Optimal Harvesting Modelling

Final Report

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Abstract

Contents

As a natural, healthy and nutritious food, with variety of species and diverse growth environments, fish seems to be a wise choice to solve some food - related crisis regarding to the human population growth around the world. On the other hand, there is a limitation for the fish population sustainability in open seas. Global high demand, resulted in over-exploiting the oceans in the past decades (Figure 1).

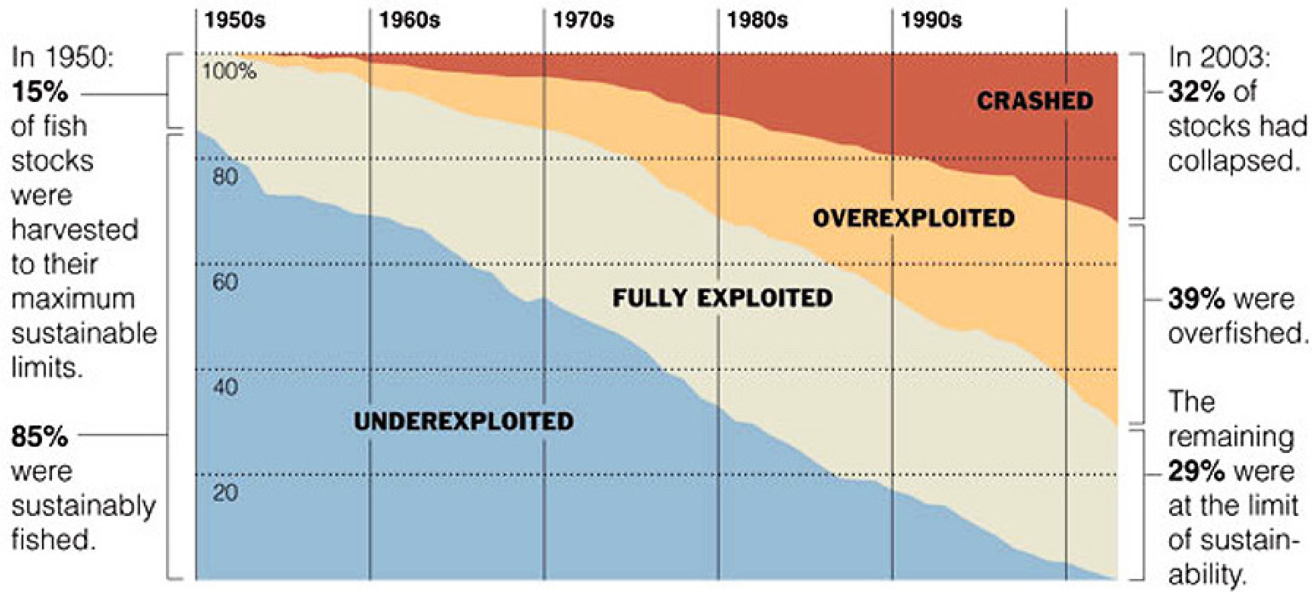


Figure 1.1: sustainable fishing between 1950 and 2003

In result, some fish populations have been severely declined during the years. Figure 1 shows the population of utilized fish population between 1970 and and 2010. As illustrated, the index for all utilized fish species indicates a 50 per cent reduction in population number globally between 1970 and 2010.

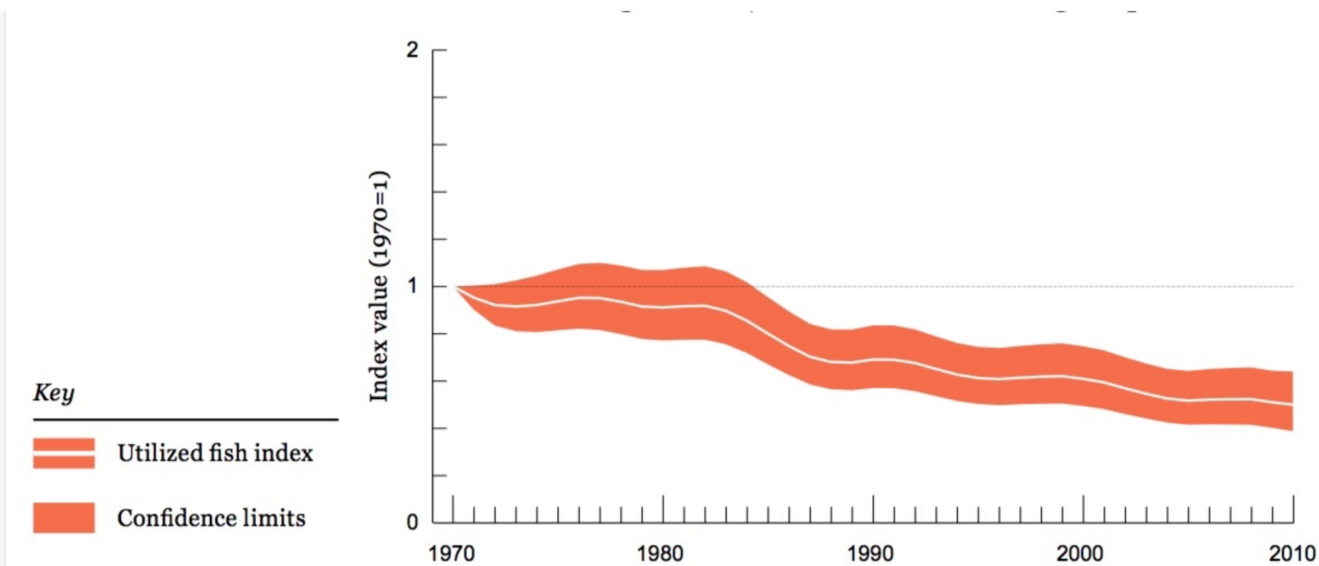


Figure 1.2: Utilized fish index value between 1970 and 2010

One of the solutions to fish population decrease problem is to shift from fish catching to fish harvesting. This strategy can help recovering fish population and size gradually beside providing human with seafood. Figure 1 and 1 shows the fish harvesting production grows in 1970 and 2010 year around the world.

