ORF 307: Lecture 2

Linear Programming: Chapter 2
The Simplex Method

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An Example.

$$
\begin{array}{lrcrcrcl}
\text{maximize} & -x_1 & + & 3x_2 & - & 3x_3 & & \\
\\
\text{subject to} & 3x_1 & - & x_2 & - & 2x_3 & \leq & 7 \\
 & -2x_1 & - & 4x_2 & + & 4x_3 & \leq & 3 \\
 & x_1 & & & - & 2x_3 & \leq & 4 \\
 & -2x_1 & + & 2x_2 & + & x_3 & \leq & 8 \\
 & 3x_1 & & & & & \leq & 5 \\
\\
 & & & & & x_1,\ x_2,\ x_3 & \geq & 0
\end{array}
$$

Rewrite with Slack Variables

$$
\begin{array}{lrcrcrcl}
\text{maximize} & -x_1 & + & 3x_2 & - & 3x_3 & & \\
\\
\text{subject to} & 3x_1 & - & x_2 & - & 2x_3 & \leq & 7 \\
 & -2x_1 & - & 4x_2 & + & 4x_3 & \leq & 3 \\
 & x_1 & & & - & 2x_3 & \leq & 4 \\
 & -2x_1 & + & 2x_2 & + & x_3 & \leq & 8 \\
 & 3x_1 & & & & & \leq & 5 \\
\\
 & & & x_1, & & x_2, \; x_3 & \geq & 0
\end{array}
$$

$$\Updownarrow$$

$$
\begin{array}{lrclcrcrcr}
\text{maximize} & \zeta & = & & & -x_1 & + & 3x_2 & - & 3x_3 \\
\\
\text{subject to} & w_1 & = & 7 & - & 3x_1 & + & x_2 & + & 2x_3 \\
 & w_2 & = & 3 & + & 2x_1 & + & 4x_2 & - & 4x_3 \\
 & w_3 & = & 4 & - & x_1 & & & + & 2x_3 \\
 & w_4 & = & 8 & + & 2x_1 & - & 2x_2 & - & x_3 \\
 & w_5 & = & 5 & - & 3x_1 & & & & \\
\\
\multicolumn{8}{r}{x_1, \; x_2, \; x_3, \; w_1, \; w_2, \; w_3, \; w_4, \; w_5 \geq} & & 0
\end{array}
$$

Rewrite with Slack Variables

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & \zeta & = & & & -x_1 & + & 3x_2 & - & 3x_3 \\
\\
\text{subject to} & w_1 & = & 7 & - & 3x_1 & + & x_2 & + & 2x_3 \\
& w_2 & = & 3 & + & 2x_1 & + & 4x_2 & - & 4x_3 \\
& w_3 & = & 4 & - & x_1 & & & + & 2x_3 \\
& w_4 & = & 8 & + & 2x_1 & - & 2x_2 & - & x_3 \\
& w_5 & = & 5 & - & 3x_1 & & & & \\
\\
& & & & & & & & &
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3,\ w_4,\ w_5 \ \geq\ 0$$

Notes:

- This layout is called a *dictionary*: the variables on the left are "defined" in terms of the variables on the right.
- We will use the Greek letter ζ for the *objective function*.
- Dependent variables, on the left, are called *basic variables*.
- Independent variables, on the right, are called *nonbasic variables*.
- Setting x_1, x_2, and x_3 to 0, we can read off the values for the other variables: $w_1 = 7$, $w_2 = 3$, etc. This specific "solution" is called a *basic solution* (aka *dictionary solution*). It's called a solution because it is one of many solutions to the system of linear equations. We are not implying that it is a solution to the optimization problem. We will call that the *optimal solution*.

Basic Solution is Feasible

We got lucky!

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad w_1 = 7, \quad w_2 = 3, \quad w_3 = 4, \quad w_4 = 8, \quad w_5 = 5$$

$$\begin{array}{lrcrcrcrcr}
\text{maximize} & \zeta & = & & & -x_1 & + & 3x_2 & - & 3x_3 \\
\text{subject to} & w_1 & = & 7 & - & 3x_1 & + & x_2 & + & 2x_3 \\
& w_2 & = & 3 & + & 2x_1 & + & 4x_2 & - & 4x_3 \\
& w_3 & = & 4 & - & x_1 & & & + & 2x_3 \\
& w_4 & = & 8 & + & 2x_1 & - & 2x_2 & - & x_3 \\
& w_5 & = & 5 & - & 3x_1 & & & & \\
\end{array}$$

$$x_1, \; x_2, \; x_3, \; w_1, \; w_2, \; w_3, \; w_4, \; w_5 \geq 0.$$

Notes:

- All the variables in the current basic solution are nonnegative.
- Such a solution is called *feasible*.
- The initial basic solution need not be feasible—we were just lucky above.

