Optimal Harvesting Modelling

Final Report

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Abstract

Contents

As a natural, healthy and nutritious food, with variety of species and diverse growth environments, fish seems to be a wise choice to solve some food - related crisis regarding to the human population growth around the world. On the other hand, there is a limitation for the fish population sustainability in open seas. Global high demand, resulted in over-exploiting the oceans in the past decades (Figure 1).

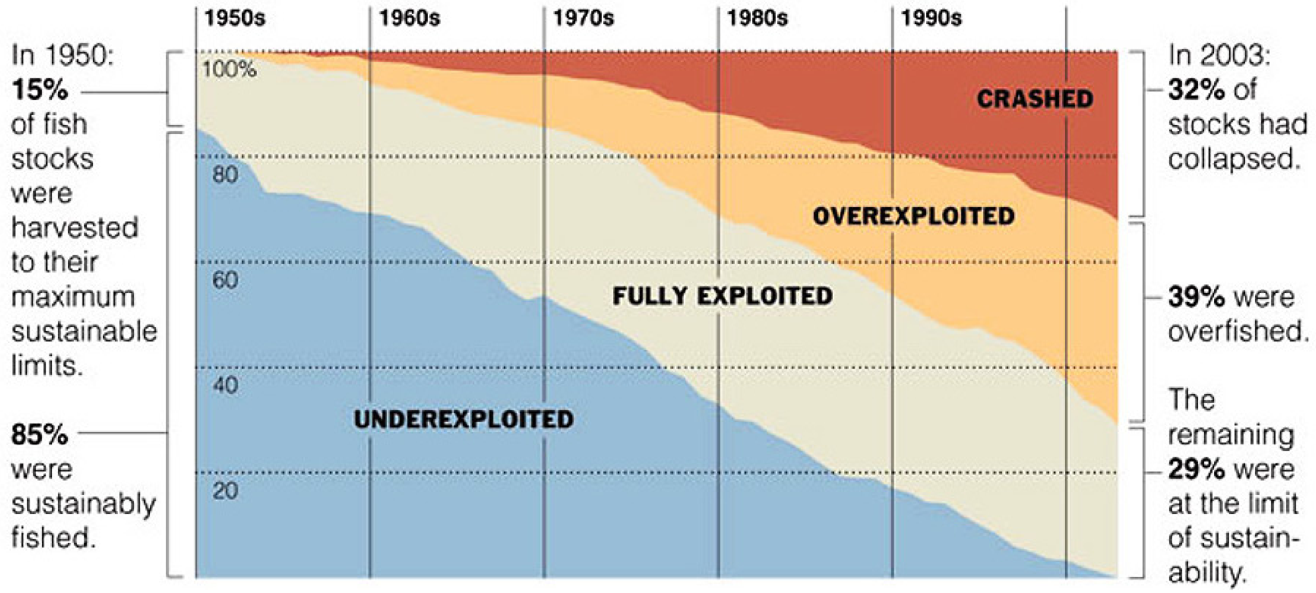


Figure 1.1: sustainable fishing between 1950 and 2003

In result, some fish populations have been severely declined during the years. Figure 1 shows the population of utilized fish population between 1970 and and 2010. As illustrated, the index for all utilized fish species indicates a 50 per cent reduction in population number globally between 1970 and 2010.

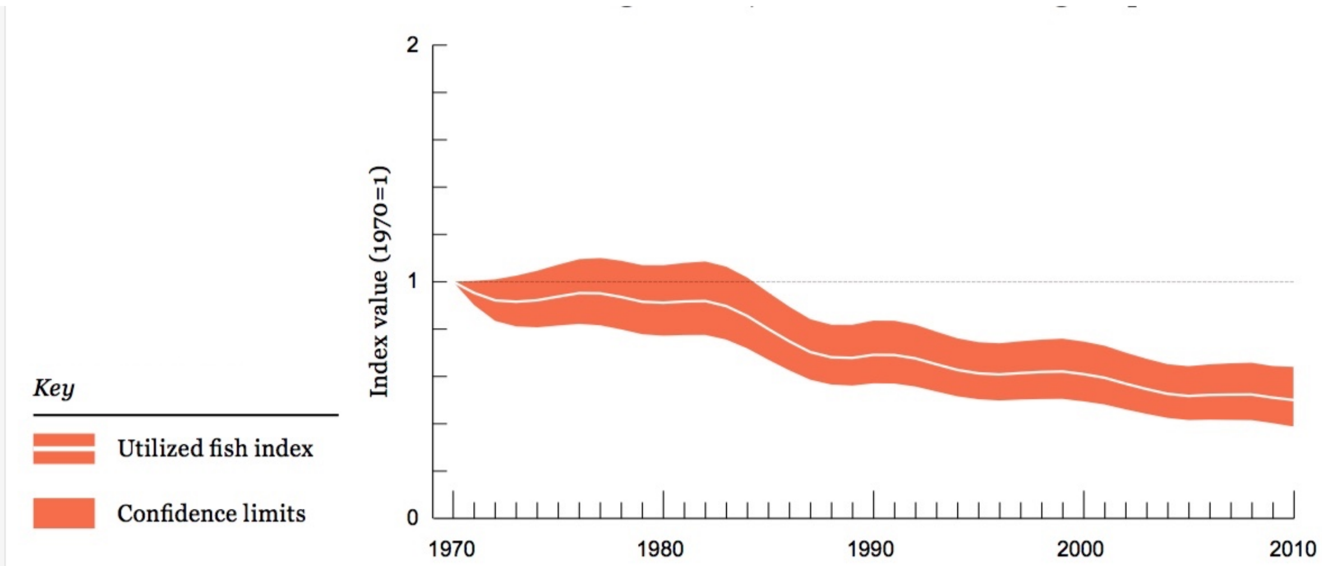


Figure 1.2: Utilized fish index value between 1970 and 2010

One of the solutions to fish population decrease problem is to shift from fish catching to fish harvesting. This strategy can help recovering fish population and size gradually beside providing human with seafood. Figure 1 and 1 shows the fish harvesting production grows in 1970 and 2010 year around the world.

