Workshop 1: Limits and Derivatives

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Month Day, Year

Contents

1 Problem-Solving Strategies

1.1 Template Plan

The book "*How to solve it*" by Polya was used to write this template [Pol45].

1. **Understand the Problem**
   - What is the unknown? What is the given data?
   - What are the conditions? Are they sufficient?
   - Can you restate the problem in your own words?
   - Can you visualize the problem with a diagram, graph, or equation?
   - Have you seen a similar problem before?
2. **Devise a Plan**
   - **Look for patterns** – Do smaller cases reveal a structure?
   - **Consider special cases** – Try simpler values or boundary cases.
   - **Work backward** – Assume the result and trace back to given information.
   - **Break it into subproblem** – Solve each part separately.
   - **Use analogies** – Have you solved a similar problem before? Can you apply the same method?
   - **Change the representation** – Convert a word problem into algebra, a formula into a diagram, etc.
   - **Consider an auxiliary problem** – Create a related problem that might be easier to solve.
3. **Carry out the Plan**
   - Follow the steps carefully, keeping track of your work.
   - If you get stuck, return to step 2 and try a different approach.
   - Verify each step logically to avoid errors.
4. **Look Back (Reflect and Verify)**
   - Check your solution – Does it satisfy all the problem's conditions?
   - Can you derive the result differently? (This might reveal deeper insights.)
   - Can you generalize the solution?
   - Can you apply the method to other problems?

1.2 Backtracking

"**Backtracking** is an important improvement over the brute force approach of exhaustive search. It provides a convenient method for generating candidate solutions while making it possible to avoid generating unnecessary candidates. The main idea is to construct solutions one component at a time and evaluate such partially constructed candidates as follows: If a partially constructed solution can be developed further without violating the problem's constraints, it is done by taking the first remaining legitimate option for the next component. If there is no legitimate option for the next component, no alternatives for *any* remaining component need to be considered. In this case, the algorithm backtracks to replace the last component of the partially constructed solution with the next option for that component." [LL11]

1.3 Decrease-and-Conquer

"The **decrease-and-conquer** strategy is based on finding a relationship between a solution to a given problem and a solution to its smaller instance. Once found, such a relationship leads naturally to a *recursive algorithm*, which reduces the problem to a sequence of its diminishing instances until it becomes small enough to be solved directly." [LL11]

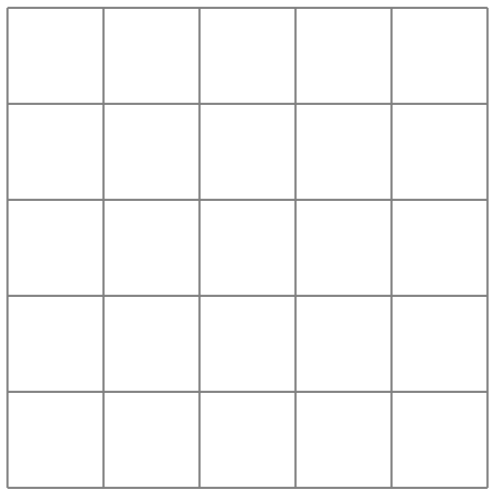
1.4 Dynamic Programming

"**Dynamic programming** is interpreted by computer scientists as a technique for solving problems with overlapping sub-problems. Rather than solving overlapping sub-problems again and again, it suggests solving each of the smaller sub-problems only once and recording the results in a table from which a solution to the original problem can then be obtained." [LL11]

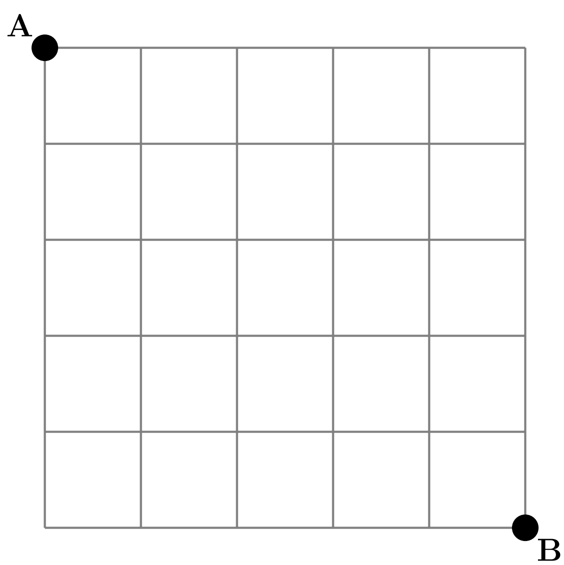
2 Puzzles

Puzzle 2.0.1. A celebrity among a group of n people is a person who knows nobody but is known by everybody else. The task is to identify a celebrity by only asking questions to people of the following form: "Do you know this person?" How many people will you need to ask before you identify the celebrity? [LL11]