Simplex Method—First Iteration

ζ	=	0	+	-1	x_1	+	3	x_2	+	-3	x_3
w_1	=	7	−	3	x_1	−	-1	x_2	−	-2	x_3
w_2	=	3	−	-2	x_1	−	-4	x_2	−	4	x_3
w_3	=	4	−	1	x_1	−	0	x_2	−	-2	x_3
w_4	=	8	−	-2	x_1	−	2	x_2	−	1	x_3
w_5	=	5	−	3	x_1	−	0	x_2	−	0	x_3

- If x_2 increases, obj goes *up*.
- How much can x_2 increase? Until w_4 decreases to zero.
- Do it. End result: $x_2 > 0$ whereas $w_4 = 0$.
- That is, x_2 must become *basic* and w_4 must become *nonbasic*.
- Algebraically rearrange equations to, in the words of Jean-Luc Picard, "Make it so."
- This is a *pivot*.

A Pivot: $x_2 \leftrightarrow w_4$

$$
\begin{array}{ccccccccccc}
\zeta & = & 0 & + & -1 & x_1 & + & 3 & x_2 & + & -3 & x_3 \\
w_1 & = & 7 & - & 3 & x_1 & - & -1 & x_2 & - & -2 & x_3 \\
w_2 & = & 3 & - & -2 & x_1 & - & -4 & x_2 & - & 4 & x_3 \\
w_3 & = & 4 & - & 1 & x_1 & - & 0 & x_2 & - & -2 & x_3 \\
w_4 & = & 8 & - & -2 & x_1 & - & 2 & x_2 & - & 1 & x_3 \\
w_5 & = & 5 & - & 3 & x_1 & - & 0 & x_2 & - & 0 & x_3
\end{array}
$$

becomes

$$
\begin{array}{ccccccccccc}
\zeta & = & 12 & + & 2 & x_1 & + & -3/2 & w_4 & + & -9/2 & x_3 \\
w_1 & = & 11 & - & 2 & x_1 & - & 1/2 & w_4 & - & -3/2 & x_3 \\
w_2 & = & 19 & - & -6 & x_1 & - & 2 & w_4 & - & 6 & x_3 \\
w_3 & = & 4 & - & 1 & x_1 & - & 0 & w_4 & - & -2 & x_3 \\
x_2 & = & 4 & - & -1 & x_1 & - & 1/2 & w_4 & - & 1/2 & x_3 \\
w_5 & = & 5 & - & 3 & x_1 & - & 0 & w_4 & - & 0 & x_3
\end{array}
$$

Simplex Method—Second Pivot

Here's the dictionary after the first pivot:

ζ	=	12	+	2	x_1	+	-3/2	w_4	+	-9/2	x_3
w_1	=	11	−	2	x_1	−	1/2	w_4	−	-3/2	x_3
w_2	=	19	−	-6	x_1	−	2	w_4	−	6	x_3
w_3	=	4	−	1	x_1	−	0	w_4	−	-2	x_3
x_2	=	4	−	-1	x_1	−	1/2	w_4	−	1/2	x_3
w_5	=	5	−	3	x_1	−	0	w_4	−	0	x_3

- Now, let x_1 increase.
- Of the basic variables, w_5 hits zero first.
- So, x_1 *enters* and w_5 *leaves* the basis.
- New dictionary is...

Simplex Method—Final Dictionary

$$
\begin{array}{rcrcrcrcrcrcr}
\zeta & = & 46/3 & + & -2/3 & w_5 & + & -3/2 & w_4 & + & -9/2 & x_3 \\
w_1 & = & 23/3 & - & -2/3 & w_5 & - & 1/2 & w_4 & - & -3/2 & x_3 \\
w_2 & = & 29 & - & 2 & w_5 & - & 2 & w_4 & - & 6 & x_3 \\
w_3 & = & 7/3 & - & -1/3 & w_5 & - & 0 & w_4 & - & -2 & x_3 \\
x_2 & = & 17/3 & - & 1/3 & w_5 & - & 1/2 & w_4 & - & 1/2 & x_3 \\
x_1 & = & 5/3 & - & 1/3 & w_5 & - & 0 & w_4 & - & 0 & x_3
\end{array}
$$

- It's optimal (no pink)!
- Click here to practice the simplex method.