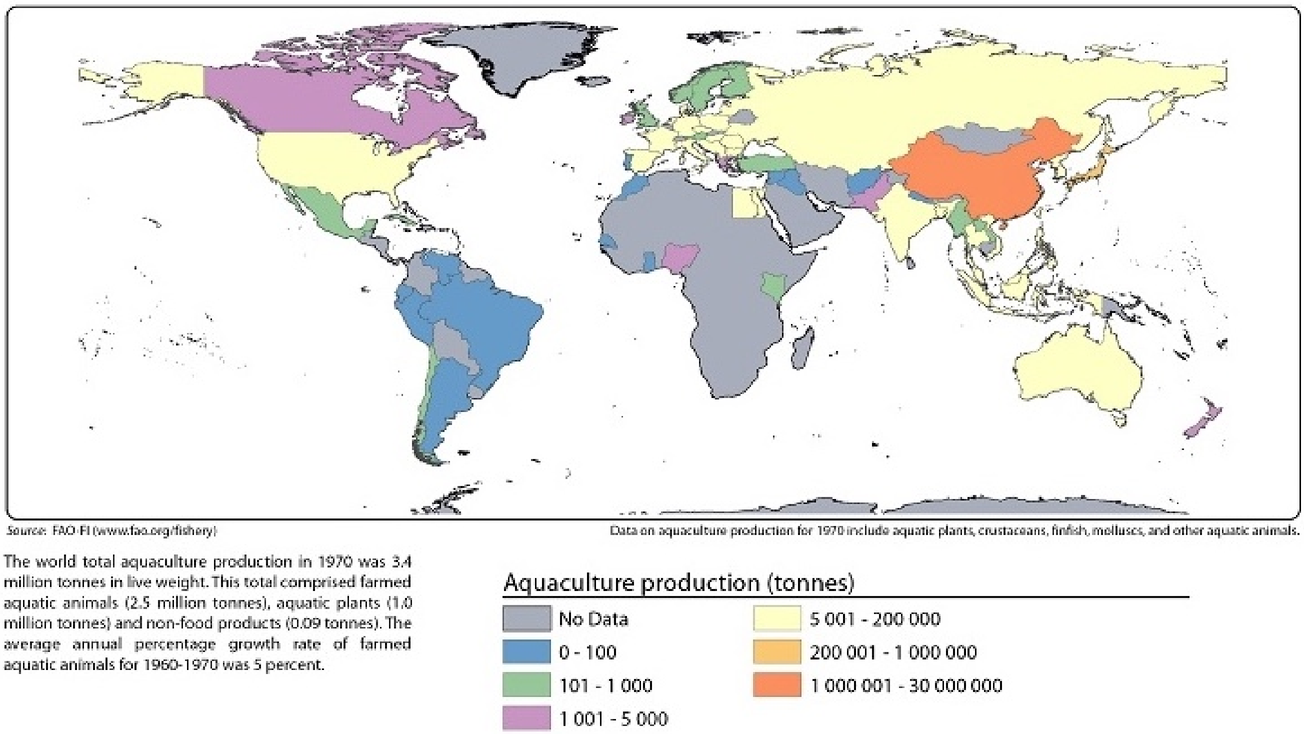


Figure 1.3: Aquaculture production in 1970 around the world

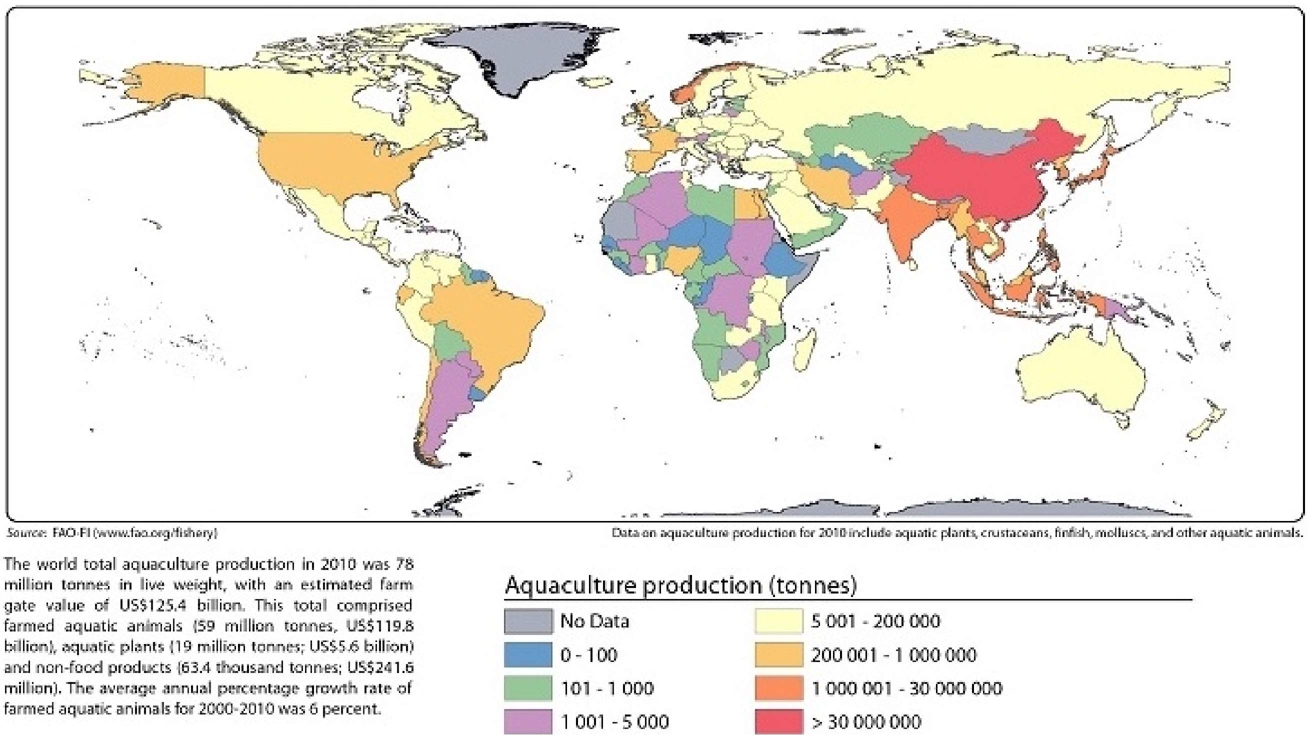


Figure 1.4: Aquaculture production in 2010 around the world

Like any other industry, It is crucial to optimize the fish harvesting procedure to have maximum -still consistent- production in fish harvesting farms. In this work, weâĂŹre trying to describe one of the fish harvesting mathematical models and achieve the optimum fish farm population to have a consistent population.

2 Mathematical Models.

Before focusing on the harvesting problem, we focus on modelling a "pure" population growth, without introducing an artificial harvesting control. We can model the population growth problem, as one dynamical system in which x represents the fish population, and the population variation with respect to time $\dot{x}$ can be described as a function of the fish population and/or time:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,t) \tag{2.1}$$

2.1 Exponential biological growth.

A first approach is to introduce a natural mortality factor m, which contributes to decreasing the fish population. To prevent the population from decaying to zero, we also introduce a constriction for the time horizon T:

$$\begin{array}{rcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & -mx \\ x(T) & = & x_T \end{array} \tag{2.2}$$

If a variable mortality due to fishing $\Phi(t)$, is also considered then the growth equation becomes,

$$\begin{array}{rcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & -(m+\Phi(t))x \\ x(T) & = & x_T \end{array} \tag{2.3}$$

We must take into account that the variable mortality factor adjusts well to an open sea population simulation, but not to a controlled population (fish farm).

2.2 Logistic Equation.

A better approach consists in the following: a biological population with plenty of food, space to grow, and no threat from predators, tends to grow at a rate that is proportional to the population- this means that per unit time a certain percentage of the individuals produce new individuals continuously:

$$F(x,t) = rx \tag{2.4}$$

where x is the population in time t, and the proportionality constant r is called the growth rate. In reality, populations are constrained by limitations on resources, so a maximum population parameter M can be introduced ("carrying capacity" of the system). The logistic growth model includes this parameter, and has the form:

$$F(x,t) = rx\left(1-\frac{x}{M}\right) \tag{2.5}$$

Equation 2.5 satisfies some basic aspects: on one hand, when the population is small relative to M, its behavior is similar to the followed by equation 2.4, and the constraint does not affect too much, but as x becomes significant compared to M, both curves diverge and the growth rate $\dot{x}$ drops to zero. On the other hand, the growth rate is only zero when $x = M$, which is what happens in reality.

We consider the behavior of the logistic equation under the presence of noise, in multiplicative way to the population. For the elements $(t,x)\in Q=(0,T)\times(0,M)$, we state the following differential equation,

$$dx=\left(rx\left(1-\frac{x}{M}\right)\right)\mathrm{d}t+\sigma x\mathrm{d}W \tag{2.6}$$

A unique solution exists if both Itó conditions hold (Fleming and Rishel, 1975). The first one is the linear growth condition, for some independent constant K,

$$\left|rx\left(1-\frac{x}{M}\right)\right|\leq K(1+|x|) \tag{2.7}$$
$$|\sigma x|\leq K(1+|x|) \tag{2.8}$$

The second one is the Lipschitz condition,$\exists L$ independent constant, and $\forall x$, $\exists B(x)$ neighborhood of x, such that $\forall x_1,x_2\in B(x)$,

$$\left|rx_2\left(1-\frac{x_2}{M}\right)-rx_1\left(1-\frac{x_1}{M}\right)\right|\leq L\,|x_2-x_1| \tag{2.9}$$
$$|\sigma(x_2-x_1)|\leq L\,|x_2-x_1| \tag{2.10}$$

Since $F(x,t)=rx\left(1-\frac{x}{M}\right)$ is continuously differentiable in x, F is Lipschitz in x then condition 2.9 is satisfied. For bounded σ, condition 2.10 is satisfied. Moreover the sufficient conditions for the Itô conditions are satisfied for all functions C^1 on the closure of any compact set Q.

Since the above conditions are satisfied, we can guarantee existence and uniqueness of the solution for the equation 2.6. Given by the equation:

$$\begin{array}{rcl} x(t) &=& x_0+\int_0^t\left(rx\left(1-\frac{x}{M}\right)\right)\mathrm{d}t+\int_0^t\sigma x\mathrm{d}W,\\ x(0) &=& x_0,\\ W(0) &=& 0. \end{array} \tag{2.11}$$

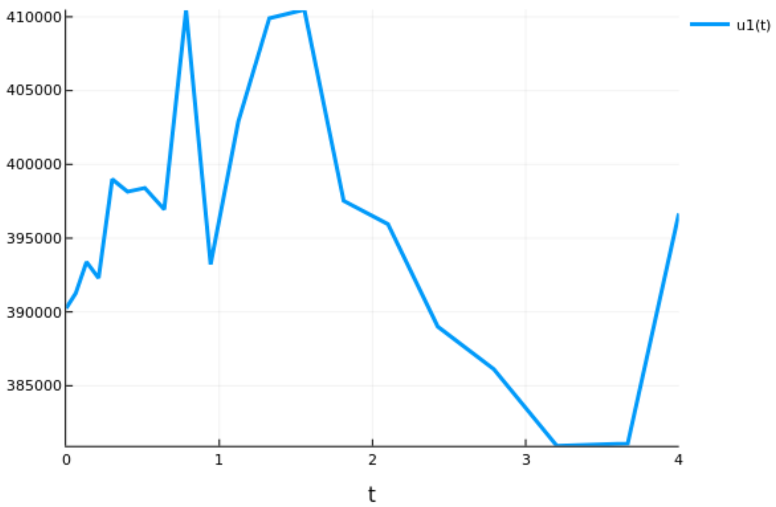


Figure 2.1: Simulation performed of logistic equation 2.6, with parameters $r=0.8$ time units, $x_0=\frac{M}{2}$, $\sigma=0.1$, noise in a multiplicative way, proportional to the population. Simulation performed during 4 months.

