LU Decomposition

Oscar S. Dalmau Cedeño
dalmau@cimat.mx

October 21, 2022

LU Decomposition Methods

- Lower-Upper (LU) decomposition/factorization allows to factor a matrix as the product of a lower triangular matrix and an upper triangular matrix
- LU decomposition can be obtained using Gaussian elimination and can be seen as the matrix form of Gaussian elimination
- LU decomposition is typically used to solve systems of linear equations,
- LU decomposition can be used for computing the determinant of a matrix.

LU Decomposition Methods

- Any matrix $\boldsymbol{A}$ can be expressed as

$$\boldsymbol{P}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U},$$

  where $\boldsymbol{L}$ is a lower triangular matrix, $\boldsymbol{U}$ is an upper triangular matrix, and $\boldsymbol{P}$ is a permutation matrix.
- The process of computing $\boldsymbol{L}$ and $\boldsymbol{U}$ (and $\boldsymbol{P}$) is called LU decomposition/factorization.

LU Decomposition Methods

- LU decomposition is not unique
- For example:

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}$$

has the following LU decompositions

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

LU Decomposition Methods

For example: Let us factorize the following 2-by-2 matrix:

$$\begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} \ell_{00} & 0 \\ \ell_{10} & \ell_{11} \end{bmatrix} \begin{bmatrix} u_{00} & u_{01} \\ 0 & u_{11} \end{bmatrix}.$$

We can find the LU decomposition by solving a non-linear system of equations with 6 unknowns and 4 equations, ie, .

$$\begin{aligned} \ell_{00} \cdot u_{00} + 0 \cdot 0 &= 1 \\ \ell_{10} \cdot u_{00} + \ell_{11} \cdot 0 &= -1 \\ \ell_{00} \cdot u_{01} + 0 \cdot u_{11} &= -1 \\ \ell_{10} \cdot u_{01} + \ell_{11} \cdot u_{11} &= 5. \end{aligned}$$

This is an underdetermined system, then, to find a unique LU decomposition, it is necessary to put some restrictions.

LU Decomposition Methods

Some constraints should be considered to obtain unique decompositions:

- Doolittle's decomposition: $\ell_{ii} = 1$, $i = 0, 1, \cdots, n-1$

$$\begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} \color{red}{1} & 0 \\ -1 & \color{red}{1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}$$

- Crout's decomposition: $u_{ii} = 1$, $i = 0, 1, \cdots, n-1$

$$\begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} \color{red}{1} & -1 \\ 0 & \color{red}{1} \end{bmatrix}$$

- Cholesky's decomposition: $\boldsymbol{L} = \boldsymbol{U}^T$ and $\ell_{ii} = u_{ii} > 0$, $i = 0, 1, \cdots, n-1$

$$\begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} \color{red}{1} & 0 \\ \color{red}{-1} & \color{red}{2} \end{bmatrix} \begin{bmatrix} \color{blue}{1} & \color{blue}{-1} \\ 0 & \color{blue}{2} \end{bmatrix}$$

LU Decomposition Methods

- The previous decompositions (Doolittle, Crout and Cholesky) do not always exist.
- If $\boldsymbol{A}$ is a square, invertible matrix and has an LU factorization with all diagonal entries of $\boldsymbol{L}$ or $\boldsymbol{U}$ equal to 1, then the factorization is unique.
- If $\boldsymbol{A}$ is a symmetric positive-definite matrix, the Cholesky decomposition always exists and is unique

LU Decomposition Methods

- For example: the following matrix $\boldsymbol{A}$ does not have Doolittle's, Crout's decomposition since $a_{0,0} = 0$.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

But we can find a permutation matrix such that $\boldsymbol{PA} = \boldsymbol{LU}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

LU Decomposition Methods

A square matrix could have infinitely many LU factorizations

- if two or more of any first $(n-1)$ columns are linearly dependent
- or any of the first $(n-1)$ columns are 0

For example: the matrix

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

has LU factorizations of the form

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & b \end{bmatrix}$$

for any a, b such that $2a + b = 1$

Solving a linear system using LU

If $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ the linear system

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

can be solved in two steps. From

$$\begin{aligned} \boldsymbol{L}\boldsymbol{U}\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{L}\left(\boldsymbol{U}\boldsymbol{x}\right) &= \boldsymbol{b} \end{aligned}$$

we can define $\boldsymbol{y} = \boldsymbol{U}\boldsymbol{x}$ and we solve the following two systems:

$$\begin{aligned} \boldsymbol{L}\boldsymbol{y} &= \boldsymbol{b}, \text{ solve using } \textbf{forward substitution} \\ \boldsymbol{U}\boldsymbol{x} &= \boldsymbol{y}, \text{ solve using } \textbf{backward substitution} \end{aligned}$$

first w.r.t. $\boldsymbol{y}$ and then w.r.t. $\boldsymbol{x}$

Computing the determinant using LU

Recall

- $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$
- If we swap two rows (columns) in $\boldsymbol{A}$, the determinant will change its sign.
- $\det(\boldsymbol{P}) = \pm 1$

