
Optimal Harvesting Modelling

Final Report

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Abstract

Contents

1	Problem Description and Framework	3
1.1	Fish harvesting models	5
2	Mathematical Models.	6
2.1	Exponential biological growth.	6
2.2	Logistic Equation.	6
2.3	Wiener Process and noise.	6
3	Fishing Strategies and Optimizing Population	8
3.1	Open Loop Strategies.	8
3.1.1	Constant Harvesting Analysis.	8
3.1.2	Time Varying Harvesting.	10
3.1.3	Optimal Harvesting. Smooth Optimal Control Problem.	11
3.2	Closed Loop Strategies.	11
3.2.1	Constant Proportional Harvesting.	11
3.2.2	Optimal Proportional Harvesting.	11
4	Economical Profit	12
4.1	Linear Costs.	12
4.2	Quadratic Costs.	12
4.3	Stochastic Analysis.	12
5	Further Research	13

1 Problem Description and Framework

As a natural, healthy and nutritious food, with variety of species and diverse growth environments, fish seems to be a wise choice to solve some food - related crisis regarding to the human population growth around the world. On the other hand, there is a limitation for the fish population sustainability in open seas. Global high demand, resulted in over-exploiting the oceans in the past decades (Figure 1).

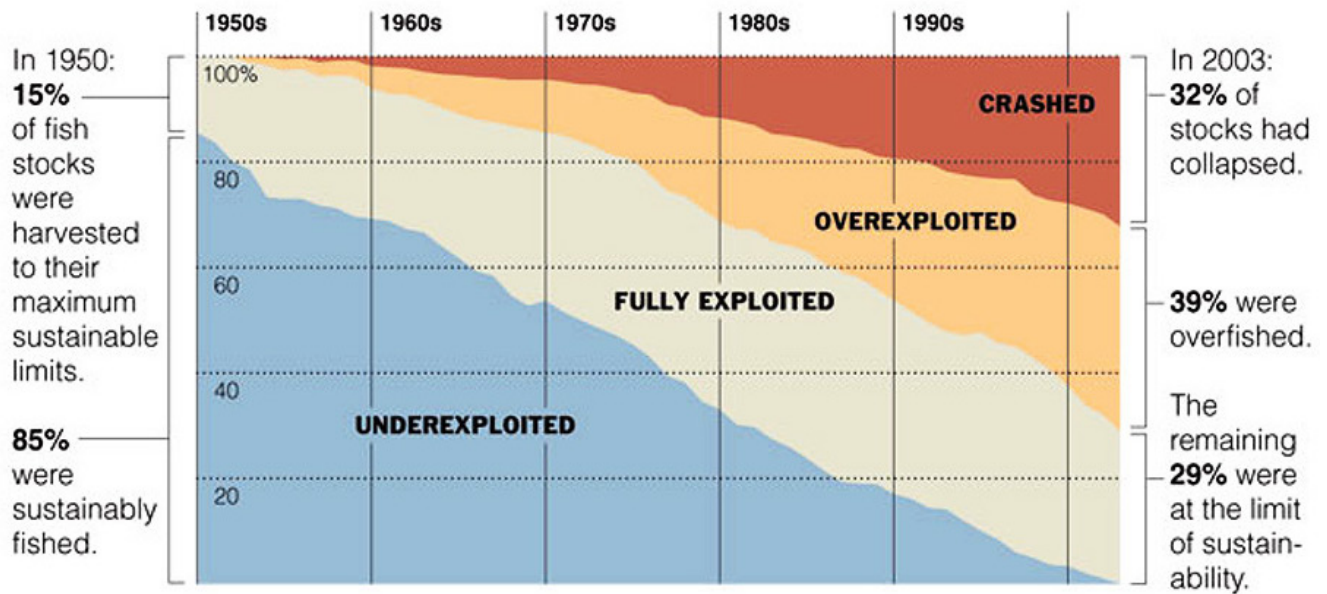


Figure 1.1: sustainable fishing between 1950 and 2003

In result, some fish populations have been severely declined during the years. Figure 1 shows the population of utilized fish population between 1970 and 2010. As illustrated, the index for all utilized fish species indicates a 50 per cent reduction in population number globally between 1970 and 2010.