Puzzle 2.0.2. Place 5 queens on a 5×5 chessboard so that no two queens attack each other by being in the same column, row, or diagonal. [LL11]



Puzzle 2.0.3. Find the number of the shortest paths from intersection A to intersection B in a city with perfectly horizontal streets and vertical avenues shown below [LL11].



3 Definitions and Notation

All definitions, notes, and theorems are sourced from Stewart's "*Calculus*" [Ste12].

3.1 Limits

Definition 3.1.1. Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Definition 3.1.2. We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of** $f(x)$ **as x approaches** a [or the **limit of** $f(x)$ **as x approaches** a **from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Definition 3.1.3. Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Definition 3.1.4. Let f be defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Definition 3.1.5. The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

- $\lim_{x \to a} f(x) = \infty$
- $\lim_{x \to a^-} f(x) = \infty$
- $\lim_{x \to a^+} f(x) = \infty$
- $\lim_{x \to a} f(x) = -\infty$
- $\lim_{x \to a^-} f(x) = -\infty$
- $\lim_{x \to a^+} f(x) = -\infty$

Definition 3.1.6. The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L.$$

Corollary 3.1.7. (Limit Laws.) Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ for some limit point a of A. Then,

(i) $\lim\limits_{x\to c} kf(x) = kL$ for all $k \in \mathbb{R}$,

(ii) $\lim\limits_{x\to c}[f(x) + g(x)] = L + M$,

(iii) $\lim\limits_{x\to c}[f(x)g(x)] = LM$, and

(iv) $\lim\limits_{x\to c} f(x)/g(x) = L/M$, provided $M \neq 0$.

Power Law:

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n, \text{ where } n \text{ is a positive integer}$$

Direct Substitution Property: If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a).$$

Theorem 3.1.8. $\lim\limits_{x\to a} f(x) = L$ if and only if $\lim\limits_{x\to a^-} f(x) = L$ and $\lim\limits_{x\to a^+} f(x) = L$.

Theorem 3.1.9. If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

Theorem 3.1.10. (The Squeeze Theorem.) If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then

$$\lim_{x\to a} g(x) = L.$$

Note 3.1.11. The Squeeze Theorem is also referred to as the Sandwich Theorem.

Definition 3.1.12. (Precise Definition of a Limit.) Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x\to a} f(x) = L$$

if for every number $\epsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon$$

3.2 Continuity

Definition 3.2.1. A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a)$$

Theorem 3.2.2. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ if $g(a) \neq 0$

Theorem 3.2.3. (Continuity of Polynomials and Rational Functions.)

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Theorem 3.2.4. If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Theorem 3.2.5. (The Intermediate Value Theorem.) Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. then there exists a number c in (a, b) such that $f(c) = N$.

Theorem 3.2.6. If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

if $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0.$$

3.3 Derivatives

Definition 3.3.1. The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Note 3.3.2. The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Note 3.3.3. If we use the **point-slope form of the equation of a line**, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

Note 3.3.4. The instantaneous rate of change $= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Note 3.3.5. The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Theorem 3.3.6. If f is differentiable at a, then f is continuous at a.

Important:

Note 3.3.7. The converse of the theorem above is false. Just because a function is continuous DOES NOT MEAN THAT IT IS DIFFERENTIABLE!

4 Practice Problems

These problems come from the *review* and *problem plus* section of Stewart's "*Calculus*" [Ste12].

4.1 Level 1: Conceptual

Problem 4.1.1. Explain what each of the following means and illustrate with a sketch.

(a) $\lim_{x \to c} f(x) = L$

(b) $\lim_{x \to c^+} f(x) = L$

(c) $\lim_{x \to c^-} f(x) = L$

(d) $\lim_{x \to c} f(x) = \infty$

(e) $\lim_{x \to \infty} f(x) = L$

Problem 4.1.2. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

Problem 4.1.3. On a separate sheet of paper, write out the following Limit Laws.

(a) Sum Law

(b) Difference Law

(c) Constant Multiple Law

(d) Product Law

(e) Quotient Law

(f) Power Law

(g) Root Law

Problem 4.1.4. Write out the Squeeze Theorem and pair it with an illustration.