Generally, there are three methods to model aquaculture mathematically:

- Open Loop harvesting Strategies: One of the simplest methods to implement, harvesting a number of fishes without regarding the state of the population.
- Closed Loop harvesting Strategies: A more complex strategy to implement, with the advantage that it is possible to introduce a control over the population in order to avoid extinction.

3.1 Open Loop Strategies.

Generally, in an open loop strategy -also called a non-feedback strategy- the process does not use a feedback to determine if its output has achieved the desired goal of the process. The implementation of open loop harvesting strategies take place without considering the impact of the extraction process. Mathematically,

$$\frac{dx}{dt} = \Psi(x,t) + C(t) \tag{3.1}$$

where $\Psi(x,t)$ is the intrinsic dynamic of the system and $C(t)$ is the control parameter, which is population independent.

3.1.1 Constant Harvesting Analysis.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{M}\right) - u \tag{3.2}$$

We introduce the following variable in order to simply calculations,

$$\beta = \frac{uM}{r} \tag{3.3}$$

Solving the differential equation,

$$\begin{aligned}
\frac{\mathrm{d}x}{rx\left(1-\frac{x}{M}\right)-u} &= \mathrm{d}t \\
\int_{x_0}^{x} \frac{\mathrm{d}\chi}{r\chi\left(1-\frac{\chi}{M}\right)-u} &= \int_0^t \mathrm{d}\tau \\
\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\chi\left(M-\chi\right)-\frac{Mu}{r}} &= t \\
-\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\chi^2 - M\chi + \beta} &= t
\end{aligned}$$

Finally, we model the above integral as one

$$-\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi-\frac{M}{2}\right)^2 - \frac{M^2}{4} + \beta} = t \tag{3.4}$$

Consider α as follows,

$$\alpha = \beta - \frac{M^2}{4} = rM\left(u - \frac{rM}{4}\right) \tag{3.5}$$

We see that the sign of α determines the nature of the solutions. Then, if $u > rM/4$ implies $\alpha > 0$,

$$\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi - \frac{M}{2}\right)^2 + \alpha} = -\frac{r}{M}t$$

$$\frac{1}{\sqrt{\beta - \frac{M^2}{4}}}\left(\arctan\left(\frac{x - M/2}{\sqrt{\beta - M^2/4}}\right) - \arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}}\right)\right) = -\frac{r}{M}t$$

Therefore, for $\alpha > 0$ the population behaves as follows,

$$x(t) = \frac{M}{2} + \sqrt{\beta - \frac{M^2}{4}}\tan\left(\arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}}\right) - \frac{r\sqrt{\beta - M^2/4}}{M}t\right) \tag{3.6}$$

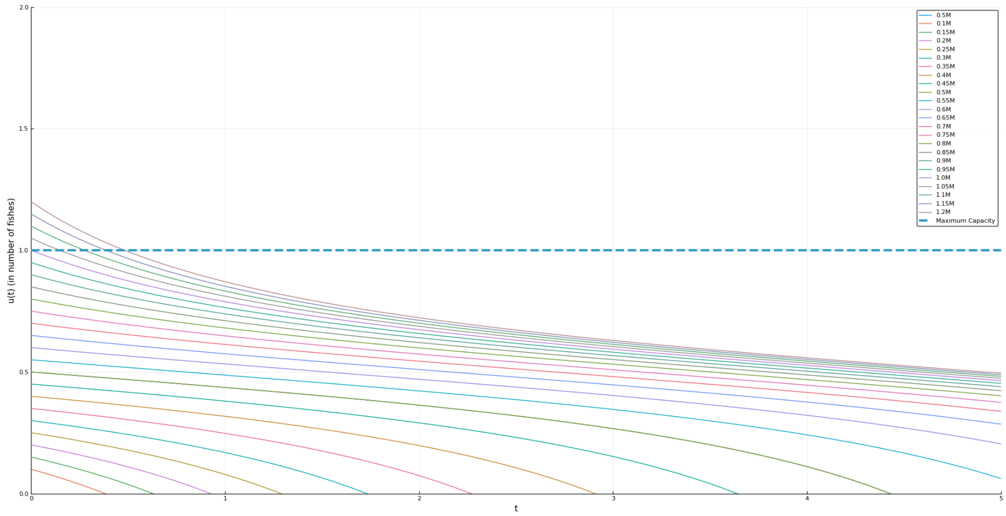


Figure 3.1: Constant harvest rate $u > \frac{rM}{4}$.

Equation 3.6 show us that for some t^*, $x(t^*) = 0$,independently of the initial condition x_0, since the argument inside the $\tan$ is monotone decreasing in t.

If $u < rM/4$ implies $-\alpha > 0$,

$$\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi - \frac{M}{2}\right)^2 - (-\alpha)} = -\frac{r}{M}t$$

Considering the zeros of the denominator, λ and $\overleftarrow{\lambda}$,

$$\begin{aligned} \lambda &= \tfrac{M}{2} + \sqrt{\tfrac{M^2}{4} - \beta} \\ \overline{\lambda} &= \tfrac{M}{2} - \sqrt{\tfrac{M^2}{4} - \beta} \end{aligned} \tag{3.7}$$

We can rewrite our expression as follows,

$$\int_{x_0}^{x}\left(\frac{1}{\chi-\lambda}-\frac{1}{\chi-\overline{\lambda}}\right)\mathrm{d}\chi=-\frac{2r\sqrt{M^2/4-\beta}}{M}t$$
$$\ln\left|\frac{x-\lambda}{x-\overline{\lambda}}\right|=\ln\left|\frac{x_0-\lambda}{x_0-\overline{\lambda}}\right|-\frac{2r\sqrt{M^2/4-\beta}}{M}t$$

For simplifying calculations, we write, $\gamma=\frac{2r\sqrt{M^2/4-\beta}}{M}$. And we obtain as result,

$$\frac{x-\lambda}{x-\overline{\lambda}}=\frac{x_0-\lambda}{x_0-\overline{\lambda}}\mathrm{e}^{-\gamma t} \tag{3.8}$$
$$x-\lambda=\left(x-\overline{\lambda}\right)\left(\frac{x_0-\lambda}{x_0-\overline{\lambda}}\right)\mathrm{e}^{-\gamma t} \tag{3.9}$$

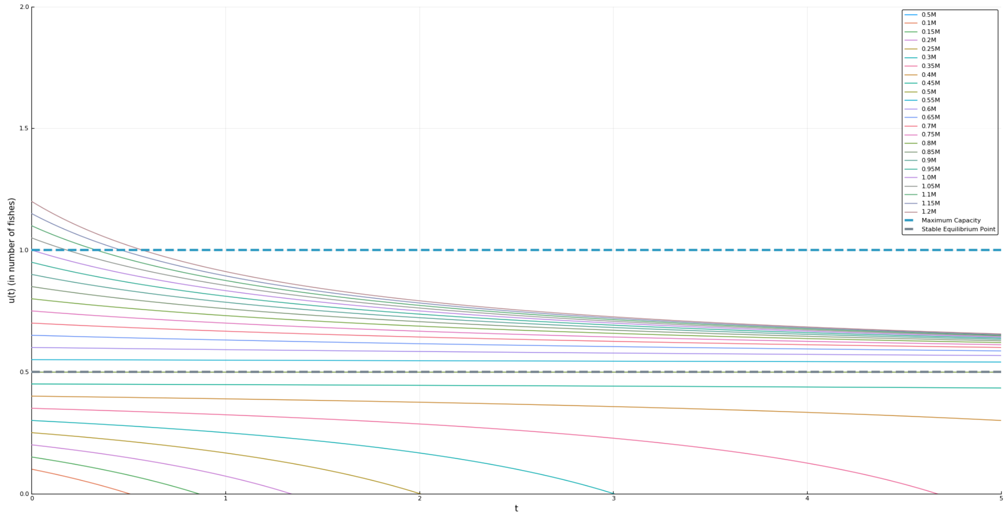
For the sake of simplicity, consider $\xi=\frac{x_0-\lambda}{x_0-\overline{\lambda}}\mathrm{e}^{-\gamma t}$. Therefore,

$$x\left(1-\xi\right)=\lambda-\overline{\lambda}\xi$$
$$x=\frac{\lambda-\overline{\lambda}\xi}{1-\xi}$$
$$x=\frac{\frac{M}{2}+\sqrt{\frac{M^2}{4}-\beta}-\left(\frac{M}{2}-\sqrt{\frac{M^2}{4}-\beta}\right)\xi}{1-\xi}$$
$$x=\frac{\frac{M}{2}+\sqrt{\frac{M^2}{4}-\beta}-\left(\frac{M}{2}-\sqrt{\frac{M^2}{4}-\beta}\right)\xi}{1-\xi}$$
$$x=\frac{\frac{M}{2}\left(1-\xi\right)+\sqrt{\frac{M^2}{4}-\beta}\left(1+\xi\right)}{1-\xi}$$
$$x=\frac{M}{2}+\sqrt{\frac{M^2}{4}-\beta}\frac{1+\xi}{1-\xi}$$

