TDDC17

First-Order Logic
First-Order Resolution
Nonmonotonic Reasoning

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Limited Expressivity using Propositional Logic

Physics of the Wumpus World:
Modeling is difficult with Propositional Logic

Schemas:

$(B_{x,y} \Leftrightarrow (P_{x,y+1} \vee P_{x,y-1} \vee P_{x+1,y} \vee P_{x-1,y}))$ Def. of breeze in pos [x,y]

$(S_{x,y} \Leftrightarrow (W_{x,y+1} \vee W_{x,y-1} \vee W_{x+1,y} \vee W_{x-1,y}))$ Def. of stench in pos [x,y]

$(W_{1,1} \vee W_{1,2} \vee \ldots \vee W_{4,4}))$ There is at least one wumpus!

..., etc. There is only one wumpus!

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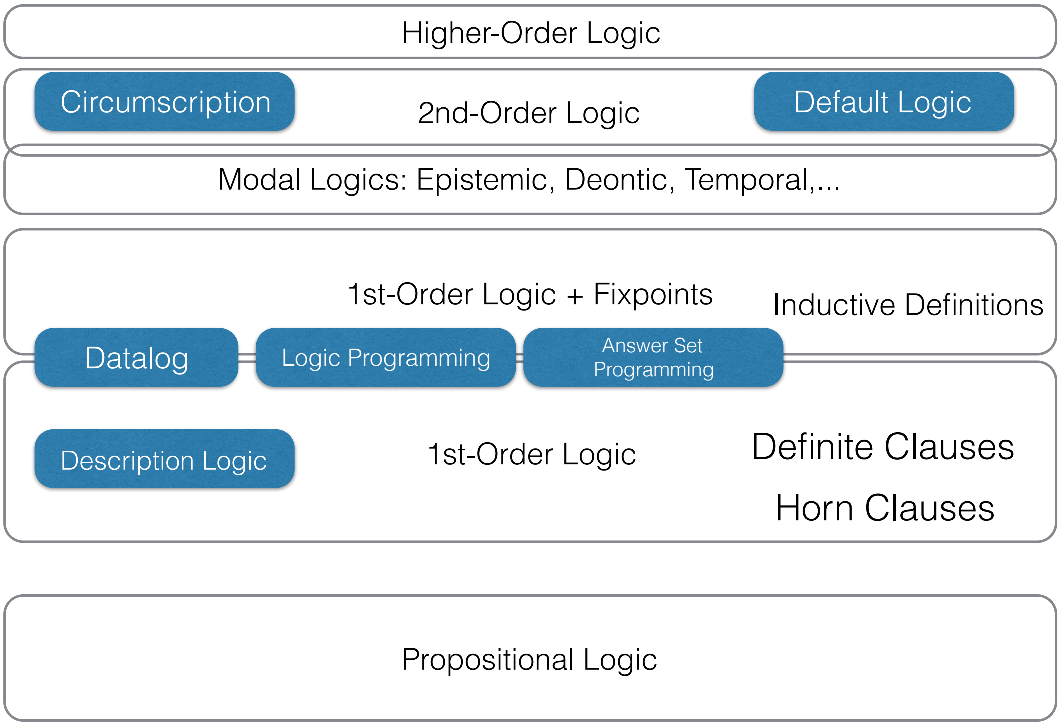
Spectrum of Logics and Languages



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Ontological and Epistemological Commitments

Language	Ontological Commitment (What exists in the world)	Epistemological Commitment (What an agent believes about facts)
Propositional Logic	facts	true/false/unknown
First-order Logic	facts, objects, relations	true/false/unknown
Temporal Logic	facts, objects, relations, times	true/false/unknown
Probability Theory	facts	degree of belief in [0,1]
Fuzzy Logic	facts with degree of truth in [0,1]	known interval value

Ontological Commitment - *what is assumed about the nature of reality*

Epistemological Commitment - *what is assumed about knowledge with respect to facts*

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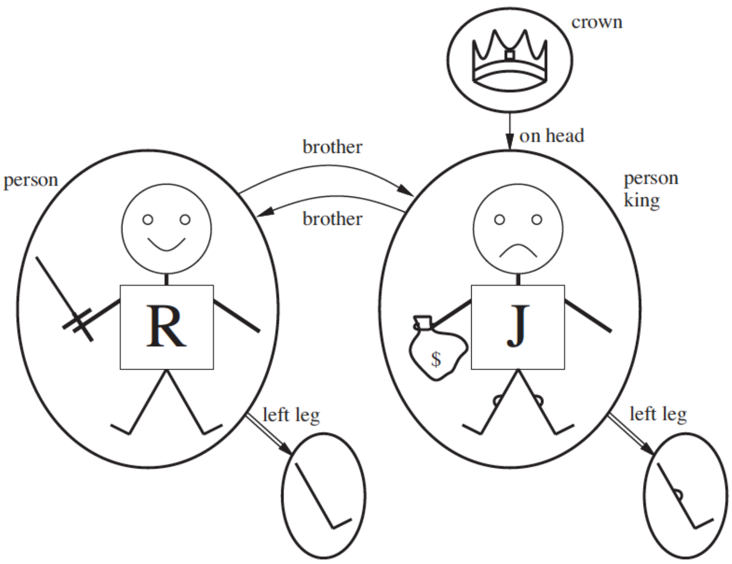
Ontological Commitments of First-order Logic