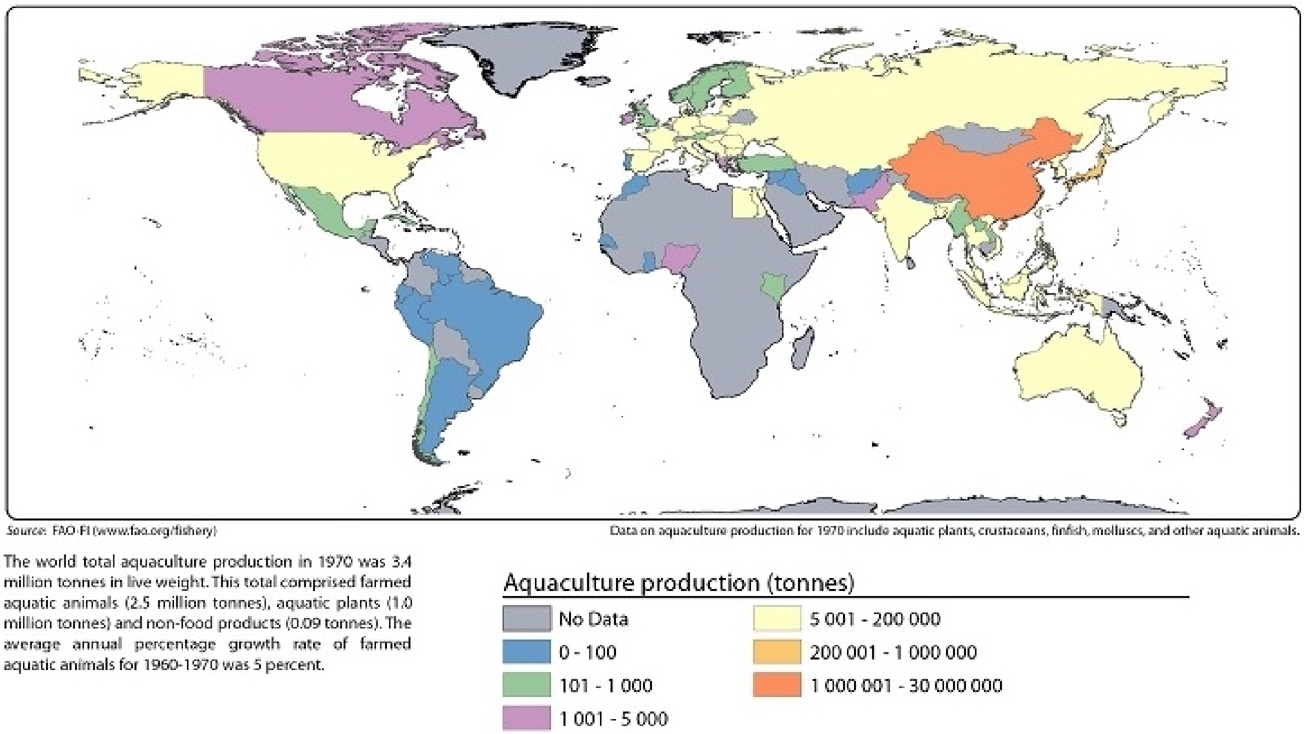


Figure 1.3: Aquaculture production in 1970 around the world

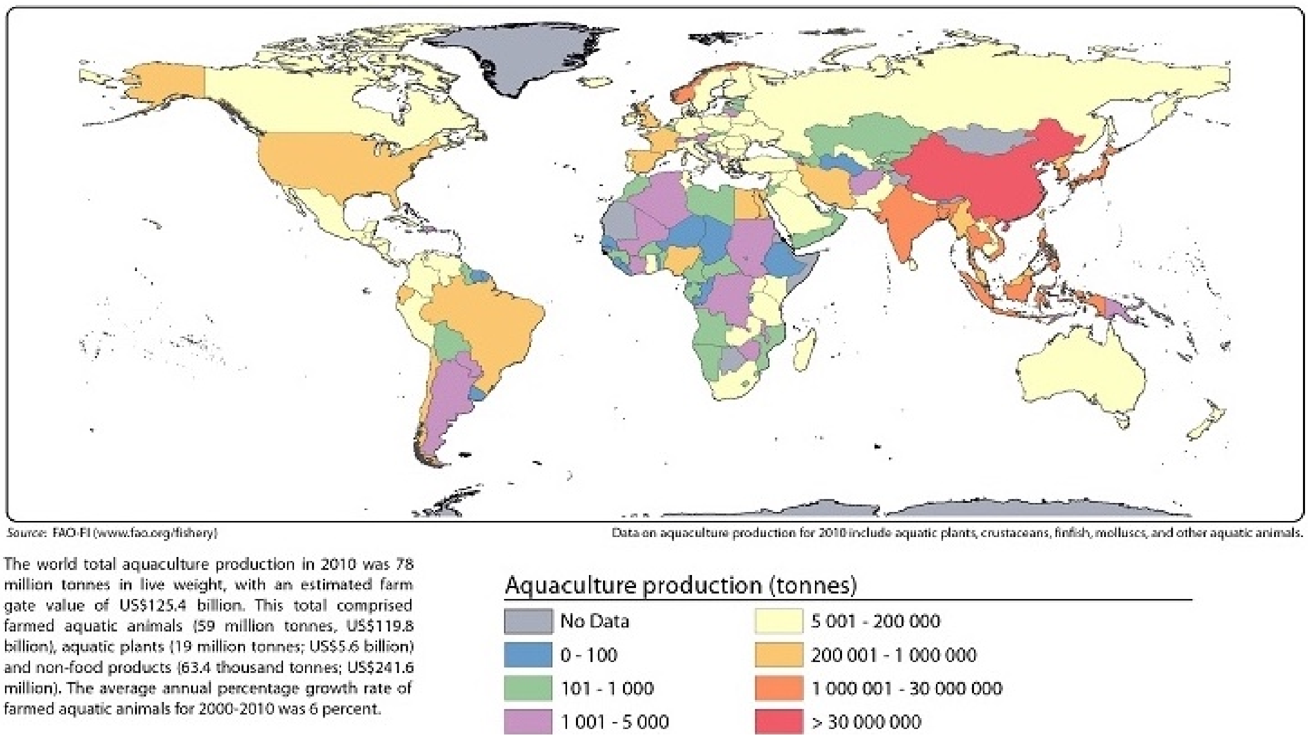


Figure 1.4: Aquaculture production in 2010 around the world

Like any other industry, It is crucial to optimize the fish harvesting procedure to have maximum -still consistent- production in fish harvesting farms. In this work, weâĂŹre trying to describe one of the fish harvesting mathematical models and achieve the optimum fish farm population to have a consistent population.