Hence, for $-\alpha>0$, we have the following result,

$$x(t)=\frac{M}{2}+\left(\sqrt{\frac{M^2}{4}-\beta}\right)\frac{(x_0-M/2)(1+\mathrm{e}^{-\gamma t})-\sqrt{M^2/4-\beta}\,(1-\mathrm{e}^{-\gamma t})}{(x_0-M/2)(1-\mathrm{e}^{-\gamma t})+\sqrt{M^2/4-\beta}\,(1+\mathrm{e}^{-\gamma t})} \tag{3.10}$$

Figure 3.2: Constant harvest rate $u = \frac{rM}{4}$.



If $u = \frac{rM}{4}$, we solve equation 3.2 as follows,

$$-\frac{M}{r}\int_{x_0}^{x}\frac{d\chi}{\left(\chi-\frac{M}{2}\right)^2} = t \tag{3.11}$$

$$\int_{x_0}^{x}\frac{d\chi}{\left(\chi-\frac{M}{2}\right)^2} = -\frac{rt}{M} \tag{3.12}$$

$$\frac{1}{x-\frac{M}{2}} = \frac{1}{x_0-\frac{M}{2}} - \frac{rt}{M} \tag{3.13}$$

$$\frac{1}{x-\frac{M}{2}} = \frac{M-\left(x_0-\frac{M}{2}\right)rt}{M\left(x_0-\frac{M}{2}\right)} \tag{3.14}$$

$$x = \frac{M}{2} + \frac{M\left(x_0-\frac{M}{2}\right)}{M-\left(x_0-\frac{M}{2}\right)rt} \tag{3.15}$$

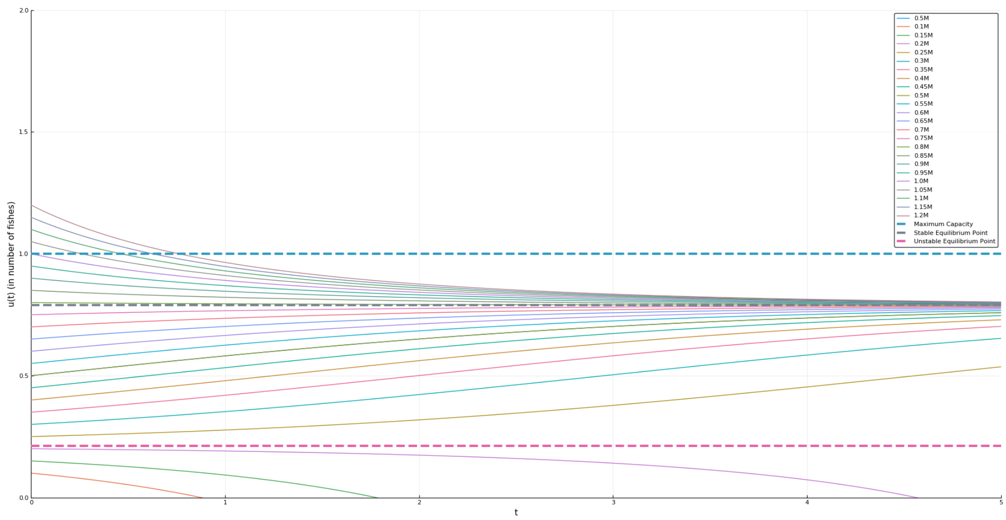


Figure 3.3: Constant harvest rate $u < \frac{rM}{4}$.

The results above stated can be explained directly from the equation 3.2, as we see in the graph 3.4, $F(x,t)$ is a paraboloid, with its maximum at $F(x^* = M/2, t) = rM^2/4$.

When $u = 0$, we have the regular logistic equation with critical points $x_{c_1} = 0$ and $x_{c_2} = M$.

We observe that the critical points x_c, such that $\frac{\mathrm{d}x_c}{\mathrm{d}t} = F(x_c, t) - u = 0$ are getting closer to each other, as u is increasing; In general, these are the solutions to the equation $F(x,t) - u = 0$,

$$x_{c_{2,1}} = \frac{M}{2} \pm \sqrt{\frac{M^2}{4} - u\frac{M}{r}} \tag{3.16}$$

Always satisfying $x_{c_2} \geq x_{c_1}$, being x_{c_2} the stable fixed point and x_{c_1} the unstable fixed point, as we see in the figure 3.3.

If our initial population x_0 lies below the unstable critical point it will lead to extinction. If our initial population lies above the unstable fixed point, it will be getting closer to the stable fixed point, we appreciate the same behavior if our population lies above the stable fixed point.

When $u = \frac{rM}{4}$ we only have one unstable fixed point. This point behaves as an attractor when $x_0 \geq \frac{M}{2}$. But when $x_0 < \frac{M}{2}$, implies $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$, for all $t > 0$ and the population decreases strictly . For $u > \frac{rM}{4}$, the dynamic has no real fixed points and $\frac{\mathrm{d}x}{\mathrm{d}t}$ is always negative, implying, that extracting constantly at a rate greater than $\frac{rM}{4}$, we will reach extinction for some given $T > 0$.

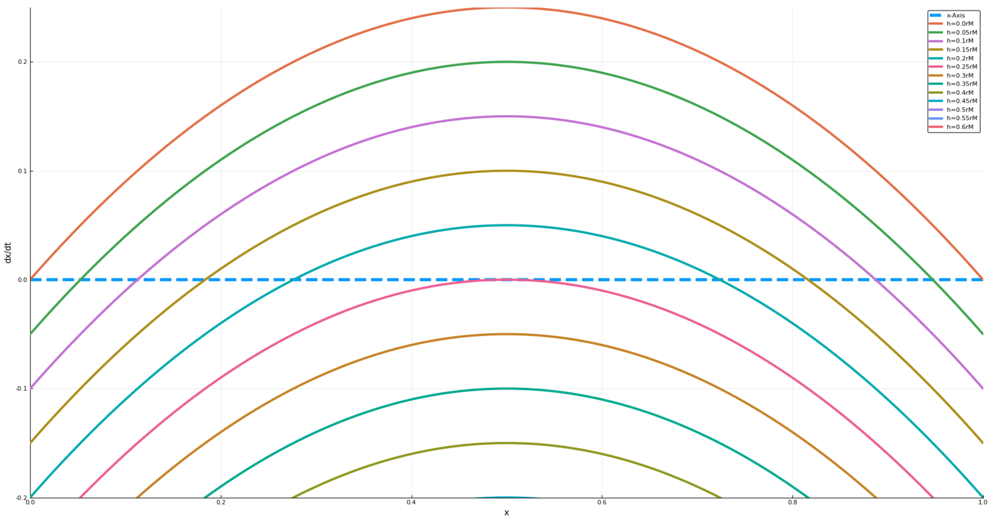


Figure 3.4: Figure representing $\frac{\mathrm{d}x}{\mathrm{d}t}$ with different harvesting rates.

3.1.2 Time Varying Harvesting.

Given a time horizon T, the amount of fish extracted for the end of this time, is given by

$$J(u;T) = \int_0^T u(t)\mathrm{d}t \tag{3.17}$$

Where $u(t)$ is the harvesting rate at time t, introduced to the population as follows,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{M}\right) - u(t) \tag{3.18}$$

We would like to know among all the possible time functions $u \in \mathbb{R}^{[0,T]}$, the functional J is maximized.

Stating in this way, is not well posed. Since we see that the functional is linear in u, implying the bigger gets u, the bigger gets J. But this approach has the disadvantage we showed in the previous discussion, that the bigger u the fastest our population can lead to extinction and therefore, we will not be able to continue harvesting the pool.

