Arcade jump trajectory

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Introduction

Without air resistance in an Euclidean referential, a projectile follows a parabolic trajectory which we express as the function f which describe the altitude of the projectile through time.

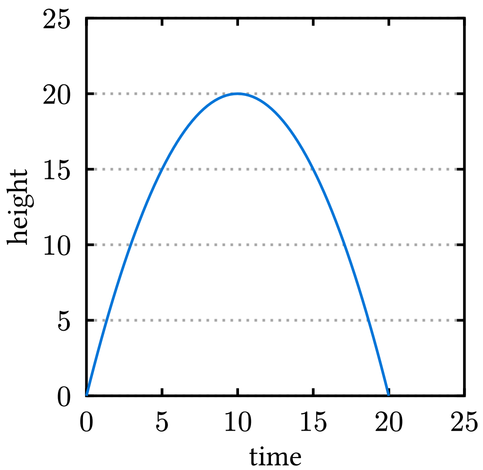

Figure 1: Ballistic trajectory

We define

- g the gravitational force
- v_0 the initial vertical velocity
- p_0 the initial altitude

$$\begin{aligned} f(t) &= \frac{1}{2}gt^2 + v_0 t + p_0 && \text{Displacement} \\ f'(t) &= gt + v_0 && \text{Velocity} \\ f''(t) &= g && \text{Acceleration} \end{aligned}$$

In the following expressions, we consider that we always start at the ground level such as $p_0 = 0$.

Parametrization

Rather than approximately fine tuning g and v_0 to generate a trajectory, we want to be able to describe it as a function of height of the peak and time to reach the peak. This will allow us to have a precise control over the trajectory of our projectile.

Since a ballistic trajectory is parabolic, when the projectile reaches its peak, its velocity is null. We have the following system where h and t_h are known.

$$\begin{aligned} f(t_h) &= \frac{1}{2}gt_h^2 + v_0 t_h &&= h \\ f'(t_h) &= gt_h + v_0 &&= 0 \end{aligned}$$

with

- h the height of the peak
- t_h the time it takes to reach that peak

We can find the value of v_0 by substitution in $f'(t_h)$.

$$\begin{aligned} f'(t_h) &= 0 \\ f'(t_h) &= gt_h + v_0 \\ -gt_h &= v_0 \end{aligned}$$

We can find an expression of g in term of h and t_h by substitution in $f(t_h)$.

$$\begin{aligned} h &= \frac{1}{2}gt_h^2 + v_0 t_h \\ &= \frac{1}{2}gt_h^2 + (-gt_h)t_h \\ &= \frac{1}{2}gt_h^2 - gt_h^2 \\ &= -\frac{1}{2}gt_h^2 \\ -\frac{2h}{t_h^2} &= g \end{aligned}$$

We can reinject the expression found for g in the equation defining v_0.

$$\begin{aligned} v_0 &= -gt_h \\ &= -\left(-\frac{2h}{t_h^2}\right)t_h \\ &= \frac{2h}{t_h} \end{aligned}$$

We can express the initial vertical velocity v_0 and the gravity g of our system as expressions of the parameters h the altitude of the peak and t_h the time to reach that peak.

$$v_0 = \frac{2h}{t_h}$$

$$g = -\frac{2h}{t_h^2}$$

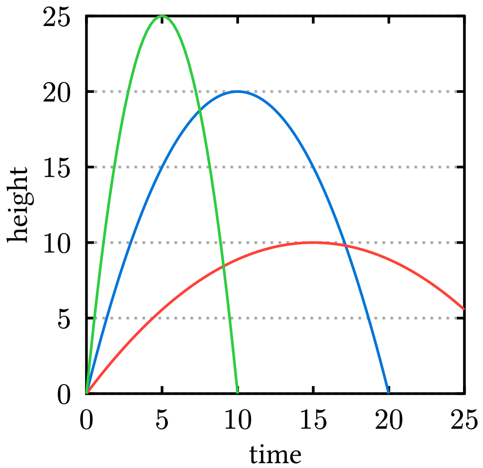

Figure 2: Parametrization of trajectories based on height and duration

The impulse 4 The gravity −0.4

The impulse 1.3333333333333333 The gravity −0.08888888888888889

The impulse 10 The gravity −2

Horizontal motion

Using t_h may not be convenient to describe a trajectory. If the projectile is also moving horizontally, we can introduce additional parameters.