Agenda

- Discuss *unboundedness;* (today)
- Discuss initialization/*infeasibility*; i.e., what if initial dictionary is not feasible. (today)
- Discuss *degeneracy.* (next lecture)

Unboundedness

Consider the following dictionary:

ζ	=	0	+	2	x_1	+	-1	x_2	+	1	x_3
w_1	=	4	−	-5	x_1	−	3	x_2	−	-1	x_3
w_2	=	10	−	-1	x_1	−	-5	x_2	−	2	x_3
w_3	=	7	−	0	x_1	−	-4	x_2	−	3	x_3
w_4	=	6	−	-2	x_1	−	-2	x_2	−	4	x_3
w_5	=	6	−	-3	x_1	−	0	x_2	−	-3	x_3

- Could increase either x_1 or x_3 to increase obj.
- Consider increasing x_1.
- Which basic variable decreases to zero first?
- Answer: none of them, x_1 can grow without bound, and obj along with it.
- This is how we detect *unboundedness* with the simplex method.

Unbounded or Not?

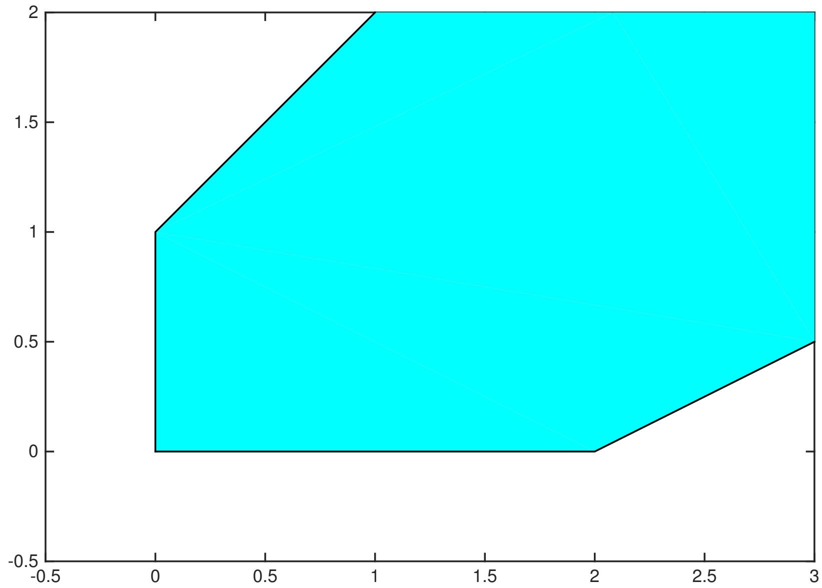
$$
\begin{array}{lrcrcl}
\text{maximize} & x_1 & + & 2x_2 & & \\
\text{subject to} & -x_1 & + & x_2 & \leq & 1 \\
 & x_1 & - & 2x_2 & \leq & 2 \\
 & & & x_1,\ x_2 & \geq & 0.
\end{array}
$$

Questions:

1. Is initial basic solution feasible or not?
2. Does the initial dictionary show the problem to be unbounded or not?
3. Is the problem unbounded or not?
4. How can we tell?

Unbounded or Not?

$$\begin{array}{lrcrcl} \text{maximize} & x_1 & + & 2x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & 1 \\ & x_1 & - & 2x_2 & \leq & 2 \\ & & & x_1,\ x_2 & \geq & 0. \end{array}$$



Unbounded or Not?

$$
\begin{array}{lrcrcl}
\text{maximize} & x_1 & + & 2x_2 & & \\
\text{subject to} & -x_1 & + & x_2 & \leq & 1 \\
 & x_1 & - & 2x_2 & \leq & 2 \\
 & & & x_1,\ x_2 & \geq & 0.
\end{array}
$$

ζ	=	0	+	1	x_1	+	2	x_2
w_1	=	1	-	-1	x_1	-	1	x_2
w_2	=	2	-	1	x_1	-	-2	x_2

ζ	=	2	+	3	x_1	+	-2	w_1
x_2	=	1	-	-1	x_1	-	1	w_1
w_2	=	4	-	-1	x_1	-	2	w_1

Initialization

Consider the following problem:

$$
\begin{array}{lrcrcr}
\text{maximize} & -3x_1 & + & 4x_2 & & \\
\text{subject to} & -4x_1 & - & 2x_2 & \leq & -8 \\
 & -2x_1 & & & \leq & -2 \\
 & 3x_1 & + & 2x_2 & \leq & 10 \\
 & -x_1 & + & 3x_2 & \leq & 1 \\
 & & & -3x_2 & \leq & -2 \\
 & & & x_1,\ x_2 & \geq & 0
\end{array}
$$

