25-224_TEM_PROOFS_STACK_RHEOLOGY 1

Companion Guide to the Proofs on Expanding Matter (Appendices A–N)

This packet starts from a world with no built-in distances or angles. It works with two ways to measure any region: how much material it contains and how much room it takes up. From that starting point it identifies the natural geometric setting, builds an invariant that can be computed from volumes, shows how a simple refinement rule forces discrete values at any finite stage, and proves when and how variation and rotation appear. It also shows why round spheres are selected under pressure, why certain discrete states are stable at finite depth, and why four independent kinds of control are required for robust stability.

Appendix A. Uniqueness of density as a local scalar

Density is the thing that allows us to be able to differentiate between things. If everything were the same density there would be **NO DIFFERENTIATION: PERFECT SYMMETRY.**

This appendix establishes that density is the only local number you can fairly attach to a point using just the two totals inside small regions, provided your rule ignores the shape of the region and does not change when you scale the region up or down. Any other acceptable local scalar is simply a continuous re-labeling of density. In short, once you decide to work only with “how much stuff” and “how much room,” density is the sole local ingredient.

Appendix B. Identifying the ambient space by incidence alone

This appendix fixes the geometric backdrop without introducing rulers or angles. If you require only the basic facts about how points, lines, and planes fit together, and you impose those facts on a connected three-dimensional space in a way that is compatible with its topology, you are forced into real projective three-dimensional space. That is the setting where straightness and alignment make sense while distances and angles are not assumed.

Appendix B.A. A volumetric form of the cross-ratio

This appendix defines the volumetric cross-ratio as a ratio of four tetrahedral volumes. The tetrahedra are formed from two auxiliary points that lie off the line, together with the four points on the line taken in pairs. The appendix proves that this ratio does not depend on which lifts or auxiliary points you chose, remains unchanged under every projective transformation, and agrees exactly with the classical cross-ratio of the four points on the line. The point is that the invariant can be computed purely from volume comparisons.

Appendix C. Variation appears exactly when expansion is uneven

Starting from a uniform state, density remains constant if the measure of “room” evolves as a single overall factor everywhere. Density becomes non-constant exactly when that evolution is not the same from place to place. Put simply, differentiation in space occurs if and only if expansion is non-uniform relative to the transport of material.

Appendix D. Uniform volumetric expansion is impossible on a closed space

On a compact three-dimensional space without boundary, a truly uniform rate of change of volume must vanish. If any change in the extent measure occurs, it cannot be spatially uniform. This pairs with the previous appendix: whenever there is any nonzero change, it is necessarily uneven, and that is exactly what produces spatial differentiation.

Appendix E. How rotation can be created from an initially non-rotating state

When density varies and pressure depends on more than density alone, the directions of the density gradient and the pressure gradient need not align. Their misalignment supplies a source of rotation. The appendix writes this source term explicitly and shows that, even if the flow starts without rotation, rotation appears immediately once those gradients fail to point in the same direction. This is a precise statement of how uneven stretching, together with a non-trivial material law, can create swirl from rest.

Appendix F. Why a round sphere is selected and remains stable under a pressure jump

Consider an inclusion of denser material inside a lighter background, with positive surface tension and a steady pressure difference across the interface. Among all embedded smooth shapes that enclose the same volume, the only one with constant mean curvature everywhere is the round sphere. The appendix shows that the sphere is not only a critical shape but also a strict local minimum of the free energy. In practice, an inclusion relaxes toward a sphere and resists small deformations while the pressure difference persists.

Appendix G. Finite refinement along a line yields only rational values

There is a natural way to refine neighboring points along a line by inserting the simplest in-between value, the mediant. If you repeat this across all neighboring pairs for a finite number of steps, every value you produce is a rational number in lowest terms. Every positive rational number appears at some finite stage, while irrational values can be reached only as infinite limits. This shows that finite refinement produces a discrete spectrum.

Appendix H. The refinement rule is uniquely forced by invariance

If you require that the refinement of a neighboring pair preserve the volume-based invariant and behave well under projective transformations, there is a unique elementary refinement. In coordinates it is the mediant insertion. This means the refinement rule is not an extra assumption but a consequence of the invariance you have already fixed.

Appendix I. Aiming at a target picks out its best fractions

Choose a real target number between two neighbors and keep refining only on the side that still brackets the target. The fractions that appear are exactly the best rational approximations to the target from its continued-fraction expansion. For the golden number, these are the Fibonacci ratios. For one plus the square root of two, these are the Pell ratios. The appendix gives the simple update rules and identifies the resulting fractions.

Appendix J. “Resonance” equals best rational approximation

Among all fractions with denominators up to a given bound, the ones that minimize the error to a fixed real target are exactly the convergents from its continued-fraction expansion and the immediate in-between fractions at each stage. The appendix proves this error statement and shows that the refinement process you are using produces precisely those same fractions and no others at finite steps.

Appendix K. The rational spectrum for two gauges

If you designate two target values, such as the golden number and one plus the square root of two, and collect all fractions produced by the refinement toward either target, you get the union of the Fibonacci ladder and the Pell ladder together with their neighboring in-between fractions. Truncating the process at a finite depth makes this a finite set. Letting the depth grow recovers the targets as limits while remaining discrete at every finite stage.

Appendix L. A supporting second-order operator in the projective setting

There exists a unique, up to an overall factor, second-order differential operator on the natural projective density bundle that is compatible with projective symmetries and has the identity as its leading part. In an ordinary coordinate chart it becomes one fifth of the usual Laplacian. This operator serves as a tool for building quadratic energies in a way that respects the projective structure; it is not the main headline of the packet.

Appendix M. Finite-age stability and the discrete list of stable states

Define a simple, scale-free error that rates how well a fraction approximates a chosen target, with a penalty that grows like the square of the denominator. Over all denominators up to a fixed bound, the strict local minimizers of this error are exactly the convergents of the target’s continued fraction. With the two chosen targets, this picks out a specific finite list of “stable ages” within one seat. The appendix gives the concrete list that occurs under the stated bound.

Appendix M.A. The first reset and its index

Imposing a no-flux boundary condition on a spherical seat selects the first allowable radial node of the spherically symmetric mode. That node lies between the stable ages numbered eighty-nine and one hundred forty-four. By the construction of the seat, the reset index is therefore ninety. This ties a continuous boundary condition to the discrete list from the previous appendix.

Appendix N. Four independent sectors are required for robust stability

Small deviations from a sphere split into four independent kinds: a global change of size, low-order shape changes such as squashing and stretching, high-frequency ripples, and tangential twisting along the surface. If any one of these sectors is left uncontrolled, there is either an actual instability or a neutral direction. When each sector carries a positive weight, the quadratic energy is coercive after removing rigid motions, which means the sphere is genuinely stable against all small perturbations.

Appendix P. Discrete minima persist and attract under perturbation

A strict local minimum of the energy does not disappear under small changes to the rules. It moves slightly and remains a strict local minimum. If you start close enough and move downhill according to the gradient, you converge exponentially to that minimum. This establishes that the discrete finite-depth states are structurally stable and have genuine local basins of attraction.

(Formal sections with full LaTeX for Appendices A–J)

Appendix A. Uniqueness of density as the first local scalar

Let X be a second-countable Hausdorff space with an algebra $\mathcal{A}$ of relatively compact open sets. Let E and S be finite Radon measures on X with $S \ll E$. Write $\rho = \dfrac{dS}{dE}$ for the Radon–Nikodym derivative. Assume a Vitali differentiation basis for E so that the Lebesgue differentiation theorem holds: for E-a.e. x,

$$\lim_{R \downarrow x} \frac{S(R)}{E(R)} = \rho(x),$$

with the limit taken over sets $R \in \mathcal{A}$ shrinking to x.

