Optimal Harvesting Modelling

Final Report

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Abstract

Contents

As a natural, healthy and nutritious food, with variety of species and diverse growth environments, fish seems to be a wise choice to solve some food - related crisis regarding to the human population growth around the world. On the other hand, there is a limitation for the fish population sustainability in open seas. Global high demand, resulted in over-exploiting the oceans in the past decades (Figure 1).

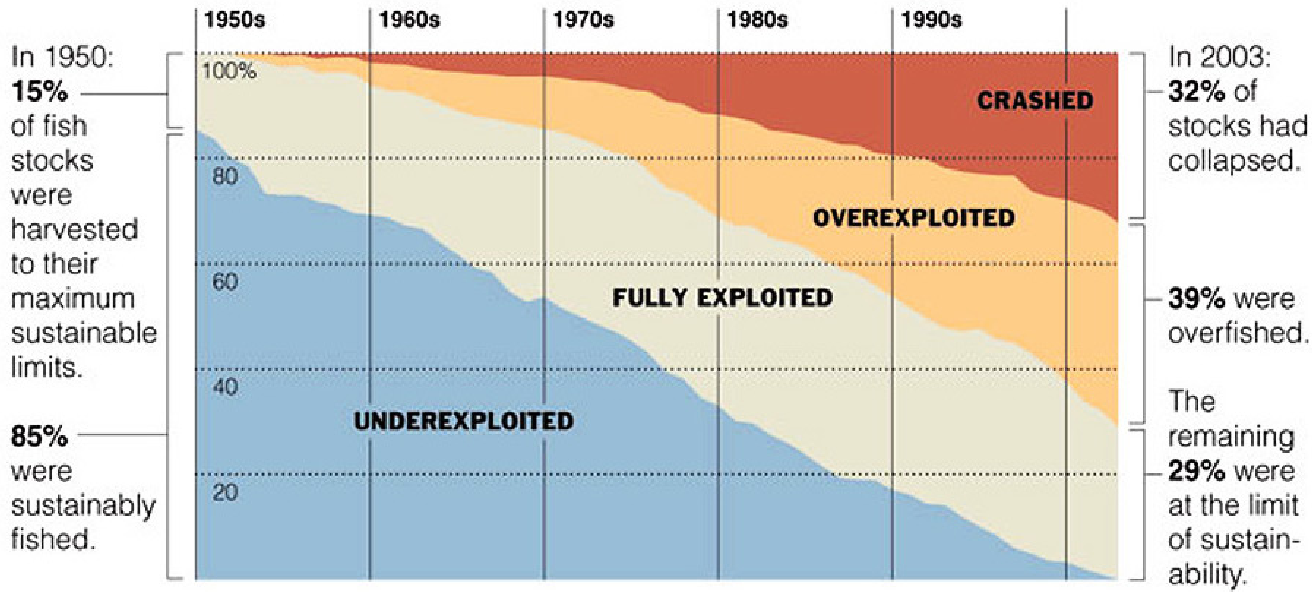

Figure 1.1: sustainable fishing between 1950 and 2003

In result, some fish populations have been severely declined during the years. Figure 1 shows the population of utilized fish population between 1970 and and 2010. As illustrated, the index for all utilized fish species indicates a 50 per cent reduction in population number globally between 1970 and 2010.