2 Mathematical Models.

Before focusing on the harvesting problem, we focus on modelling a "pure" population growth, without introducing an artificial harvesting control. We can model the population growth problem, as one dynamical system in which x represents the fish population, and the population variation with respect to time $\dot{x}$ can be described as a function of the fish population and/or time:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,t) \tag{2.1}$$

2.1 Exponential biological growth.

A first approach is to introduce a natural mortality factor m, which contributes to decreasing the fish population. To prevent the population from decaying to zero, we also introduce a constriction for the time horizon T:

$$\begin{aligned} \tfrac{\mathrm{d}x}{\mathrm{d}t} &= -mx \\ x(T) &= x_T \end{aligned} \tag{2.2}$$

If a variable mortality due to fishing $\Phi(t)$, is also considered then the growth equation becomes,

$$\begin{aligned} \tfrac{\mathrm{d}x}{\mathrm{d}t} &= -(m+\Phi(t))x \\ x(T) &= x_T \end{aligned} \tag{2.3}$$

We must take into account that the variable mortality factor adjusts well to an open sea population simulation, but not to a controlled population (fish farm).

2.2 Logistic Equation.

A better approach consists in the following: a biological population with plenty of food, space to grow, and no threat from predators, tends to grow at a rate that is proportional to the population- this means that per unit time a certain percentage of the individuals produce new individuals continuously:

$$F(x,t) = rx \tag{2.4}$$

where x is the population in time t, and the proportionality constant r is called the growth rate. More realistically, populations are constrained by limitations on resources, so a maximum population parameter M can be introduced ("carrying capacity" of the system). The logistic growth model includes this parameter, and has the form:

$$F(x,t) = rx\left(1 - \frac{x}{M}\right) \tag{2.5}$$

Equation 2.5 satisfies some basic aspects: on one hand, when the population is small relative to M, its behavior is similar to the followed by equation 2.4, and the constraint does not affect too much, but as x becomes significant compared to M, both curves diverge and the growth rate $\dot{x}$ drops to zero. On the other hand, the growth rate is only zero when $x = M$, which is what happens in reality.

We consider the behavior of the logistic equation under the presence of noise, in multiplicative way to the population. For the elements $(t,x) \in Q = (0,T) \times (0,M)$, we state the following differential equation,

$$dx = \left(rx\left(1 - \frac{x}{M}\right)\right) \mathrm{d}t + \sigma x \mathrm{d}W \tag{2.6}$$

A unique solution exists if both Itó conditions hold (Fleming and Rishel, 1975). The first one is the linear growth condition, for some independent constant K,

$$\left|rx\left(1 - \frac{x}{M}\right)\right| \leq K(1 + |x|) \tag{2.7}$$

$$|\sigma x| \leq K(1 + |x|) \tag{2.8}$$

The second one is the Lipschitz condition,$\exists L$ independent constant, and $\forall x$, $\exists B(x)$ neighborhood of x, such that $\forall x_1, x_2 \in B(x)$,

$$\left|rx_2\left(1 - \frac{x_2}{M}\right) - rx_1\left(1 - \frac{x_1}{M}\right)\right| \leq L|x_2 - x_1| \tag{2.9}$$

$$|\sigma(x_2 - x_1)| \leq L|x_2 - x_1| \tag{2.10}$$

Since $F(x,t) = rx\left(1 - \frac{x}{M}\right)$ is continuously differentiable in x, F is Lipschitz in x then condition 2.9 is satisfied. For bounded σ, condition 2.10 is satisfied. Moreover the sufficient conditions for the Itô conditions are satisfied for all functions C^1 on the closure of any compact set Q.

Since the above conditions are satisfied, we can guarantee existence and uniqueness of the solution for the equation 2.6. Given by the equation:

$$\begin{array}{rcl} x(t) & = & x_0 + \int_0^t \left(rx\left(1 - \frac{x}{M}\right)\right) \mathrm{d}t + \int_0^t \sigma x \mathrm{d}W, \\ x(0) & = & x_0, \\ W(0) & = & 0. \end{array} \tag{2.11}$$

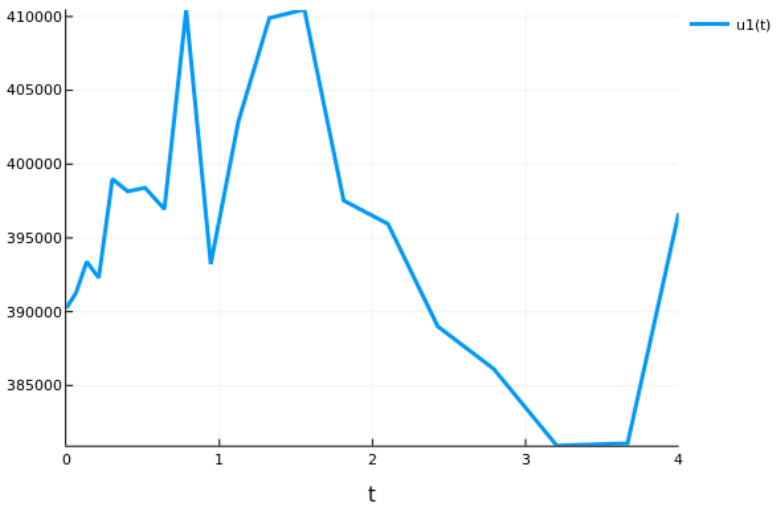


Figure 2.1: Simulation performed of logistic equation 2.6, with parameters $r = 0.8$, $x_0 = \frac{M}{2}$, for a population in natural conditions (harvest exploitation $u = 0$.), with presence of noise proportional to the population, with $\sigma = 0.1$. Performed during 4 months.

3 Fishing Strategies and Optimizing Population

Generally, there are three methods to model aquaculture mathematically:

- Open Loop harvesting Strategies: One of the simplest methods to implement, harvesting a number of fishes without regarding the state of the population.
- Closed Loop harvesting Strategies: A more complex strategy to implement, with the advantage that it is possible to introduce a control over the population in order to avoid extinction.

3.1 Open Loop Strategies.

Generally, in an open loop strategy -also called a non-feedback strategy- the process does not use a feedback to determine if its output has achieved the desired goal of the process. The implementation of open loop harvesting strategies take place without considering the impact of the extraction process. Mathematically,

$$\frac{dx}{dt} = \Psi(x,t) + C(t) \tag{3.1}$$

where $\Psi(x,t)$ is the intrinsic dynamic of the system and $C(t)$ is the control parameter, which is population independent.