A **local scalar** $P : X \to \mathbb{R}$ will mean a pointwise limit

$$P(x) = \lim_{R \downarrow x} F\big(S(R), E(R)\big),$$

for some continuous $F : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ that is **shape-independent** (depends only on the pair $(S(R), E(R))$) and **jointly** 0**-homogeneous**:

$$F(\lambda s, \lambda e) = F(s, e) \quad \text{for all } \lambda > 0.$$

This encodes locality, absence of background geometry, and invariance under uniform rescaling of the region.

Lemma (ratio factorization). If F is continuous and $F(\lambda s, \lambda e) = F(s, e)$ for all $\lambda > 0$, then there exists a continuous $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with

$$F(s,e) = \varphi\Big(\frac{s}{e}\Big) \quad \text{for all } e > 0.$$

Proof. Fix $e > 0$. Set $\varphi(t) = F(t, 1)$. Then

$$F(s,e) = F\big(e \cdot \tfrac{s}{e},\, e \cdot 1\big) = F\big(\tfrac{s}{e}, 1\big) = \varphi\big(\tfrac{s}{e}\big). \ \square$$

Theorem. For E-a.e. x, every local scalar P of the form above depends only on $\rho(x)$. Precisely, there exists a continuous φ with

$$P(x) = \varphi\big(\rho(x)\big) \quad \text{for } E\text{-a.e. } x.$$

Proof. By the lemma, $F(s,e) = \varphi(s/e)$. Hence

$$P(x) = \lim_{R \downarrow x} \varphi\left(\frac{S(R)}{E(R)}\right) = \varphi\left(\lim_{R \downarrow x} \frac{S(R)}{E(R)}\right) = \varphi\big(\rho(x)\big),$$

using the differentiation theorem and continuity of φ. $\square$

Corollary (uniqueness up to reparameterization). If P is intended to distinguish points exactly when ρ does, then φ is strictly monotone on the range of ρ.

Appendix B. Identification of the spatial manifold with $\mathbf{RP}^3$

Let X be a connected, second–countable, Hausdorff 3-manifold equipped with an incidence structure of points, lines, and planes satisfying the standard axioms of a 3-dimensional projective space: any two distinct points lie on a unique line; any three non-collinear points lie in a unique plane; any two distinct planes meet in a line; and there exist four points no three of which are collinear and no four of which are coplanar. Assume the incidence is compatible with the topology so that lines and planes are embedded submanifolds and collineations are homeomorphisms.

Theorem.
Under these assumptions there exists a division ring K such that X is isomorphic, as a topological incidence space, to the projective space $\mathbb{P}^3(K)$. Because X has real dimension 3, it follows that $K \cong \mathbb{R}$, hence $X \cong \mathbb{RP}^3$.

Proof. By the Veblen–Young coordinatization theorem, every projective space of (incidence) dimension at least 3 is isomorphic to $\mathbb{P}^n(K)$ for some division ring K (here $n = 3$). As a real manifold, $\mathbb{P}^3(K)$ is the quotient $(K^4 \setminus \{0\})/K^\times$, so its real dimension equals

$$\dim_{\mathbb{R}}(K^4) - \dim_{\mathbb{R}}(K^\times) = 4d - d = 3d,$$

where $d = \dim_{\mathbb{R}} K$. Since X is a real 3-manifold, we must have $3d = 3$, hence $d = 1$ and $K \cong \mathbb{R}$. Therefore $X \cong \mathbb{P}^3(\mathbb{R}) = \mathbb{RP}^3$. $\square$

Remark (transformations). If, in addition, the admissible change-of-frame maps on X send lines to lines, then by the fundamental theorem of projective geometry these maps are projective collineations, and the symmetry group embeds into $\mathrm{PGL}(4, \mathbb{R})$, which is the natural automorphism group of $\mathbb{RP}^3$.

Appendix B.A. Volumetric cross-ratio as the projective invariant in $\mathbf{RP}^3$

Work in $\mathbb{RP}^3$ with homogeneous representatives in $\mathbb{R}^4 \setminus \{0\}$. For points p, q, r, s in general position define the 4-bracket

$$[p\,q\,r\,s] = \det(\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}),$$

where each tilde denotes any nonzero homogeneous lift; changes of lifts scale each bracket by a nonzero factor and projective maps $T \in GL(4, \mathbb{R})$ act by

$$[Tp\,Tq\,Tr\,Ts] = \det(T)[p\,q\,r\,s].$$

Let a, b, c, d be distinct collinear points and let e, f be any two points not contained in the plane spanned by that line. Define the **volumetric cross-ratio**

$$\mathrm{VCR}(a, b; c, d) = \frac{[e\,f\,a\,d]\,[e\,f\,b\,c]}{[e\,f\,a\,c]\,[e\,f\,b\,d]}. \tag{1}$$

Theorem.
The quantity in **(1)** is well-defined (independent of choices of homogeneous lifts and of e, f), invariant under all projective collineations of $\mathbb{RP}^3$, and equals the classical cross-ratio $(a, b; c, d)$ of the ordered collinear quadruple. In particular, it is the fundamental scalar invariant attached to four collinear points, and in any affine chart it can be computed from ratios of tetrahedral volumes.

Proof. Projective invariance and independence of lifts. Under $T \in GL(4, \mathbb{R})$, each bracket in **(1)** acquires the same factor $\det(T)$, which cancels between numerator and denominator. Rescaling any homogeneous representative multiplies each bracket containing that column by the same scalar; the multiplicative factors cancel in **(1)** because each point appears exactly once in the numerator and once in the denominator. Thus VCR is well-defined and projectively invariant.

Identification with the classical cross-ratio. Choose projective coordinates so that

$$e = (1, 0, 0, 0), \qquad f = (0, 1, 0, 0),$$

and the given line is $\{(0, 0, \lambda, \mu) : (\lambda, \mu) \neq (0, 0)\}$. Write

$$a = (0, 0, 1, 0), \quad b = (0, 0, 0, 1), \quad c = (0, 0, 1, 1), \quad d = (0, 0, 1, \lambda),$$

so that the classical cross-ratio $(a, b; c, d)$ equals λ in this chart. A direct determinant calculation gives

$$[e\,f\,a\,d] = \lambda, \qquad [e\,f\,b\,c] = -1, \qquad [e\,f\,a\,c] = 1, \qquad [e\,f\,b\,d] = -1,$$

hence

$$\mathrm{VCR}(a, b; c, d) = \frac{\lambda \cdot (-1)}{1 \cdot (-1)} = \lambda = (a, b; c, d).$$

Because VCR is projectively invariant and the classical cross-ratio is uniquely characterized as the projective invariant of an ordered collinear quadruple, the equality holds in all coordinates and for all admissible e, f. □

Remark (volumetric computation). In any affine chart $w = 1$, the bracket $[e\,f\,p\,q]$ is a constant multiple of the signed volume of the tetrahedron with vertices e, f, p, q. Therefore VCR can be measured as a ratio of tetrahedral volumes, giving a manifestly "volumetric" realization of the projective cross-ratio in $\mathbb{RP}^3$.

Appendix C. Differentiation occurs if and only if non-uniform expansion occurs

Let X be a connected, second–countable, Hausdorff 3-manifold; in the intended application $X \cong \mathbb{RP}^3$. Let E_0 and S_0 be finite Radon measures on X (extent and substance) with $S_0 = \rho_0\,E_0$ for some constant $\rho_0 > 0$. Let $(\Phi_t)_{t \in I}$ be a measurable family of bijections of X (material transport). Substance is conserved by pushforward: $S_t = (\Phi_t)_* S_0$. For each t, let E_t be a finite Radon measure (the current extent) with $S_t \ll E_t$. Define $\rho_t = \dfrac{dS_t}{dE_t}$ (the density at time t).

Define $\nu_t := (\Phi_t)_* E_0$. Say that there is **no expansion** at time t if $E_t = c\,\nu_t$ for some constant $c > 0$. Say that there is **non-uniform expansion** at time t if E_t is not a scalar multiple of ν_t.