We introduce

- v_x as the horizontal velocity
- d as the range of the jump
- $r \in [0, 1]$ as the ratio between the ascending and descending phases of the jump

The values of v_0 and g can be reexpressed trivially as expressions of h, v_x, d and r.

$$t_h = \frac{dr}{v_x}$$

$$v_0 = \frac{2hv_x}{dr}$$

$$g = -\frac{2hv_x^2}{d^2r^2}$$

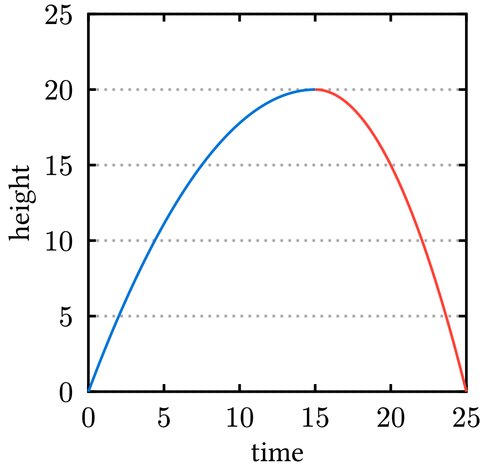

Figure 3: Different ascending and descending phases

Variation

In video games, common tropes are being able to vary the height of a jump after initiating it. Those behavior don't occur in the real world but provide better controls. Since we have four parameters, constraining two of them gives us the result for the other two.

We start by expressing g in terms of v_0 and t_h.

$$\begin{aligned} g &= -\frac{2h}{t_h^2} \\ &= -\frac{2h}{t_h} \times \frac{1}{t_h} \\ &= -\frac{v_0}{t_h} \end{aligned}$$

We can trivially deduce $v_0 = -gt_h$.

Which gives us an expression of h in terms of g and t_h.

$$\frac{2h}{t_h} = -gt_h$$

$$h = -\frac{1}{2}gt_h^2$$

Then we can find an expression of h in terms of v_0 and t_h.

$$\begin{aligned} h &= -\frac{1}{2}\left(-\frac{v_0}{t_h}\right)t_h^2 \\ &= \frac{1}{2}v_0t_h \end{aligned}$$

From $v_0 = -gt_h$, we have $t_h = -\frac{v_0}{g}$ which we use to find an expression of g in terms of h and v_0.

$$\begin{aligned}
g &= -\frac{2h}{t_h^2} \\
&= -\frac{2h}{\left(-\frac{v_0}{g}\right)^2} \\
&= -\frac{2h}{\frac{v_0^2}{g^2}} \\
&= -\frac{2hg^2}{v_0^2} \\
1 &= -\frac{2hg}{v_0^2} \\
-\frac{v_0^2}{2h} &= g
\end{aligned}$$

We can deduce an expression of v_0 in terms of g and h. But in that case we have two complex numbers as solutions.

$$\begin{aligned}
-\frac{v_0^2}{2h} &= g \\
v_0^2 &= -2hg \\
v_0 &= \pm i\sqrt{2hg}
\end{aligned}$$

We also have h in terms of v_0 and g.

$$\begin{aligned}
g &= -\frac{v_0^2}{2h} \\
h &= -\frac{v_0^2}{2g}
\end{aligned}$$

Finally, we can find expressions of t_h in terms of v_0 and g.

$$\begin{aligned}
-gt_h &= v_0 \\
t_h &= -\frac{v_0}{g}
\end{aligned}$$

Then in terms of h and v_0.

$$\begin{aligned}
v_0 &= \frac{2h}{t_h} \\
t_h &= \frac{2h}{v_0}
\end{aligned}$$

And in terms of h and g. Again we get complex numbers as solutions.