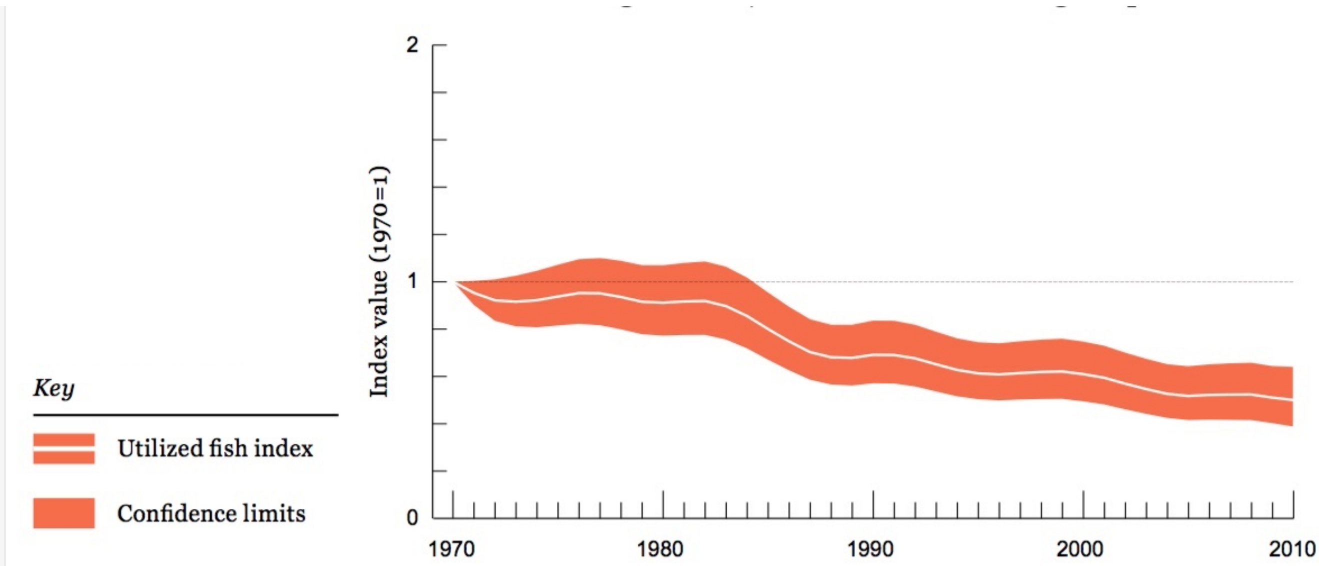

Figure 1.2: Utilized fish index value between 1970 and 2010

One of the solutions to fish population decrease problem is to shift from fish catching to fish harvesting. This strategy can help recovering fish population and size gradually beside providing human with seafood. Figure 1 and 1 shows the fish harvesting production grows in 1970 and 2010 year around the world.