3.1.1 Constant Harvesting Analysis.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{M}\right) - u \tag{3.2}$$

We introduce the following variable in order to simply calculations,

$$\beta = \frac{uM}{r} \tag{3.3}$$

Solving the differential equation,

$$\begin{aligned}
\frac{\mathrm{d}x}{rx\left(1-\frac{x}{M}\right)-u} &= \mathrm{d}t \\
\int_{x_0}^{x} \frac{\mathrm{d}\chi}{r\chi\left(1-\frac{\chi}{M}\right)-u} &= \int_0^t \mathrm{d}\tau \\
\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\chi\left(M-\chi\right)-\frac{Mu}{r}} &= t \\
-\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\chi^2 - M\chi + \beta} &= t
\end{aligned}$$

Finally, we model the above integral as one

$$-\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi-\frac{M}{2}\right)^2 - \frac{M^2}{4} + \beta} = t \tag{3.4}$$

Consider α as follows,

$$\alpha = \beta - \frac{M^2}{4} = rM\left(u - \frac{rM}{4}\right) \tag{3.5}$$

We see that the sign of α determines the nature of the solutions. Then, if $u > rM/4$ implies $\alpha > 0$,

$$\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi - \frac{M}{2}\right)^2 + \alpha} = -\frac{r}{M}t$$

$$\frac{1}{\sqrt{\beta - \frac{M^2}{4}}}\left(\arctan\left(\frac{x - M/2}{\sqrt{\beta - M^2/4}}\right) - \arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}}\right)\right) = -\frac{r}{M}t$$

Therefore, for $\alpha > 0$ the population behaves as follows,

$$x(t) = \frac{M}{2} + \sqrt{\beta - \frac{M^2}{4}}\tan\left(\arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}}\right) - \frac{r\sqrt{\beta - M^2/4}}{M}t\right) \tag{3.6}$$

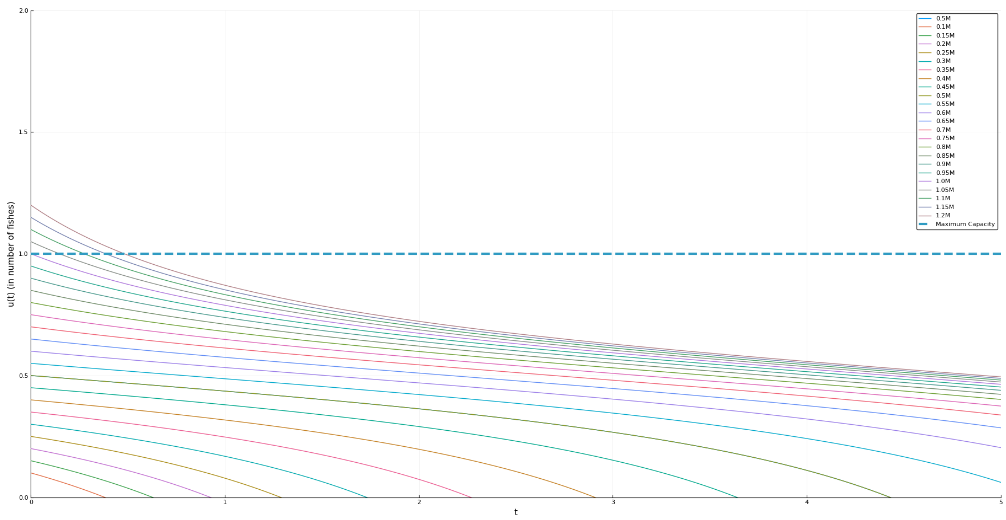


Figure 3.1: Constant harvest rate $u > \frac{rM}{4}$.

Equation 3.6 show us that for some t^*, $x(t^*) = 0$,independently of the initial condition x_0, since the argument inside the $\tan$ is monotone decreasing in t.

If $u < rM/4$ implies $-\alpha > 0$,

$$\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi - \frac{M}{2}\right)^2 - (-\alpha)} = -\frac{r}{M}t$$

Considering the zeros of the denominator, λ and $\overleftarrow{\lambda}$,

$$\begin{aligned} \lambda &= \tfrac{M}{2} + \sqrt{\tfrac{M^2}{4} - \beta} \\ \bar{\lambda} &= \tfrac{M}{2} - \sqrt{\tfrac{M^2}{4} - \beta} \end{aligned} \tag{3.7}$$

We can rewrite our expression as follows,

$$
\begin{aligned}
\int_{x_0}^{x}\left(\frac{1}{\chi-\lambda}-\frac{1}{\chi-\overline{\lambda}}\right)\mathrm{d}\chi &= -\frac{2r\sqrt{M^2/4-\beta}}{M}t \\
\ln\left|\frac{x-\lambda}{x-\overline{\lambda}}\right| &= \ln\left|\frac{x_0-\lambda}{x_0-\overline{\lambda}}\right| - \frac{2r\sqrt{M^2/4-\beta}}{M}t
\end{aligned}
$$

For simplifying calculations, we write, $\gamma = \frac{2r\sqrt{M^2/4-\beta}}{M}$. And we obtain as result,

$$
\frac{x-\lambda}{x-\overline{\lambda}} = \frac{x_0-\lambda}{x_0-\overline{\lambda}}\mathrm{e}^{-\gamma t} \tag{3.8}
$$

$$
x-\lambda = \left(x-\overline{\lambda}\right)\left(\frac{x_0-\lambda}{x_0-\overline{\lambda}}\right)\mathrm{e}^{-\gamma t} \tag{3.9}
$$

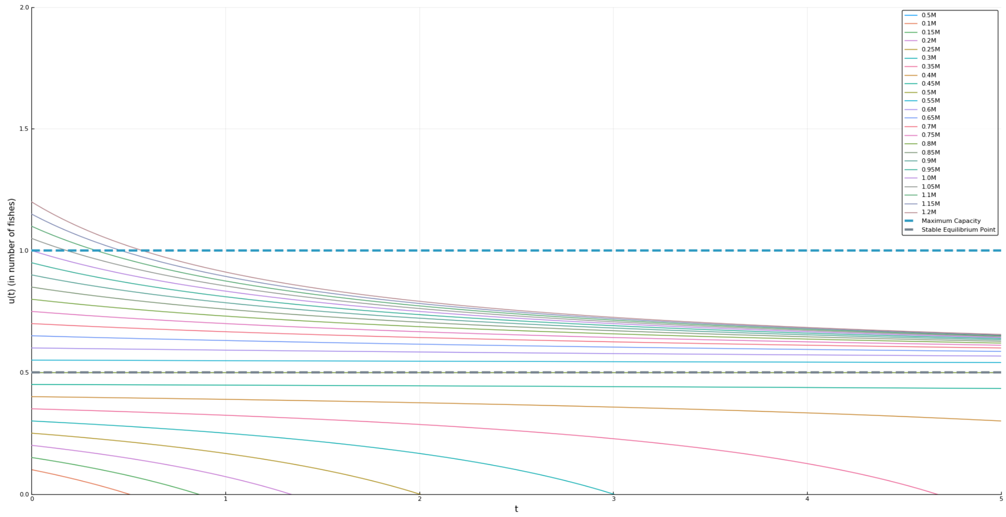
For the sake of simplicity, consider $\xi = \frac{x_0-\lambda}{x_0-\overline{\lambda}}\mathrm{e}^{-\gamma t}$. Therefore,