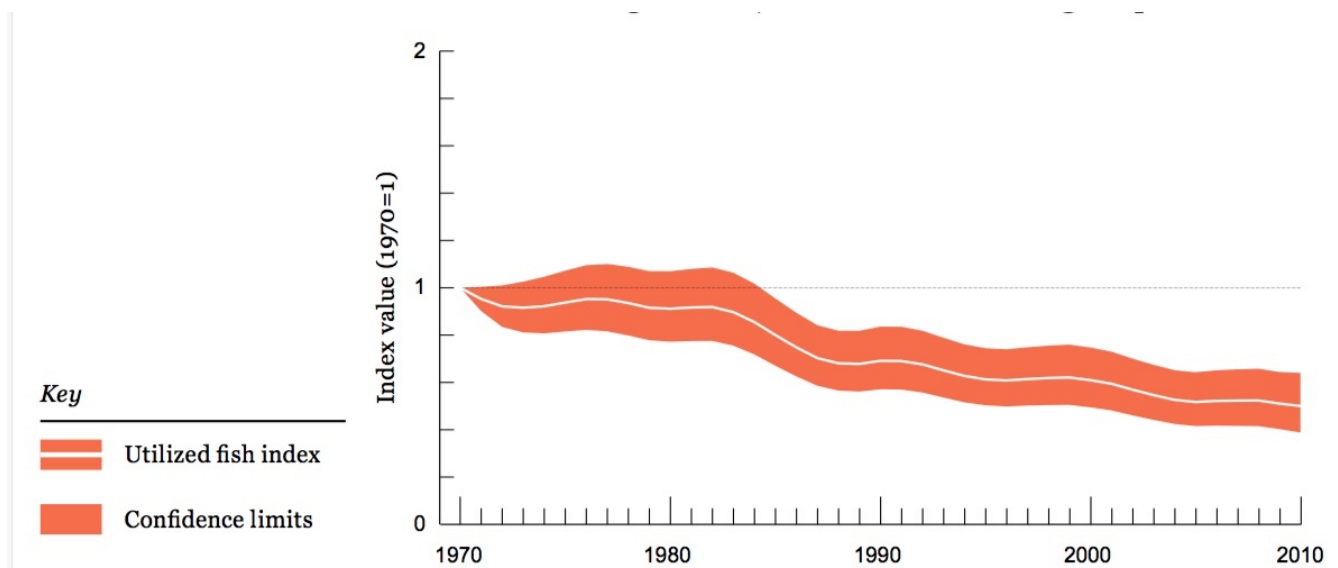


Figure 1.2: Utilized fish index value between 1970 and 2010

One of the solutions to fish population decrease problem is to shift from fish catching to fish harvesting. This strategy can help recovering fish population and size gradually beside providing human with seafood. Figure 1 and 1 shows the fish harvesting production grows in 1970 and 2010 year around the world.