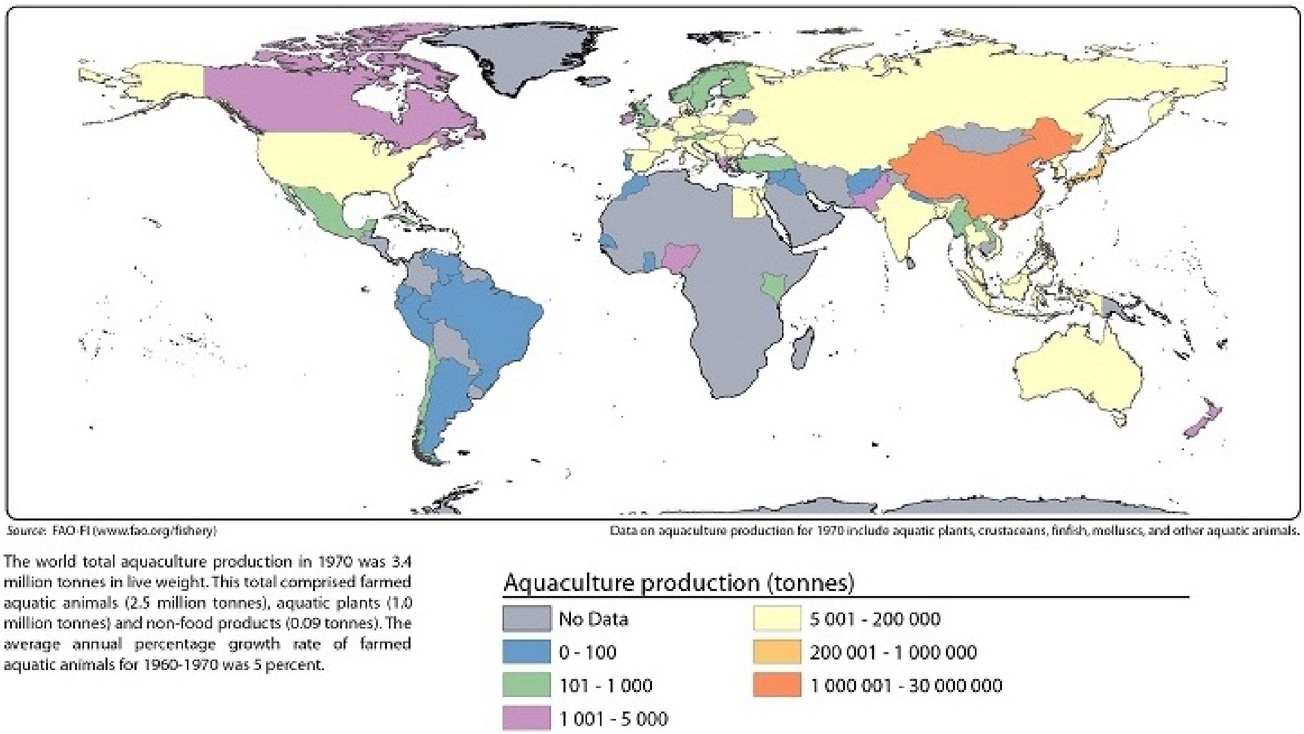

Figure 1.3: Aquaculture production in 1970 around the world

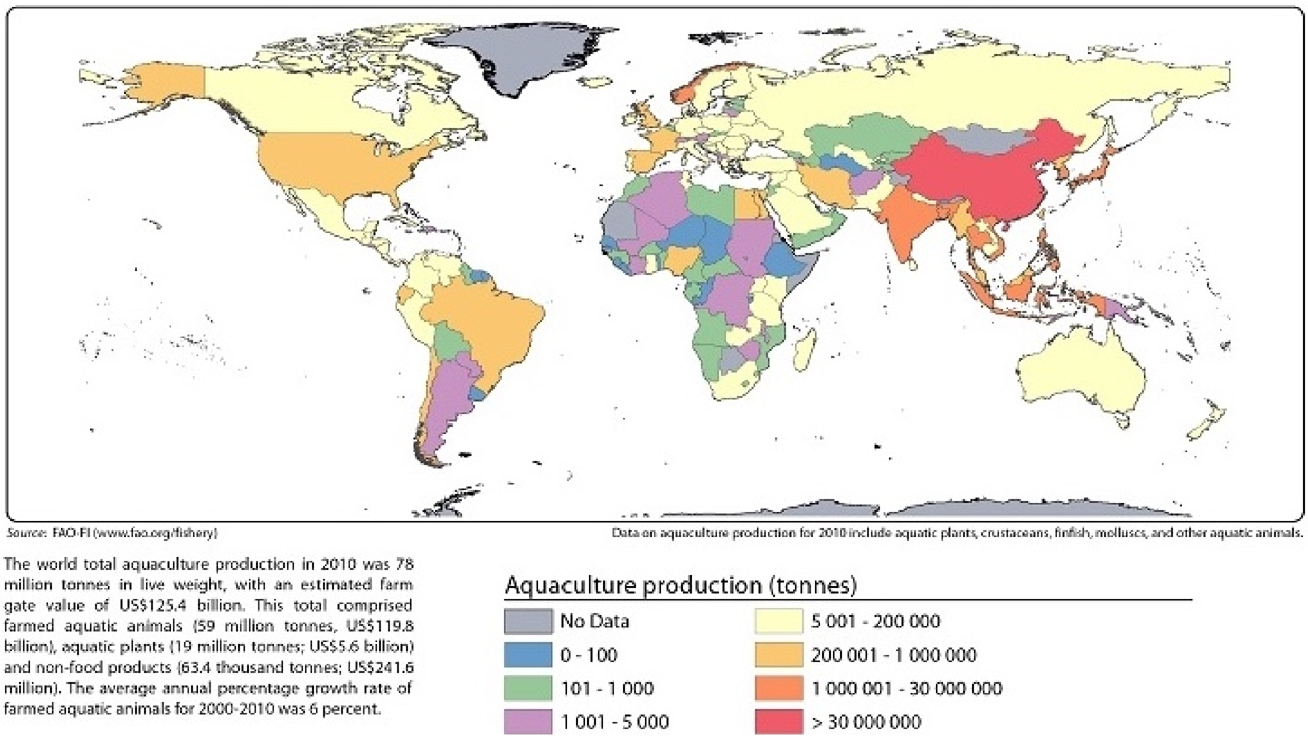

Figure 1.4: Aquaculture production in 2010 around the world

Like any other industry, It is crucial to optimize the fish harvesting procedure to have maximum -still consistent- production in fish harvesting farms. In this work, weâĂŹre trying to describe one of the fish harvesting mathematical models and achieve the optimum fish farm population to have a consistent population.

1.1 Fish harvesting models

Generally, there are three methods to model aquaculture mathematically:

- constant harvesting: One of the simplest methods is the idea of harvesting where a constant number of fish removing from the main population over a given time interval
- proportional harvesting: Another common form of harvesting that the quantity of harvested fish is proportional to the population.
- periodic harvesting :Another very used form of harvesting is when it is done during periods of time within a year, so the fish wonâĂŹt become extinct during fishing time and in some periods fishing is stopped, the population of fish might be able to increase again.

2 Mathematical Models.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,t) \tag{2.1}$$

2.1 Exponential biological growth.

Assuming the natural fish mortality to be a constant M, we get the growth dynamics as,

$$\begin{aligned} \tfrac{\mathrm{d}x}{\mathrm{d}t} &= -mx \\ x(T) &= x_T \end{aligned} \tag{2.2}$$

If a variable mortality due to fishing $\Phi(t)$, is also considered then the growth equation becomes,

$$\begin{aligned} \tfrac{\mathrm{d}x}{\mathrm{d}t} &= -(m+\Phi(t))x \\ x(T) &= x_T \end{aligned} \tag{2.3}$$

2.2 Logistic Equation.

Logistic equation.

$$F(x,t) = rx\left(1-\frac{x}{M}\right) \tag{2.4}$$

2.3 Wiener Process and noise.

We consider the behavior of the logistic equation under the presence of noise, in multiplicative way to the population. For the elements $(t,x) \in Q = (0,T) \times (0,M)$, we state the following differential equation,

$$dx = \left(rx\left(1-\frac{x}{M}\right)\right)\mathrm{d}t + \sigma x \mathrm{d}W \tag{2.5}$$