Phase-I Problem

- Modify problem by subtracting a new variable, x_0, from each constraint and
- replacing objective function with $-x_0$

Phase-I Problem

$$
\begin{array}{ll}
\text{maximize} & -x_0 \\
\text{subject to} & \begin{array}{rcrcrcr}
-x_0 & - & 4x_1 & - & 2x_2 & \leq & -8 \\
-x_0 & - & 2x_1 & & & \leq & -2 \\
-x_0 & + & 3x_1 & + & 2x_2 & \leq & 10 \\
-x_0 & - & x_1 & + & 3x_2 & \leq & 1 \\
-x_0 & & & & -3x_2 & \leq & -2 \\
 & & & & x_0,\ x_1,\ x_2 & \geq & 0
\end{array}
\end{array}
$$

- Current basic solution is infeasible. But...
- Problem is clearly feasible: pick x_0 large, $x_1 = 0$ and $x_2 = 0$.
- If optimal solution has obj $= 0$, then original problem is feasible.
- Final phase-I dictionary can be used as initial *phase-II* dictionary (ignoring x_0 thereafter).
- If optimal solution has obj < 0, then original problem is infeasible.

Initialization—First Pivot

Applet depiction shows both the Phase-I and the Phase-II objectives:

ζ_2	=	0	+	0	x_0	+	-3	x_1	+	4	x_2
ζ_1	=	0	+	-1	x_0	+	0	x_1	+	0	x_2
w_1	=	-8	−	-1	x_0	−	-4	x_1	−	-2	x_2
w_2	=	-2	−	-1	x_0	−	-2	x_1	−	0	x_2
w_3	=	10	−	-1	x_0	−	3	x_1	−	2	x_2
w_4	=	1	−	-1	x_0	−	-1	x_1	−	3	x_2
w_5	=	-2	−	-1	x_0	−	0	x_1	−	-3	x_2

- Dictionary is infeasible even for Phase-I.
- One pivot needed to get feasible.
- Entering variable is x_0.
- Leaving variable is one whose current value is most negative, i.e. w_1.
- After first pivot...

Initialization—Second Pivot

Going into second pivot:

ζ_2	=	0	+	0		w_1	+	-3	x_1	+	4	x_2
ζ_1	=	-8	+	-1		w_1	+	4	x_1	+	2	x_2
x_0	=	8	−	-1		w_1	−	4	x_1	−	2	x_2
w_2	=	6	−	-1		w_1	−	2	x_1	−	2	x_2
w_3	=	18	−	-1		w_1	−	7	x_1	−	4	x_2
w_4	=	9	−	-1		w_1	−	3	x_1	−	5	x_2
w_5	=	6	−	-1		w_1	−	4	x_1	−	-1	x_2

- Feasible!
- Focus on the yellow highlights.
- Let x_1 enter.
- Then w_5 must leave.
- After second pivot...

Initialization—Third Pivot

Going into third pivot:

ζ_2 =	-9/2	+	-3/4	w_1	+	3/4	w_5	+	13/4	x_2
ζ_1 =	-2	+	0	w_1	+	-1	w_5	+	3	x_2
x_0 =	2	−	0	w_1	−	-1	w_5	−	3	x_2
w_2 =	3	−	-1/2	w_1	−	-1/2	w_5	−	5/2	x_2
w_3 =	15/2	−	3/4	w_1	−	-7/4	w_5	−	23/4	x_2
w_4 =	9/2	−	-1/4	w_1	−	-3/4	w_5	−	23/4	x_2
x_1 =	3/2	−	-1/4	w_1	−	1/4	w_5	−	-1/4	x_2

- x_2 must enter.
- x_0 must leave.
- After third pivot...

End of Phase-I

Current dictionary:

ζ_2 =	-7/3	+	-3/4	w_1	+	11/6	w_5	+	0	x_0
ζ_1 =	0	+	0	w_1	+	0	w_5	+	0	x_0
x_2 =	2/3	−	0	w_1	−	-1/3	w_5	−	0	x_0
w_2 =	4/3	−	-1/2	w_1	−	1/3	w_5	−	0	x_0
w_3 =	11/3	−	3/4	w_1	−	1/6	w_5	−	0	x_0
w_4 =	2/3	−	-1/4	w_1	−	7/6	w_5	−	0	x_0
x_1 =	5/3	−	-1/4	w_1	−	1/6	w_5	−	0	x_0

- Optimal for Phase-I (no yellow highlights).
- obj $= 0$, therefore original problem is feasible.

