Optimal Harvesting Modelling

Final Report

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Abstract

Contents

$$r = 0.8 \tag{2.1}$$

$$M = 780500 \tag{2.2}$$

3 Mathematical Models.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,t) \tag{3.1}$$

3.1 Exponential biological growth.

Assuming the natural fish mortality to be a constant M, we get the growth dynamics as,

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -mx \\ x(T) &= x_T \end{aligned} \tag{3.2}$$

If a variable mortality due to fishing $\Phi(t)$, is also considered then the growth equation becomes,

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -(m+\Phi(t))x \\ x(T) &= x_T \end{aligned} \tag{3.3}$$

3.2 Logistic Equation.

Logistic equation.

$$F(x,t) = rx\left(1-\frac{x}{M}\right) \tag{3.4}$$

3.3 Wiener Process and noise.

$$dx = \left(rx\left(1-\frac{x}{M}\right)-u\right)\mathrm{d}t + \sigma x \mathrm{d}W \tag{3.5}$$

A unique solution exists if both Itó conditions hold (Fleming and Rishel, 1975). The first one is the linear growth condition, for some independent constant K,

$$\left|rx\left(1-\frac{x}{M}\right)-u\right| \leq K(1+|x|) \tag{3.6}$$

$$|\sigma x| \leq K(1+|x|) \tag{3.7}$$

We see that $\sup_{x\in\mathbb{R}} rx(1-x/M)$ is reached $x^* = M/2$, and it is unique, with value $F(x^*,t) = \frac{rM}{4} - u$. Therefore for bounded u, r and σ conditions 3.6 and 3.7 are satisfied. The second one is the Lipschitz condition,$\exists L$ independent constant, and $\forall x$, $\exists B(x)$ neighborhood of x, such that $\forall x_1, x_2 \in B(x)$,

$$\left|rx_2\left(1-\frac{x_2}{M}\right)-rx_1\left(1-\frac{x_1}{M}\right)\right| \leq L|x_2-x_1| \tag{3.8}$$

$$|\sigma(x_2-x_1)| \leq L|x_2-x_1| \tag{3.9}$$

Since $F(x,t) = rx(1-x/M)-u$ is continuously differentiable in x, F Lipschitz in x then condition 3.8 is satisfied. For bounded σ, condition 3.9 is satisfied.

Since the above conditions are satisfied, we can guarantee existence and uniqueness of the solution for the equation 3.5. Given by the equation:

$$\begin{array}{rcl} x(t) & = & x_0 + \int_0^t \left(rx\left(1 - \frac{x}{M}\right) - u\right) \mathrm{d}t + \int_0^t \sigma x \mathrm{d}W, \\ x(0) & = & x_0, \\ W(0) & = & 0. \end{array} \tag{3.10}$$

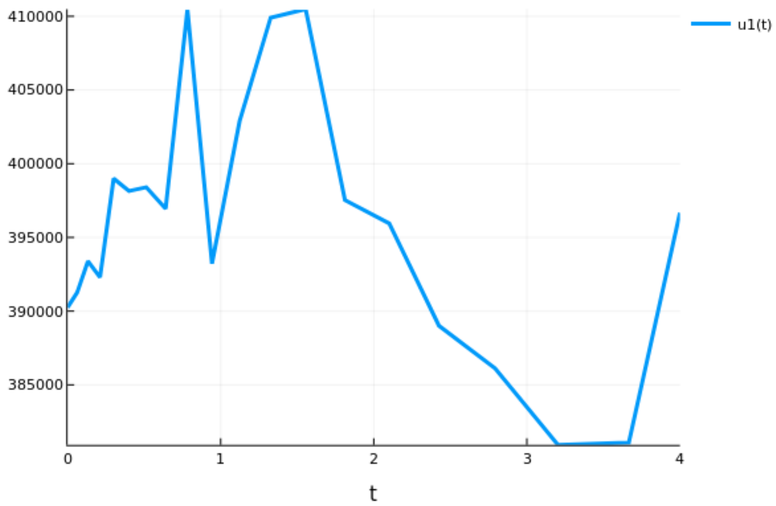

Figure 3.1: Simulation performed of logistic equation 3.5, with parameters $r = 0.8\frac{1}{\text{month}\times\text{fish}}$, $x_0 = \frac{M}{2}$, for a population in natural conditions (harvest exploitation $u = 0.$), with presence of noise proportional to the population, with $\sigma = 0.1$. Performed during 4 months.

4 Fishing Strategies and Optimizing Population

4.1 Open Loop Strategies.

4.1.1 Constant Harvesting Analysis.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{M}\right) - u \tag{4.1}$$

$$\beta = \frac{uM}{r} \tag{4.2}$$

$$\begin{aligned}
\frac{\mathrm{d}x}{rx\left(1-\frac{x}{M}\right)-u} &= \mathrm{d}t \\
\int_{x_0}^{x} \frac{\mathrm{d}\chi}{r\chi\left(1-\frac{\chi}{M}\right)-u} &= \int_0^t \mathrm{d}\tau \\
\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\chi(M-\chi)-\frac{Mu}{r}} &= t \\
-\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\chi^2 - M\chi + \beta} &= t
\end{aligned}$$

$$-\frac{M}{r}\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi - \frac{M}{2}\right)^2 - \frac{M^2}{4} + \beta} = t \tag{4.3}$$

$$\alpha = \beta - \frac{M^2}{4} = rM\left(u - \frac{rM}{4}\right) \tag{4.4}$$

If $u > rM/4$ implies $\alpha > 0$

$$\begin{aligned}
\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi - \frac{M}{2}\right)^2 + \alpha} &= -\frac{r}{M}t \\
\frac{1}{\sqrt{\beta - \frac{M^2}{4}}}\left(\arctan\left(\frac{x - M/2}{\sqrt{\beta - M^2/4}}\right) - \arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}}\right)\right) &= -\frac{r}{M}t
\end{aligned}$$

$$x = \frac{M}{2} + \sqrt{\beta - \frac{M^2}{4}}\tan\left(\arctan\left(\frac{x_0 - M/2}{\sqrt{\beta - M^2/4}}\right) - \frac{r\sqrt{\beta - M^2/4}}{M}t\right) \tag{4.5}$$