A unique solution exists if both Itó conditions hold (Fleming and Rishel, 1975). The first one is the linear growth condition, for some independent constant K,

$$\left|rx\left(1-\frac{x}{M}\right)\right| \leq K(1+|x|) \tag{2.6}$$

$$|\sigma x| \leq K(1+|x|) \tag{2.7}$$

The second one is the Lipschitz condition,$\exists L$ independent constant, and $\forall x$, $\exists B(x)$ neighborhood of x, such that $\forall x_1, x_2 \in B(x)$,

$$\left|rx_2\left(1-\frac{x_2}{M}\right) - rx_1\left(1-\frac{x_1}{M}\right)\right| \leq L|x_2-x_1| \tag{2.8}$$

$$|\sigma(x_2-x_1)| \leq L|x_2-x_1| \tag{2.9}$$

Since $F(x,t) = rx\left(1-\frac{x}{M}\right)$ is continuously differentiable in x, F is Lipschitz in x then condition 2.8 is satisfied. For bounded σ, condition 2.9 is satisfied. Moreover the sufficient conditions for the Itô conditions are satisfied for all functions C^1 on the closure of any compact set Q.

Since the above conditions are satisfied, we can guarantee existence and uniqueness of the solution for the equation 2.5. Given by the equation:

$$\begin{aligned} x(t) &= x_0 + \int_0^t \left(rx\left(1 - \frac{x}{M}\right)\right) \mathrm{d}t + \int_0^t \sigma x \mathrm{d}W, \\ x(0) &= x_0, \\ W(0) &= 0. \end{aligned} \tag{2.10}$$

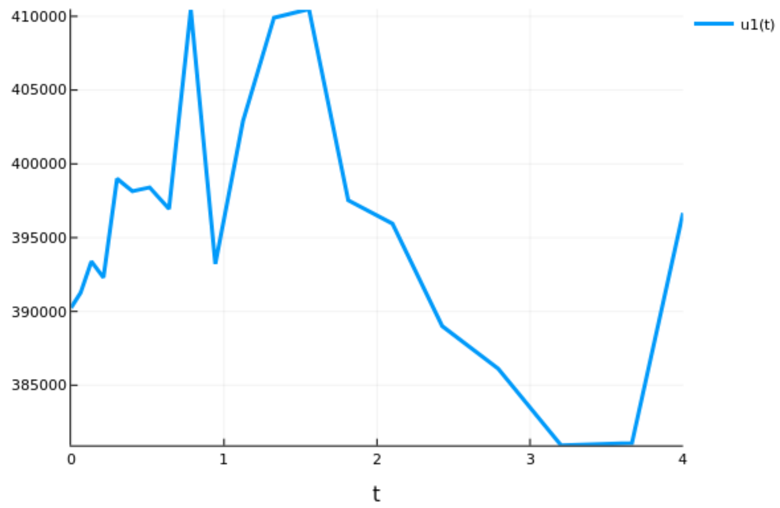

Figure 2.1: Simulation performed of logistic equation 2.5, with parameters $r = 0.8 \frac{1}{\text{month} \times \text{fish}}$, $x_0 = \frac{M}{2}$, for a population in natural conditions (harvest exploitation $u = 0.$), with presence of noise proportional to the population, with $\sigma = 0.1$. Performed during 4 months.

3.1 Open Loop Strategies.

Generally, in an open loop strategy -also called a non-feedback strategy- the process does not use a feedback to determine if its output has achieved the desired goal of the process. The implementation of open loop harvesting strategies take place without considering the impact of the extraction process. Mathematically,

$$\frac{dx}{dt} = \Psi(x,t) + C(t) \tag{3.1}$$

where $\Psi(x,t)$ is the intrinsic dynamic of the system and $C(t)$ is the control parameter, which is population independent.

3.1.1 Constant Harvesting Analysis.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{M}\right) - u \tag{3.2}$$

We introduce the following variable in order to simply calculations,

$$\beta = \frac{uM}{r} \tag{3.3}$$

Solving the differential equation,

$$\begin{aligned}
\frac{\mathrm{d}x}{rx\left(1-\frac{x}{M}\right)-u} &= \mathrm{d}t \\
\int_{x_0}^{x} \frac{\mathrm{d}\chi}{r\chi\left(1-\frac{\chi}{M}\right)-u} &= \int_0^t \mathrm{d}\tau \\
\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\chi\,(M-\chi)-\frac{Mu}{r}} &= t \\
-\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\chi^2 - M\chi + \beta} &= t
\end{aligned}$$

Finally, we model the above integral as one

$$-\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi - \frac{M}{2}\right)^2 - \frac{M^2}{4} + \beta} = t \tag{3.4}$$

Consider α as follows,

$$\alpha = \beta - \frac{M^2}{4} = rM\left(u - \frac{rM}{4}\right) \tag{3.5}$$

We see that the sign of α determines the nature of the solutions. Then, if $u > rM/4$ implies $\alpha > 0$,

$$\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi - \frac{M}{2}\right)^2 + \alpha} = -\frac{r}{M} t$$

$$\frac{1}{\sqrt{\beta - \frac{M^2}{4}}} \left(\arctan\left(\frac{x - M/2}{\sqrt{\beta - M^2/4}} \right) - \arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}} \right) \right) = -\frac{r}{M} t$$

Therefore, for $\alpha > 0$ the population behaves as follows,

$$x(t) = \frac{M}{2} + \sqrt{\beta - \frac{M^2}{4}} \tan\left(\arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}} \right) - \frac{r\sqrt{\beta - M^2/4}}{M} t \right) \tag{3.6}$$

Equation 3.6 show us that for some t^*, $x(t^*) = 0$,independently of the initial condition x_0, since the argument inside the $\tan$ is monotone decreasing in t.