Phase-II

Current dictionary:

ζ_2	=	-7/3	+	-3/4	w_1	+	11/6	w_5	+	0	x_0
ζ_1	=	0	+	0	w_1	+	0	w_5	+	0	x_0
x_2	=	2/3	−	0	w_1	−	-1/3	w_5	−	0	x_0
w_2	=	4/3	−	-1/2	w_1	−	1/3	w_5	−	0	x_0
w_3	=	11/3	−	3/4	w_1	−	1/6	w_5	−	0	x_0
w_4	=	2/3	−	-1/4	w_1	−	7/6	w_5	−	0	x_0
x_1	=	5/3	−	-1/4	w_1	−	1/6	w_5	−	0	x_0

For Phase-II:

- Ignore column with x_0 in Phase-II.
- Ignore Phase-I objective row.

w_5 must enter. w_4 must leave...

Optimal Solution

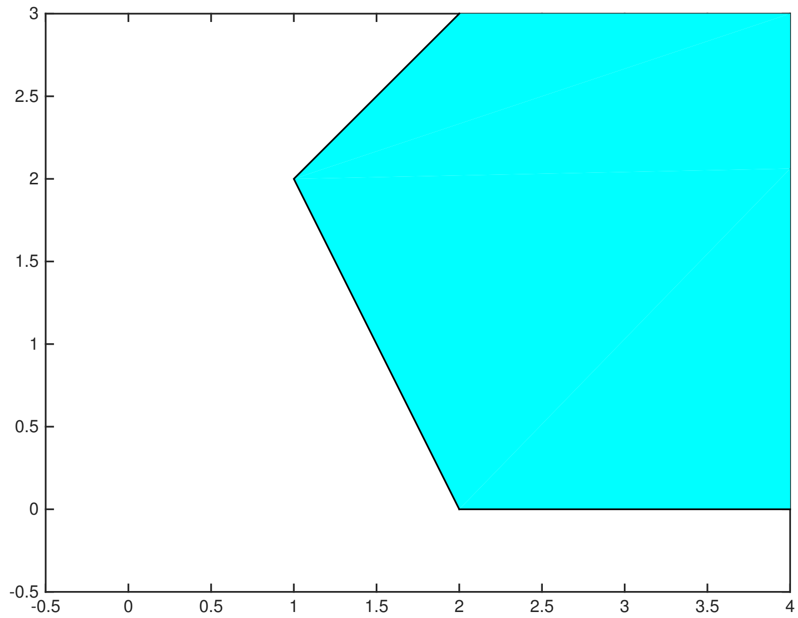
$$
\begin{array}{rcrcrcrcrcrcrl}
\zeta_2 & = & -9/7 & + & -5/14 & w_1 & + & -11/7 & w_4 & + & 0 & x_0 \\
\zeta_1 & = & 0 & + & 0 & w_1 & + & 0 & w_4 & + & 0 & x_0 \\
 & & & & & & & & & & & \\
x_2 & = & 6/7 & - & -1/14 & w_1 & - & 2/7 & w_4 & - & 0 & x_0 \\
w_2 & = & 8/7 & - & -3/7 & w_1 & - & -2/7 & w_4 & - & 0 & x_0 \\
w_3 & = & 25/7 & - & 11/14 & w_1 & - & -1/7 & w_4 & - & 0 & x_0 \\
w_5 & = & 4/7 & - & -3/14 & w_1 & - & 6/7 & w_4 & - & 0 & x_0 \\
x_1 & = & 11/7 & - & -3/14 & w_1 & - & -1/7 & w_4 & - & 0 & x_0
\end{array}
$$

- Optimal!
- Click here to practice the simplex method on problems that may have infeasible first dictionaries.

Solve This Problem

$$
\begin{array}{llcrcrcr}
\text{maximize} & -2x_1 & + & x_2 & & \\
\text{subject to} & -x_1 & + & x_2 & \leq & 1 \\
 & -2x_1 & - & x_2 & \leq & -4 \\
 & & & x_1,\ x_2 & \geq & 0.
\end{array}
$$

Use the two-phase simplex method to solve this problem.


3
2.5
2
1.5
1
0.5
0
-0.5
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4