$$
\begin{aligned}
x\left(1-\xi\right) &= \lambda-\overline{\lambda}\xi \\
x &= \frac{\lambda-\overline{\lambda}\xi}{1-\xi} \\
x &= \frac{\frac{M}{2}+\sqrt{\frac{M^2}{4}-\beta}-\left(\frac{M}{2}-\sqrt{\frac{M^2}{4}-\beta}\right)\xi}{1-\xi} \\
x &= \frac{\frac{M}{2}+\sqrt{\frac{M^2}{4}-\beta}-\left(\frac{M}{2}-\sqrt{\frac{M^2}{4}-\beta}\right)\xi}{1-\xi} \\
x &= \frac{\frac{M}{2}\left(1-\xi\right)+\sqrt{\frac{M^2}{4}-\beta}\left(1+\xi\right)}{1-\xi} \\
x &= \frac{M}{2}+\sqrt{\frac{M^2}{4}-\beta}\frac{1+\xi}{1-\xi}
\end{aligned}
$$

Hence, for $-\alpha > 0$, we have the following result,

$$
x(t) = \frac{M}{2}+\left(\sqrt{\frac{M^2}{4}-\beta}\right)\frac{\left(x_0-M/2\right)\left(1+\mathrm{e}^{-\gamma t}\right)-\sqrt{M^2/4-\beta}\left(1-\mathrm{e}^{-\gamma t}\right)}{\left(x_0-M/2\right)\left(1-\mathrm{e}^{-\gamma t}\right)+\sqrt{M^2/4-\beta}\left(1+\mathrm{e}^{-\gamma t}\right)} \tag{3.10}
$$

Figure 3.2: Constant harvest rate $u = \frac{rM}{4}$.



If $u = \frac{rM}{4}$, we solve equation 3.2 as follows,

$$-\frac{M}{r}\int_{x_0}^{x}\frac{d\chi}{\left(\chi-\frac{M}{2}\right)^2} = t \tag{3.11}$$

$$\int_{x_0}^{x}\frac{d\chi}{\left(\chi-\frac{M}{2}\right)^2} = -\frac{rt}{M} \tag{3.12}$$

$$\frac{1}{x-\frac{M}{2}} = \frac{1}{x_0-\frac{M}{2}} - \frac{rt}{M} \tag{3.13}$$

$$\frac{1}{x-\frac{M}{2}} = \frac{M-\left(x_0-\frac{M}{2}\right)rt}{M\left(x_0-\frac{M}{2}\right)} \tag{3.14}$$

$$x = \frac{M}{2} + \frac{M\left(x_0-\frac{M}{2}\right)}{M-\left(x_0-\frac{M}{2}\right)rt} \tag{3.15}$$

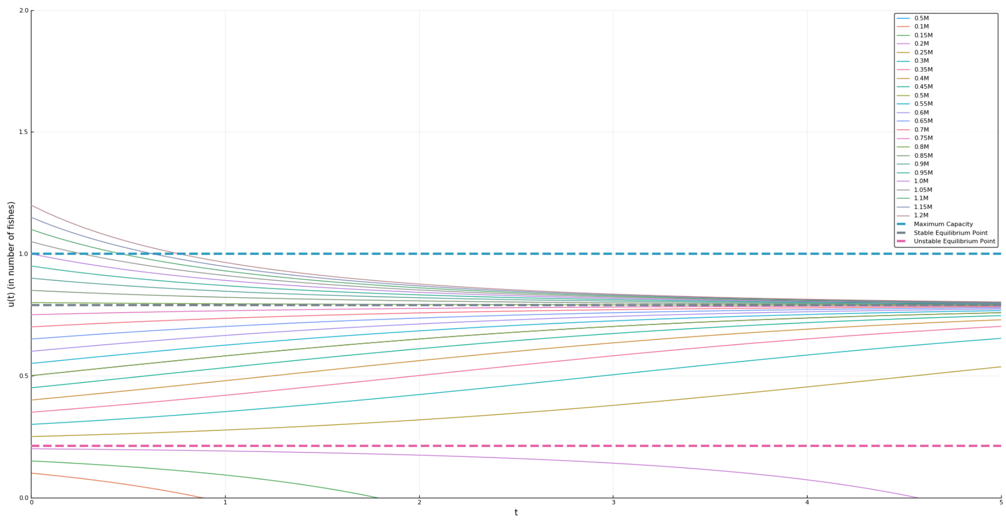


Figure 3.3: Constant harvest rate $u < \frac{rM}{4}$.

The results above stated can be explained directly from the equation 3.2, as we see in the graph 3.4, $F(x,t)$ is a paraboloid, with its maximum at $F(x^* = M/2, t) = rM^2/4$.

When $u = 0$, we have the regular logistic equation with critical points $x_{c_1} = 0$ and $x_{c_2} = M$.

We observe that the critical points x_c, such that $\frac{\mathrm{d}x_c}{\mathrm{d}t} = F(x_c, t) - u = 0$ are getting closer to each other, as u is increasing; In general, these are the solutions to the equation $F(x,t) - u = 0$,

$$x_{c_{2,1}} = \frac{M}{2} \pm \sqrt{\frac{M^2}{4} - u\frac{M}{r}} \tag{3.16}$$

Always satisfying $x_{c_2} \geq x_{c_1}$, being x_{c_2} the stable fixed point and x_{c_1} the unstable fixed point, as we see in the figure 3.3.

If our initial population x_0 lies below the unstable critical point it will lead to extinction. If our initial population lies above the unstable fixed point, it will be getting closer to the stable fixed point, we appreciate the same behavior if our population lies above the stable fixed point.

When $u = \frac{rM}{4}$ we only have one unstable fixed point. This point behaves as an attractor when $x_0 \geq \frac{M}{2}$. But when $x_0 < \frac{M}{2}$, implies $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$, for all $t > 0$ and the population decreases strictly . For $u > \frac{rM}{4}$, the dynamic has no real fixed points and $\frac{\mathrm{d}x}{\mathrm{d}t}$ is always negative, implying, that extracting constantly at a rate greater than $\frac{rM}{4}$, we will reach extinction for some given $T > 0$.