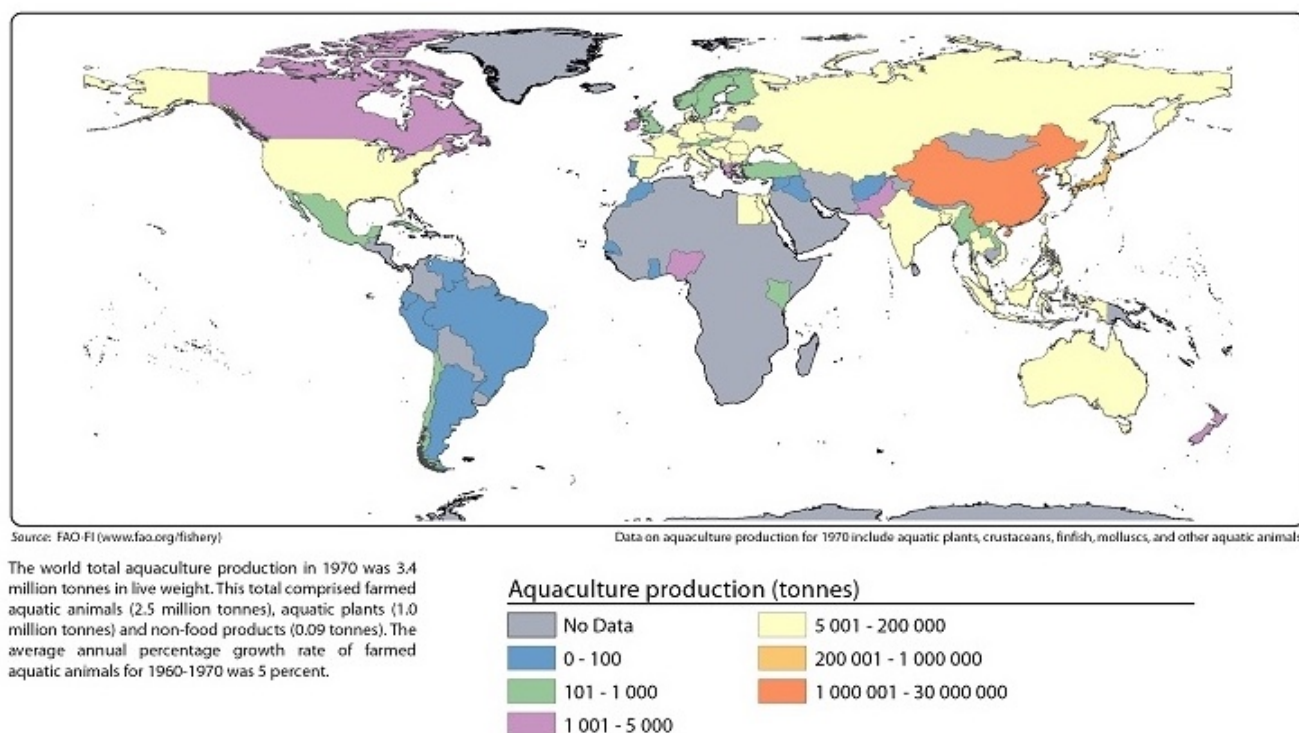


Figure 1.3: Aquaculture production in 1970 around the world

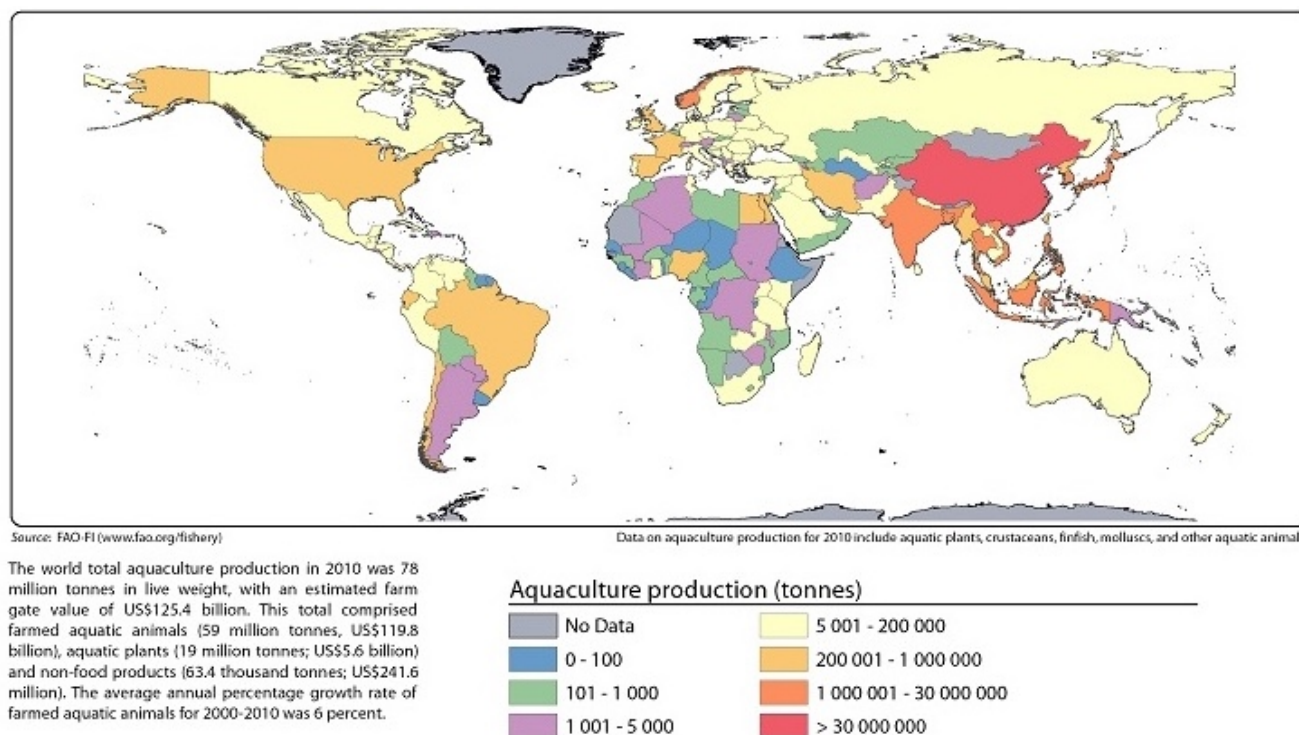


Figure 1.4: Aquaculture production in 2010 around the world

Like any other industry, It is crucial to optimize the fish harvesting procedure to have maximum -still consistent- production in fish harvesting farms. In this work, weâ€™re trying to describe one of the fish harvesting mathematical models and achieve the optimum fish farm population to have a consistent population.

1.1 Fish harvesting models

Generally, there are three methods to model aquaculture mathematically:

- constant harvesting: One of the simplest methods is the idea of harvesting where a constant number of fish removing from the main population over a given time interval
- proportional harvesting: Another common form of harvesting that the quantity of harvested fish is proportional to the population.
- periodic harvesting :Another very used form of harvesting is when it is done during periods of time within a year, so the fish wonâ€™t become extinct during fishing time and in some periods fishing is stopped, the population of fish might be able to increase again.

2 Mathematical Models.

$$\frac{dx}{dt} = F(x, t) \quad (2.1)$$

2.1 Exponential biological growth.

Assuming the natural fish mortality to be a constant M , we get the growth dynamics as,

$$\begin{aligned} \frac{dx}{dt} &= -mx \\ x(T) &= x_T \end{aligned} \quad (2.2)$$

If a variable mortality due to fishing $\Phi(t)$, is also considered then the growth equation becomes,