Problem 4.1.5. Answer the following questions:

(a) What does it mean for f to be continuous at a?

(b) What does it mean for f to be continuous on $\mathbb{R}$? What can you say about the graph of such a function?

Problem 4.1.6. Write out the Intermediate Value Theorem.

Problem 4.1.7. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

Problem 4.1.8. If $y = f(x)$ and x changes from x_1 to x_2, write expressions for the following

(a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$.

(b) The instantaneous rate of change of y with respect to x at $x = x_1$.

Problem 4.1.9. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

Problem 4.1.10. Define the second derivative of f. If $f(t)$ is the position function of a particle, how can you interpret the second derivative?

Problem 4.1.11. (a) What does it mean for f to be differentiable at a?

(b) What is the relation between the differentiablity and continuity of a function?

(c) Sketch the graph of a function that is continuous but not differentiable at $a = 2$.

4.2 Level 2: Basics

Problem 4.2.1. Find the limit.

1. $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x^2+2x-3}$
2. $\displaystyle\lim_{x\to 1^+} \frac{x^2-9}{x^2+2x-3}$
3. $\displaystyle\lim_{t\to 2} \frac{t^2-4}{t^3-8}$
4. $\displaystyle\lim_{v\to 4^+} \frac{4-v}{|4-v|}$
5. $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2}$
6. $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{x^2-9}}{2x-6}$
7. $\displaystyle\lim_{x\to -\infty} \frac{1-2x^2-x^4}{5+x-3x^4}$
8. $\displaystyle\lim_{x\to \infty} e^{x-x^2}$
9. $\displaystyle\lim_{x\to 1} \left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)$
10. $\displaystyle\lim_{x\to 1} e^{x^3-x}$
11. $\displaystyle\lim_{x\to -3} \frac{x^2-9}{x^2+2x-3}$
12. $\displaystyle\lim_{h\to 0} \frac{(h-1)^3+1}{h}$
13. $\displaystyle\lim_{r\to 9} \frac{\sqrt{r}}{(r-9)^4}$
14. $\displaystyle\lim_{u\to 1} \frac{u^4-1}{u^3+5u^2-6u}$
15. $\displaystyle\lim_{x\to \infty} \frac{\sqrt{x^2-9}}{2x-6}$
16. $\displaystyle\lim_{x\to \pi^-} \ln(\sin(x))$
17. $\displaystyle\lim_{x\to \infty} (\sqrt{x^2+4x+1}-x)$
18. $\displaystyle\lim_{x\to 0^+} \tan^{-1}\left(\frac{1}{x}\right)$

Problem 4.2.2. If $2x-1 \le f(x) \le x^2$ for $0 < x < 3$, find $\lim_{x\to 1} f(x)$.

Problem 4.2.3. Prove the statement using the precise definition of a limit.

1. $\displaystyle\lim_{x\to 2}(14-5x) = 4$
2. $\displaystyle\lim_{x\to 0}(\sqrt[3]{x}) = 0$

Problem 4.2.4. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leq x < 3 \\ (x-3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\lim_{x \to 0^+} f(x)$

(ii) $\lim_{x \to 0^-} f(x)$

(iii) $\lim_{x \to 0} f(x)$

(iv) $\lim_{x \to 3^-} f(x)$

(v) $\lim_{x \to 3^+} f(x)$

(vi) $\lim_{x \to 3} f(x)$

(b) Where is f discontinuous?

Problem 4.2.5. Use the Intermediate Value Theorem to show that there is a root of the equation in the given interval.

$$x^5 - x^3 + 3x - 5 = 0, \quad (1, 2)$$

Problem 4.2.6. (a) Find the slope of the tangent line to the curve $y = 9 - 2x^2$ at the point $(2, 1)$.

(b) Find an equation of this tangent line.

Problem 4.2.7. The displacement (in meters) of an object moving in a straight line is given by $s = 1 + 2t + \frac{1}{4}t^2$, where t is measured in seconds

(a) Find the average velocity over each time period.