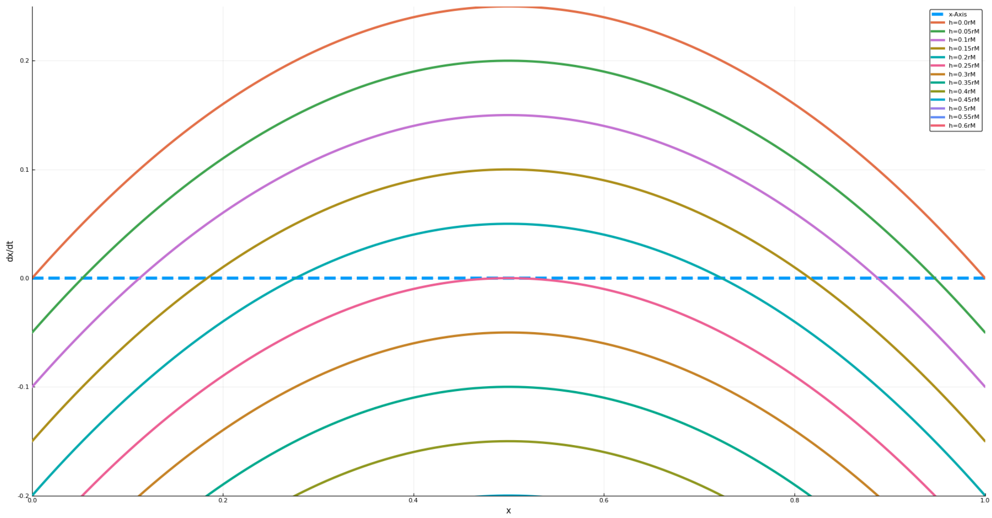


Figure 3.4: Figure representing $\frac{\mathrm{d}x}{\mathrm{d}t}$ with different harvesting rates.

3.1.2 Time Varying Harvesting.

Given a time horizon T we want to extract the maximum amount of fishes, during this time.

$$H = \int_0^T u(t)\mathrm{d}t \tag{3.17}$$

Therefore,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{M}\right) - u(t) \tag{3.18}$$

From the above result we see that for $u(t) > \frac{rM}{4}$, we lead the population to extinction.

From the above analysis we have that,

$$0 < u \leq \frac{rM}{4} \tag{3.19}$$

3.1.3 Optimal Harvesting. Smooth Optimal Control Problem.

For Optimal Control, we reduce the problem to the following,

$$\min_{\substack{x\in X\\ u\in U}} J(x,u) \tag{3.20}$$

subject to,

$$e(x,u)=0 \tag{3.21}$$

$h(t)=\frac{rM}{4}-u(t)$

$$J(x,u)=\frac{\zeta}{2}\left(x(T)-\frac{M}{2}\right)^2+\frac{1}{2}\left\|x-\frac{M}{2}\right\|^2_{L^2([0,T])}+\frac{\eta}{2}\|h\|^2_{L^2([0,T])} \tag{3.22}$$

subject to,

$$\frac{\mathrm{d}x}{\mathrm{d}t}=rx\left(1-\frac{x}{M}\right)-\frac{rM}{4}+h(t) \tag{3.23}$$

3.2 Closed Loop Strategies.

Closed loop strategies implement feedback as part of the dynamic, in order to improve the dynamic of the system. In this section, two different closed loop fish harvesting strategies have been developed, and all incorporate a population dependent harvesting type.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\Psi(x,t)+C(x,t) \tag{3.24}$$

where $\Psi(x,t)$ is the intrinsic dynamic of the system, and $C(x,t)$ is the implemented control, which is population dependent.

3.2.1 Constant Proportional Harvesting.

Consider the closed loop strategy, we propose harvest ina a proportional way to the population, for this case instead of a constant, we take a constant multiplying out population in this way:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=rx\left(1-\frac{x}{M}\right)-px \tag{3.25}$$

Taking out rx as common factor we obtain:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=rx\left(1-\frac{p}{r}-\frac{x}{M}\right) \tag{3.26}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=r\left(1-\frac{p}{r}\right)\left(1-\frac{x}{M\left(1-\frac{p}{r}\right)}\right)x \tag{3.27}$$

Making this change of variables, $\gamma=r\left(1-\frac{p}{r}\right)$, $K=M\left(1-\frac{p}{r}\right)$. With $\frac{p}{r}<1$ the equation is reduce to:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\gamma x\left(1-\frac{x}{K}\right) \tag{3.28}$$

Calculating the integral of this we obtain:

$$x=\frac{Kx_0}{x_0+(K-x_0)\,\mathrm{e}^{-\gamma t}} \tag{3.29}$$

so our population in function of t is:

$$x(t)=\frac{M\left(1-\frac{p}{r}\right)x_0}{x_0+\left(M-\frac{Mp}{r}-x_0\right)\mathrm{e}^{-\gamma t}}. \tag{3.30}$$

3.2.2 Optimal Proportional Harvesting.

Since our harvesting control is proportional to our population, given a finite time horizon T, the amount of fishes we have extracted from our pool is given by,

$$J(x;p,T) = \int_0^T px\,\mathrm{d}t$$
$$= \int_0^T \frac{Mp(r-p)x_0}{rx_0 + (M(r-p) - rx_0)\,\mathrm{e}^{-(r-p)t}}\mathrm{d}t$$

The equation 3.30 determines the population of fishes at time t. Consider the transformations $y = x/M$, $\tau = rt$, $\bar{p} = rp$. Therefore the equation 3.25, is transformed into:

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = (1-\bar{p})y\left(1 - \frac{y}{1-\bar{p}}\right) \tag{3.31}$$

with initial condition $y(0) = y_0 = x_0/M$. And solution,

$$y(\tau) = \frac{(1-\overline{p})y_0}{y_0 + (1-\overline{p}-y_0)\,\mathrm{e}^{(\overline{p}-1)\tau}} \tag{3.32}$$

Then our function in the time horizon $\bar{T} = rT$

$$J\left(y;\bar{p},\bar{T}\right) = \frac{1}{rM}\int_0^{\bar{T}} \bar{p}y(\tau)\mathrm{d}\tau \tag{3.33}$$
$$= \frac{\bar{p}}{rM}\left(\ln\left(1-\bar{p}+y_0\left(\mathrm{e}^{(1-\bar{p})\bar{T}}-1\right)\right) - \ln(1-\bar{p})\right) \tag{3.34}$$

We would like to know the constant $\bar{p}^*$ that for a given time horizon $\bar{T}$ maximizes J. Therefore $\bar{p}$ should satisfy the necessary condition,

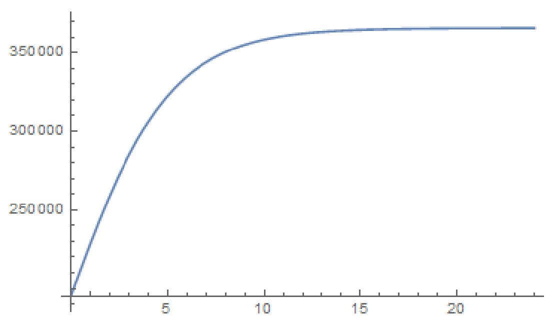
$$\left.\frac{\partial J\left(y;\bar{p},\bar{T}\right)}{\partial p}\right|_{\bar{p}=\bar{p}^*} = 0 \tag{3.35}$$