$$\begin{aligned} \frac{dx}{dt} &= -(m + \Phi(t))x \\ x(T) &= x_T \end{aligned} \quad (2.3)$$

2.2 Logistic Equation.

Logistic equation.

$$F(x, t) = rx \left(1 - \frac{x}{M}\right) \quad (2.4)$$

2.3 Wiener Process and noise.

We consider the behavior of the logistic equation under the presence of noise, in multiplicative way to the population. For the elements $(t, x) \in Q = (0, T) \times (0, M)$, we state the following differential equation,

$$dx = \left(rx \left(1 - \frac{x}{M}\right) \right) dt + \sigma x dW \quad (2.5)$$

A unique solution exists if both Itô conditions hold (Fleming and Rishel, 1975). The first one is the linear growth condition, for some independent constant K ,

$$\left| rx \left(1 - \frac{x}{M}\right) \right| \leq K(1 + |x|) \quad (2.6)$$

$$|\sigma x| \leq K(1 + |x|) \quad (2.7)$$

The second one is the Lipschitz condition, $\exists L$ independent constant, and $\forall x, \exists B(x)$ neighborhood of x , such that $\forall x_1, x_2 \in B(x)$,

$$\left| rx_2 \left(1 - \frac{x_2}{M}\right) - rx_1 \left(1 - \frac{x_1}{M}\right) \right| \leq L |x_2 - x_1| \quad (2.8)$$

$$|\sigma(x_2 - x_1)| \leq L |x_2 - x_1| \quad (2.9)$$

Since $F(x, t) = rx \left(1 - \frac{x}{M}\right)$ is continuously differentiable in x , F is Lipschitz in x then condition 2.8 is satisfied. For bounded σ , condition 2.9 is satisfied. Moreover the sufficient conditions for the Itô conditions are satisfied for all functions C^1 on the closure of any compact set Q .