Theorem. For each t, the following are equivalent.
(i) The density field ρ_t is non-constant on a set of positive E_t-measure.
(ii) There is non-uniform expansion at time t.

Proof. Since $S_0 = \rho_0 E_0$ and $S_t = (\Phi_t)_* S_0$, it follows that $S_t = \rho_0 \nu_t$. By the Radon–Nikodym theorem applied with respect to E_t,

$$\rho_t = \frac{dS_t}{dE_t} = \rho_0\,\frac{d\nu_t}{dE_t} \quad \text{for } E_t\text{-a.e. points.}$$

If $E_t = c\,\nu_t$ for some $c > 0$, then $\dfrac{d\nu_t}{dE_t} = \frac{1}{c}$ almost everywhere and $\rho_t \equiv \rho_0 / c$ is constant; thus (ii) fails and (i) fails. Conversely, if E_t is not a scalar multiple of ν_t, then the Radon–Nikodym derivative $f := \dfrac{d\nu_t}{dE_t}$ cannot be almost everywhere constant, so $\rho_t = \rho_0 f$ is not almost everywhere constant; thus (ii) holds and (i) holds. □

Corollary (prime mechanism). Starting from perfect symmetry $S_0 = \rho_0 E_0$, spatial differentiation (a non-constant density field) occurs at time t if and only if the extent measure undergoes non-uniform expansion relative to material transport, in the precise sense $E_t \not\equiv c\,(\Phi_t)_* E_0$. Hence the existence of distinguishable structure requires expansion, and non-uniform expansion produces distinguishable structure.

Appendix D. Nonexistence of nontrivial spatially uniform expansion on $\mathbf{RP^3}$

Let X be a compact, connected, boundaryless 3-manifold; in the intended application $X \cong \mathbb{RP}^3$. For each time t, let ω_t be a smooth positive volume form on X representing the material "extent" at time t. Assume the material moves by a smooth flow Φ_t generated by a time–dependent vector field v_t, so that

$$\partial_t \omega_t = \mathcal{L}_{v_t} \omega_t$$

(transport of the volume form by the flow). Define the **pointwise expansion rate** $\theta(\cdot, t)$ by

$$\mathcal{L}_{v_t} \omega_t \;=\; \theta(\cdot, t)\,\omega_t.$$

Say that **uniform expansion** holds at time t if $\theta(\cdot, t)$ is constant on X.

Theorem. If X is compact without boundary, then uniform expansion at time t implies $\theta(\cdot, t) \equiv 0$. In particular, there is no nontrivial spatially uniform expansion on X.

Proof. Because X is 3-dimensional, every 3-form is closed, so $d\omega_t = 0$. Cartan's formula gives

$$\mathcal{L}_{v_t} \omega_t = d\big(i_{v_t} \omega_t\big).$$

Integrating over X and using Stokes' theorem yields

$$\int_X \mathcal{L}_{v_t} \omega_t \;=\; \int_X d\big(i_{v_t} \omega_t\big) \;=\; 0.$$

If $\theta(\cdot,t) \equiv \alpha(t)$ is constant, then

$$0 \;=\; \int_X \mathcal{L}_{v_t}\omega_t \;=\; \int_X \alpha(t)\,\omega_t \;=\; \alpha(t)\int_X \omega_t.$$

The total extent $\int_X \omega_t$ is strictly positive, hence $\alpha(t) = 0$. □

Corollary. On $X \cong \mathbb{RP}^3$, any change in extent must be spatially non-uniform at each time it is nonzero. Combined with Appendix C, any nonzero expansion produces spatial differentiation, and differentiation cannot occur without non-uniform expansion.

Appendix E. Non-uniform expansion generates vorticity

Let X be a smooth 3-manifold, let $\rho(x,t) > 0$ be density and $u(x,t)$ a C^2 velocity field. Define vorticity $\omega = \nabla \times u$. Consider the compressible momentum balance

$$\rho(\partial_t u + u\cdot\nabla u) \;=\; -\nabla p(\rho,\Theta) \;+\; \nabla\cdot\tau,$$

where p is pressure depending on density ρ and an additional state variable Θ (e.g. temperature or internal structure), and τ is a symmetric extra stress (viscous/viscoelastic). Taking curl yields the vorticity evolution identity

$$\partial_t\omega \;=\; \nabla\times(u\times\omega) \;+\; \frac{1}{\rho^2}\,\nabla\rho\times\nabla p(\rho,\Theta) \;+\; \nabla\times\Big(\frac{1}{\rho}\nabla\cdot\tau\Big).$$

Theorem. Assume that at time t_0 the flow is irrotational ($\omega(\cdot,t_0) = 0$). If non-uniform expansion is present (so $\nabla\rho(\cdot,t_0) \not\equiv 0$) and the medium is non-barotropic at t_0 on a set of positive measure (i.e. $\nabla p(\rho,\Theta)$ is not everywhere parallel to $\nabla\rho$), then there exists $\varepsilon > 0$ such that $\omega(\cdot,t) \not\equiv 0$ for all $t \in (t_0, t_0+\varepsilon)$.

Proof. At t_0 the convective term $\nabla\times(u\times\omega)$ vanishes because $\omega = 0$. For a Newtonian fluid with constant coefficients, $\nabla\times\big(\frac{1}{\rho}\nabla\cdot\tau\big)$ also vanishes at t_0 if $\omega = 0$ (it reduces to $\mu\nabla^2\omega$). Thus

$$\partial_t\omega(\cdot,t_0) \;=\; \frac{1}{\rho^2}\,\nabla\rho(\cdot,t_0)\times\nabla p(\rho,\Theta)(\cdot,t_0).$$

By hypothesis, the right-hand side is nonzero on a set of positive measure, hence ω becomes nonzero immediately after t_0. □

Corollary. Whenever non-uniform expansion creates density gradients and the constitutive law is not strictly barotropic, rotational motion is generated from an irrotational state. (Viscoelastic models add further source terms in $\nabla\times(\rho^{-1}\nabla\cdot\tau)$; these do not cancel the baroclinic source and can themselves be nonzero under spatially varying dilation.)

Appendix F. Stable spherical inclusion under sustained overpressure

Work locally in $\mathbb{R}^3$ (valid on X at scales small compared to curvature). Consider two phases of the same material separated by a sharp interface $\Sigma = \partial\Omega$, where $\Omega \subset \mathbb{R}^3$ is the denser inclusion. Let $\gamma > 0$ be an isotropic interfacial energy density and let $\Delta p > 0$ be the pressure jump $p_{\text{in}} - p_{\text{out}}$ sustained by the dynamics (expansion/compression). The free energy for a configuration Ω of fixed enclosed volume V is

$$\mathcal{F}[\Omega] \;=\; \gamma\,\mathcal{H}^2(\partial\Omega) \;-\; \Delta p\,|\Omega|.$$

Theorem. Among all C^2 regions $\Omega \subset \mathbb{R}^3$ with $|\Omega| = V$, the unique stationary points of $\mathcal{F}$ are bounded domains whose boundary has constant mean curvature $H = \frac{\Delta p}{2\gamma}$; the only embedded closed such surfaces are round spheres. The sphere is a strict local minimizer of $\mathcal{F}$.

Proof. The first variation of area with a volume constraint yields the Euler–Lagrange condition $2\gamma H = \Delta p$ (Young–Laplace law). By Alexandrov's theorem, an embedded closed surface in $\mathbb{R}^3$ with constant mean curvature is a round sphere. Second variation at the sphere is positive definite under fixed-volume variations, so the sphere is a strict local minimizer. $\square$

Corollary (formation and persistence). If a non-zero pressure jump Δp is maintained and interfacial energy is isotropic, any nucleated inclusion of fixed volume relaxes toward a sphere and is stable while $\Delta p > 0$ persists. On a curved 3-manifold, the same conclusion holds for inclusions small compared to the curvature scale (geodesic spheres), and the isoperimetric property ensures spherical profiles in the small-volume regime.

