfish" set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

We are not interested in solving this problem here (we encourage you to try it on you own, it is quite approachable as Putnam problems go), but rather in pointing out that there is nothing inherently wrong with the apparent self-reference in a set containing its own cardinality.

And yet there is clearly something wrong with *defining* a set in terms of its own cardinality, as our "paradox" above demonstrates. It is a symptom of the broader phenomenon of self reference, which often leads to puzzlement. One resolution is to simply not allow this, but where is the fun in that. Is it possible for a paper to use itself as one of its references? At least one paper has [3]. That's this paper, so should it count as an example?

To be clear, we are not claiming that we have found an error in mathematics. Defining a set by listing its elements belongs squarely within *naive set theory*. This is set theory as it is taught and used in most undergraduate mathematics courses, and truthfully, how most mathematicians think of sets. Generally, it is sufficient for day-to-day mathematics, but from a foundations of mathematics perspective, is necessarily lacking.

The standard illustration of the problem with naive set theory is some variation of Russell's paradox: Is the set of all sets that are not elements of themselves an element of itself? Here, the "set" $A = \{x \ : \ x \notin x\}$ is the problem: $A \in A$ if and only if $A \notin A$. This is a lovely paradox, but not especially convincing to someone encountering sets for the first time. Why would you think you could define a set of sets like that to start with? Sets are collections of numbers! Our paradox above illustrates that even with finite sets of numbers, naive set theory can be problematic.

The fix: limit what a set is by giving axioms that say exactly what sets you can build. The common axioms used today are those of Zermelo-Fraenkel set theory (ZFC consists of these axioms plus the axiom of choice; see [2] for a gentle introduction). The relevant axiom for both Russell's paradox and our paradox of finite cardinality is *comprehension*. This is the axiom that allows us to construct sets through "set builder notation." Essentially, it says that sets exist of the form

$$\{x \in z \ : \ \varphi(x)\},$$

where z is a set that already exists and $\varphi(x)$ is a formula with one free variable x. Requiring a superset z, thus defining subsets of sets we have already built, resolves Russell's paradox. But not ours, which can be defined using set builder notation as,

$$A = \{x \in \mathbb{Z} \ : \ x = 2 \text{ or } x = |A|\}.$$

Rather, we need to pay attention to the other caveat of the axiom scheme, that $\varphi(x)$ does not contain free variables other than x (or in particular, that A does not occur freely). Phew! Mathematics survives.

So the resolution to our paradox is that sets like ours don't actually exist. But this shouldn't stop us from playing with them. We are reminded of the liar paradox: "this statement is false" can neither be true nor false and thus is not a statement at all. However, allowing statements to talk about their own truth values leads to the enjoyable *knights and knaves* puzzles popularized by Raymond Smullyan and others (for example, see [5] or [6]). In what follows we will see that similar puzzles can arise by defining sets in terms of their own cardinalities. We will investigate when these *puzzles of cardinality* have solutions and determine exactly which sets can be solutions to such puzzles. Finally, we will take a closer look at how puzzles of cardinality compare to knights and knaves puzzles.

1 Puzzles and solutions

Let's solve some puzzles.

To fix terminology, we say that a *cardinality puzzle* is a description of a set that explicitly mentions the cardinality of itself. When we write a cardinality puzzle, we often use $a = |A|$ as a notational convenience. For instance, our example from the introduction can be written as $A = \{2, a\}$. This is a paradox, so as a puzzle, it has no solution. We will see that other outcomes are possible as well.

Try solving the following puzzles before looking at the solutions and discussions about the lessons we can learn from these examples.

Puzzle 1. What is the cardinality of the set $A = \{2, 3, a\}$?

Puzzle 2. What is the cardinality of the set $A = \{1, 3, a\}$?

Puzzle 3. What is the cardinality of the set $A = \{4, a, 2a\}$?

Puzzle 4. What is the cardinality of the set $A = \{4, 5, a, a + 1, 2a - 1\}$?

As a general hint, a good place to start is to determine what the potential cardinalities might be. If you viewed each set as a *multiset*, then the cardinality would just be the number of terms displayed (which makes these puzzles a lot less entertaining). But as a set, some or all of the terms containing a may coincide with constants or each other, reducing the cardinality as a set. Once you know the reasonable values that a can take on, you can check each for consistency or contradictions.