From the previous results we see that for $u(t) > \frac{rM}{4}$, we lead the population to extinction. Therefore, we restrict our harvesting rate u to,

$$0 \leq u(t) \leq \frac{rM}{4}. \tag{3.19}$$

Please note that, the equation 3.18 can be rewritten in this way,

$$u(t) = rx\left(1 - \frac{x}{M}\right) - \frac{\mathrm{d}x}{\mathrm{d}t} \tag{3.20}$$

and the functional J now, as a map of x and $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$ instead of u.

$$J(x, \dot{x}; T, r, M) = \int_0^T \left(rx\left(1 - \frac{x}{M}\right) - \dot{x}\right)\mathrm{d}t \tag{3.21}$$

We can restate the problem as follows, we would like to find the population $x(t)$ among all the possible populations[1], such that J is maximized. Once we know the population, we can construct a control $u(t)$ such that equation 3.18 is satisfied. Then, we ask the condition for $x \in C^1([0, T])$.

Therefore, the pair x^*, $\dot{x}^*$ that maximizes the functional $J = \int_0^T L(x, \dot{x}, t)\mathrm{d}t$, should satisfy the Euler-Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^*}\right) - \frac{\partial L}{\partial x^*} = 0 \tag{3.22}$$

where $L : [0, T] \times C^1([0, T]) \mapsto \mathbb{R}$. In our case $L(x, \dot{x}, t) = u(t)$, as stated in equation 3.20. Therefore,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u}{\partial \dot{x}^*}\right) - \frac{\partial u}{\partial x^*} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}(1) - r + 2r\frac{x}{M} &= 0 \\ \implies x - \frac{M}{2} &= 0 \end{aligned} \tag{3.23}$$

Hence J has one stationary point $x^*(t) = \frac{M}{2}$. Moreover, L is concave for the pair $(x, \dot{x})$, therefore the function $x^*(t)$ is a maximum, and is unique.

From equation 3.18, when $x^*(t) = \frac{M}{2}$ we need to construct the control $u(t)$, as follows,

$$u(t) = r\frac{M}{2}\left(1 - \frac{x}{M}\right) = \frac{rM}{4} \tag{3.24}$$

We observe that, when $x(t) = x^*(t)$, the control is the maximum of the constrain 3.19, we have imposed from the analysis of the fixed points of the dynamic.

[1] We see from equation 3.21, that the space where we are trying to find the optimum is $\mathrm{L}^2([0, T]) \cap \mathrm{BV}([0, T])$, where $\mathrm{BV}(\Omega)$ is the space of bounded variations over some open set Ω.

Knowing the fact, that most of the times the initial population is not always $\frac{M}{2}$, we implement the following harvesting strategy.

$$u(t) = \begin{cases} 0 & x(t) < x^*(t) \\ \frac{rM}{4} & x(t) = x^*(t) \\ \omega > \frac{rM}{4} & x(t) > x^*(t) \end{cases} \tag{3.25}$$

Where ω is some rate a bit greater than $rM/4$, fixed according to the requirements of the implementation. The aim of this strategy is to approach the fastest as possible to the stationary point of J. This can be seen as a raw closed loop dynamic, since our control is a function of the population x, as explained in the section 3.2.

3.1.3 Noise Effect.

Even we obtained a result for the optimum for J, realistically, this is a bad harvesting strategy, since we are considering the controls and the populations ideal. And small deviations in the dynamic.

In order to have a better understanding of the reality, we add the effect of noise to the dynamic.

Consider the presence of white noise, normally distributed, added in multiplicative way to the population. That is the higher the population we have, the noise increases proportional. Then, we can write the logistic equation under the optimal harvesting rate as a Weiner process,

$$dx = \left(rx\left(1 - \frac{x}{M}\right) - \frac{rM}{4}\right)\mathrm{d}t + \sigma x \mathrm{d}W \tag{3.26}$$

Figure 3.5 shows a simulation of the behavior of the ideal dynamic under the presence of noise with $\sigma = 0.1$.

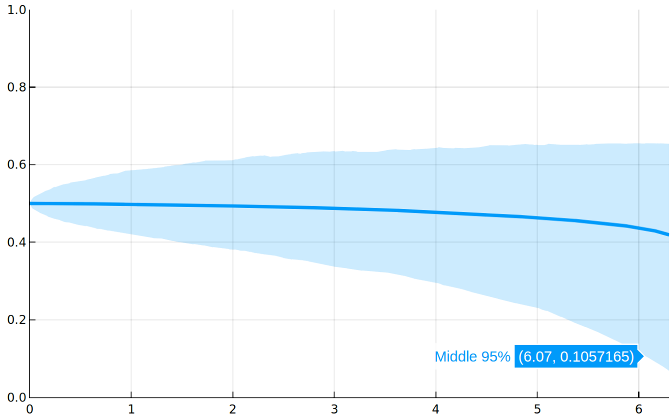


Figure 3.5: Ideal Optimum x^* under presence of noise. Population in M-units of and time in units of r-units.

We can interpret figure 3.5, as follows, with probability 0.95, our dynamic will lie in the space covered by the light blue area, with median shown by the darker line. What is really bad, since we can appreciate we have the probability to lead our population to extinction. Moreover, we can see from the graphic that for approximately six r units our population will descend from $0.5M$ to around $0.4M$ in the middle situation and will decrease to around $0.1M$ in a bad situation.

In addition to the discussion, figure 3.6 shows the 95% and 50% quantiles for the dynamic, (blue and orange area respectively).

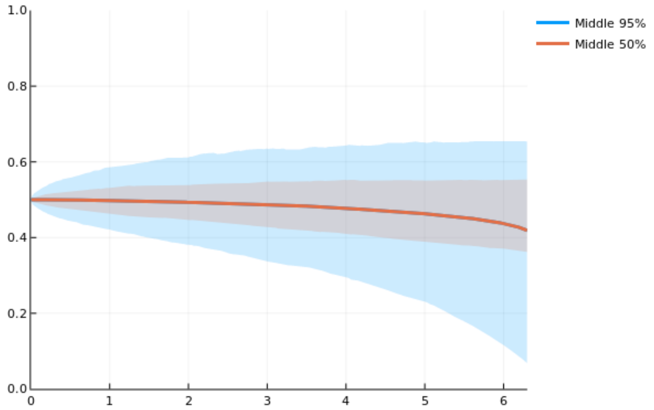
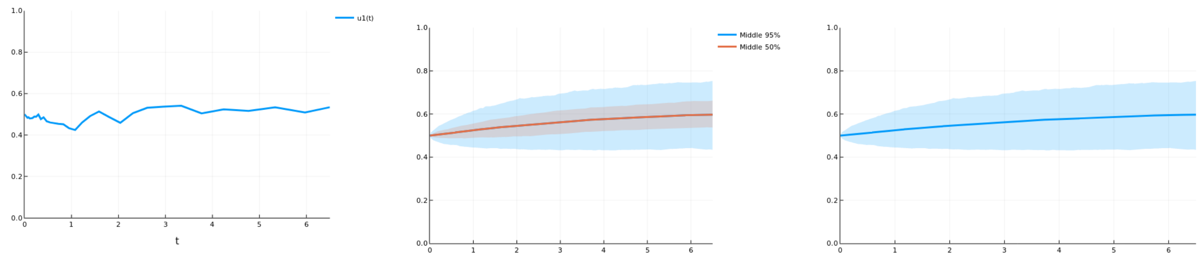


Figure 3.6: Quantiles for the ideal dynamic.

Our suggestion is to use a smarter constant harvest rate: $\bar{u} < u^*$, but close to the dynamic of (x^*, u^*). Since this for $u < u^*$ generates two fixed points, with one of them stable, most probably will not decreases until extinction. Moreover, in a really bad case it will decrease to the unstable fixed point. Also note that the middle dynamic will increase to the stable fixed point.

We show the following results for $u = 0.9u^* = \frac{9}{40}rM$ and time horizon of six r-units.

Figure 3.7: Noise Tolerant Harvesting.



(a) One simulation performed of the dynamic for $\bar{u} = 0.9u^*$.

(b) 0.95 and 0.5 quantiles

(c) 0.95 quantile

3.1.4 Optimal Harvesting. Smooth Optimal Control Problem.

As we have seen the pair $C^1([0,T]) \times C([0,T]) \ni z = (x^*, u^*) : \mathbb{R}_+ \ni t \mapsto \left(\frac{M}{2}, \frac{rM}{4}\right)$, is a stationary point, for the harvested fish functional, given a time horizon T.

Since the initial populations can be different from the stationary point, i.e. $x_0 \neq \frac{M}{2}$, we would like to implement an efficient strategy of harvesting, when we start from different initial conditions.

We propose as strategy the pair $(\hat{x}, \hat{u})$ close to the ideal dynamic (x^*, u^*). In order to do a comparison between the distance of two different pairs, we propose the L^2 norm, i.e.

$$\|\cdot\|_{\mathrm{L}^2([0,T])} : \mathrm{L}^2([0,T]) \times \mathrm{L}^2([0,T]) \to \mathbb{R}_+ \tag{3.27}$$

$$: (x,u) \mapsto \int_0^T x^2(t)\mathrm{d}t + \int_0^T u^2(t)\mathrm{d}t \tag{3.28}$$

.

