
Optimal Harvesting Modelling

Final Report

UAB
Universitat Autònoma
de Barcelona

Centre de Recerca Matemàtica
Universitat Autònoma de Barcelona
Group number 4.



Abstract

Contents

1	Preliminary Concept	3
2	Problem Framework	4
3	Mathematical Models.	5
3.1	Exponential biological growth.	5
3.2	Logistic Equation.	5
3.3	Wiener Process and noise.	5
4	Fishing Strategies and Optimizing Population	6
4.1	Open Loop Strategies.	6
4.1.1	Constant Harvesting Analysis.	6
4.1.2	Time Varying Harvesting.	8
4.1.3	Optimal Harvesting. Smooth Optimal Control Problem.	8
4.2	Closed Loop Strategies.	8
4.2.1	Constant Proportional Harvesting.	8
4.2.2	Optimal Proportional Harvesting.	8
5	Economical Profit	9
5.1	Linear Costs.	9
5.2	Quadratic Costs.	9
5.3	Stochastic Analysis.	9
6	Further Research	10

1 Preliminary Concept

2 Problem Framework

$$r = 0.8 \tag{2.1}$$

$$M = 780500 \tag{2.2}$$

3 Mathematical Models.

$$\frac{dx}{dt} = F(x, t) \quad (3.1)$$

3.1 Exponential biological growth.

$$F(x, t) = rx \quad (3.2)$$

3.2 Logistic Equation.

Logistic equation.

$$F(x, t) = rx \left(1 - \frac{x}{M}\right) \quad (3.3)$$

3.3 Wiener Process and noise.

$$dx = \left(rx \left(1 - \frac{x}{M}\right) - u \right) dt + \sigma x dW \quad (3.4)$$

A unique solution exists if both Itô conditions hold (Fleming and Rishel, 1975). The first is the linear growth condition.

$$\left| rx \left(1 - \frac{x}{M}\right) - u \right| \leq K(1 + |x|) \quad (3.5)$$

$$|\sigma x| \leq K(1 + |x|) \quad (3.6)$$

and the Lipschitz condition,

$$\left| rx_2 \left(1 - \frac{x_2}{M}\right) - rx_1 \left(1 - \frac{x_1}{M}\right) \right| \leq L|x_2 - x_1| \quad (3.7)$$

$$|\sigma(x_2 - x_1)| \leq L|x_2 - x_1| \quad (3.8)$$

For some independent constants L and K .

4 Fishing Strategies and Optimizing Population

4.1 Open Loop Strategies.

4.1.1 Constant Harvesting Analysis.

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{M}\right) - u \quad (4.1)$$

$$\beta = \frac{uM}{r} \quad (4.2)$$

$$\begin{aligned} \frac{dx}{rx \left(1 - \frac{x}{M}\right) - u} &= dt \\ \int_{x_0}^x \frac{d\chi}{r\chi \left(1 - \frac{\chi}{M}\right) - u} &= \int_0^t d\tau \\ \frac{M}{r} \int_{x_0}^x \frac{d\chi}{\chi(M - \chi) - \frac{Mu}{r}} &= t \\ -\frac{M}{r} \int_{x_0}^x \frac{d\chi}{\chi^2 - M\chi + \beta} &= t \\ -\frac{M}{r} \int_{x_0}^x \frac{d\chi}{\left(\chi - \frac{M}{2}\right)^2 - \frac{M^2}{4} + \beta} &= t \end{aligned} \quad (4.3)$$

$$\alpha = \beta - \frac{M^2}{4} = rM \left(u - \frac{rM}{4}\right) \quad (4.4)$$

If $u > rM/4$ implies $\alpha > 0$

$$\begin{aligned} \int_{x_0}^x \frac{d\chi}{\left(\chi - \frac{M}{2}\right)^2 + \alpha} &= -\frac{r}{M} t \\ \frac{1}{\sqrt{\beta - \frac{M^2}{4}}} \left(\arctan\left(\frac{x - M/2}{\sqrt{\beta - M^2/4}}\right) - \arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}}\right) \right) &= -\frac{r}{M} t \end{aligned}$$

$$x = \frac{M}{2} + \sqrt{\beta - \frac{M^2}{4}} \tan\left(\arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}}\right) - \frac{r\sqrt{\beta - M^2/4}}{M} t\right) \quad (4.5)$$