Solution to Puzzle 1. We are given $A = \{2, 3, a\}$. A quick inspection shows us that the possible cardinalities are 2 and 3. If $a = 2$, then $A = \{2, 3\}$. This set has cardinality 2, as assumed. If $a = 3$, then $A = \{2, 3\}$. Again this has cardinality 2, but that contradicts the assumption $a = 3$. We conclude that the cardinality of A is 2. ◇

Since we found a cardinality that works, we can say this cardinality puzzle has a *solution*, namely the resulting set $A = \{2, 3\}$ with cardinality 2. In general, we will say that a cardinality puzzle has a *solution* if there is a set of numbers that agrees with the description. Whether we call the set or its cardinality the solution is immaterial: if you have either of these, the other is completely determined.

Solution to Puzzle 2. Here, we solve $A = \{1, 3, a\}$. This puzzle has no solution, despite the title of this paragraph. As with Puzzle 1, the only possible cardinalities for A are 2 and 3. However, this time if $a = 2$, then by the definition of A, we have $A = \{1, 3, 2\}$. So in that case we get a set of cardinality 3, which contradicts our assumption that $a = 2$. Likewise, if $a = 3$, then $A = \{1, 3\}$, a set of cardinality 2, leading to another contradiction. $\diamond$

This puzzle can be generalized. Fix an integer $k \geq 1$ and consider,

$$A = \{1, 2, 3, \ldots, k, k + 2, a\}.$$

By inspection, the set A has cardinality $k + 1$ or $k + 2$. But if $a = k + 1$, then A has $k + 2$ elements, and if $a = k + 2$ then A has $k + 1$ elements (a contradiction in both cases).

So far, we have witnessed puzzles with a unique solution or no solution at all. Could there be puzzles with more than one solution?

Solution to Puzzle 3. This time $A = \{4, a, 2a\}$, so a could reasonably be 1, 2, or 3. If $a = 1$, then A contains 1 and 4, forcing $a > 1$, a contradiction. If $a = 2$, then $A = \{2, 4\}$. This indeed has cardinality 2. So perhaps we have found the solution? Not so fast: if $a = 3$, then $A = \{4, 3, 6\}$ and we get a set of cardinality 3. So there are two solutions. $\diamond$

Is there a puzzle with three solutions? Consider $A = \{1, 2, a, a - 1\}$. The possible cardinalities are $a = 2, 3, 4$, which yield the solutions $\{1, 2\}$, $\{1, 2, 3\}$, and $\{1, 2, 4, 3\}$, respectively. So all three potential cardinalities lead to solutions.

We can keep going, to get puzzles with four solutions, five solutions, and so on. For $n \geq 2$, the following cardinality puzzle has n solutions:

$$A_n = \{n, n + 1, \ldots, 2n - 2,\ a, a + 1, \ldots, a + n - 1\}.$$

For example, let $n = 6$. Then $A_6 = \{6, 7, 8, 9, 10,\ a, a + 1, \ldots, a + 5\}$. The six possible cardinalities and their corresponding solutions are:

$$\begin{aligned} a = 6&: \quad \{6, 7, \ldots, 11\} \\ a = 7&: \quad \{6, 7, \ldots, 12\} \\ &\ \vdots \\ a = 11&: \quad \{6, 7, \ldots, 16\}. \end{aligned}$$

Moving forward, we will restrict our attention to cardinality puzzles with *unique* solutions. After all, in some moral sense, a *puzzle* should have exactly one solution. We will see that there can still be quite a variety of these puzzles, as our final solution of this section illustrates.

Solution to Puzzle 4. We are looking at $A = \{4, 5, a, a + 1, 2a - 1\}$. There are four possible cardinalities: $a = 2, 3, 4, 5$. The only one that works is $a = 3$, which gives us $\{3, 4, 5\}$ as the unique solution. $\diamond$

2 Unique solutions and puzzle complexity

So far, we have seen two cardinality puzzles with unique solutions. Puzzle 1 has the unique solution $\{2, 3\}$. Puzzle 4 has the unique solution $\{3, 4, 5\}$. What other sets might be the unique solution to a cardinality puzzle?