If $u < rM/4$ implies $\alpha < 0$,

$$\int_{x_0}^{x} \frac{\mathrm{d}\chi}{\left(\chi - \frac{M}{2}\right)^2 - \alpha} = -\frac{r}{M}t$$

$$\begin{aligned} \lambda &= \tfrac{M}{2} + \sqrt{\tfrac{M^2}{4} - \beta} \\ \overline{\lambda} &= \tfrac{M}{2} - \sqrt{\tfrac{M^2}{4} - \beta} \end{aligned} \tag{4.6}$$

$$\begin{aligned} \int_{x_0}^{x} \left(\frac{1}{\chi - \lambda} - \frac{1}{\chi - \overline{\lambda}}\right) \mathrm{d}\chi &= -\frac{2r\sqrt{M^2/4 - \beta}}{M}t \\ \ln\left|\frac{x - \lambda}{x - \overline{\lambda}}\right| &= \ln\left|\frac{x_0 - \lambda}{x_0 - \overline{\lambda}}\right| - \frac{2r\sqrt{M^2/4 - \beta}}{M}t \end{aligned}$$

$\gamma = \frac{2r\sqrt{M^2/4-\beta}}{M}$

$$\frac{x - \lambda}{x - \overline{\lambda}} = \frac{x_0 - \lambda}{x_0 - \overline{\lambda}} \mathrm{e}^{-\gamma t} \tag{4.7}$$

$$x - \lambda = \left(x - \overline{\lambda}\right)\left(\frac{x_0 - \lambda}{x_0 - \overline{\lambda}}\right)\mathrm{e}^{-\gamma t} \tag{4.8}$$

$\xi = \frac{x_0-\lambda}{x_0-\overline{\lambda}}\mathrm{e}^{-\gamma t}$

$$\begin{aligned} x(1 - \xi) &= \lambda - \overline{\lambda}\xi \\ x &= \frac{\lambda - \overline{\lambda}\xi}{1 - \xi} \\ x &= \frac{\frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta} - \left(\frac{M}{2} - \sqrt{\frac{M^2}{4} - \beta}\right)\xi}{1 - \xi} \\ x &= \frac{\frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta} - \left(\frac{M}{2} - \sqrt{\frac{M^2}{4} - \beta}\right)\xi}{1 - \xi} \\ x &= \frac{\frac{M}{2}(1 - \xi) + \sqrt{\frac{M^2}{4} - \beta}\,(1 + \xi)}{1 - \xi} \\ x &= \frac{M}{2} + \sqrt{\frac{M^2}{4} - \beta}\,\frac{1 + \xi}{1 - \xi} \end{aligned}$$

$$x(t) = \frac{M}{2} + \left(\sqrt{\frac{M^2}{4} - \beta}\right) \frac{(x_0 - M/2)(1 + \mathrm{e}^{-\gamma t}) - \sqrt{M^2/4 - \beta}\,(1 - \mathrm{e}^{-\gamma t})}{(x_0 - M/2)(1 - \mathrm{e}^{-\gamma t}) + \sqrt{M^2/4 - \beta}\,(1 + \mathrm{e}^{-\gamma t})} \tag{4.9}$$

4.1.2 Time Varying Harvesting.

For time varying Harvest,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1-\frac{x}{M}\right)-u(t) \tag{4.10}$$

4.1.3 Optimal Harvesting. Smooth Optimal Control Problem.

For Optimal Control we reduce the problem to the following,

$$\min_{\substack{x\in X\\ u\in U}} J(x,u) \tag{4.11}$$

subject to,

$$e(x,u)=0 \tag{4.12}$$

$h(t)=\frac{rM}{4}-u(t)$

$$J(x,u)=\frac{\zeta}{2}\left(x(T)-\frac{M}{2}\right)^2+\frac{1}{2}\left\|x-\frac{M}{2}\right\|^2_{L^2([0,T])}+\frac{\eta}{2}\|h\|^2_{L^2([0,T])} \tag{4.13}$$

subject to,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1-\frac{x}{M}\right)-\frac{rM}{4}+h \tag{4.14}$$

4.2 Closed Loop Strategies.

4.2.1 Constant Proportional Harvesting.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1-\frac{x}{M}\right)-px \tag{4.15}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1-\frac{p}{r}-\frac{x}{M}\right) \tag{4.16}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r\left(1-\frac{p}{r}\right)\left(1-\frac{x}{M\left(1-\frac{p}{r}\right)}\right)x \tag{4.17}$$

$\gamma=r\left(1-\frac{p}{r}\right)$, $K=M\left(1-\frac{p}{r}\right)$. With $\frac{p}{r}<1$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \gamma x\left(1-\frac{x}{K}\right) \tag{4.18}$$

$$x=\frac{Kx_0}{x_0+(K-x_0)\,\mathrm{e}^{-\gamma t}} \tag{4.19}$$

$$x(t)=\frac{M\left(1-\frac{p}{r}\right)x_0}{x_0+\left(M-\frac{Mp}{r}-x_0\right)\mathrm{e}^{-\gamma t}} \tag{4.20}$$

4.2.2 Optimal Proportional Harvesting.

5 Economical Profit

5.1 Linear Costs.

5.2 Quadratic Costs.

5.3 Stochastic Analysis.

6 Further Research