Since the above conditions are satisfied, we can guarantee existence and uniqueness of the solution for the equation 2.5. Given by the equation:

$$\begin{aligned} x(t) &= x_0 + \int_0^t \left(rx \left(1 - \frac{x}{M} \right) - u \right) dt + \int_0^t \sigma x dW, \\ x(0) &= x_0, \\ W(0) &= 0. \end{aligned} \tag{2.10}$$

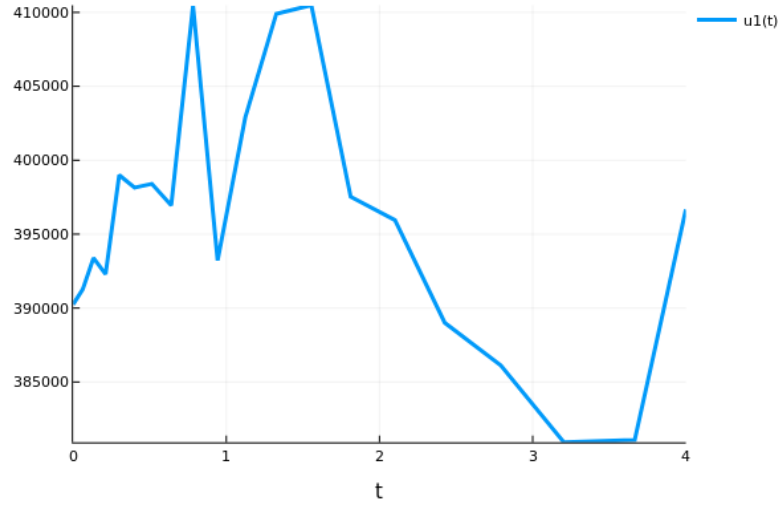


Figure 2.1: Simulation performed of logistic equation 2.5, with parameters $r = 0.8 \frac{1}{\text{month} \times \text{fish}}$, $x_0 = \frac{M}{2}$, for a population in natural conditions (harvest exploitation $u = 0$), with presence of noise proportional to the population, with $\sigma = 0.1$. Performed during 4 months.

3 Fishing Strategies and Optimizing Population

3.1 Open Loop Strategies.

3.1.1 Constant Harvesting Analysis.

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{M}\right) - u \quad (3.1)$$

$$\beta = \frac{uM}{r} \quad (3.2)$$

$$\begin{aligned} \frac{dx}{rx \left(1 - \frac{x}{M}\right) - u} &= dt \\ \int_{x_0}^x \frac{d\chi}{r\chi \left(1 - \frac{\chi}{M}\right) - u} &= \int_0^t d\tau \\ \frac{M}{r} \int_{x_0}^x \frac{d\chi}{\chi(M - \chi) - \frac{Mu}{r}} &= t \\ -\frac{M}{r} \int_{x_0}^x \frac{d\chi}{\chi^2 - M\chi + \beta} &= t \\ -\frac{M}{r} \int_{x_0}^x \frac{d\chi}{\left(\chi - \frac{M}{2}\right)^2 - \frac{M^2}{4} + \beta} &= t \end{aligned} \quad (3.3)$$

$$\alpha = \beta - \frac{M^2}{4} = rM \left(u - \frac{rM}{4}\right) \quad (3.4)$$

If $u > rM/4$ implies $\alpha > 0$

$$\begin{aligned} \int_{x_0}^x \frac{d\chi}{\left(\chi - \frac{M}{2}\right)^2 + \alpha} &= -\frac{r}{M} t \\ \frac{1}{\sqrt{\beta - \frac{M^2}{4}}} \left(\arctan\left(\frac{x - M/2}{\sqrt{\beta - M^2/4}}\right) - \arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}}\right) \right) &= -\frac{r}{M} t \end{aligned}$$

$$x(t) = \frac{M}{2} + \sqrt{\beta - \frac{M^2}{4}} \tan\left(\arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}}\right) - \frac{r\sqrt{\beta - M^2/4}}{M} t\right) \quad (3.5)$$