First note that there is not a bijection between puzzles and solutions. Consider the puzzle $A = \{a, a+1, a+2\}$. We have a possibly 1, 2, or 3. When $a = 1$ or $a = 2$, we get a contradiction. If $a = 3$, then $A = \{3, 4, 5\}$. This set does indeed have 3 elements, so it is the unique solution. But this is the same solution as Puzzle 4. This says that if we want to somehow classify the sets that you get as a solution to some puzzle, you must work backwards from the solutions.

Let's start with an example. Consider the set

$$S = \{1, 3, 4, 6\}.$$

Here is a puzzle that has S as its unique solution.

Puzzle 5. What is the cardinality of the set $A = \{1, 3, 6, a,\ 18 - 3a\}$?

This puzzle really does have S as its solution: The only reasonable choices for a are 3, 4, or 5. If $a = 3$, we get $A = \{1, 3, 6, 9\}$, too big. If $a = 5$, we get $A = \{1, 3, 6, 5\}$, too small. If $a = 4$, we get $\{1, 3, 4, 6\}$, just right.

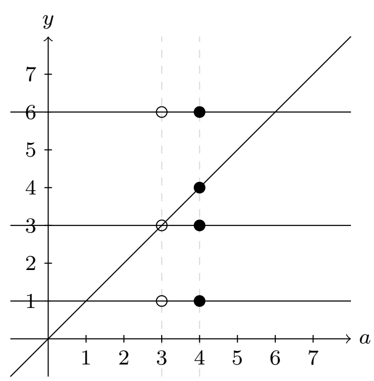

Figure 1: Two solutions

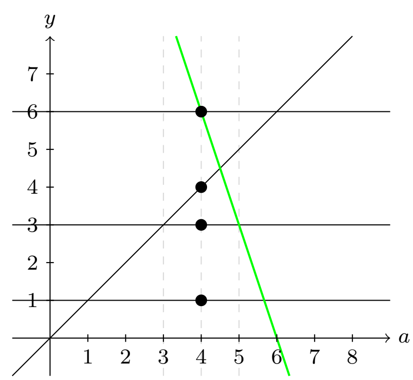

Figure 2: Puzzle 5 graphically.

But where did the puzzle come from? We replaced 4 with a (after all, we want $a = 4$ to give the unique solution). This is not enough though, since $A = \{1, 3, a, 6\}$ has two solutions: $a = 3$ and $a = 4$. You can visualize this in Figure 1. To fix this, we add a new line $f(a) = 18 - 3a$, as shown in Figure 2. This was chosen so that it passes through $(4, 6)$ (so at $a = 4$, $f(a)$ coincides with the largest constant, 6), and $(5, 3)$ (so at $a = 5$, $f(a)$ coincides with the second largest constant, 3). Since this will give a line with negative slope, we know at $a < 4$, we will have $f(a)$ distinct from all the constants and line $y = a$.

Thus $a = 3$ is no longer a solution. And because f intersects another line at $a = 5$, that cannot be a solution either.

Now generalize. Notice that we will not be able to get every set, since we insist a be an element of the set. That is, any solution will necessarily contain its own cardinality, i.e., be selfish. The surprising result is that this is also sufficient.

Theorem 1. *A set S of natural numbers is the unique solution to a cardinality puzzle if and only if S is selfish.*

Proof. To establish the nontrivial direction, we describe a method for building a cardinality puzzle from any selfish set S. The only selfish set of cardinality 1 is $\{1\}$, and this is the unique solution to the (admittedly trivial) puzzle $A = \{a\}$. Next consider selfish sets of cardinalty 2. That is, $S = \{2, c\}$ for some constant $c \neq 2$. If $c \neq 1$, consider the puzzle $A = \{c, a\}$. Here the two potential solutions are $a = 1$ and $a = 2$. If $a = 2$, we get our desired set as a solution; while if $a = 1$, we get a set of size 2, a contraction. In the case that $c = 1$, we can take the puzzle $\{a, 3 - a\}$. This again has potential solutions $a = 1$ and $a = 2$. If $a = 1$, we get our desired set $\{1, 2\}$, but it has cardinality 2, a contradiction. On the other hand, if $a = 2$, we also get $\{1, 2\}$, so it is the unique solution.