Facts
Objects
Relations

Model with:

- *5 objects*
- *2 binary relations*
  - brother
  - on-head
- 3 unary relations
  - person
  - crown
  - king
- *1 unary function*
  - left-leg()

The Syntax of First-Order Logic

Different applications have different first-order languages, but certain components are common to all languages

Propositional Connectives: $\wedge, \vee, \rightarrow, \neg, \leftrightarrow$

Propositional Constants: $\bot, \top$

Quantifiers:

$\forall$ (for all, the universal quantifier)

$\exists$ (there exists, the existential quantifier)

Punctuation: ")", "(", ",",...

Variables: $v_1, v_2, \ldots$ (which we write informally as $x, y, z, \ldots$)

First-Order Language

Definition 1. A *first-order language*, $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$, is determined by specifying:

(1) A finite or countable set $\mathbf{R}$ of *relation symbols*, or *predicate symbols*, each of which has a positive integer associated with it denoting its arity.
(2) A finite or countable set $\mathbf{F}$ of *function symbols*, each of which has a positive integer associated with it denoting its arity.
(3) A finite or countable set $\mathbf{C}$ of *constant symbols*.

Example 1. Let $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ be the first-order language where,

(1) $\mathbf{R} = \{\langle Brother, 2\rangle, \langle OnHead, 2\rangle, \langle Person, 1\rangle, \langle Crown, 1\rangle, \langle King, 1\rangle\}$.
(2) $\mathbf{F} = \{\langle leftleg, 1\rangle\}$.
(3) $\mathbf{C} = \{John, Richard, Crown\}$.

First-Order Language

Terms name individuals

Definition 2. The family of terms of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ is the smallest set meeting the conditions:

(1) Any variable is a term of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$.
(2) Any constant symbol (member of $\mathbf{C}$) is a term of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$.
(3) If f is an n-place function symbol (member of $\mathbf{F}$) and $t_1, \ldots, t_n$ are terms of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$, then $f(t_1, \ldots, t_n)$ is a term of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$.

Examples: $Richard, John, \mathit{leftleg}(x), \mathit{leftleg}(John), Crown$

First-Order Language

Definition 3. An *atomic formula* of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ is any string of the form $R(t_1, \ldots, t_n)$ where $R \in \mathbf{R}$ is an n-place relation symbol and $t_1, \ldots, t_n$ are terms of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$; also $\bot$ and $\top$ are taken to be atomic formulas of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$.

Examples:

$$Brother(Richard, John), King(John), OnHead(Crown, x)$$

Definition 4. The family of formulas of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ is the smallest set meeting the following conditions:

(1) Any atomic formula of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ is a formula of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$.
(2) If A is a formula of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$, so is $\neg A$.
(3) For a binary connective $\circ$, if A and B are formulas of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$, so is $(A \circ B)$.
(4) If A is a formula of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ and x is a variable, then $(\forall x)A$ and $(\exists x)A$ are formulas of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$.

Examples:

$$(\forall x)(\forall y)Brother(x, y) \rightarrow Brother(y, x)$$

$$Brother(Richard, John) \land King(John)$$

$$(\exists x)King(x) \land OnHead(Crown, x)$$

First-order Semantics

In order to give meaning to a formula in a first-order language, we must:

- Specify a **Domain**
  - what individuals are involved for the quantifiers to quantify over
- Specify an **Interpretation**
  - how we are interpreting the constant, function and relation symbols with respect to the domain
- Specify an **Assignment**
  - how we interpret free variables in formulas

The two items, *domain* and *interpretation*, specify a **model**

Models & Assignments

Definition 5. A *model* for the first-order language $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ is a pair $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ where:

- **:** $\mathbf{D}$ is a non-empty set, called the *domain* of $\mathbf{M}$.
- **:** $\mathbf{I}$ is a mapping, called an *interpretation* that associates:
  - **:** To every constant symbol $c \in \mathbf{C}$ some member $c^{\mathbf{I}} \in \mathbf{D}$.
  - **:** To every n-place function symbol $f \in \mathbf{F}$, some n-ary function $f^{\mathbf{I}} : \mathbf{D}^n \to \mathbf{D}$.
  - **:** To every n-ary relation symbol $P \in \mathbf{R}$, some n-ary relation $P^{\mathbf{I}} \subseteq \mathbf{D}^n$.