From

$$\boldsymbol{PA} \;=\; \boldsymbol{LU},$$

and $\det(\boldsymbol{P}) = (-1)^k$, where k is the number of row exchanges in the previous decomposition, then

$$\det(\boldsymbol{A}) \;=\; (-1)^k \det(\boldsymbol{L})\det(\boldsymbol{U}) = (-1)^k \prod \ell_{ii} \prod u_{ii}$$

Comment

If $\boldsymbol{E}_i$ is an permutation of two rows of the identity matrix (ie, a permutation of the form $\boldsymbol{P}_{ij}$), then after k row exchanges.

$$\boldsymbol{P} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \cdots \boldsymbol{E}_1 \boldsymbol{I}$$

since $\det(\boldsymbol{E}_i) = -1$ then

$$\det(\boldsymbol{P}) = (-1)^k$$

Comment

Note also that $\boldsymbol{E}_i\boldsymbol{E}_i = \boldsymbol{I}$, $\boldsymbol{E}_i = \boldsymbol{E}_i^{-1} = \boldsymbol{E}_i^T$ these are properties of two-row permutation matrices of the form $\boldsymbol{P}_{ij}$, then

$$\boldsymbol{P}\boldsymbol{P}^T = \boldsymbol{E}_k\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{E}_1^T\boldsymbol{E}_2^T\cdots\boldsymbol{E}_{k-1}^T\boldsymbol{E}_k^T = \boldsymbol{I}$$

and again

$$\begin{aligned} \det(\boldsymbol{P})\det(\boldsymbol{P}^T) &= 1 \\ \det(\boldsymbol{P})^2 &= 1 \text{ since } \det(\boldsymbol{P}) = \det(\boldsymbol{P}^T) \\ \det(\boldsymbol{P}) &= \pm 1 \end{aligned}$$

Solve systems of linear equations using

We can use LU decomposition to solve **systems of linear equations** using:

$$\boldsymbol{AX} = \boldsymbol{B}$$

where $\boldsymbol{X}$ and $\boldsymbol{B}$ are matrices of the same size n-by-p.
We can rewrite the previous systems

$$\boldsymbol{LUX} = \boldsymbol{B}$$

Solve systems of linear equations using

We can also adapt the forward and backward substitutions methods to find the solution matrix $\boldsymbol{X}$ of the following systems of linear equations

$$\boldsymbol{L}\,(\boldsymbol{U}\boldsymbol{X}) = \boldsymbol{B}$$

by solving

$$\begin{aligned} \boldsymbol{L}\boldsymbol{Y} &= \boldsymbol{B} \\ \boldsymbol{U}\boldsymbol{X} &= \boldsymbol{Y} \end{aligned}$$

First, we solve for $\boldsymbol{Y}$ using forward substitution and then we solve for $\boldsymbol{X}$ using backward substitution.

Inverting a matrix

Inverting a matrix: In particular, if $\boldsymbol{B}$ is the identity matrix, then the solution $\boldsymbol{X}$ of

$$\boldsymbol{L}\boldsymbol{U}\boldsymbol{X} = \boldsymbol{I}$$

is the inverse of $\boldsymbol{A}$ (use forward-backward substitution).

Another alternative is to find both the inverse of $\boldsymbol{L}$ and $\boldsymbol{U}$ by solving

$$\begin{aligned} \boldsymbol{U}\boldsymbol{X} &= \boldsymbol{I} \\ \boldsymbol{L}\boldsymbol{Y} &= \boldsymbol{I} \end{aligned}$$

then $\boldsymbol{X} = \boldsymbol{U}^{-1}$ and $\boldsymbol{Y} = \boldsymbol{L}^{-1}$ therefore

$$\boldsymbol{A}^{-1} = (\boldsymbol{L}\boldsymbol{U})^{-1} = \boldsymbol{U}^{-1}\boldsymbol{L}^{-1} = \boldsymbol{X}\boldsymbol{Y}$$

LU Decomposition Methods: Doolittle's Method

Doolittle's Decomposition Method 3×3 matrix

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{1,0} & 1 & 0 \\ \ell_{2,0} & \ell_{2,1} & 1 \end{bmatrix}, \ \boldsymbol{U} = \begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} \\ 0 & u_{1,1} & u_{1,2} \\ 0 & 0 & u_{2,2} \end{bmatrix}$$

LU Decomposition Methods: Doolittle's Method

Doolittle's Decomposition Method using a 3×3 matrix

$$
\begin{aligned}
\boldsymbol{L} &= \begin{bmatrix} 1 & 0 & 0 \\ \ell_{1,0} & 1 & 0 \\ \ell_{2,0} & \ell_{2,1} & 1 \end{bmatrix}, \; \boldsymbol{U} = \begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} \\ 0 & u_{1,1} & u_{1,2} \\ 0 & 0 & u_{2,2} \end{bmatrix} \\
\boldsymbol{LU} &= \begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} \\ u_{0,0}\ell_{1,0} & u_{0,1}\ell_{1,0} + u_{1,1} & u_{0,2}\ell_{1,0} + u_{1,2} \\ u_{0,0}\ell_{2,0} & u_{0,1}\ell_{2,0} + u_{2,1} & u_{0,2}\ell_{2,0} + u_{2,2} \end{bmatrix}
\end{aligned}
$$