Now suppose $|S| \geq 3$. Let k be the (true) cardinality of the set S (to avoid confusion with the constant a that represents the cardinality in the puzzle). Thus $S = \{k, c_1, \ldots, c_{k-1}\}$ for some constants $c_1 < c_2 < \cdots < c_{k-1}$, all distinct from k.

Consider the cardinality puzzle,

$$A = \{c_1, \ldots, c_{k-1}, a, f(a)\},$$

where f is the linear function passing through the points (k, c_{k-1}) and $(k + 1, c_{k-2})$. Since $c_{k-2} < c_{k-1}$, this line has slope $c_{k-2} - c_{k-1} < 0$, so in particular, $f(k - 1) > c_{k-1}$ and thus not equal to any c_i.

This cardinality puzzle has three potential solutions: $a = k - 1$, $a = k$, and $a = k+1$. If $a = k-1$, we get the set $A = \{c_1, \ldots, c_{k-1}, k-1, f(k-1)\}$, which has at least k elements, a contradiction. (Notice, it has exactly k elements if some c_i is $k - 1$.) Similarly, if $a = k + 1$, then $A = \{c_1, \ldots, c_{k-1}, k+1, f(k+1) = c_{k-2}\}$, which has at most k elements, again a contradiction. However, $a = k$ produces a solution: $A = \{c_1, \ldots, c_{k-1}, k, f(k) = c_{k-1}\}$, which really does contain k elements and is the desired selfish set. □

2.1 Increasing complexity

Some puzzles are going to be more interesting than others. One way to measure this is to consider how many reasonable values a might take on. We will call this the *complexity* of the puzzle. The puzzles generated by the proof of Theorem 1 will all have complexity 3 (as long as S has size at least 3). We could make these puzzles more complex by adding additional terms that rely on a. In fact, we can make puzzles as complex as we like.

Theorem 2. *For any selfish set S with at least three elements and any $n \geq 3$, there is a cardinality puzzle with complexity n that has S as its unique solution.*

Instead of giving a full proof, we will illustrate the idea by increasing the complexity of Puzzle 5, which has solution $S = \{1, 3, 4, 6\}$. Here is a puzzle with that solution, with complexity 5:

Puzzle 6. What is the cardinality of the set $A = \{1, a, 17 - \frac{11}{4}a, \frac{1}{4}a + 2, \frac{1}{4}a + 5\}$?

To see that this puzzle really has S as its unique solutions, you could consider the five cases $a = 1, 2, \ldots, 5$. This is easier to do graphically, as seen in Figure 3.

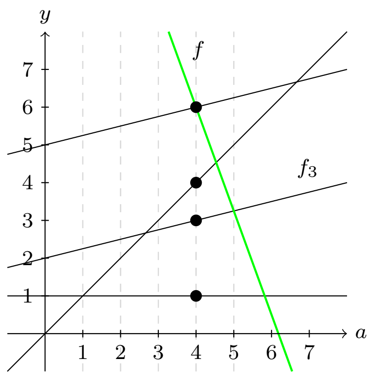

Figure 3: Puzzle 6 graphically.

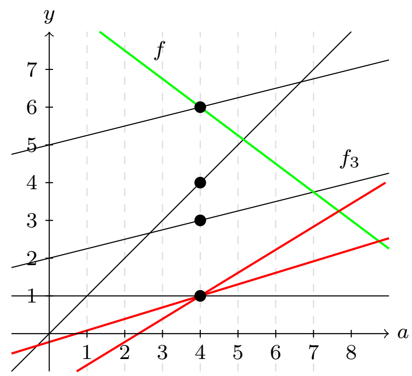

Figure 4: Upping complexity by 2.

Note that at any $a < 4$, the negatively sloped line f ensures there are at least 4 elements in the set. This is in part due to our choices for the slopes of the other lines. We have *tilted* some of the constants in S to be lines with slope $1/4$ (in general, this would be slope $1/k$), so none of these lines will intersect any other line (other than $y = a$, which they already intersected as constants) to the right of $a = 0$. For example, the constant $y = 3$ became the line $f_3(x) = \frac{1}{4}x + 2$. Note, this will require us to modify the line f so it (still) intersects f_3 at $a = 5$.