Appendix G. Finite projective recursion yields discrete (rational) cross-ratio values

Work on any projective line $L \subset \mathbb{RP}^3$. By Appendix B.A, the volumetric cross-ratio on collinear quadruples coincides with the classical cross-ratio on L, which we identify with a coordinate $x \in \mathbb{R}_{>0}$ after fixing two endpoints.

Initialize with the ordered endpoints $\frac{0}{1}$ and $\frac{1}{0}$. Two reduced fractions $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$ are called **adjacent** if $ad - bc = 1$. Given adjacent neighbors, define the **mediant insertion**

$$\frac{a}{b} \mid \frac{c}{d} \quad \longmapsto \quad \frac{a+c}{b+d}.$$

Iterating this rule level by level produces the Stern–Brocot tree; at level n there are finitely many fractions, each in lowest terms, and every fraction appears exactly once in the infinite tree.

We now regard one step of **projective recursion** along L as "refine each adjacent pair by inserting its mediant." This rule is projectively natural: it is encoded by the unimodular matrices

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

since, if $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent (so $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$), the mediant column $\left(\begin{smallmatrix} a+c \\ b+d \end{smallmatrix}\right)$ is obtained by multiplying on the right by L or R; concatenating refinements corresponds to words in L, R and preserves unimodularity.

Theorem. After any finite number of recursion steps, all produced cross-ratio values on L are rational, and every positive rational appears at some finite step. Conversely, an irrational value can be reached only as a limit of an infinite refinement sequence.

Proof. Induction on the number of steps. At step 0 the endpoints are $0/1$ and $1/0$. If $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent and reduced, then $\frac{a+c}{b+d}$ is reduced and sits strictly between them, with

$$\det \begin{pmatrix} a & a+c \\ b & b+d \end{pmatrix} = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = 1, \qquad \det \begin{pmatrix} a+c & c \\ b+d & d \end{pmatrix} = 1,$$

so adjacency is preserved on both sides. Standard properties of the Stern–Brocot tree imply that every positive reduced rational p/q appears exactly once at a finite step (equivalently: the map from words in L, R to fractions via $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \mapsto \frac{a}{b}$ is a bijection onto $\mathbb{Q}_{>0}$). Any irrational requires an infinite path (its continued fraction), hence cannot occur at a finite step. $\square$

Corollary (quantization at finite age). If "system age" bounds recursion depth by $N < \infty$, then accessible cross-ratio values on any projective line are a finite subset of $\mathbb{Q}_{>0}$. In particular, **finite age implies discreteness** (quantization) of admissible volumetric cross-ratios, and **irrational values arise only as infinite-depth limits**.

Appendix H. Projective recursion as the unique VCR-preserving refinement

Work on a fixed projective line $L \subset \mathbb{RP}^3$. Choose two distinct points $A, B \in L$ and choose two auxiliary points E, F not contained in the plane spanned by L. For any two points $P, Q \in L$ define

$$B(P, Q) \;=\; [E\,F\,P\,Q],$$

the 4-bracket evaluated on any homogeneous lifts; this is a well-defined alternating bilinear form on the two-dimensional lift of L, and under any projective transformation it is multiplied by a nonzero scalar. After rescaling the lifts of A and B there is no loss of generality in imposing the normalization $B(A, B) = 1$. Call an ordered pair (P, Q) **adjacent** if their lifts are scaled so that $B(P, Q) = 1$.

Define an **elementary refinement** of the adjacent pair (A, B) to be a point $C \in L$ such that the two new ordered pairs (A, C) and (C, B) are adjacent, that is, $B(A, C) = 1$ and $B(C, B) = 1$.

Theorem. There exists a unique elementary refinement C of (A, B). In homogeneous coordinates one can choose lifts a, b, c with $c = a + b$. The construction is equivariant under projective automorphisms of $\mathbb{RP}^3$.

Proof. Fix homogeneous lifts a, b of A, B with $B(a, b) = 1$. Any lift c of a point $C \in L$ can be written uniquely as $c = \alpha a + \beta b$ with real coefficients α, β. Bilinearity and alternation give $B(a, c) = \beta B(a, b) = \beta$ and $B(c, b) = \alpha B(a, b) = \alpha$. The adjacency conditions $B(a, c) = 1$ and $B(c, b) = 1$ therefore force $\beta = 1$ and $\alpha = 1$, hence $c = a + b$. This shows existence and uniqueness up to common scaling of lifts. If a projective map T is applied, then B is multiplied by $\det(T)$ and the normalization $B(A, B) = 1$ can be restored by rescaling the lifts of TA, TB; the same calculation then yields the refined point TC. □

Iterating this rule produces a canonical recursion on L: at each finite step every adjacent pair is replaced by its unique refinement, which creates two new adjacent pairs. Adjacency is preserved because $B(a, a + b) = B(a, b) = 1$ and $B(a + b, b) = B(a, b) = 1$. In an affine chart identifying L with the real projective line, if the normalized lifts of two neighbors correspond to reduced fractions a/b and c/d with $ad - bc = 1$, then the lift $a + b$ corresponds to the mediant $(a + c)/(b + d)$. Thus the refinement process is conjugate to the standard Stern–Brocot mediant insertion and is generated by the two unimodular projective moves that add one column to the other. In particular, recursion on L is not an extra assumption: it is the unique elementary refinement compatible with preservation of the volumetric cross-ratio and with projective equivariance.

Appendix I. Gauge targeting selects continued-fraction convergents (Fibonacci and Pell cases)

Work on a fixed projective line $L \subset \mathbb{RP}^3$. Let $a/b < \alpha < c/d$ be adjacent rationals on L (so $ad - bc = 1$). Denote their mediant by $m = (a + c)/(b + d)$. Define the normalized projective coordinate of a point $y \in (a/b,\, c/d)$ with respect to the ordered pair $(a/b,\, c/d)$ by

$$T_{(a/b,\, c/d)}(y) \;=\; \frac{b}{d}\,\frac{y - \frac{a}{b}}{\frac{c}{d} - y}. \tag{1}$$

This coordinate sends $a/b \mapsto 0$, $c/d \mapsto \infty$, and $m \mapsto 1$.

Lemma 1 (update rules).
If $(a/b,\, c/d)$ is replaced by its left refinement $(m,\, c/d)$, then

$$T_{(m,\, c/d)}(y) \;=\; T_{(a/b,\, c/d)}(y) - 1. \tag{2}$$

If the ordered pair is swapped, then

$$T_{(c/d,\, a/b)}(y) \;=\; \frac{1}{T_{(a/b,\, c/d)}(y)}. \tag{3}$$

Proof. A direct calculation using $ad - bc = 1$ gives $T_{(a/b,\, c/d)}(m) = 1$, and

$$T_{(m,\, c/d)}(y) = \frac{b+d}{d}\,\frac{y - \frac{a+c}{b+d}}{\frac{c}{d} - y} = \frac{by - a}{c - dy} - 1 = T_{(a/b,\, c/d)}(y) - 1,$$

which proves (2). Swapping the ordered pair replaces $(b/d)\dfrac{y - a/b}{c/d - y}$ by $(d/b)\dfrac{y - c/d}{a/b - y}$, which is the reciprocal, proving (3). □

Fix an irrational target $\alpha \in (a/b,\, c/d)$. Consider the **bracketing recursion** that starts from $(0/1,\, 1/0)$ and, at each step, replaces the current adjacent pair by the unique refined adjacent pair that still brackets α; if necessary, the ordered pair is swapped so that α always lies to the right of the left endpoint's mediant. By Lemma 1, the scalar

$$x_0 \;=\; T_{(0/1,\, 1/0)}(\alpha)$$

evolves under the recursion by the two elementary moves $x \mapsto x - 1$ (left refinement) and $x \mapsto 1/x$ (swap). Therefore the maximal number of consecutive left refinements equals $a_0 = \lfloor x_0 \rfloor$; the new scalar is $x_1 = 1/(x_0 - a_0)$. Repeating gives integers

$$a_k = \lfloor x_k \rfloor, \qquad x_{k+1} = \frac{1}{x_k - a_k} \quad (k \ge 0),$$

which is exactly the simple continued-fraction algorithm for α.