If $u < rM/4$ implies $-\alpha > 0$,

$$\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi - \frac{M}{2}\right)^2 - (-\alpha)} = -\frac{r}{M} t$$

Considering the zeros of the denominator, λ and $\overleftarrow{\lambda}$,

$$\begin{aligned} \lambda &= \frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta} \\ \bar{\lambda} &= \frac{M}{2} - \sqrt{\frac{M^2}{4} - \beta} \end{aligned} \tag{3.7}$$

We can rewrite our expression as follows,

$$\int_{x_0}^{x} \left(\frac{1}{\chi - \lambda} - \frac{1}{\chi - \bar{\lambda}} \right) \mathrm{d}\chi = -\frac{2r\sqrt{M^2/4 - \beta}}{M} t$$

$$\ln\left| \frac{x - \lambda}{x - \bar{\lambda}} \right| = \ln\left| \frac{x_0 - \lambda}{x_0 - \bar{\lambda}} \right| - \frac{2r\sqrt{M^2/4 - \beta}}{M} t$$

For simplifying calculations, we write, $\gamma = \frac{2r\sqrt{M^2/4 - \beta}}{M}$. And we obtain as result,

$$\frac{x - \lambda}{x - \bar{\lambda}} = \frac{x_0 - \lambda}{x_0 - \bar{\lambda}} \mathrm{e}^{-\gamma t} \tag{3.8}$$

$$x - \lambda = \left(x - \bar{\lambda} \right) \left(\frac{x_0 - \lambda}{x_0 - \bar{\lambda}} \right) \mathrm{e}^{-\gamma t} \tag{3.9}$$

For the sake of simplicity, consider $\xi = \frac{x_0-\lambda}{x_0-\overline{\lambda}}e^{-\gamma t}$. Therefore,

$$
\begin{aligned}
x(1-\xi) &= \lambda - \overline{\lambda}\xi \\
x &= \frac{\lambda - \overline{\lambda}\xi}{1-\xi} \\
x &= \frac{\frac{M}{2} + \sqrt{\frac{M^2}{4}-\beta} - \left(\frac{M}{2} - \sqrt{\frac{M^2}{4}-\beta}\right)\xi}{1-\xi} \\
x &= \frac{\frac{M}{2} + \sqrt{\frac{M^2}{4}-\beta} - \left(\frac{M}{2} - \sqrt{\frac{M^2}{4}-\beta}\right)\xi}{1-\xi} \\
x &= \frac{\frac{M}{2}(1-\xi) + \sqrt{\frac{M^2}{4}-\beta}\,(1+\xi)}{1-\xi} \\
x &= \frac{M}{2} + \sqrt{\frac{M^2}{4}-\beta}\,\frac{1+\xi}{1-\xi}
\end{aligned}
$$

Hence, for $-\alpha > 0$, we have the following result,

$$
x(t) = \frac{M}{2} + \left(\sqrt{\frac{M^2}{4}-\beta}\right)\frac{(x_0-M/2)(1+e^{-\gamma t}) - \sqrt{M^2/4-\beta}\,(1-e^{-\gamma t})}{(x_0-M/2)(1-e^{-\gamma t}) + \sqrt{M^2/4-\beta}\,(1+e^{-\gamma t})} \tag{3.10}
$$

If $u = \frac{rM}{4}$, we solve equation 3.2 as follows,

$$
-\frac{M}{r}\int_{x_0}^{x}\frac{d\chi}{\left(\chi-\frac{M}{2}\right)^2} = t \tag{3.11}
$$

$$
\int_{x_0}^{x}\frac{d\chi}{\left(\chi-\frac{M}{2}\right)^2} = -\frac{rt}{M} \tag{3.12}
$$

$$
\frac{1}{x-\frac{M}{2}} = \frac{1}{x_0-\frac{M}{2}} - \frac{rt}{M} \tag{3.13}
$$

$$
\frac{1}{x-\frac{M}{2}} = \frac{M-\left(x_0-\frac{M}{2}\right)rt}{M\left(x_0-\frac{M}{2}\right)} \tag{3.14}
$$

$$
x = \frac{M}{2} + \frac{M\left(x_0-\frac{M}{2}\right)}{M-\left(x_0-\frac{M}{2}\right)rt} \tag{3.15}
$$

The results above stated can be explained directly from the equation 3.2, as we see in the graph 3.1, $F(x,t)$ is a paraboloid, with its maximum at $F(x^* = M/2, t) = rM^2/4$.