We can tilt as few or as many of the constants as we wish, and each constant we tilt increases the complexity by 1. (In general, when tilting constants, start with the largest c_i and work down.) To further increase complexity, we add new lines (not replacing constants).

Each of the new lines (as seen in Figure 4) will pass through $(4, 1)$ (or (k, c_1) in general) and have positive slope small enough so they do not intersect any other line before $a = j$, where j is the number of lines including constants (the multiset cardinality). This will ensure that for every $4 < a < 7$ (or $k < a < j$ in general), there are $j = 7$ lines. We then slide the intersection of f with f_3 to occur at $a = 7$, ensuring that at $a = 7$ there are only 6 lines.

In general, only two lines can intersect at any point with $a > k$ (because we tilted larger constants first), so this process will work unless two lines happen to intersect at $a = j - 1$. In this case, we would have $j - 1$ points at $a = j - 1$, a second solution. We remedy this by putting the first new line through the second smallest original line at $a = j-1$. This results in $j-2$ points at $a = j-1$. We would still have at most $j-1$ points at j, and at least $j-1$ points at any $k < a < j-1$.

3 Symbiotic sets, knights, and knaves

Let's now consider another way to make puzzles of cardinality more complex: make them about multiple sets at the same time. Instead of *selfish* sets, we will take a pair (or more) of sets each containing the cardinality of another. One might call these *symbiotic sets*. As before, let $|A| = a$, but now also $|B| = b$ and so on. Here are a few puzzles to try.

Puzzle 7. What are the cardinalities of $A = \{1, 2, b\}$ and $B = \{2, a\}$?

Puzzle 8. What are the cardinalities of $A = \{3, b\}$ and $B = \{1, a, b\}$?

Puzzle 9. What are the cardinalities of $A = \{2, 3, 6, a, b+3\}$ and $B = \{a, 5\}$?

As with the one-set puzzles, a good strategy for solving the puzzles is to consider the possible cardinalities of each set and proceed by cases. Usually, it is enough to consider the possible cases for just one of the sets, as this will give information about the other automatically.

Solution to Puzzle 7. Given $A = \{1, 2, b\}$ and $B = \{2, a\}$, the cardinality of A is either 2 or 3. In the case that $a = 3$, the set B becomes $B = \{2, 3\}$, which makes $b = 2$. But then $A = \{1, 2\}$, which only has cardinality 2, a contradiction. On the other hand, if $a = 2$, then $B = \{2\}$, so $b = 1$. This is consistent, given $A = \{1, 2\}$ of cardinality 2.

Thus $a = 2$ and $b = 1$. ◇

Note that this puzzle has a unique solution. In fact, all the puzzles in this section do, but this is not at all necessary. Consider $A = \{1, b\}$ and $B = \{2, a\}$. If $a = 2$ then $B = \{2\}$ but then $a = 1$. On the other hand, if $a = 1$ then $B = \{1, 2\}$ making $b = 2$ and thus $a = 2$.

Symbiotic sets, even more so than one-set puzzles, remind us of the classic *knights and knaves* logic puzzles. In these puzzles, you encounter a number of trolls, each of whom is either a knight, who always tells the truth, or a knave, who always lies. You must determine the clan of each speaker based only on their statements.

The simplest possible puzzles involve a single speaker who either says, "I am a knight" or "I am a knave." In the first case, there are two possible solutions; in the latter there are none (the statement is inconsistent). This reminds us of $A = \{1, a\}$ and $A = \{2, a\}$, in at least as far as the first has two solutions ($a = 1$ or $a = 2$ are both consistent), and the latter has none.

Most knights and knaves (K&K) puzzles involve more than one troll. Here is a simple example. Suppose you meet two trolls, who make the following statements.

Troll 1: Troll 2 is a knave.

Troll 2: We are both knights.