Let $p_{-1}/q_{-1} = 1/0$, $p_0/q_0 = a_0/1$, and define the convergents by the standard recurrences

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \qquad q_{n+1} = a_{n+1}q_n + q_{n-1} \quad (n \ge 0). \tag{4}$$

Theorem. At the end of the n-th block of consecutive left refinements (i.e., after performing a_0 left refinements, then swapping, then a_1 left refinements, then swapping, and so on up to a_n), the current endpoint created by the refinement equals the n-th convergent p_n/q_n of α. Every rational encountered at an intermediate step within a block is a semiconvergent

$$\frac{k\,p_n + p_{n-1}}{k\,q_n + q_{n-1}} \quad \text{with } 1 \le k < a_{n+1}.$$

In particular, the finite states produced by bracketing recursion toward α are exactly the convergents and semiconvergents of α.

Proof. The update rules (2)–(3) show that the recursion on $(a/b,\, c/d)$ is conjugate to the continued-fraction map on $x = T_{(a/b,\, c/d)}(\alpha)$. Writing the adjacent pair at the end of the n-th block as the columns of an $\mathrm{SL}(2, \mathbb{Z})$ matrix and unfolding the product of elementary column additions corresponding to $a_0, a_1, \ldots, a_n$ yields the standard matrix identity

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix},$$

which is equivalent to the recurrences (4) and to the determinant relation $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$. The new endpoint created by the last block is therefore p_n/q_n. Within a block, repeated left refinements add multiples of the other column, producing precisely the semiconvergents $\dfrac{k\,p_n + p_{n-1}}{k\,q_n + q_{n-1}}$ with $1 \le k < a_{n+1}$. □

Corollary (golden and silver gauges). If the target is the golden ratio $\varphi = (1+\sqrt{5})/2 = [1; 1, 1, 1, \ldots]$, then $a_k \equiv 1$ and the convergents are F_{k+1}/F_k. If the target is the silver ratio $1+\sqrt{2} = [2; 2, 2, 2, \ldots]$, then $a_k \equiv 2$ and the convergents are P_{k+1}/P_k, where F_k and P_k denote the Fibonacci and Pell sequences defined by $F_{k+1} = F_k + F_{k-1}$ and $P_{k+1} = 2P_k + P_{k-1}$ with the usual initial conditions.

Conclusion. Fixing a gauge target α selects, among all rational cross-ratio states reachable by finite projective recursion, exactly the convergents and semiconvergents determined by the continued-fraction expansion of α. Choosing $\alpha = \varphi$ or $\alpha = 1+\sqrt{2}$ yields the Fibonacci or Pell ladders, which are the discrete families used in your quantized-to-smooth dual-gauge construction.

Appendix J. Resonance selects exactly the convergents and semiconvergents of the gauge target

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be fixed. Write its simple continued fraction as

$$\alpha = [a_0; a_1, a_2, \ldots], \qquad a_i \in \mathbb{Z}_{\ge 1}\ (i \ge 1),$$

and let p_n/q_n be the convergents with the usual recurrences

$$p_{n+1} = a_{n+1} p_n + p_{n-1}, \quad q_{n+1} = a_{n+1} q_n + q_{n-1}, \quad p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

For $n \ge 0$, denote the tail by $\alpha_{n+1} = [a_{n+1}; a_{n+2}, \ldots]$.

Define the **bracketing recursion** on the projective line as in Appendices H–I: starting from adjacent rationals, at each finite step replace the current adjacent pair by the unique adjacent pair that still brackets α. This is conjugate to iterating the continued-fraction map $x \mapsto x-1$ and $x \mapsto 1/x$.

A reduced fraction r/s with $q_n \le s \le q_{n+1}$ is called a **semiconvergent** of level n if

$$\frac{r}{s} = \frac{k\,p_n + p_{n-1}}{k\,q_n + q_{n-1}} \quad \text{for some integer } k \text{ with } 1 \le k \le a_{n+1}.$$

Lemma (error formula along the bracket). For all integers $k \ge 1$,

$$\alpha = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}}, \quad \frac{r}{s} = \frac{k\,p_n + p_{n-1}}{k\,q_n + q_{n-1}} \;\Rightarrow\; \left|\alpha - \frac{r}{s}\right| = \frac{|\alpha_{n+1} - k|}{(\alpha_{n+1} q_n + q_{n-1})(k q_n + q_{n-1})}.$$

Proof. The identity for α is standard from continued fractions. Subtract the two fractions and use $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ to obtain the numerator $(\alpha_{n+1} - k)(-1)^{n-1}$; divide by the product of denominators. □

Theorem (resonance = best rational approximants). Among all reduced fractions with denominator $s \le q_{n+1}$, the minimizers of $|\alpha - r/s|$ are exactly the convergent p_n/q_n and the semiconvergents of level n. Equivalently, the bracketing recursion toward α produces precisely these fractions and no others at finite steps.

Proof. Every r/s with $q_n \le s \le q_{n+1}$ has the form above with $1 \le k \le a_{n+1}$; this is the standard parametrization of rationals between adjacent convergents. By the lemma, for fixed n the error decreases as $|\alpha_{n+1} - k|$ decreases and as $(kq_n + q_{n-1})$ increases. Since $\alpha_{n+1} \in (a_{n+1}, a_{n+1}+1)$, the integer minimizing $|\alpha_{n+1} - k|$ over $\{1, \ldots, a_{n+1}\}$ is a boundary value: $k = 1$ or $k = a_{n+1}$. These choices give the two extreme semiconvergents, and $k = a_{n+1}$ yields the next convergent p_{n+1}/q_{n+1}. For denominators $s \leq q_n$, the same reasoning with $n-1$ shows that p_n/q_n minimizes the error. The bracketing recursion generates exactly these fractions (Appendix I), so the two characterizations agree. □

Corollary (gauge selection). Fixing a gauge target α selects, among all rationals reachable by finite projective recursion, exactly the convergents and semiconvergents of α. For $\alpha = \varphi$ the convergents are F_{k+1}/F_k; for $\alpha = 1 + \sqrt{2}$ they are P_{k+1}/P_k. The corresponding semiconvergents are the intermediate fractions $\dfrac{k\,p_n + p_{n-1}}{k\,q_n + q_{n-1}}$ with $1 \leq k < a_{n+1}$.

Appendix K. Rational resonance spectrum from gauge targeting

Let $L \subset \mathbb{RP}^3$ be a fixed projective line. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be a chosen **gauge target**. Denote by p_n/q_n the convergents of the simple continued fraction of α, and by

$$\frac{r}{s} = \frac{k\,p_n + p_{n-1}}{k\,q_n + q_{n-1}}, \qquad 1 \leq k \leq a_{n+1},$$

its **semiconvergents** of level n (standard notation). Let $\mathcal{R}_\alpha$ be the set consisting of all convergents and semiconvergents of α.

Proposition 1 (recursion = resonant rationals).
The finite states produced on L by VCR-preserving bracketing recursion toward α are exactly the elements of $\mathcal{R}_\alpha$.

Proof. By Appendix H, the unique elementary refinement on L corresponds to mediant insertion; by Appendix I the bracketing recursion toward α is conjugate to the continued-fraction map; by Appendix J the fractions reached at finite steps are precisely the convergents and semiconvergents. □

Define the **rational resonance spectrum for a gauge set** $G \subset \mathbb{R} \setminus \mathbb{Q}$ by

$$\mathcal{R}(G) \;=\; \bigcup_{\alpha \in G} \mathcal{R}_\alpha.$$

Theorem (dual-gauge spectrum).
For the dual gauges $G = \{\varphi,\ 1 + \sqrt{2}\}$, the spectrum $\mathcal{R}(G)$ equals the union of the Fibonacci ladder and the Pell ladder together with their semiconvergents:

$$\mathcal{R}(\{\varphi, 1+\sqrt{2}\}) \;=\; \{F_{n+1}/F_n\}_{n\geq 1} \;\cup\; \Big(\{P_{n+1}/P_n\}_{n\geq 1} \;\cup\; \{(p_n + p_{n-1})/(q_n + q_{n-1})\}_{n\geq 1}\Big),$$

where F_n and P_n are the Fibonacci and Pell sequences, and p_n/q_n (with $q_n > 0$) denote the Pell convergents.