$$\begin{aligned}
-\frac{1}{2}gt_h^2 &= h \\
t_h^2 &= -\frac{2h}{g} \\
t_h &= \pm i\sqrt{\frac{2h}{g}}
\end{aligned}$$

Conclusion

To summarize, we get the following expressions for h, t_h, v_0 and g.

$$\begin{aligned}
h &= \frac{1}{2}v_0 t_h \\
&= -\frac{1}{2}gt_h^2 \\
&= -\frac{v_0^2}{2g}
\end{aligned}$$

$$\begin{aligned}
t_h &= \frac{2h}{v_0} \\
&= \pm i\sqrt{\frac{2h}{g}} \\
&= -\frac{v_0}{g}
\end{aligned}$$

$$\begin{aligned}
v_0 &= \frac{2h}{t_h} \\
&= \pm i\sqrt{2hg} \\
&= -gt_h
\end{aligned}$$

$$\begin{aligned}
g &= -\frac{2h}{t_h^2} \\
&= -\frac{v_0^2}{2h} \\
&= -\frac{v_0}{t_h}
\end{aligned}$$

From those formulas, we can implement a variable height jump based on how long we press on the “jump” button. In that case, the initial vertical impulse is the same but the gravity will change when the player release the button.

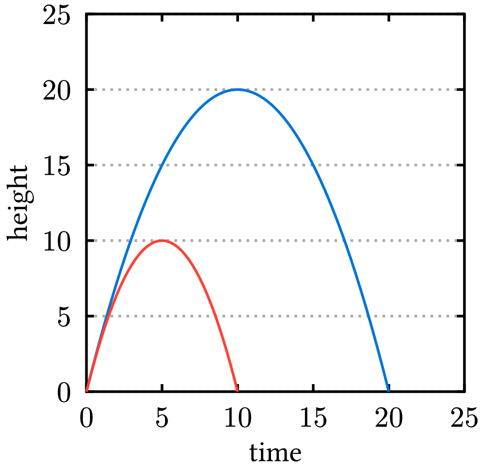

Figure 4: Different heights with a vertical velocity constraint

We can also implement a double-jump which use the same gravity as the main jump but should be able to reach a smaller height from the point where e started the double-jump.

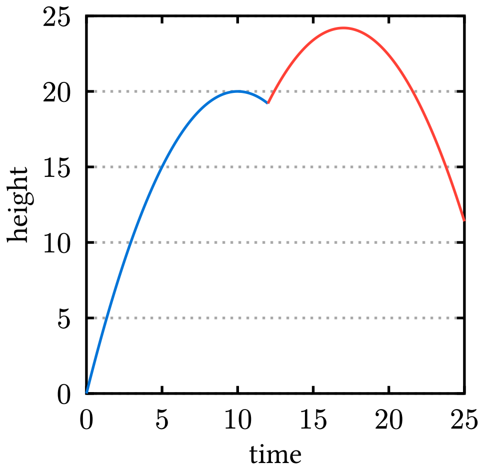

Figure 5: Double-jump with a smaller second jump

And we can implement a wall-jump but instead of picking a height, we want to select the distance we can reach.

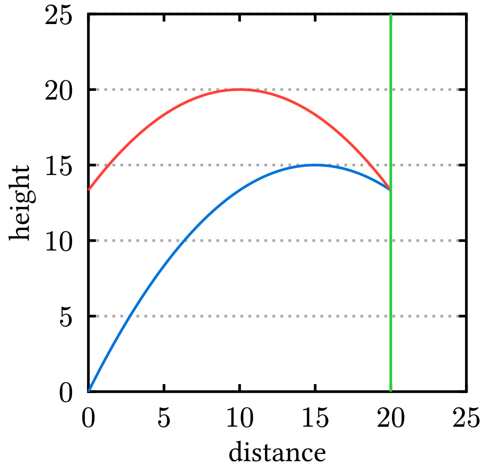

Figure 6: Wall-jump with a predefined reach

References

- GDC Building a Better Jump by Kyle Pittman