This puzzle has a unique solution: Troll 1 is a knight and Troll 2 is a knave. To see this, suppose first that Troll 1 is a knave. That would make Troll 2 a knight (based on what Troll 1 says). But looking at the statement of Troll 2, we see that it must be false, a contradiction. So Troll 1 must be a knight. This means Troll 2 really is a knave, which is consistent (his statement is false).

Now let's return to our cardinality puzzles. Compare the proof of the K&K puzzle to the solution of Puzzle 8.

Solution to Puzzle 8. The cardinality puzzle $A = \{3, b\}$ and $B = \{1, a, b\}$ has a unique solution: $A = \{2, 3\}$ and $B = \{1, 2\}$ (so $a = 2$ and $b = 2$). To see this, suppose $a = 1$. That would mean that $b = 3$ (based on what A is defined as). But looking at B, we would then only have two elements, a contradiction. So a must be 2. This means it cannot be that $b = 3$ (that would leave A with only one element) so then we have $B = \{1, 2, b\}$, so $b = 2$. ◇

It is not just that the K&K puzzle and cardinality puzzle both have unique solutions, or that their solutions match up in some way. The proofs for those solutions basically match up as well.

Let's explore this connection further. The K&K puzzle had Troll 1 a knight and Troll 2 a knave. In the solution to the cardinality puzzle, the set A has the "expected" cardinality, while B does not. That is, A is shown with 2 elements, and those elements turn out to be distinct, while B is shown with 3 elements, but one of those elements overlaps with another. So B is *tricky* in a way: not what it claims to be. You might even say that the set B is lying to us.

In fact, we constructed Puzzle 8 by starting with the K&K puzzle. Here's how: Troll 2 says that both trolls are knights. He is making a claim about both trolls, so the set B should mention both cardinalities. Further, if Troll 2 is going to be telling the truth, then both these cardinalities should be equal to the number of elements displayed by their sets. We pick A in a way that would make $a = 2$ and $b = 3$. However, Troll 1 claims that Troll 2 is a knave, so the set A should mention b and should, if all its elements were to be distinct, make b not the number of elements displayed by B. So put 3 and b both into A, since B is displaying three elements.

What about going the other way? Can we find a K&K puzzle to match a given cardinality puzzle? Take Puzzle 9 for example. First, here is the solution.

Solution to Puzzle 9. We are given $A = \{2, 3, 6, a, b + 3\}$ and $B = \{a, 5\}$. If $b = 2$, then it must be that $a \neq 5$. But then there will be 5 elements in A, as it must be that $A = \{2, 3, 6, 4, 5\}$, a contradiction. The other possibility is that $b = 1$, which happens exactly when $a = 5$. Then the set $A = \{2, 3, 6, 5, 4\}$, which does indeed have 5 elements. ◇

We want a K&K puzzle that will have the following solution, translated from the Puzzle 9's solution above:

If Troll 2 is a knight, then Troll 1 will be a knave, but that would make Troll 1's statement true. The other possibility is that Troll 2 is a knave, which happens exactly when Troll 1 is a knight, and we see that his statement is indeed true.

We can capture all this with the following puzzle.

> Troll 1: If Troll 2 is a knight, then I am a knave.
>
> Troll 2: Troll 1 is a knave.

We could also have derived this puzzle by looking purely at the cardinality puzzle, not its solution. Notice that the set A *claims* to have 5 elements, and the set B claims 2. But for $b = 2$ it must be that $a \neq 5$, so B is claiming that A's claim is false. To get Troll 1's statement, A claims that $a \neq b + 3$. The claim is that if $b = 2$ then $a \neq 5$, and that if $b \neq 2$ then $a = 5$. Translated: if B is a knight, then A is a knave, and if B is a knave, then A is a knight. But A is also claiming that $a = 5$, so we really only need the first half of the conjunction.

We are making a pretty bold insinuation: that every cardinality puzzle has a corresponding K&K puzzle, and that every K&K puzzle has a corresponding cardinality puzzle. Perhaps there is a *solution-preserving* isomorphism between puzzles of the two sorts, where *solution* is meant to include the proof. But surely this is false: there are infinitely many cardinality puzzles that correspond to *I am a knave*. Or perhaps we would say that there are infinitely many ways to restate *I am a knave*?