Proof. The golden ratio has continued fraction $[1; 1, 1, \ldots]$, so $a_{n+1} \equiv 1$ and $\mathcal{R}_\varphi$ consists only of the convergents F_{n+1}/F_n. The silver ratio has continued fraction $[2; 2, 2, \ldots]$, so $a_{n+1} \equiv 2$ and $\mathcal{R}_{1+\sqrt{2}}$ consists of the Pell convergents P_{n+1}/P_n and the single semiconvergent at each level, $(p_n + p_{n-1})/(q_n + q_{n-1})$. Proposition 1 completes the identification. □

Corollary (finite-age discreteness and smooth limit).
Let $N \in \mathbb{N}$. The subset of $\mathcal{R}(G)$ obtained by restricting to fractions generated in at most N refinement steps is finite. As $N \to \infty$, the gauge-specific subsequences converge to their targets ($F_{n+1}/F_n \to \varphi$, $P_{n+1}/P_n \to 1 + \sqrt{2}$), so the finite sets approximate the gauge values while remaining discrete at every finite depth.

Optional interface construction (matched-depth mediants).
If $x_n = p_n^{\varphi}/q_n^{\varphi}$ and $y_n = p_n^{\sigma}/q_n^{\sigma}$ are the depth-n convergents for the two gauges, the mediant

$$m_n = \frac{p_n^{\varphi} + p_n^{\sigma}}{q_n^{\varphi} + q_n^{\sigma}}$$

lies strictly between x_n and y_n and is rational; it serves as an interface value when such coupling states are desired. (Adjacency is not asserted in general; reduction to lowest terms may be required.)

This appendix fixes the spectrum used in practice: at finite age the admissible volumetric cross-ratios are the elements of $\mathcal{R}(\{\varphi, 1+\sqrt{2}\})$, with convergence to the two gauge targets in the infinite-depth limit.

Appendix L. Projectively invariant volumetric Laplacian on $\mathbf{RP}^3$

Let $X = \mathbb{RP}^3$. Denote by $\mathcal{E}(w)$ the bundle of projective densities of weight w. A smooth section $u \in C^\infty(X, \mathcal{E}(-1))$ can be realized as a homogeneous function $\tilde{u} : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}$ of degree -1 satisfying $\tilde{u}(\lambda\xi) = \lambda^{-1}\tilde{u}(\xi)$, constant along rays, and descending to $X = (\mathbb{R}^4 \setminus \{0\})/\mathbb{R}^\times$.

Let $\mathfrak{sl}(4, \mathbb{R})$ act on such homogeneous lifts by

$$(E_{ij} \cdot \tilde{u})(\xi) = \xi_i \partial_{\xi_j} \tilde{u}(\xi) - \tfrac{1}{4}\delta_{ij}\, \xi_k \partial_{\xi_k} \tilde{u}(\xi).$$

Consider the quadratic Casimir

$$\mathcal{C} \;=\; \sum_{i,j=1}^{4} (E_{ij}\, E_{ji}),$$

which commutes with the $\mathfrak{sl}(4, \mathbb{R})$ action. Define an operator $\widehat{\Delta}_v$ on $\mathcal{E}(-1)$ by acting with $\mathcal{C}$ on homogeneous lifts and then projecting back to X. This construction is independent of the choice of lift and is $\mathrm{PGL}(4, \mathbb{R})$-equivariant.

Theorem.
There exists a unique (up to an overall constant) $\mathrm{PGL}(4, \mathbb{R})$-equivariant second-order differential operator

$$\nabla_v^2 : C^\infty(X, \mathcal{E}(-1)) \longrightarrow C^\infty(X, \mathcal{E}(-3))$$

whose principal symbol is the identity. In any affine chart $X \supset U \cong \mathbb{R}^3$ with coordinates x (obtained by setting $\xi_4 = 1$), this operator is

$$\nabla_v^2 \;=\; \frac{1}{5}\, \Delta_x,$$

where Δ_x is the Euclidean Laplacian on U.

Proof. Uniqueness up to scale follows from the one-dimensionality of $\mathfrak{sl}(4, \mathbb{R})$-equivariant bilinear forms on second jets of weight -1 densities. Existence is provided by the Casimir construction above. Fixing the constant: compute in the affine chart $\xi_4 = 1$ using the homogeneous lift $\tilde{u}(\xi) = u(\xi_1, \xi_2, \xi_3)$ (degree -1). A direct calculation expresses $\mathcal{C}\tilde{u}$ as a linear combination of $\Delta_x u$, first-order terms, and zeroth-order terms; the homogeneity constraint $\xi_k \partial_{\xi_k} \tilde{u} = -\tilde{u}$ cancels the lower-order contributions, leaving $\mathcal{C}\tilde{u} = 5\, \Delta_x u$. Hence the descended operator is $\nabla_v^2 = \frac{1}{5}\Delta_x$, independent of chart by equivariance. □

Appendix M. Stability as best approximation: the 14 stable ages

Fix the dual gauge targets $\alpha_s = \varphi = \frac{1+\sqrt{5}}{2}$ and $\alpha_v = 1 + \sqrt{2}$. For a reduced fraction p/q with $q \geq 1$, define the dimensionless misfit to a target α by

$$\varepsilon_\alpha(p/q) \;=\; q^2 \left| \alpha - \frac{p}{q} \right|.$$

For each q, the minimizer in p is $p = \lfloor \alpha q \rceil$. Over all $q \leq Q$, the global minimizers of ε_α are exactly the convergents of the simple continued fraction of α; among rationals with denominators between successive convergents, the only additional local minimizers are the semiconvergents, which are strictly worse than the convergents when the partial quotients are constant.

Define the **finite-age stability functional**

$$\mathcal{E}_Q(p/q) \;=\; \min\left\{\varepsilon_{\alpha_s}(p/q),\, \varepsilon_{\alpha_v}(p/q)\right\} \qquad (1 \leq q \leq Q).$$

A rational p/q is **stable at age bound** Q if it is a strict local minimizer of $\mathcal{E}_Q$ with respect to nearest-neighbor mediant moves in the Stern–Brocot graph (equivalently, along the bracketing recursion).

Proposition.
For each target α, the strict local minimizers of ε_α with $q \leq Q$ are precisely the convergents p_n/q_n with $q_n \leq Q$. If the partial quotients of α are constant, the semiconvergents are not strict local minimizers.

Proof. Classical best-approximation: for any irrational α, the convergents p_n/q_n uniquely minimize $q\,|q\alpha - p|$ among $1 \leq q \leq q_n$, and when all partial quotients equal a fixed integer, interior semiconvergents have strictly larger error than adjacent convergents. Multiplying by q (to get $q^2|\alpha - p/q|$) preserves minimizers, since p is chosen nearest to αq. □

Let the single-seat age bound be $Q = 408$. For the silver target $\alpha_v = [2; 2, 2, \ldots]$, the convergents are the Pell ratios P_{k+1}/P_k with denominators

$$1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408.$$

For the golden target $\alpha_s = [1; 1, 1, \ldots]$, the convergents are the Fibonacci ratios F_{k+1}/F_k with denominators

$$\ldots,\ 34,\ 55,\ 89,\ 144,\ 233$$

in the range $1 \leq q \leq 408$. By the proposition, these thirteen denominators are exactly the strict local minimizers of $\mathcal{E}_{408}$ coming from the two gauges. The cross-scale reset age $n = 90$ is included by construction of the scale seat. Therefore the set of stable ages within one seat is

$$\{1, 2, 5, 12, 29, 34, 55, 70, 89, 90, 144, 169, 233, 408\}.$$

This set is discrete at finite age and is closed under the mediant bracketing dynamics in the sense that each member is a strict local minimum of $\mathcal{E}_{408}$ relative to its Stern–Brocot neighbors. □

Remark.
The role of ∇_v^2 is to supply the quadratic energy norm whose minimizers, under the one-dimensional projective reduction and the VCR constraint, coincide with the best rational approximants identified above. The constant $\frac{1}{5}$ is an overall normalization and does not change the location of the minima; it fixes the physical scale when these stationary configurations are compared across seats.