From equation 3.5 we observe that independently the initial condition x_0 , for some t^* , $x(t^*) = 0$, since the argument inside the \tan is monotone decreasing in t . If $u < rM/4$ implies $\alpha < 0$,

$$\int_{x_0}^x \frac{d\chi}{\left(\chi - \frac{M}{2}\right)^2 - \alpha} = -\frac{r}{M}t$$

$$\begin{aligned}\lambda &= \frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta} \\ \bar{\lambda} &= \frac{M}{2} - \sqrt{\frac{M^2}{4} - \beta}\end{aligned}\tag{3.6}$$

$$\int_{x_0}^x \left(\frac{1}{\chi - \lambda} - \frac{1}{\chi - \bar{\lambda}} \right) d\chi = -\frac{2r\sqrt{M^2/4 - \beta}}{M}t$$

$$\ln \left| \frac{x - \lambda}{x - \bar{\lambda}} \right| = \ln \left| \frac{x_0 - \lambda}{x_0 - \bar{\lambda}} \right| - \frac{2r\sqrt{M^2/4 - \beta}}{M}t$$

$$\gamma = \frac{2r\sqrt{M^2/4 - \beta}}{M}$$

$$\frac{x - \lambda}{x - \bar{\lambda}} = \frac{x_0 - \lambda}{x_0 - \bar{\lambda}} e^{-\gamma t}\tag{3.7}$$

$$x - \lambda = (x - \bar{\lambda}) \left(\frac{x_0 - \lambda}{x_0 - \bar{\lambda}} \right) e^{-\gamma t}\tag{3.8}$$

$$\xi = \frac{x_0 - \lambda}{x_0 - \bar{\lambda}} e^{-\gamma t}$$

$$\begin{aligned}x(1 - \xi) &= \lambda - \bar{\lambda}\xi \\ x &= \frac{\lambda - \bar{\lambda}\xi}{1 - \xi} \\ x &= \frac{\frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta} - \left(\frac{M}{2} - \sqrt{\frac{M^2}{4} - \beta} \right) \xi}{1 - \xi} \\ x &= \frac{\frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta} - \left(\frac{M}{2} - \sqrt{\frac{M^2}{4} - \beta} \right) \xi}{1 - \xi} \\ x &= \frac{\frac{M}{2}(1 - \xi) + \sqrt{\frac{M^2}{4} - \beta}(1 + \xi)}{1 - \xi} \\ x &= \frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta} \frac{1 + \xi}{1 - \xi}\end{aligned}$$

$$x(t) = \frac{M}{2} + \left(\sqrt{\frac{M^2}{4} - \beta} \right) \frac{(x_0 - M/2)(1 + e^{-\gamma t}) - \sqrt{M^2/4 - \beta}(1 - e^{-\gamma t})}{(x_0 - M/2)(1 - e^{-\gamma t}) + \sqrt{M^2/4 - \beta}(1 + e^{-\gamma t})}\tag{3.9}$$

If $u = \frac{rM}{4}$, the solution for the equation 3.1

$$-\frac{M}{r} \int_{x_0}^x \frac{d\chi}{\left(\chi - \frac{M}{2}\right)^2} = t \quad (3.10)$$