Unfortunately, our insinuations are only backed up by circumstantial evidence at this point. For every K&K puzzle we have looked at, we have successfully been able to create a corresponding cardinality puzzle (we think this is the more interesting direction, as it suggests a method for constructing new cardinality puzzles). But while we have some heuristics for this conversion, there seems to be too much *tweaking* that needs to be done for us to concisely describe a full algorithm.

The more substantive problem is that it is difficult to classify what makes something a K&K puzzle. It might be possible to get some partial correspondence by restricting K&K puzzles to a specific type, but we will not explore this further here. Instead, we invite the reader to consider the *meta-puzzle* of taking a given K&K puzzle and constructing a cardinality puzzle to match. To give some suggestions at a strategy, here is a suitably complicated K&K puzzle to try.

> Troll 1: Only one of us is a knave.
>
> Troll 2: No, only one of us is a knight.
>
> Troll 3: We are all knaves.

Where to start? You will want to use three sets A, B and C. The set A will mention b and c (to say only one of us is a knave is to say that exactly one of the other trolls is a knight). The set B will mention a and c (Troll 2 is essentially saying that both of the other trolls are knaves). The set C will mention a, b, and c, since it must claim that all three sets are mistaken.

It is easier when everything can fit together without unwanted interference. To this end, "pad" the sets with some extra constants to ensure that the potential cardinalities of each are disjoint. For example, you could add constants to the sets to ensure that $5 \leq a \leq 7$, $2 \leq b \leq 4$, and $8 \leq c \leq 11$. These ranges are not obvious, except they will happen to work in this case.

So now we have the number of constants in each set, but what are they? Start with B. To claim that both of the other sets are knaves is to claim that their cardinalities are not $a = 7$ and not $c = 11$ (the maximum in each range will correspond to the set being truthful). This tells us that we want $B = \{7, 11, a, c\}$.

That was the easy one. The set C should claim that all three sets are untruthful, so include the constants 4, 7, and 11 (plus five others as part of the padding).

The set A is even more of a challenge, as its troll makes a disjunctive statement: either B or C is not its maximum cardinality. So we want either $b = 4$ but $c \neq 11$ or vice versa. One way to ensure this is for A to include both b and $c - 7$ (the claim of A being that these are distinct, so they cannot both be 4). But again, we need to play with the other constants to make enough room for the consistent solution to really be consistent.

Ready for the solution to the meta-puzzle? Here you go.

Puzzle 10. What are the cardinalities of the following sets?

$$A = \{1, 3, 5, 6, 7, b, c - 7\}$$

$$B = \{7, 11, a, c\}$$

$$C = \{4, 7, 11, 12, 13, 14, 15, 16, a, b, c\}.$$

4 Open questions

To conclude, we share a few directions an interested reader could explore.

All of the sets we have described above are presented by listing elements. If we allow set builder notation, we can still get some interesting puzzles. Consider,

$$A = \{x \in \mathbb{Z}^+ \ : \ x \leq a\}.$$

This puzzles has infinitely many solutions! In fact, A can have any finite cardinality. Even more interesting: $A = \{x \in \mathbb{Z}^+ \ : \ x = 13 \text{ or } x \leq a\}$ has infinitely many solutions, but only for $a > 12$. Here is a similar puzzle: $A = \{x \in \mathbb{N} \ : \ x \leq a\}$, where we take $\mathbb{N} = \{0, 1, 2, \ldots\}$. This has no solutions for a finite. However, if $a = \aleph_0$ (the cardinality of $\mathbb{N}$) then $A = \mathbb{N}$ is a solution.

Question 1. *Which (infinite) sets of cardinalities are possible for solutions to a single cardinality puzzle? What happens if we allow infinite cardinals, or even ordinals, to be elements of the set or solutions?*

Notice that none of the paradoxes or puzzles in this paper would be possible if we consider A to be a multiset instead of a set. The number of elements listed in a multiset is simply the cardinality of that multiset. However, multisets do open the door for a suite of puzzles with a similar feel. Let $m(k)$ denote the *multiplicity* of the element k (i.e., the number of times it appears in the multiset). What is the multiplicity of 1 in the following multiset?

$$A = \{1, m(1)\}.$$

If $m(1) = 1$, then A contains 1 twice (so $m(1) = 2$). If $m(1) = 2$, then $A = \{1, 2\}$, so $m(1) = 1$.