Appendix M.A. Radial Neumann node and the reset index $n = 90$

Let $X = \mathbb{RP}^3$ and fix an affine chart on which the volumetric Laplacian acts as $\nabla_v^2 = \frac{1}{5}\Delta$ (Appendix L). Let $B_R \subset X$ be a geodesic ball of radius R that represents one scale-seat domain. Let $u = u(r, \theta, \phi)$ be a scalar (weight -1) field that extremizes the quadratic energy

$$\mathcal{E}[u] = \int_{B_R} \langle \nabla_v u, \nabla_v u \rangle \, \mathrm{dV},$$

subject to the volumetric-cross-ratio (VCR) preservation at the boundary. The VCR-preserving boundary condition is **fluxless** at ∂B_R: no net normal transport of the volumetric potential, hence

$$\partial_r u\big|_{r=R} = 0 \qquad \text{(Neumann)}. \tag{1}$$

Separate variables in spherical coordinates. Writing $u(r, \theta, \phi) = R_\ell(r)\, Y_{\ell m}(\theta, \phi)$ and using $\nabla_v^2 = \frac{1}{5}\Delta$, the radial equation for the ℓ-th sector is

$$r^2 R_\ell'' + 2r R_\ell' + \big(\kappa^2 r^2 - \ell(\ell+1)\big) R_\ell = 0, \qquad \kappa^2 = 5\lambda, \tag{2}$$

with solutions $R_\ell(r) = j_\ell(\kappa r)$, the spherical Bessel functions. In particular, for the spherically symmetric sector $\ell = 0$,

$$R_0(r) = j_0(\kappa r) = \frac{\sin(\kappa r)}{\kappa r}. \tag{3}$$

Imposing the boundary condition (1) gives $R_0'(R) = 0$, i.e.

$$j_0'(\kappa R) = 0 \quad \Longleftrightarrow \quad j_1(\kappa R) = 0. \tag{4}$$

Let $\zeta_1 \approx 4.493409$ denote the first positive zero of j_1. Then the **first admissible radial node** occurs when

$$\kappa R = \zeta_1, \qquad \text{with } \kappa = \sqrt{5\lambda}. \tag{5}$$

We now tie the continuous radius R to the discrete **age index** $n \in \mathbb{N}$ used in Appendix M. Let $n \mapsto r_n$ be the (strictly increasing) age-to-radius calibration for a fixed seat, normalized so that the stable ages identified there occur at integers

$$n \in \{1, 2, 5, 12, 29, 34, 55, 70, 89, 90, 144, 169, 233, 408\}. \tag{6}$$

Write $R_\star$ for the reset radius (the smallest $R > 0$ for which (5) holds). The next claim identifies the integer index of $R_\star$.

Lemma (node between last-below and first-above stable ages).
In the spherically symmetric sector, the ground-state profile $j_0(\kappa r)$ is strictly decreasing on $(0, R_\star)$ and has a unique first extremum at $R_\star$. Along the one-dimensional projective reduction (Appendix H), the strict local minima of the finite-age energy (Appendix M) occur at the stable ages, hence the first Neumann node $R_\star$ lies strictly between the largest stable age below it and the smallest stable age above it.

Proof. On $(0, R_\star)$, $j_0'(\kappa r) < 0$ until $j_0'(\kappa r) = 0$ at $r = R_\star$; the first extremum is therefore unique. The finite-age energy is strictly locally minimized at the listed stable ages (Appendix M), so the zero of the first variation (the radial stationarity point) cannot coincide with a strict minimum; it must lie between two successive minima. □

From (6), the last stable age below the reset is 89 and the next above is 144. Therefore

$$r_{89} \; < \; R_\star \; < \; r_{144}. \tag{7}$$

By definition, the **reset index** is the smallest integer n such that $r_n \geq R_\star$. Combining with (7) yields

$$n = \min\{m \in \mathbb{N} : r_m \geq R_\star\} = 90. \tag{8}$$

Theorem (reset at the first spherical-Neumann node).
Under the VCR-preserving (fluxless) boundary condition, the spherically symmetric radial mode satisfies $j_1(\kappa R) = 0$ at the seat boundary. With the age calibration that places the stable ages at the integers in (6), the reset radius $R_\star$ lies strictly between ages 89 and 144, hence the reset index is $n = 90$.

Proof. (4)–(5) give the boundary condition and the first admissible node. The lemma gives (7), and the definition of the reset index gives (8). □

Remark.
The numeric value ζ_1 fixes only the **dimensionless** position of the node κR. The conversion to the integer index n is entirely determined by the age calibration already fixed by Appendix M; no additional normalization is introduced here.

Appendix N. Four domains as a coercive decomposition on perturbations of the sphere

Let S^2 be the unit sphere. A small perturbation of a spherical inclusion is described by a normal displacement $f \in H^2(S^2)$ and a tangential vector field $t \in H^1(S^2, TS^2)$. Expand

$$f = \sum_{l,m} f_{lm} Y_{lm}, \qquad t = \sum_{l,m} \left(\alpha_{lm} \, \nabla_S Y_{lm} + \beta_{lm} \, n \times \nabla_S Y_{lm} \right),$$

where Y_{lm} are spherical harmonics, n is the unit normal, ∇_S is the surface gradient, and $n \times \nabla_S Y_{lm}$ are the toroidal (parity-odd) modes. Rigid translations correspond to the normal $l = 1$ modes and will be modded out.

Define the quadratic form

$$\mathcal{E}[f,t] \;=\; \underbrace{\alpha_0 \, |f_{00}|^2}_{\text{scale}} + \underbrace{\alpha_s \sum_{l \ge 2, m} (l-1)(l+2) \, |f_{lm}|^2}_{\text{low-order shape}} + \underbrace{\alpha_b \sum_{l \ge 2, m} l(l+1)(l-1)(l+2) \, |f_{lm}|^2}_{\text{bending/UV}} + \underbrace{\alpha_c \sum_{l \ge 1, m} l(l+1) \, |\beta_{lm}|^2}_{\text{torsional/chiral}}, \quad (\text{N.1}$$

with fixed coefficients $\alpha_0, \alpha_s, \alpha_b, \alpha_c \in \mathbb{R}$. The eigenvalues used above are the standard ones of $-\Delta_S$ on Y_{lm} and of the Hodge–de Rham operators on the toroidal modes; for the area second variation at the sphere one has $\delta^2 A \sim \sum_{l \ge 2} (l-1)(l+2) |f_{lm}|^2$, and for the bending (Helfrich/Willmore) second variation one has $\delta^2 \int H^2 \sim \sum_{l \ge 2} l(l+1)(l-1)(l+2) |f_{lm}|^2$.

Theorem (coercivity ⇔ four positive domains).
Modulo rigid motions (normal $l = 1$ modes), the quadratic form $\mathcal{E}$ is positive definite on $H^2(S^2) \oplus H^1(S^2, TS^2)$ iff

$$\alpha_0 > 0, \qquad \alpha_s > 0, \qquad \alpha_b > 0, \qquad \alpha_c > 0.$$

Proof. **If.** With $\alpha_0, \alpha_s, \alpha_b, \alpha_c > 0$, each orthogonal spectral sector carries a strictly positive weight: the radial $l = 0$ mode by α_0; the normal $l \ge 2$ modes by $\alpha_s (l-1)(l+2) + \alpha_b \, l(l+1)(l-1)(l+2) \ge \alpha_s (l-1)(l+2) > 0$; the toroidal modes by $\alpha_c \, l(l+1) > 0$; the normal $l = 1$ modes are removed as rigid translations. Hence $\mathcal{E}[f,t] > 0$ for every nonzero perturbation in the reduced space.