When $u = 0$, we have the regular logistic equation with critical points $x_{c_1} = 0$ and $x_{c_2} = M$. With x_{c_2} being an stable fixed point and x_{c_1} an unstable fixed point. In general, these are the solutions to the equation $F(x,t) - u = 0$,

$$
x_{c_{2,1}} = \frac{M}{2} \pm \sqrt{\frac{M^2}{4} - u\frac{M}{r}} \tag{3.16}
$$

We observe that the critical points x_c, such that $\frac{dx_c}{dt} = F(x_c,t) - u = 0$ are getting closer to each other, as u is increasing; when $u = \frac{rM}{4}$ we only have one critical unstable point. That behaves as an attractor when $x_0 \geq \frac{M}{2}$. But when $x_0 < \frac{M}{2}$ the population strictly decreases. For $u > \frac{rM}{4}$, the population $x(t)$ has

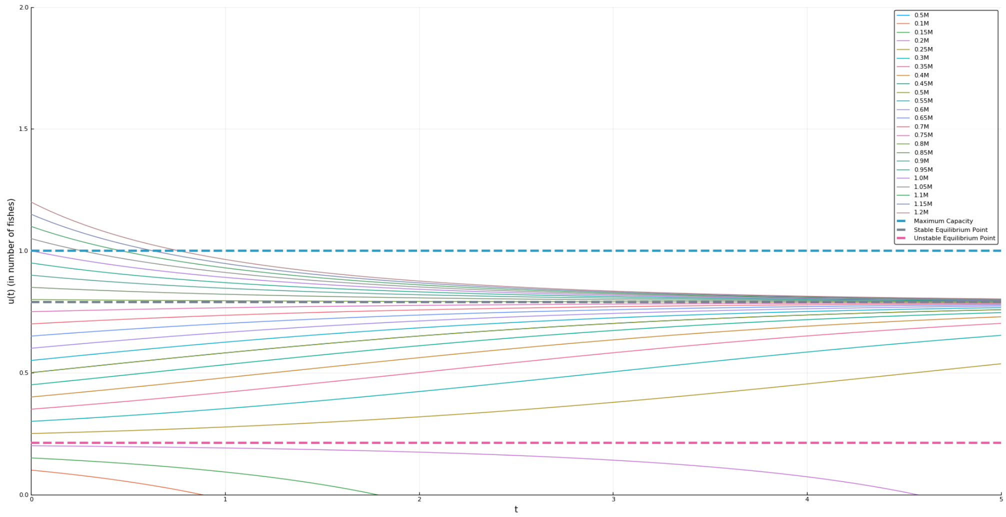

Figure 3.2: Constant harvest rate $u < \frac{rM}{4}$.

no real critical points and the derivative $\frac{\mathrm{d}x}{\mathrm{d}t}$ is always negative, implying, that we will lead always the population to extinction, extracting constantly at a rate greater than $\frac{rM}{4}$.

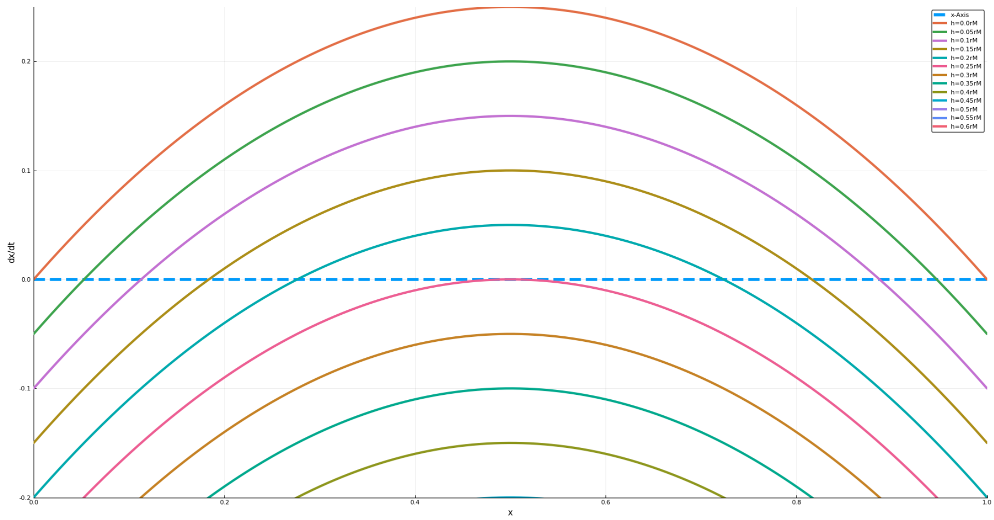

Figure 3.1: Figure representing $\frac{\mathrm{d}x}{\mathrm{d}t}$ with different harvesting rates.