Using this norm as a measure of distance between functions, has all the properties of a norm, it is Fréchet differentiable and convex. It also has as advantage, that the functions belonging to this space form a Hilbert Space giving us a geometry for the space and reflexivity of the dual. All of them great advantages at the moment of implementing algorithms and developing theory for the process of finding optimal solutions of subsets in this spaces.

Consider the transformation $h = \frac{rM}{4} - u$, then equation 3.18 transform into,

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{M}\right) - \frac{rM}{4} + h(t) \\ x(0) &= x_0 \end{aligned} \tag{3.29}$$

We see, $h \to 0$ if, and only if $u \to \frac{rM}{4}$, under the L^2-norm.[2]. Therefore, we would like to know the pair (x^*, h^*), subject to the constraining equations 3.29, such that the distance δ, between itself and the optimal ideal pair for the harvested fish, i.e. $\left(\frac{M}{2}, 0\right)$, is the smallest possible among all the feasible pairs. i.e.

$$\delta = \left\| \begin{pmatrix} x^* \\ h^* \end{pmatrix} - \begin{pmatrix} \frac{M}{2} \\ 0 \end{pmatrix} \right\|_{L^2([0,T])} = \left\| x^* - \frac{M}{2} \right\|_{L^2([0,T])} + \|h^*\|_{L^2([0,T])} \tag{3.30}$$

Could be useful to manipulate the strategy, in order to give more priority either to get closer to the optimal harvest rate or closer to the optimal population. Therefore, we add a term η to the control to manipulate this desired "priority".

Also it is necessary to add a penalty for the functions, that after a greedy extraction get further from the stationary population in a given time horizon T. By the previous requirements we propose as a fishing strategy the control that minimizes the following functional,

$$\mathscr{J}(x,u) = \frac{1}{2}\left(x(T) - \frac{M}{2}\right)^2 + \frac{1}{2}\left\| x - \frac{M}{2} \right\|^2_{L^2([0,T])} + \frac{\eta}{2}\|h\|^2_{L^2([0,T])} \tag{3.31}$$

Since the operator $e : (x,h) \mapsto \frac{d}{dt}x - rx(1 - x/M) - h$ is continuous Fréchet differentiable, by the implicit function theorem, $e(x,h) = 0$ has solution. And $\mathscr{J}$ is convex and lower-semicontinuous. Hence the existence and uniqueness of the optimal is the minimizer is assured.[3]

To solve this problem we will use an approach, first optimize then discretize. Which consists on finding analytically an expression that should be satisfied for infinite dimensional problem and then solve for it using a discrete numerical method, one advantage of this approach is that the solution found by the numerical method carry only the uncertainties proper of the implemented method.

By the structure of the problem we can apply the Pontryagin's Theorem, in order to find the expression for the minimizer element.

That is, if U is the set of values of permissible controls then the states that the optimal control h must satisfy:

$$H(x^*, u^*, \lambda^*, t) \leq H(x^*, u, \lambda^*, t), \qquad \forall u \in U,\ t \in [0,T] \tag{3.32}$$

where $x^* \in C^1[0,T]$ is the optimal state trajectory and $\lambda^* \in \mathrm{BV}[0,T]$ is the optimal costate trajectory. With the proper Hamiltonian for this problem,

$$H(x,u,\lambda,t) = \lambda\left(rx\left(1 - \frac{x}{M}\right) - \frac{rM}{4} + h\right) + \frac{1}{2}\left(x - \frac{M}{2}\right)^2 + \frac{\eta}{2}h^2 \tag{3.33}$$

[2] In general in any L^p space.
[3] Add Hinze's Bibliography here.

We see that this Hamiltonian is well defined since, our solution should satisfy $e(x,h) = 0$, and the penalty imposed is consistent with the shape of the Hamiltonian.

$$H(x(T),h(T),\lambda(T),T) = \Upsilon(x(T)) = \frac{1}{2}\left(x(T) - \frac{M}{2}\right)^2 \tag{3.34}$$

Therefore, the tuple (x^*,h^*,λ^*), should satisfy the necessary conditions,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\lambda^*) &= -\frac{\partial H}{\partial x^*} \\ \lambda^*(T) &= \frac{\mathrm{d}\Upsilon}{\mathrm{d}x}\Big|_{x^*=x(T)} \end{aligned} \tag{3.35}$$

and,

$$\frac{\partial H}{\partial h^*} = 0 \tag{3.36}$$

By the above equations our problem is reduce to find the solutions to,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\lambda^*) &= -r\lambda^* + \frac{2r}{M}x^*\lambda^* - x^* + \frac{M}{2} \\ \lambda^*(T) &= x^*(T) - \frac{M}{2} \end{aligned} \tag{3.37}$$

and

$$0 = \lambda + \eta h \tag{3.38}$$

$$\Longrightarrow \lambda = -\eta h \tag{3.39}$$

Therefore, we need to solve the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \lambda^* \\ x^* \end{pmatrix} = \begin{pmatrix} -r\lambda^* + \frac{2r}{M}x^*\lambda^* - x^* + \frac{M}{2} \\ rx^*\left(1 - \frac{1}{M}x^*\right) - \frac{rM}{4} - \frac{1}{\eta}\lambda(t) \end{pmatrix} \tag{3.40}$$

$$\lambda^*(T) = x^*(T) - M/2 \tag{3.41}$$

$$x^*(0) = x_0 \tag{3.42}$$

In order to implement the discrete solution, we need to of the above expression, we propose a Foward-Backward-Method, starting for some $x^{[0]}$, $\lambda^{[0]}$, we construct a sequence $\left(x^{[n]}, \lambda^{[n]}\right)_{n\in\mathbb{N}}$solve iteratively for,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \lambda^{[n+1]} \\ x^{[n+1]} \end{pmatrix} = \begin{pmatrix} -r\lambda^{[n+1]} + x^{[n+1]}\lambda^{[n+1]}\frac{2r}{M} - x^{[n+1]} + \frac{M}{2} \\ rx^{[n+1]}\left(1 - \frac{1}{M}x^{[n+1]}\right) - \frac{rM}{4} - \frac{1}{\eta}\lambda^{[n]}(t) \end{pmatrix} \tag{3.43}$$

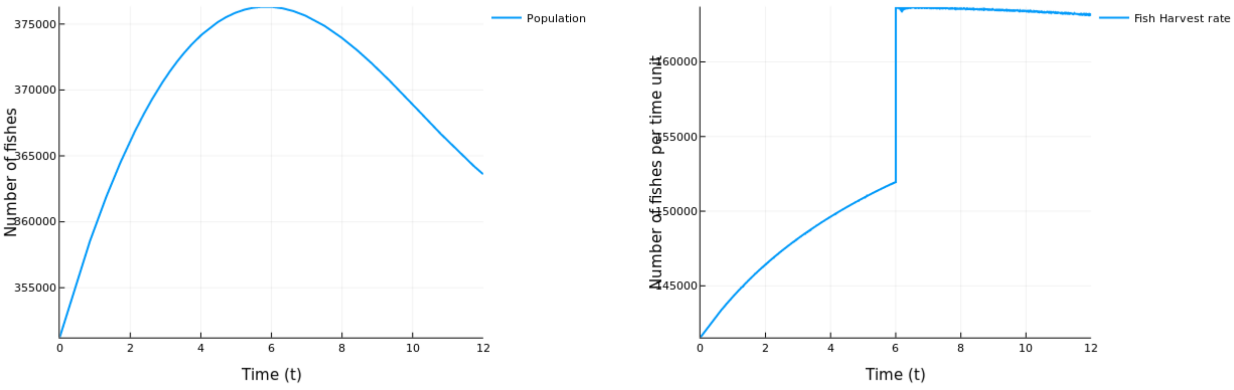
$$\lambda^{[n+1]}(T) = x^{[n+1]}(T) - M/2 \tag{3.44}$$

$$x^{[n+1]}(0) = x_0 \tag{3.45}$$

And after each iteration we substitute the value of $\lambda^{[n+1]} \leftarrow 0.5\lambda^{[n+1]} + 0.5\lambda^{[n]}$.

The idea is that after many iterations the contributions of the initials conditions are getting exponentially less significant, and the converging value of λ^* is solution for the equations3.41, 3.40 and 3.42. We present, the result of executing the above algorithm for, $T = 12$, $\eta = 20.0$, $M = 780500$, $r = 0.8$, and $x_0 = 0.45M$.

Figure 3.9: Smooth Control for Open Loop Strategy.



(a) Population under open loop control.