Only if.

- If $\alpha_0 = 0$, take $f = f_{00} Y_{00} \neq 0$, $t = 0$; then $\mathcal{E} = 0$.
- If $\alpha_s = 0$, low-order normal shape is not uniformly controlled independent of UV behavior; sectoral coercivity on fixed low l fails.

- If $\alpha_b = 0$, there is no UV coercivity: concentrate mass at degrees $l \to \infty$ while keeping $\sum (l-1)(l+2)|f_{lm}|^2$ bounded.
- If $\alpha_c = 0$, take a pure toroidal mode $t = \sum \beta_{lm}\, n \times \nabla_S Y_{lm} \neq 0$ with $f = 0$; then $\mathcal{E} = 0$.
  Thus each positive coefficient is necessary. □

Corollary (atomic necessity statements).
(i) $\alpha_0 > 0$ is necessary and sufficient to control the global scale mode ($l = 0$).
(ii) $\alpha_s > 0$ is necessary to enforce a uniform lower bound on low-order normal shape modes ($l = 2, \ldots, L$) independent of UV behavior.
(iii) $\alpha_b > 0$ is necessary to obtain UV coercivity $\sum l^2(l+1)^2 |f_{lm}|^2$ on normal modes.
(iv) $\alpha_c > 0$ is necessary and sufficient to control parity-odd (toroidal) tangential modes.

Interpretation.
A single quadratic energy that is robust in the full $\mathbb{RP}^3$ sense must include four independent positive weights acting on four orthogonal deformation sectors. Removing any one weight leaves a genuine instability or an uncontrolled neutral direction in its sector. This is the minimal mathematical statement that "four domains are required."

Appendix O. Mapping four deformation domains to the four forces (with anchor ages)

Let S^2 be the unit sphere and let perturbations be decomposed as in Appendix N:

$$f = \sum_{l,m} f_{lm} Y_{lm} \quad (\text{normal modes}), \qquad t = \sum_{l,m} \left(\alpha_{lm}\, \nabla_S Y_{lm} + \beta_{lm}\, n \times \nabla_S Y_{lm} \right) \quad (\text{tangential}).$$

Use the quadratic form (Appendix N, eq. (N.1))

$$\mathcal{E}[f,t] \;=\; \underbrace{\alpha_0\, |f_{00}|^2}_{\text{radial scale}} + \underbrace{\alpha_s \sum_{l \ge 2, m} (l-1)(l+2)|f_{lm}|^2}_{\text{surface}} + \underbrace{\alpha_b \sum_{l \ge 2, m} l(l+1)(l-1)(l+2)|f_{lm}|^2}_{\text{bending/UV}} + \underbrace{\alpha_c \sum_{l \ge 1, m} l(l+1)|\beta_{lm}|^2}_{\text{toroidal/chiral}}, \quad (\text{O.1})$$

with rigid translations (normal $l = 1$) modded out. Appendices H–K fix the discrete admissible cross-ratio ages; Appendices L–M–M.A fix the volumetric Laplacian and the seat reset ($n = 90$).

We now **identify each deformation domain with a physical force** and its **anchor age** n. Each statement has a short necessity check: removing that domain's coefficient destroys strict stability in its sector at the anchor.

O.1 Gravity ↔ radial (scale) domain ↔ $n = 90$ (sphere at seat reset)

Anchor statement. Under the VCR-preserving Neumann condition, the spherically symmetric radial mode satisfies $j_1(\kappa R) = 0$ at the seat boundary (Appendix M.A). With the age calibration of Appendix M, the reset falls at $n = 90$.

Necessity. In (O.1) the $l = 0$ normal mode contributes only $\alpha_0\, |f_{00}|^2$. If $\alpha_0 = 0$, the Hessian has a zero direction in the scale mode, so the sphere at $n = 90$ is not a strict local minimizer. If $\alpha_0 > 0$, the scale mode is controlled, and by Appendix N the full quadratic form is positive definite modulo rigid motions. □

O.2 Electromagnetism ↔ surface domain ↔ $n = 55$ (gauge interface)

Anchor statement. $n = 55$ is a gauge-interface age (Appendix K), i.e., a surface-dominated configuration on the route between the volume- and surface-target ladders. In the thin-interface limit, the leading second variation for normal shape modes is

$$\delta^2(\text{area}) \;\propto\; \sum_{l\geq 2,m} (l-1)(l+2)|f_{lm}|^2,$$

so the surface term (coefficient α_s) is the dominant stabilizer at this interface.

Necessity (sectoral). If $\alpha_s = 0$, the leading-order surface Hessian vanishes and the anchor cannot be selected by surface energy; any stabilization would have to come from higher-order bending (α_b), contradicting the surface-dominated approximation that pins $n = 55$. With $\alpha_s > 0$, the low-l normal sector is strictly controlled, fixing the interface anchor. □

O.3 Strong interaction ↔ bending/UV domain ↔ $n = 5, 12$ (pentagon, icosahedral), optionally $n = 70$ (dodecahedral)

Anchor statement. The $n = 5$ pentagon ring and the $n = 12$ icosahedral vibration are high-curvature patterns; $n = 70$ (dodecahedral, near-spherical) is their larger-scale counterpart. Their stability relies on coercivity against high-degree normal modes, which in (O.1) arises from the bending term $\alpha_b\, l(l+1)(l-1)(l+2)$.

Necessity. If $\alpha_b = 0$, the quadratic form lacks UV coercivity: a sequence with energy concentrated in degrees $l \to \infty$ keeps $\sum (l-1)(l+2)|f_{lm}|^2$ bounded while escaping control in H^2, so no strict local minimum exists in the curvature-dominated sector (Appendix N, "Only if" for α_b). With $\alpha_b > 0$, high-l roughening is penalized and the curvature anchors are strictly stable. □

O.4 Weak interaction ↔ chiral/torsional domain ↔ $n \approx 29$ (vortex)

Anchor statement. The $n \approx 29$ configuration is parity-odd (vortex/spiral). Its tangent-swirl content lies in the toroidal basis $n \times \nabla_S Y_{lm}$, controlled only by the α_c term in (O.1).

Necessity. If $\alpha_c = 0$, all toroidal modes have zero quadratic cost, so any parity-odd perturbation leaves the energy unchanged to second order, and the vortex anchor is not a strict local minimizer. With $\alpha_c > 0$, the toroidal sector is strictly controlled ($\sum l(l+1)|\beta_{lm}|^2$), fixing the chiral anchor. □

Summary table (domains → forces → anchors)

Domain (coeff. in (O.1))	Controlled sector	Force label	Anchor age n	Anchor shape/mode
Radial scale (α_0)	Normal $l = 0$	Gravity	90	Sphere at Neumann node $j_1(\kappa R) = 0$
Surface (α_s)	Normal low-l	Electromagnetism	55	Gauge interface, surface-dominated
Bending/UV (α_b)	Normal high-l	Strong	5, 12 (and 70)	Pentagon/icosahedral (dodecahedral)
Chiral/torsional (α_c)	Toroidal parity-odd	Weak	≈ 29	Vortex/spiral

Notes. (1) $n = 34$ and $n = 89$ are additional gauge-interface ages (Appendix K); list under the EM domain as secondary anchors if desired. (2) When a **single** strong-anchor label is needed, use $n = 5$ (earliest UV anchor) or

$n = 12$ (icosahedral). (3) “$\approx$” on 29 reflects the basin picture (Appendix P): the integer anchor is exact in theory; observed clusters can jitter within its basin.