Therefore, for given y_0 we need to solve for $\bar{p}^*$ the following equation,

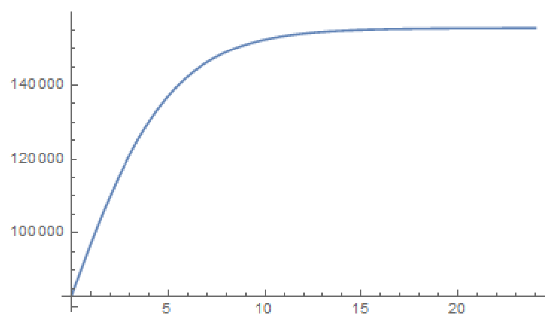
$$\bar{p}^*\left(\frac{1+Ty_0\mathrm{e}^{(1-\bar{p}^*)\bar{T}}}{\bar{p}^*+y_0-1-y_0\mathrm{e}^{(1-\bar{p}^*)\bar{T}}} + \frac{1}{1-\bar{p}^*}\right) + \ln\left(1-\bar{p}^*-y_0+y_0\mathrm{e}^{(1-\bar{p}^*)\bar{T}}\right) - \ln(1-\bar{p}^*) = 0 \tag{3.36}$$

This expression has no closed form solution, but we can estimate it numerically, if we know y_0 and T. For example for $y_0 = 0.75$ and $\bar{T} = 20$, we have $\bar{p}^* \approx 0.541881$.

Given a time horizon $\bar{T} = 24$ time units (usually given in months), above with parameters $M = 780500$ fishes, $r = 0.8$ inverse time units and initial population $x_0 = 390250$. The following numerical results are done with $\bar{p}^* \approx 0.531176$ given by the equation 3.36.

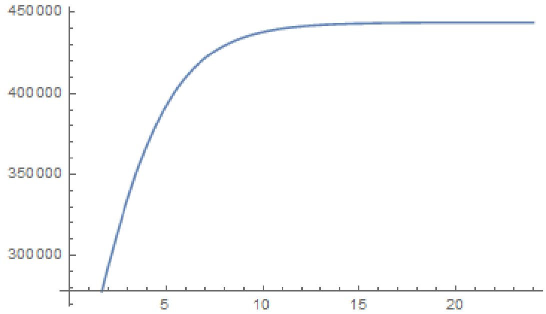


(a) Population with optimal p^*.

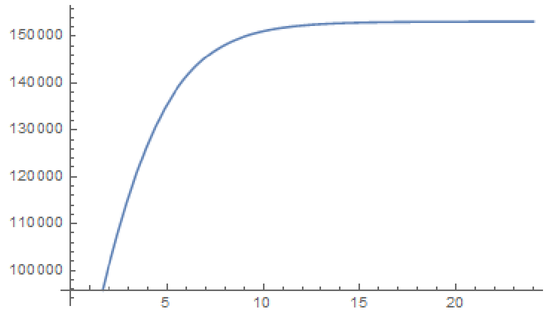


(b) Harvest Fishes taken with optimal p^*.

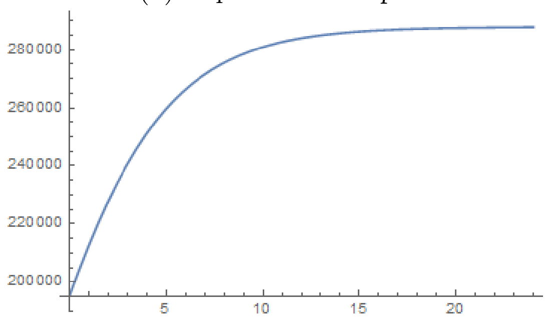
The first graph show the total population of fishes during the time, and the second one represent the amount of fishes that are harvest from the model. As we can see the population goes to an stability point that is equal to $\left(1-\frac{p}{r}\right)M$ and in the same way the amount of fishes that we harvest goes to $\left(1-\frac{p}{r}\right)M\bar{p}^{*}$. The following graphs shows the variation $\bar{p}^{*} \pm 0.1$ an the effect in the amount of harvested fish, please note how the stable fixed points lie below $M/2$:



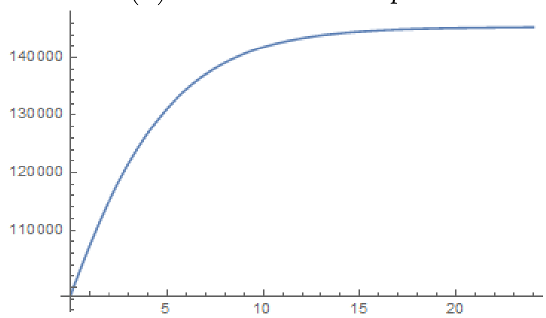
(a) Population with $\bar{p}^{*}-0.1$



(b) Fishes taken with $\bar{p}^{*}-0.1$



(c) Population with $\bar{p}^{*}+0.1$



(d) Fishes taken with $\bar{p}^{*}+0.1$

In the first two there is less harvesting, then the population is bigger, but since the harvesting coefficient is smaller the amount of fishes is smaller. In the other hand, when the harvesting coefficient is bigger the population is smaller, since the harvested fish is proportional to the population, then we harvest less fish. We see the above explained behavior in table 3.1 in contrast with the optimum, corresponding to some specific parameters.

$\bar{p}$	$J(x;T,p,r,M,x0)$ (in Millions of fishes harvested)
$\bar{p}^{*}$	4.698153484016761
$\bar{p}^{*}-0.001$	4.698138002580389
$\bar{p}^{*}+0.001$	4.698138014054768
$\bar{p}^{*}-0.01$	4.696602091505628
$\bar{p}^{*}+0.01$	4.696609852932111
$\bar{p}^{*}-0.1$	4.540022027777682
$\bar{p}^{*}+0.1$	4.547918150118066

Table 3.1: Harvested fish $J(x;\bar{p},T,M,r,x_0)$ in units of million of fishes with different $\bar{p}$, for $T=24$, $r=0.8$, $M=780500$, $x_0=390250$

We can know estimate the value of $\bar{p}^{*}$, for long time horizons. Consider,

$$\lim_{T\to\infty} \ln(a+b\mathrm{e}^{cT}) \approx cT \tag{3.37}$$

For any $a, b \in \mathbb{R}$. And

$$\lim_{T-\infty} \frac{a + bT\mathrm{e}^{cT}}{r + d\mathrm{e}^{cT}} \approx \frac{b}{d}T \tag{3.38}$$

For any constants $a, b, c, d, r \in \mathbb{R}$.