3.1.2 Time Varying Harvesting.

Given a time horizon T we want to extract the maximum amount of fishes, during this time.

$$H = \int_0^T u(t)\mathrm{d}t \tag{3.17}$$

Therefore,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{M}\right) - u(t) \tag{3.18}$$

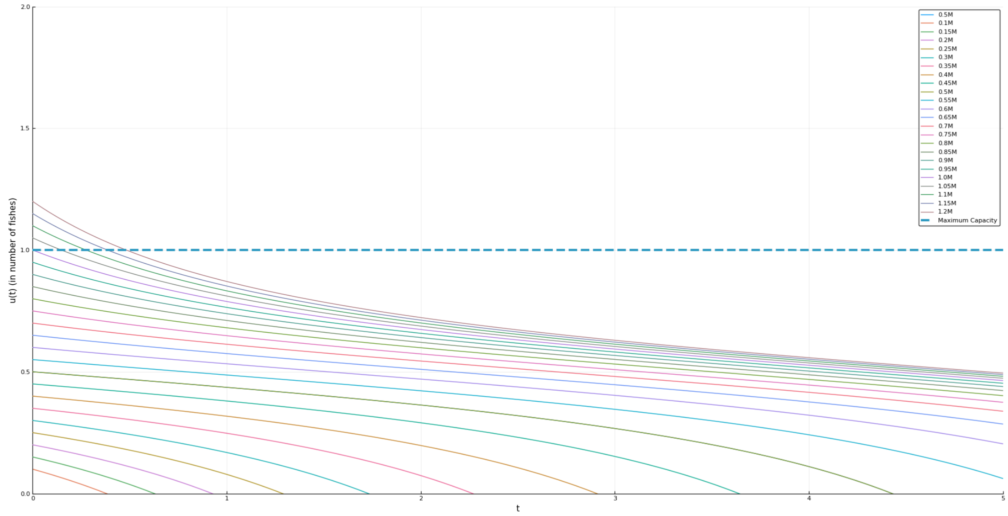

Figure 3.3: Constant harvest rate $u > \frac{rM}{4}$.

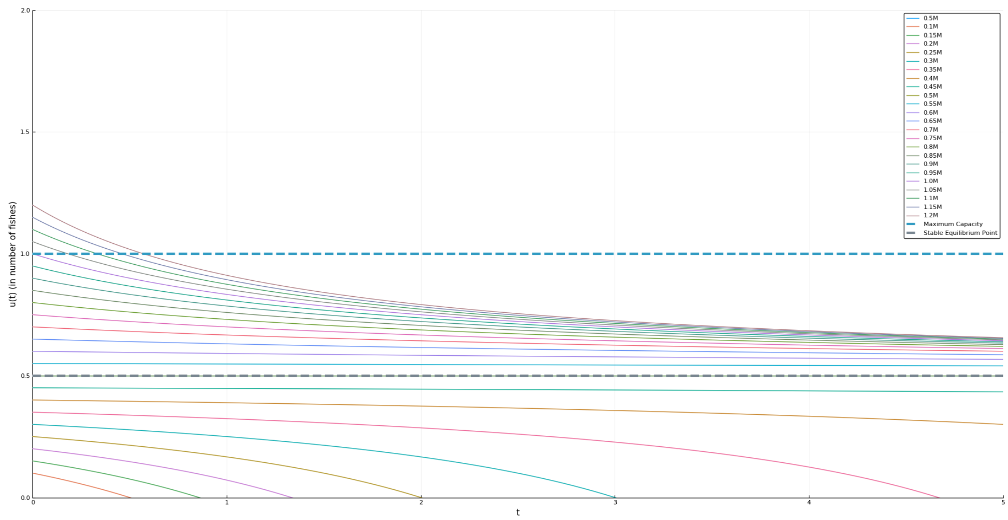

Figure 3.4: Constant harvest rate $u = \frac{rM}{4}$.

From the above result we see that for $u(t) > \frac{rM}{4}$, we lead the population to extinction.

From the above analysis we have that,

$$0 < u \leq \frac{rM}{4} \tag{3.19}$$

3.1.3 Optimal Harvesting. Smooth Optimal Control Problem.

For Optimal Control we reduce the problem to the following,

$$\min_{\substack{x \in X \\ u \in U}} J(x,u) \tag{3.20}$$

subject to,

$$e(x,u) = 0 \tag{3.21}$$

$h(t) = \frac{rM}{4} - u(t)$

$$J(x,u) = \frac{\zeta}{2}\left(x(T) - \frac{M}{2}\right)^2 + \frac{1}{2}\left\|x - \frac{M}{2}\right\|^2_{L^2([0,T])} + \frac{\eta}{2}\|h\|^2_{L^2([0,T])} \tag{3.22}$$

subject to,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{M}\right) - \frac{rM}{4} + h(t) \tag{3.23}$$