If $u < rM/4$ implies $\alpha < 0$,

$$\int_{x_0}^x \frac{d\chi}{\left(\chi - \frac{M}{2}\right)^2 - \alpha} = -\frac{r}{M}t$$

$$\begin{aligned}\lambda &= \frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta} \\ \bar{\lambda} &= \frac{M}{2} - \sqrt{\frac{M^2}{4} - \beta}\end{aligned}\tag{4.6}$$

$$\begin{aligned}\int_{x_0}^x \left(\frac{1}{\chi - \lambda} - \frac{1}{\chi - \bar{\lambda}} \right) d\chi &= -\frac{2r\sqrt{M^2/4 - \beta}}{M}t \\ \ln \left| \frac{x - \lambda}{x - \bar{\lambda}} \right| &= \ln \left| \frac{x_0 - \lambda}{x_0 - \bar{\lambda}} \right| - \frac{2r\sqrt{M^2/4 - \beta}}{M}t\end{aligned}$$

$$\gamma = \frac{2r\sqrt{M^2/4 - \beta}}{M}$$

$$\frac{x - \lambda}{x - \bar{\lambda}} = \frac{x_0 - \lambda}{x_0 - \bar{\lambda}} e^{-\gamma t}\tag{4.7}$$

$$x - \lambda = (x - \bar{\lambda}) \left(\frac{x_0 - \lambda}{x_0 - \bar{\lambda}} \right) e^{-\gamma t}\tag{4.8}$$

$$\xi = \frac{x_0 - \lambda}{x_0 - \bar{\lambda}} e^{-\gamma t}$$

$$\begin{aligned}x(1 - \xi) &= \lambda - \bar{\lambda}\xi \\ x &= \frac{\lambda - \bar{\lambda}\xi}{1 - \xi} \\ x &= \frac{\frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta} - \left(\frac{M}{2} - \sqrt{\frac{M^2}{4} - \beta} \right) \xi}{1 - \xi} \\ x &= \frac{\frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta} - \left(\frac{M}{2} - \sqrt{\frac{M^2}{4} - \beta} \right) \xi}{1 - \xi} \\ x &= \frac{\frac{M}{2}(1 - \xi) + \sqrt{\frac{M^2}{4} - \beta}(1 + \xi)}{1 - \xi} \\ x &= \frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta} \frac{1 + \xi}{1 - \xi}\end{aligned}$$

$$x = \frac{M}{2} + \left(\sqrt{\frac{M^2}{4} - \beta} \right) \frac{(x_0 - M/2)(1 + e^{-\gamma t}) - \sqrt{M^2/4 - \beta}(1 - e^{-\gamma t})}{(x_0 - M/2)(1 - e^{-\gamma t}) + \sqrt{M^2/4 - \beta}(1 + e^{-\gamma t})}\tag{4.9}$$

4.1.2 Time Varying Harvesting.

For time varying Harvest,

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{M}\right) - u(t) \quad (4.10)$$

4.1.3 Optimal Harvesting. Smooth Optimal Control Problem.

For Optimal Control we reduce the problem to the following,

$$\min_{\substack{x \in X \\ u \in U}} J(x, u) \quad (4.11)$$

subject to,

$$e(x, u) = 0 \quad (4.12)$$

$$h(t) = \frac{rM}{4} - u(t)$$

$$J(x, u) = \frac{\zeta}{2} \left(x(T) - \frac{M}{2}\right)^2 + \frac{1}{2} \left\|x - \frac{M}{2}\right\|_{L^2([0, T])}^2 + \frac{\eta}{2} \|h\|_{L^2([0, T])}^2 \quad (4.13)$$

subject to,

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{M}\right) - \frac{rM}{4} + h \quad (4.14)$$

4.2 Closed Loop Strategies.

4.2.1 Constant Proportional Harvesting.

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{M}\right) - px \quad (4.15)$$

$$\frac{dx}{dt} = rx \left(1 - \frac{p}{r} - \frac{x}{M}\right) \quad (4.16)$$

$$\frac{dx}{dt} = r \left(1 - \frac{p}{r}\right) \left(1 - \frac{x}{M \left(1 - \frac{p}{r}\right)}\right) x \quad (4.17)$$

$\gamma = r \left(1 - \frac{p}{r}\right)$, $K = M \left(1 - \frac{p}{r}\right)$. With $\frac{p}{r} < 1$

$$\frac{dx}{dt} = \gamma x \left(1 - \frac{x}{K}\right) \quad (4.18)$$

$$x = \frac{Kx_0}{x_0 + (K - x_0)e^{-\gamma t}} \quad (4.19)$$

$$x = \frac{M \left(1 - \frac{p}{r}\right) x_0}{x_0 + \left(M - \frac{Mp}{r} - x_0\right) e^{-\gamma t}} \quad (4.20)$$

4.2.2 Optimal Proportional Harvesting.

5 Economical Profit

5.1 Linear Costs.

5.2 Quadratic Costs.

5.3 Stochastic Analysis.

6 Further Research