(b) Harvest rate solution for 3.40

Under the above conditions, the harvested population is given by $\int_0^T u(t)\mathrm{d}t \approx 1.8670261 \times 10^6$, in contrast with the optimal $J^*(\frac{M}{2}, \frac{rM}{4}) = \frac{rM}{4}T \approx 1.8732 \times 10^6$.

3.2 Closed Loop Strategies.

Closed loop strategies implement feedback as part of the dynamic, in order to improve the dynamic of the system. We write mathematically, the resulting dynamic of implementing a closed loop strategy as follows,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Psi(x,t) + C(x,t) \tag{3.46}$$

where $\Psi(x,t)$ is the intrinsic dynamic of the system, and $C(x,t)$ is the implemented control, which is a function of the population an time.

3.2.1 Constant Proportional Harvesting.

Consider the closed loop strategy, we propose to harvest in proportional way to the population, for this case instead of a constant, we take a constant multiplying out population in this way:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{M}\right) - px \tag{3.47}$$

Taking out rx as common factor we obtain:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{p}{r} - \frac{x}{M}\right) \tag{3.48}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r\left(1 - \frac{p}{r}\right)\left(1 - \frac{x}{M\left(1 - \frac{p}{r}\right)}\right)x \tag{3.49}$$

Making this change of variables, $\gamma = r\left(1 - \frac{p}{r}\right)$, $K = M\left(1 - \frac{p}{r}\right)$. With $\frac{p}{r} < 1$ the equation is reduce to:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \gamma x\left(1 - \frac{x}{K}\right) \tag{3.50}$$

Calculating the integral of this we obtain:

$$x = \frac{Kx_0}{x_0 + (K - x_0)\,\mathrm{e}^{-\gamma t}} \tag{3.51}$$

so our population in function of t is:

$$x(t) = \frac{M\left(1 - \frac{p}{r}\right)x_0}{x_0 + \left(M - \frac{Mp}{r} - x_0\right)\mathrm{e}^{-\gamma t}}. \tag{3.52}$$

3.2.2 Optimal Proportional Harvesting.

Since our harvesting control is proportional to our population, given a finite time horizon T, the amount of fishes we have extracted from our pool is given by,

$$\begin{aligned} J(x; p, T) &= \int_0^T px\,\mathrm{d}t \\ &= \int_0^T \frac{Mp\,(r-p)\,x_0}{rx_0 + (M\,(r-p) - rx_0)\,\mathrm{e}^{-(r-p)t}}\mathrm{d}t \end{aligned}$$

The equation 3.52 determines the population of fishes at time t. Consider the transformations $y = x/M$, $\tau = rt$, $\bar{p} = rp$. Therefore the equation 3.47, is transformed into:

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = (1-\bar{p})y\left(1 - \frac{y}{1-\bar{p}}\right) \tag{3.53}$$

with initial condition $y(0) = y_0 = x_0/M$. And solution,

$$y(\tau) = \frac{(1-\overline{p})y_0}{y_0 + (1 - \overline{p} - y_0)\,\mathrm{e}^{(\overline{p}-1)\tau}} \tag{3.54}$$

Then our function in the time horizon $\bar{T} = rT$

$$J\left(y; \bar{p}, \bar{T}\right) = \frac{1}{rM}\int_0^{\bar{T}} \bar{p}\,y(\tau)\mathrm{d}\tau \tag{3.55}$$

$$= \frac{\bar{p}}{rM}\left(\ln\left(1 - \bar{p} + y_0\left(\mathrm{e}^{(1-\bar{p})\bar{T}} - 1\right)\right) - \ln(1-\bar{p})\right) \tag{3.56}$$

We would like to know the constant $\bar{p}^*$ that for a given time horizon $\bar{T}$ maximizes J. Therefore $\bar{p}$ should satisfy the necessary condition,

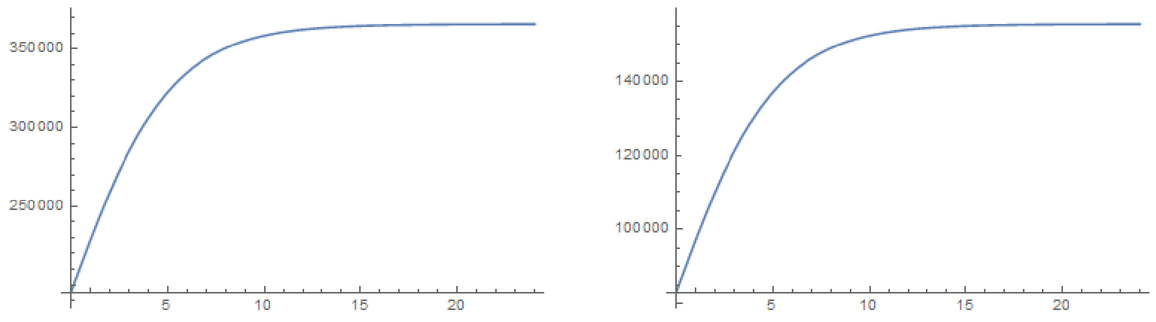
$$\left.\frac{\partial J\left(y; \bar{p}, \bar{T}\right)}{\partial p}\right|_{\bar{p}=\bar{p}^*} = 0 \tag{3.57}$$

Therefore, for given y_0 we need to solve for $\bar{p}^*$ the following equation,

$$\bar{p}^*\left(\frac{1 + Ty_0\mathrm{e}^{\left(1-\bar{p}^*\right)\bar{T}}}{\bar{p}^* + y_0 - 1 - y_0\mathrm{e}^{(1-\bar{p}^*)\bar{T}}} + \frac{1}{1-\bar{p}^*}\right) + \ln\left(1 - \bar{p}^* - y_0 + y_0\mathrm{e}^{(1-\bar{p}^*)\bar{T}}\right) - \ln(1-\bar{p}^*) = 0 \tag{3.58}$$

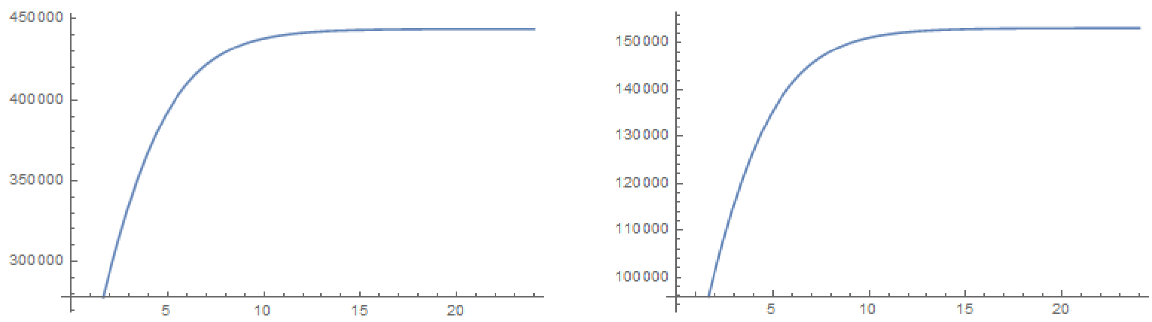
This expression has no closed form solution, but we can estimate it numerically, if we know y_0 and T. For example for $y_0 = 0.75$ and $\bar{T} = 20$, we have $\bar{p}^* \approx 0.541881$.

Given a time horizon $\bar{T} = 24$ time units (usually given in months), above with parameters $M = 780500$ fishes, $r = 0.8$ inverse time units and initial population $x_0 = 390250$. The following numerical results are done with $\bar{p}^* \approx 0.531176$ given by the equation 3.58.

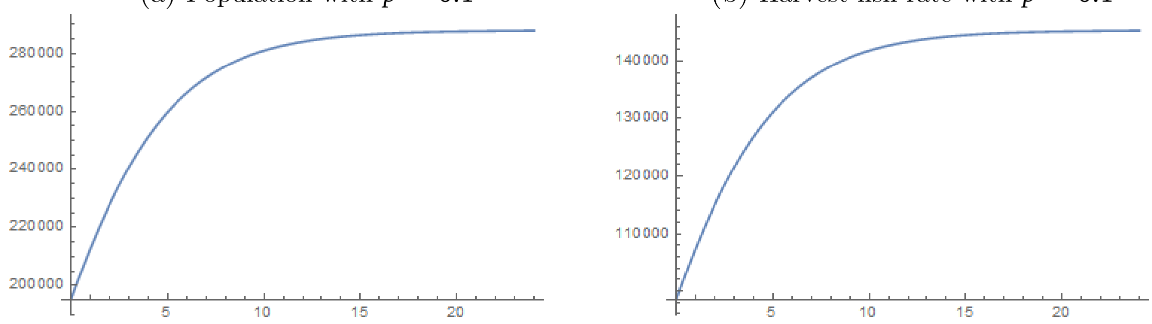


(a) Population with optimal p^*. (b) Harvest Fishes taken with optimal p^*.