3.2 Closed Loop Strategies.

3.2.1 Constant Proportional Harvesting.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{M}\right) - px \tag{3.24}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{p}{r} - \frac{x}{M}\right) \tag{3.25}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r\left(1 - \frac{p}{r}\right)\left(1 - \frac{x}{M\left(1 - \frac{p}{r}\right)}\right)x \tag{3.26}$$

$\gamma = r\left(1 - \frac{p}{r}\right)$, $K = M\left(1 - \frac{p}{r}\right)$. With $\frac{p}{r} < 1$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \gamma x\left(1 - \frac{x}{K}\right) \tag{3.27}$$

$$x = \frac{Kx_0}{x_0 + (K - x_0)\,\mathrm{e}^{-\gamma t}} \tag{3.28}$$

$$x(t) = \frac{M\left(1 - \frac{p}{r}\right)x_0}{x_0 + \left(M - \frac{Mp}{r} - x_0\right)\mathrm{e}^{-\gamma t}} \tag{3.29}$$

Since our harvesting control is proportional to our population, given a finite time horizon T, the amount of fishes we have extracted from our pool is given by,

$$\begin{aligned} J(x;p,T) &= \int_0^T px\mathrm{d}t \\ &= \int_0^T \frac{Mp(r-p)x_0}{rx_0+(M(r-p)-rx_0)\mathrm{e}^{-(r-p)t}}\mathrm{d}t \end{aligned}$$

The equation 3.29 determines the population of fishes at time t. Consider the transformations $y = x/M$, $\tau = rt$, $\overline{p} = rp$. Therefore the equation 3.24, is transformed into:

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = (1-\overline{p})y\left(1-\frac{y}{1-\overline{p}}\right) \tag{3.30}$$

with initial condition $y(0) = y_0 = x_0/M$. And solution,

$$y(\tau) = \frac{(1-\overline{p})y_0}{y_0+(1-\overline{p}-y_0)\mathrm{e}^{(\overline{p}-1)\tau}} \tag{3.31}$$

Then our function in the time horizon $\overline{T} = rT$

$$J\left(y;\overline{p},\overline{T}\right) = \frac{1}{rM}\int_0^{\overline{T}} \overline{p}y(\tau)\mathrm{d}\tau \tag{3.32}$$

$$= \frac{\overline{p}}{rM}\left(\ln\left(1-\overline{p}+y_0\left(\mathrm{e}^{(1-\overline{p})\overline{T}}-1\right)\right)-\ln(1-\overline{p})\right) \tag{3.33}$$

We would like to know the constant $\overline{p}^*$ that for a given time horizon $\overline{T}$ maximizes J. Therefore $\overline{p}$ should satisfy the necessary condition,

$$\left.\frac{\partial J\left(y;\overline{p},\overline{T}\right)}{\partial p}\right|_{\overline{p}=\overline{p}^*} = 0 \tag{3.34}$$

Therefore, for given y_0 we need to solve for $\overline{p}^*$ the following equation,

$$\overline{p}^*\left(\frac{1+Ty_0\mathrm{e}^{(1-\overline{p}^*)\overline{T}}}{\overline{p}^*+y_0-1-y_0\mathrm{e}^{(1-\overline{p}^*)\overline{T}}}+\frac{1}{1-\overline{p}^*}\right)+\ln\left(1-\overline{p}^*-y_0+y_0\mathrm{e}^{(1-\overline{p}^*)\overline{T}}\right)-\ln\left(1-\overline{p}^*\right)=0 \tag{3.35}$$

This expression has no closed form solution, but we can estimate it numerically, if we know y_0 and T. For example for $y_0 = 0.75$ and $\overline{T} = 20$, we have $\overline{p}^* \approx 0.541881$.

We can know the value of $\overline{p}^*$, for $T \to \infty$, Consider,

$$\lim_{T\to\infty} \ln(a+b\mathrm{e}^{cT}) \approxeq cT \tag{3.36}$$

For any $a, b \in \mathbb{R}$. And

$$\lim_{T-\infty} \frac{a+bT\mathrm{e}^{cT}}{r+d\mathrm{e}^{cT}} \approxeq \frac{b}{d}T \tag{3.37}$$

For any constants $a, b, c, d, r \in \mathbb{R}$.

Therefore, for big enough T, the contribution for fixed $\overline{p}^*$, we can write equation 3.35 in small o notation as follows,

$$\lim_{T\to\infty} \left.\frac{\partial J}{\partial p}\right|_{p=\overline{p}^*} = (1-\overline{p}^*)\overline{T}-\overline{p}^*\overline{T}+o(T)+o(T^2)+\cdots = 0 \tag{3.38}$$

Hence, when $T \to \infty$

$$(1-2\overline{p}^*)\overline{T} = 0 \implies \overline{p}^* = \frac{1}{2} \tag{3.39}$$

4 Economical Profit

4.1 Linear Costs.

4.2 Dynamic Programming.

4.3 Stochastic Analysis.