Therefore, for big enough T, the contribution for fixed $\bar{p}^*$, we can write equation 3.36 in small o notation as follows,

$$\lim_{T\to\infty} \left.\frac{\partial J}{\partial p}\right|_{p=\bar{p}^*} = (1-\bar{p}^*)\bar{T} - \bar{p}^*\bar{T} + o(T) + o(T^2) + \cdots = 0 \tag{3.39}$$

Hence, when $T \to \infty$

$$(1-2\bar{p}^*)\bar{T} = 0 \implies \bar{p}^* = \frac{1}{2} \tag{3.40}$$

This result was expected, since $\lim\limits_{T\to\infty} x(t) = \bar{p}(1-\bar{p})M$ for any initial condition x_0; it is a concave function of $\bar{p}^*$. Therefore for long time horizons T, the maximum is reached at the point $\bar{p}^* = \frac{1}{2}$.

4 Economical Profit

In the previous discussion, we have focused on optimizing the amount of harvested fish. A more useful aim, is to optimize the economical profits obtained from selling the fish. In the following sections we will use calculus of variations, to find an optimum profit from the harvesting problem.

4.1 Linear Costs.

In this section we will use calculus of variations theory to maximize the long-term profit. The general mark is to search for the functions that maximize or minimize given a functional.

For solving the former, the $J(x)$ has to be maximized with respect to x, as follows:

$$J(x) = \int_0^T g(t, x, \dot{x})dt \tag{4.1}$$

$$\left.\begin{aligned} x(0) &= x_0 \\ x(T) &= x_T \end{aligned}\right\} \tag{4.2}$$

Where $g()$ is a differentiable function and $\dot{x}$ denote the derivative of the x respect to the time. $x(t)$ is the x in specific time whereas x (without t) shows the entire x path. Any x that satisfy the boundary conditions in equation 4.1 is admissible.

Analysis of x in infinite small variations in admissible range (4.1) to optimize the 4.1 function guides to the Euler-Lagrange equation:

$$\frac{\partial g}{\partial x} = \frac{d}{dt}\frac{\partial g}{\partial \dot{x}} \tag{4.3}$$

4.1.1 Costs

As the goal is maximizing the profit, we are going to use the following functional used by Clark and Hannensson in their studies about this topic:

$$J(x) = \int_0^\infty e^{-pt}[p - c(x)]hdt \tag{4.4}$$

note that:

$$\begin{gathered} \dot{x} = \tfrac{dx}{dt} = f(x) - h \\ x(0) = x_0 \end{gathered} \tag{4.5}$$

where $x(t)$ is population, $h(t)$ is the harvest rate, $f(x)$ is biological growth, $c(x)$ is the unit cost of farming and p is the unit price of harvested aquaculture. we assumed p as a constant.

Maximized function can show as:

$$J* = max_h \int_0^\infty e^{-\rho t}[p - c(x)]hdt \quad (4.6)$$

Because the value of money unit decrease (with the rate ρ) by the time, if we have n time of money value decrease in time unit, then t units of time amount to nt discount periods. So we can obtain the present value of money unit by:

$$\lim_{n\to\infty}(1 - \frac{\rho}{n})^{nt} = e^{-\rho t} \quad (4.7)$$

Equation 4.1 and 4.1.1 gives:

$$J* = max_x \int_0^\infty e^{-\rho t}[p - c(x)][f(x) - \dot{x}]dt \quad (4.8)$$

Now, using Euler-Lagrange condition, we can write:

$$f'(x) - \frac{c'(x)f(x)}{p - c(x)} = \rho \quad (4.9)$$

note that primes are differentiations with respect to x. This equation also can be written as:

$$\frac{\partial}{\partial x}[(p - c(x))f(x)] = \rho[p - c(x)] \quad (4.10)$$

If we define $x*$ as the optimal population level that maximize the profit, it can be obtain by solving followed equation:

$$\frac{\partial}{\partial x*}[(p - c(x*))f(x*)] = \rho[p - c(x*)] \quad (4.11)$$

This obtained using equation 4.1.1.

By applying $x*$ in 4.1.1, $h*$ can obtain. It is possible that the last equation might have more than one root and $x*$ not be unique.

4.1.2 Logistic growth

Assume that: $f(x) = rx(1-x/k)$ $h = qEx$ $c(x) = c/(qx)$ in which $f(x)$ is growth, h is harvest rate and $c(x)$ is cost function with constant cost per unit (c). Then, the equation 4.1.1 can be written as:

$$J* = max_h \int_0^\infty e^{-\rho t}[p - \frac{c}{qx}]hdt = max_E \int_0^\infty e^{-\rho t}(pqx - c)Edt \quad (4.12)$$

in which following must be satisfied:

$$\begin{gathered} \dot{x} = rx(1 - \tfrac{x}{K}) - qEx \\ x(0) = x_0 \end{gathered} \quad (4.13)$$

So, utilizing equation 4.1.2, effort (E) can written as

$$E = \frac{rx(1-\frac{x}{K})-\dot{x}}{qx} \tag{4.14}$$

Using and , maximized objective can be written as:

$$J* = max_x \int_0^\infty e^{-\rho t}(p-\frac{c}{qx})[rx(1-\frac{x}{K})-\dot{x}]dt \tag{4.15}$$

By applying Euler-Lagrange condition, following equation will be obtain to calculate optimal population $x*$

$$x* = \frac{K}{4}\left[(1+\frac{c}{pKq}-\frac{\rho}{r})+\sqrt{(1+\frac{c}{pKq}-\frac{p}{r})^2+\frac{8cp}{pKqr}}\right] \tag{4.16}$$

So when the population yields to that value the rate of harvest is equal to the biological growth rate and the population is established at that optimal point.

As now we know the optimal population $x*$ we can develop a harvesting policy to drive the population to that value as fast as possible and keep it there. We assume that the effort is constrained as $0 \leq E \leq E_{max}$ because there is a top in the effort that we can apply in the harvest. We define this policy as follows:

$$E*(t) = \begin{cases} E_{max}, & \text{x(t) > x*,} \\ \frac{rx*(1-\frac{x*}{K})}{qx*}, & \text{x(t)=x*,} \\ 0, & \text{x(t) < x*.} \end{cases} \tag{4.17}$$

With that policy we help the system to be always at the optimal point by making the maximum harvesting effort when the population is to big and letting the population growth free when it's below the optimal point.

With that

4.2 Dynamic Programming.

4.3 Stochastic Analysis.