(i) $[1, 3]$

(ii) $[1, 2]$

(iii) $[1, 1.5]$

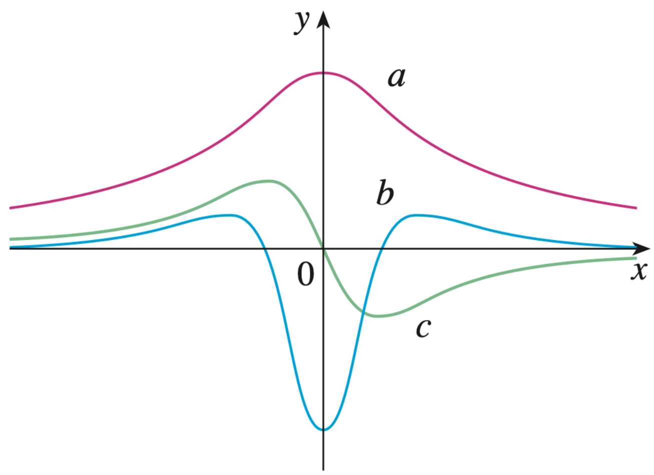
(iv) $[1, 1.1]$

(b) Find the instantaneous velocity when $t = 1$.

Problem 4.2.8. (a) Use the definition of a derivative to find $f'(2)$, where $f(x) = x^3 - 2x$.

(b) Find an equation of the tangent line to the curve $y = x^3 - 2x$ at the point $(2, 4)$.

Problem 4.2.9. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



4.3 Level 3: Intermediate

Problem 4.3.1. Prove that $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$.

Problem 4.3.2. Prove the statement using the precise definition of a limit.

1. $\lim\limits_{x\to 2}(x^2 - 3x) = -2$
2. $\lim\limits_{x\to 4^+} \dfrac{2}{\sqrt{x-4}} = \infty$

Problem 4.3.3. Use the Intermediate Value Theorem to show that there is a root of the equation in the given interval.

$$\cos(\sqrt{x}) = e^x - 2, \quad (0, 1)$$

Problem 4.3.4. According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the pressure P and the volume V is a constant. Suppose that, for a certain gas, $PV = 800$, Where P is measure in pounds per square inch and V is measured in cubic inches.

(a) Find the average rate of change of P as V increases from 200 in^3 to 250 in^3.

(b) Express V as a function of P and show that the instantaneous rate of change of V with respect to P is inversely proportional to the square of P.

Problem 4.3.5. Find a function f and a number a such that

$$\lim_{h\to 0} \frac{(2+h)^6 - 64}{h} = f'(a)$$

Problem 4.3.6. Suppose that $|f(x)| \leq g(x)$ for all x, where $\lim_{x\to a} g(x) = 0$. Find $\lim_{x\to a} f(x)$.

Problem 4.3.7. Let $f(x) = \llbracket x \rrbracket + \llbracket -x \rrbracket$. (The function $\llbracket x \rrbracket$ returns the greatest integer less than or equal to x.)

(a) For what values of a does $\lim_{x\to a} f(x)$ exists?

(b) At what numbers of f discontinuous?

4.4 Level 4: Advanced

Problem 4.4.1. Evaluate $\displaystyle\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$.

Problem 4.4.2. Find numbers a and b such that $\displaystyle\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} = 1$.

Problem 4.4.3. Evaluate $\displaystyle\lim_{x\to 0} \frac{|2x-1|-|2x+1|}{x}$.

Problem 4.4.4. Evaluate the following limits, if they exist, where $\llbracket x \rrbracket$ denotes the greatest integer function.

(a) $\displaystyle\lim_{x\to 0} \frac{\llbracket x \rrbracket}{x}$

(b) $\displaystyle\lim_{x\to 0} x\llbracket 1/x \rrbracket$

Problem 4.4.5. Find all values of a such that f is continuous on $\mathbb{R}$:

$$f(x) = \begin{cases} x+1 & \text{if } x \leq a \\ x^2 & \text{if } x > a \end{cases}$$

Problem 4.4.6. If $\lim_{x\to a}[f(x)+g(x)] = 2$ and $\lim_{x\to a}[f(x)-g(x)] = 1$, find $\lim_{x\to a}[f(x)g(x)]$.

Problem 4.4.7. If f is differentiable function and $g(x) = xf(x)$, use the definition of a derivative to show that $g'(x) = xf'(x) + f(x)$.

Problem 4.4.8. Suppose f is a function that satisfies the equation

$$f(x+y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1$$

(a) Find $f(0)$.

(b) Find $f'(0)$.

(c) Find $f'(x)$.

Problem 4.4.9. Suppose f is a function with the property that $|f(x)| \leq x^2$ for all x. Show that $f(0) = 0$. Then show that $f'(0) = 0$.

References

[LL11] Anany Levitin and Maria Levitin. *Algorithmic Puzzles*. Oxford University Press, 2011. ISBN: 978-0199740444.

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