Question 2. *What sorts of interesting puzzles can we make that involve multiplicity? Moreover, take any statistic about a set or multiset, and allow it to be listed as an element in the set/multiset. Are there puzzles that refer to the mean? The maximum?*

Returning to our puzzles of cardinality, we argued above that any selfish set is the unique solution to a cardinality puzzle, in fact with arbitrarily large complexity. The functions of a we used were always linear, but when we increased complexity, the lines had non-integer slopes. Perhaps you find such a puzzle inelegant.

Question 3. *For every selfish set A and every integer $n \geq 3$, is there a cardinality puzzle of complexity n and unique solution A that uses only linear functions with integer coefficients? If not, which sets have such puzzles?*

While thinking about the elegance of a puzzle, we also wonder how difficult it would be to construct puzzles where, at every potential cardinality, you get a cardinality that is only one more or one less than that supposed solution. This is one way in which a puzzle might be less *obvious*, but there might be other ways to measure this as well. For example, the solution to a puzzle might be much smaller than the number of elements displayed in A (i.e., the multiset cardinality).

Question 4. *Can the difference between the cardinality of the solution and the multiset cardinality of the puzzle be controlled independently of the complexity?*

In terms of the knights and knaves connection, the difference between the solution's cardinality and the puzzle's multiset cardinality is a measure of how much the set is lying. This doesn't make much sense in classical logic: a troll is either a knight or a knave. This suggests a way in which cardinality puzzles are more expressive than K&K puzzles, so perhaps cardinality puzzles are better suited to model non-classical logic. See [4] for some examples of how K&K puzzles behave in this realm. Similarly, Smullyan's books contain many variations on K&K puzzles; perhaps there is a more general sort of puzzle to which cardinality puzzles correspond.

Question 5. *What kinds of cardinality puzzles are there that simulate the different versions of knights and knaves puzzles?*

There is still more to do in order to prove that *every* K&K puzzle corresponds to a cardinality puzzle. This would require a formal definition of a K&K puzzle, and likely require a method to capture logical connectives by cardinality puzzles. This would be a fun project for a student interested in logic.

We have given one suggestion of how the K&K puzzles relate to cardinality puzzles, but perhaps there are other ways to match these up. Is there a single-set puzzle that models a K&K puzzle involving two (or more) trolls? We would want to have the variable expressions overlap, or not, in some combination that agrees with the possible solutions to the K&K puzzle.

Question 6. *What is the best way to match up cardinality puzzles with K&K puzzles, and is there a bijective correspondence between these?*

Finally, the multiple set cardinality puzzles we used to model K&K puzzles would be interesting to study in their own right.

Question 7. *Is every pair of symbiotic sets the unique solution to a two-set cardinality puzzle?*

Or forget about self-reference entirely and do a new Putnam-like problem:

Question 8. *How many minimal symbiotic sets are subsets of* $\{1, 2, \ldots, n\}$*?*

Abstract

We investigate a new sort of puzzle of self-reference, in which the puzzler is asked to find the cardinality of a set defined in terms of its own cardinality. We discover which sets are the unique solution to such puzzles and see how puzzles for such sets can be made arbitrarily complex. Then we compare our new cardinality puzzles to the classic puzzles of knights and knaves.

References

[1] L. F. Klosinski, G. L. Alexanderson and L. C. Larson, The Fifty-Seventh William Lowell Putnam Mathematical Competition, *Amer. Math. Monthly* **104** (1997), no. 8, 744–754. MR1543074

[2] K. Kunen, *The foundations of mathematics*, Studies in Logic (London), 19, College Publications, London, 2009. MR2667506

[3] O. Levin, T. Markkanen, Puzzles of Cardinality, *College Math. J.*, to appear.

[4] J. Rosenhouse, Knights, knaves, normals, and neutrals, *College Math. J.* **45** (2014), no. 4, 297–306. MR3317053

[5] R. Smullyan, *The Lady or the Tiger?*, Random House, 1992.

[6] R. Smullyan, *What Is the Name of This Book?*, Penguin Books, 1990.