The first graph show the total population of fishes during the time, and the second one represent the amount of fishes that are harvest from the model. As we can see the population goes to an stability point that is equal to $\left(1 - \frac{p}{r}\right) M$ and in the same way the amount of fishes that we harvest goes to $\bar{p}^* (1 - \bar{p}) M$. The following graphs shows the variation $\bar{p}^* \pm 0.1$ an the effect in the amount of harvested fish, please note how the stable fixed points lie below $M/2$:



(a) Population with $\bar{p}^* - 0.1$ (b) Harvest fish rate with $\bar{p}^* - 0.1$



(c) Population with $\bar{p}^* + 0.1$ (d) Harvest fish rate with $\bar{p}^* + 0.1$

In the first two there is less harvesting, then the population is bigger, but since the harvesting coefficient is smaller the amount of fishes is smaller. In the other hand, when the harvesting coefficient is bigger the population is smaller, since the harvested fish is proportional to the population, then we harvest less fish. We see the above explained behavior in table 3.1 in contrast with the optimum, corresponding to some specific parameters.

$\bar{p}$	$J(x;T,p,r,M,x0)$ (in Millions of fishes harvested)
$\bar{p}^*$	4.698153484016761
$\bar{p}^*-0.001$	4.698138002580389
$\bar{p}^*+0.001$	4.698138014054768
$\bar{p}^*-0.01$	4.696602091505628
$\bar{p}^*+0.01$	4.696609852932111
$\bar{p}^*-0.1$	4.540022027777682
$\bar{p}^*+0.1$	4.547918150118066

Table 3.1: Harvested fish $J(x;\bar{p},T,M,r,x_0)$ in units of million of fishes with different $\bar{p}$, for $T=24$, $r=0.8$, $M=780500$, $x_0=390250$

We can know estimate the value of $\bar{p}^*$, for long time horizons. Consider,

$$\lim_{T\to\infty}\ln(a+be^{cT})\approxeq cT \tag{3.59}$$

For any $a,b\in\mathbb{R}$. And

$$\lim_{T-\infty}\frac{a+bTe^{cT}}{r+de^{cT}}\approxeq\frac{b}{d}T \tag{3.60}$$

For any constants $a,b,c,d,r\in\mathbb{R}$.

Therefore, for big enough T, the contribution for fixed $\bar{p}^*$, we can write equation 3.58 in small-o notation as follows,

$$\lim_{T\to\infty}\left.\frac{\partial J}{\partial p}\right|_{p=\bar{p}^*}=(1-\bar{p}^*)\bar{T}-\bar{p}^*\bar{T}+o(T)+o(T^2)+\cdots=0 \tag{3.61}$$

Hence, when $T\to\infty$

$$(1-2\bar{p}^*)\bar{T}=0\implies\bar{p}^*=\frac{1}{2} \tag{3.62}$$

This result was expected, since $\lim_{T\to\infty}x(t)=r\bar{p}(1-\bar{p})M$, for any initial condition x_0. Therefore, for long time horizons $T\to\infty$, the functional $J(x;T,\bar{p},r,M)\to r\bar{p}(1-\bar{p})MT$, becomes a concave function of $\bar{p}^*$, whose maximum is reached at the point $\bar{p}^*=\frac{1}{2}$.

In the previous discussion, we have focused on optimizing the amount of harvested fish. A more useful aim, is to optimize the economical profits obtained from selling the fish. In the following sections we will use calculus of variations, to find an optimum profit from the harvesting problem.

4.1 Linear Costs.

In this section we will use calculus of variations theory to maximize the long-term profit. The general mark is to search for the functions that maximize or minimize given a functional.

For solving the former, the $J(x)$ has to be maximized with respect to x, as follows:

$$J(x) = \int_0^T g(t, x, \dot{x}) dt \tag{4.1}$$

$$\left.\begin{aligned} x(0) &= x_0 \\ x(T) &= x_T \end{aligned}\right\} \tag{4.2}$$

Where $g()$ is a differentiable function and $\dot{x}$ denote the derivative of the x respect to the time. $x(t)$ is the x in specific time whereas x (without t) shows the entire x path. Any x that satisfy the boundary conditions in equation 4.1 is admissible.

Analysis of x in infinite small variations in admissible range (4.1) to optimize the 4.1 function guides to the Euler-Lagrange equation:

$$\frac{\partial g}{\partial x} = \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} \tag{4.3}$$

4.1.1 Costs

As the goal is maximizing the profit, we are going to use the following functional used by Clark and Hannensson in their studies about this topic:

$$J(x) = \int_0^\infty e^{-pt}[p - c(x)]h dt \tag{4.4}$$

note that:

$$\begin{gathered} \dot{x} = \tfrac{dx}{dt} = f(x) - h \\ x(0) = x_0 \end{gathered} \tag{4.5}$$

where $x(t)$ is population, $h(t)$ is the harvest rate, $f(x)$ is biological growth, $c(x)$ is the unit cost of farming and p is the unit price of harvested aquaculture. we assumed p as a constant.

Maximized function can show as:

$$J* = max_h \int_0^\infty e^{-\rho t}[p - c(x)]hdt \tag{4.6}$$

Because the value of money unit decrease (with the rate ρ) by the time, if we have n time of money value decrease in time unit, then t units of time amount to nt discount periods. So we can obtain the present value of money unit by:

$$\lim_{n\to\infty}(1 - \frac{\rho}{n})^{nt} = e^{-\rho t} \tag{4.7}$$

Equation 4.1 and 4.1.1 gives:

$$J* = max_x \int_0^\infty e^{-\rho t}[p - c(x)][f(x) - \dot{x}]dt \tag{4.8}$$

Now, using Euler-Lagrange condition, we can write:

$$f'(x) - \frac{c'(x)f(x)}{p - c(x)} = \rho \tag{4.9}$$

note that primes are differentiations with respect to x. This equation also can be written as:

$$\frac{\partial}{\partial x}[(p - c(x))f(x)] = \rho[p - c(x)] \tag{4.10}$$

If we define $x*$ as the optimal population level that maximize the profit, it can be obtain by solving followed equation:

$$\frac{\partial}{\partial x*}[(p - c(x*))f(x*)] = \rho[p - c(x*)] \tag{4.11}$$

This obtained using equation 4.1.1.

By applying $x*$ in 4.1.1, $h*$ can obtain. It is possible that the last equation might have more than one root and $x*$ not be unique.

4.1.2 Logistic growth

Assume that: $f(x) = rx(1-x/k)$ $h = qEx$ $c(x) = c/(qx)$ in which $f(x)$ is growth, h is harvest rate and $c(x)$ is cost function with constant cost per unit (c). Then, the equation 4.1.1 can be written as:

$$J* = max_h \int_0^\infty e^{-\rho t}[p - \frac{c}{qx}]hdt = max_E \int_0^\infty e^{-\rho t}(pqx - c)Edt \tag{4.12}$$

in which following must be satisfied:

$$\begin{gathered} \dot{x} = rx(1 - \tfrac{x}{K}) - qEx \\ x(0) = x_0 \end{gathered} \tag{4.13}$$

So, utilizing equation 4.1.2, effort (E) can written as

$$E = \frac{rx(1-\frac{x}{K}) - \dot{x}}{qx} \tag{4.14}$$

Using and , maximized objective can be written as:

$$J* = max_x \int_0^{\infty} e^{-\rho t}(p - \frac{c}{qx})[rx(1-\frac{x}{K}) - \dot{x}]dt \tag{4.15}$$

By applying Euler-Lagrange condition, following equation will be obtain to calculate optimal population $x*$

$$x* = \frac{K}{4}\left[(1 + \frac{c}{pKq} - \frac{\rho}{r}) + \sqrt{(1 + \frac{c}{pKq} - \frac{p}{r})^2 + \frac{8cp}{pKqr}}\right] \tag{4.16}$$

So when the population yields to that value the rate of harvest is equal to the biological growth rate and the population is established at that optimal point.

As now we know the optimal population $x*$ we can develop a harvesting policy to drive the population to that value as fast as possible and keep it there. We assume that the effort is constrained as $0 \leq E \leq E_{max}$ because there is a top in the effort that we can apply in the harvest. We define this policy as follows:

$$E*(t) = \begin{cases} E_{max}, & \text{x(t) > x*,} \\ \frac{rx*(1-\frac{x*}{K})}{qx*}, & \text{x(t)=x*,} \\ 0, & \text{x(t) < x*.} \end{cases} \tag{4.17}$$

With that policy we help the system to be always at the optimal point by making the maximum harvesting effort when the population is to big and letting the population growth free when it's below the optimal point.

With that

4.2 Dynamic Programming.

4.3 Stochastic Analysis.