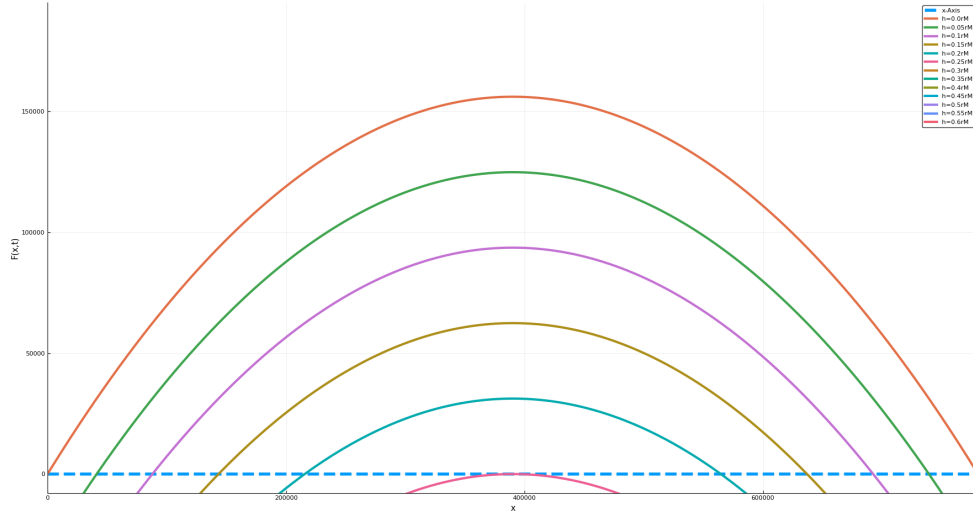
$$\int_{x_0}^x \frac{d\chi}{\left(\chi - \frac{M}{2}\right)^2} = -\frac{rt}{M} \quad (3.11)$$

$$\frac{1}{x - \frac{M}{2}} = \frac{1}{x_0 - \frac{M}{2}} - \frac{rt}{M} \quad (3.12)$$

$$\frac{1}{x - \frac{M}{2}} = \frac{M - \left(x_0 - \frac{M}{2}\right)rt}{M\left(x_0 - \frac{M}{2}\right)} \quad (3.13)$$

$$x = \frac{M}{2} + \frac{M\left(x_0 - \frac{M}{2}\right)}{M - \left(x_0 - \frac{M}{2}\right)rt} \quad (3.14)$$

The results above stated can be explained directly from the equation 3.1.



From the above analysis we have that,

$$0 < u \leq \frac{rM}{4} \quad (3.15)$$

3.1.2 Time Varying Harvesting.

Given a time horizon T we want to extract the maximum amount of fishes, during this time.

$$H = \int_0^T u(t)dt \quad (3.16)$$

Therefore,

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{M}\right) - u(t) \quad (3.17)$$

From the above result we see that for $u(t) > \frac{rM}{4}$, we lead the population to extinction.

3.1.3 Optimal Harvesting. Smooth Optimal Control Problem.

For Optimal Control we reduce the problem to the following,

$$\min_{\substack{x \in X \\ u \in U}} J(x, u) \quad (3.18)$$

subject to,

$$e(x, u) = 0 \quad (3.19)$$

$$h(t) = \frac{rM}{4} - u(t)$$

$$J(x, u) = \frac{\zeta}{2} \left(x(T) - \frac{M}{2} \right)^2 + \frac{1}{2} \left\| x - \frac{M}{2} \right\|_{L^2([0, T])}^2 + \frac{\eta}{2} \|h\|_{L^2([0, T])}^2 \quad (3.20)$$

subject to,

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{M} \right) - \frac{rM}{4} + h(t) \quad (3.21)$$

3.2 Closed Loop Strategies.

3.2.1 Constant Proportional Harvesting.

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{M} \right) - px \quad (3.22)$$

$$\frac{dx}{dt} = rx \left(1 - \frac{p}{r} - \frac{x}{M} \right) \quad (3.23)$$

$$\frac{dx}{dt} = r \left(1 - \frac{p}{r} \right) \left(1 - \frac{x}{M \left(1 - \frac{p}{r} \right)} \right) x \quad (3.24)$$

$\gamma = r \left(1 - \frac{p}{r} \right)$, $K = M \left(1 - \frac{p}{r} \right)$. With $\frac{p}{r} < 1$

$$\frac{dx}{dt} = \gamma x \left(1 - \frac{x}{K} \right) \quad (3.25)$$

$$x = \frac{Kx_0}{x_0 + (K - x_0)e^{-\gamma t}} \quad (3.26)$$

$$x(t) = \frac{M \left(1 - \frac{p}{r} \right) x_0}{x_0 + \left(M - \frac{Mp}{r} - x_0 \right) e^{-\gamma t}} \quad (3.27)$$

3.2.2 Optimal Proportional Harvesting.

4 Economical Profit

4.1 Linear Costs.

4.2 Quadratic Costs.

4.3 Stochastic Analysis.

5 Further Research