Model Example

Example 1. Let $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ be the first-order language where,

(1) $\mathbf{R} = \{\langle Brother, 2\rangle, \langle OnHead, 2\rangle, \langle Person, 1\rangle, \langle Crown, 1\rangle, \langle King, 1\rangle\}$.
(2) $\mathbf{F} = \{\langle leftleg, 1\rangle\}$.
(3) $\mathbf{C} = \{John, Richard, Crown\}$.

Example 2. A *model* for the first-order language $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ is a pair $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ where:

- **:** $\mathbf{D} = \{\mathsf{john}, \mathsf{richard}, \mathsf{crown1}, \mathsf{leftleg1}, \mathsf{leftleg2}\}$.
- **:** $\mathbf{I}(John) = \mathsf{john}$, $\mathbf{I}(Richard) = \mathsf{richard}$, $\mathbf{I}(Crown) = \mathsf{crown}$.
- **:** $\mathbf{I}(leftleg) = \mathsf{leftleg1}$, where
  - **:** $\mathsf{leftleg1(john)} \mapsto \mathsf{leftleg1}$, $\mathsf{leftleg1(richard)} \mapsto \mathsf{leftleg2}, \ldots$
- **:** $\mathbf{I}(Brother) = \mathsf{Brother1}$, where
  - **:** $\mathsf{Brother1} = \{\langle \mathsf{john}, \mathsf{richard}\rangle, \langle \mathsf{richard}, \mathsf{john}\rangle\}$.
- **:** $\mathbf{I}(OnHead) = \mathsf{OnHead1}$, where
  - **:** $\mathsf{OnHead1} = \{\langle \mathsf{crown}, \mathsf{john}\rangle\}$.
- **:** $\mathbf{I}(Person) = \mathsf{Person1}$, where
  - **:** $\mathsf{Person1} = \{\langle \mathsf{john}\rangle, \langle \mathsf{richard}\rangle\}$.
- **:** $\mathbf{I}(Crown) = \mathsf{Crown1}$, where
  - **:** $\mathsf{Crown1} = \{\langle \mathsf{crown}\rangle\}$.
- **:** $\mathbf{I}(King) = \mathsf{King1}$, where
  - **:** $\mathsf{King1} = \{\langle \mathsf{richard}\rangle\}$.

Assignments

Definition 6. An *assignment* in a model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ is a mapping $\mathbf{A}$ from the set of variables to the set $\mathbf{D}$. We denote the image of a variable v under an assignment $\mathbf{A}$ by $v^{\mathbf{A}}$.

Definition 7. Let x be a variable. The assignment $\mathbf{B}$ in the model $\mathbf{M}$ is an *x-variant* of the assignment $\mathbf{A}$, provided $\mathbf{A}$ and $\mathbf{B}$ assign the same values to every variable except possibly x.

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Interpreting Terms

Given an interpretation and an assignment, we can calculate values for arbitrary terms

Definition 8. Let $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ be a model of the language $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$, and let $\mathbf{A}$ be an assignment in this model. To each term t of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$, we associate a value $t^{\mathbf{I},\mathbf{A}}$ in $\mathbf{D}$ as follows:

(1) For a constant symbol c, $c^{\mathbf{I},\mathbf{A}} = c^{\mathbf{I}}$.
(2) For a variable symbol v, $v^{\mathbf{I},\mathbf{A}} = v^{\mathbf{A}}$.
(3) For a function symbol f, $[f(t_1, \ldots, t_n)]^{\mathbf{I},\mathbf{A}} = f^{\mathbf{I}}(t_1^{\mathbf{I},\mathbf{A}}, \ldots, t_n^{\mathbf{I},\mathbf{A}})$.

If a term is closed (no variables) then its value does not depend on an assignment for $\mathbf{A}$

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Interpreting Formulas

Definition 9. Let $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ be a model of the language $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$, and let $\mathbf{A}$ be an assignment in this model. To each formula Φ of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$, we associate a truth value $\Phi^{\mathbf{I},\mathbf{A}}$ ($\mathbf{t}$ or $\mathbf{f}$) as follows:

(1) For the atomic cases,
  (a) $[P(t_1, \ldots, t_n)]^{\mathbf{I},\mathbf{A}} = \mathbf{t} \iff \langle t_1^{\mathbf{I},\mathbf{A}}, \ldots, t_n^{\mathbf{I},\mathbf{A}} \rangle \in P^{\mathbf{I}}$.
  (b) $\top^{\mathbf{I},\mathbf{A}} = \mathbf{t}$.
  (c) $\bot^{\mathbf{I},\mathbf{A}} = \mathbf{f}$.
(2) $[\neg X]^{\mathbf{I},\mathbf{A}} = \neg[X^{\mathbf{I},\mathbf{A}}]$.
(3) $[X \circ Y]^{\mathbf{I},\mathbf{A}} = X^{\mathbf{I},\mathbf{A}} \circ Y^{\mathbf{I},\mathbf{A}}$.
(4) $[(\forall x)\Phi]^{\mathbf{I},\mathbf{A}} = \mathbf{t} \iff \Phi^{\mathbf{I},\mathbf{B}} = \mathbf{t}$ for every assignment $\mathbf{B}$ in $\mathbf{M}$ that is an x-variant of $\mathbf{A}$.
(5) $[(\exists x)\Phi]^{\mathbf{I},\mathbf{A}} = \mathbf{t} \iff \Phi^{\mathbf{I},\mathbf{B}} = \mathbf{t}$ for some assignment $\mathbf{B}$ in $\mathbf{M}$ that is an x-variant of $\mathbf{A}$.

If a formula is closed (no variables) then its value does not depend on an assignment for $\mathbf{A}$

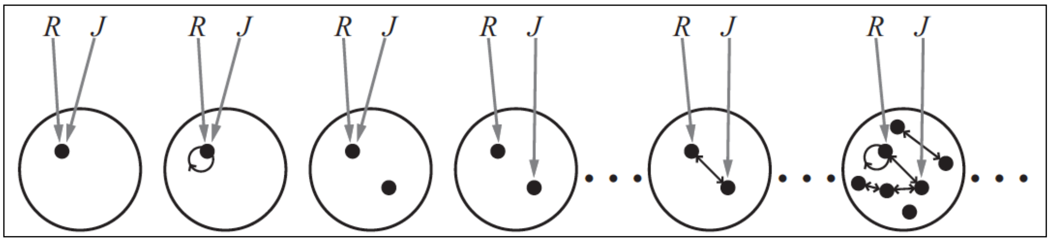
Validity and Satisfiability

Definition 10.

(1) A formula Φ of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ is *true in the model* $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ for $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ provided $\Phi^{\mathbf{I},\mathbf{A}} = \mathbf{t}$ for all assignments $\mathbf{A}$.
(2) A formula Φ is *valid* if Φ is true in all models for the language.
(3) A set S of formulas is *satisfiable* in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$, provided there is some assignment $\mathbf{A}$ (called the *satisfying assignment*) such that $\Phi^{\mathbf{I},\mathbf{A}} = \mathbf{t}$ for for all $\Phi \in S$. S is *satisfiable* if it is satisfiable in some model.

Domains, Models & Naming

- Language with 2 constants, R,J and one binary relation



Some models

Naming individuals:
- More than one name
- No name

Under-constrained in FOL!

Richard has two brothers?

$$Brother(John, Richard) \land Brother(Geoffrey, Richard)$$

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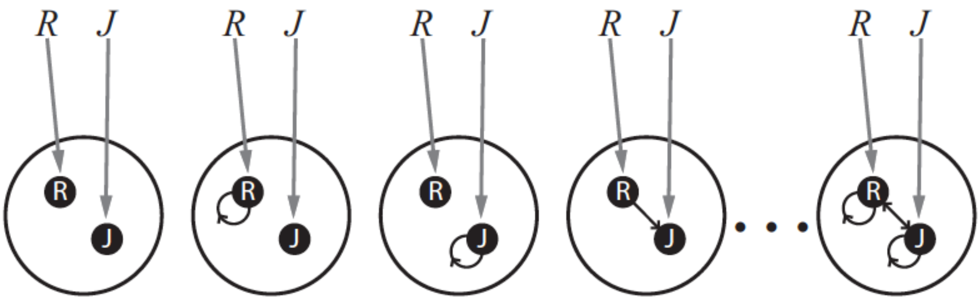
Domains, Models & Naming

- Language with 2 constants, R,J and one binary relation



Some models

Naming individuals:
- Unique Names Assumptions
- Domain Closure

Database Semantics

Many applications assume UNA, DC

$$Brother(John, Richard) \land Brother(Geoffrey, Richard) \land$$
$$John \neq Geoffrey \land John \neq Richard \land Geoffrey \neq Richard \land$$
$$\forall x\ Brother(x, Richard) \Rightarrow (x = John \lor x = Geoffrey)$$

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Models for FOL: Too many!!

Entailment in Propositional Logic can be computed by enumerating models

How about enumerating models in FOL for a given KB Vocabulary?

For each number of domain elements n from 1 to infinity
  For each k-ary predicate P_k in the vocabulary
    For each constant symbol C in the vocabulary
      For each choice of referent for C from n objects…

Computing entailment by complete enumeration in FOL is not practically possible!!

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First-Order Resolution

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Will be added later!!

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Reasoning about Action and Change Nonmonotonic Reasoning

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Will be added later!!

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