LU Decomposition Methods: Doolittle's Method

Since $\boldsymbol{LU} = \boldsymbol{A}$, we obtain the following nonlinear system of equations

$$\begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} \\ u_{0,0}\ell_{1,0} & u_{0,1}\ell_{1,0} + u_{1,1} & u_{0,2}\ell_{1,0} + u_{1,2} \\ u_{0,0}\ell_{2,0} & u_{0,1}\ell_{2,0} + u_{2,1} & u_{0,2}\ell_{2,0} + u_{2,2} \end{bmatrix} = \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ a_{1,0} & a_{1,1} & a_{1,2} \\ a_{2,0} & a_{2,1} & a_{2,2} \end{bmatrix}$$

with 9 equations and 9 unknowns.

We can find a solution by solving this system. In this case, we observe that the entries of $\boldsymbol{U}$ corresponds to the Gaussian elimination process.

LU Decomposition Methods: Doolittle's Method

For example: $\boldsymbol{u}_{0,\cdot} = \boldsymbol{a}_{0,\cdot}$, ie, the first row of $\boldsymbol{U}$ matches with the first row of $\boldsymbol{A}$. Note that: $u_{0,0}\ell_{1,0} = a_{1,0}$ and $u_{0,0}\ell_{2,0} = a_{2,0}$ then

$$\ell_{1,0} = \frac{a_{1,0}}{a_{0,0}};\ \ell_{2,0} = \frac{a_{2,0}}{a_{0,0}}$$

and

$$\begin{aligned} \boldsymbol{u}_{1,.} &= \boldsymbol{a}_{1,.} - \ell_{1,0}\boldsymbol{a}_{0,.} \\ \boldsymbol{u}_{2,.} &= \boldsymbol{a}_{2,.} - \ell_{2,0}\boldsymbol{a}_{0,.} \end{aligned}$$

this suggest to directly use Gaussian Elimination to compute the entries of $\boldsymbol{L}$ and $\boldsymbol{U}$

LU Decomposition Methods: Doolittle's Method

Let us now apply Gauss elimination to the previous matrix

$$\begin{array}{c} R_0: \\ R_1: \\ R_2: \end{array} \begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} \\ u_{0,0}\ell_{1,0} & u_{0,1}\ell_{1,0} + u_{1,1} & u_{0,2}\ell_{1,0} + u_{1,2} \\ u_{0,0}\ell_{2,0} & u_{0,1}\ell_{2,0} + u_{1,1}\ell_{2,1} & u_{0,2}\ell_{2,0} + u_{1,2}\ell_{2,1} + u_{2,2} \end{bmatrix}$$

if $u_{0,0} \neq 0$ we can compute a unique $\ell_{1,0} = \frac{a_{1,0}}{a_{0,0}} = \frac{u_{0,0}\ell_{1,0}}{u_{0,0}}$, $\ell_{2,0} = \frac{a_{2,0}}{a_{0,0}} = \frac{u_{0,0}\ell_{2,0}}{u_{0,0}}$ and apply the Gauss elimination that allows us to compute $u_{1,1}, u_{1,2}$

$$\begin{aligned} R_1 &\leftarrow R_1 - \ell_{1,0} R_0 \\ R_2 &\leftarrow R_2 - \ell_{2,0} R_0 \end{aligned}$$

$$\begin{array}{c} R_0: \\ R_1: \\ R_2: \end{array} \begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} \\ \boxed{0} & u_{1,1} & u_{1,2} \\ \boxed{0} & u_{1,1}\ell_{2,1} & u_{1,2}\ell_{2,1} + u_{2,2} \end{bmatrix}$$

LU Decomposition Methods: Doolittle's Method

if $u_{1,1} \neq 0$ then we can compute a unique $\ell_{2,1} = \frac{a_{2,1}}{a_{1,1}} = \frac{u_{1,1}\ell_{2,1}}{u_{1,1}}$, and this allows us to compute $u_{2,2}$

$$\begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} \\ \boxed{0} & u_{1,1} & u_{1,2} \\ \boxed{0} & u_{1,1}\ell_{2,1} & u_{1,2}\ell_{2,1} + u_{2,2} \end{bmatrix} \xrightarrow{R_2 - \ell_{2,1} R_1} \begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} \\ \boxed{0} & u_{1,1} & u_{1,2} \\ \boxed{0} & \boxed{0} & u_{2,2} \end{bmatrix}$$

LU Decomposition Methods: Doolittle's Method

With the previous procedure, we obtain $\boldsymbol{L}$ and $\boldsymbol{U}$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{1,0} & 1 & 0 \\ \ell_{2,0} & \ell_{2,1} & 1 \end{bmatrix}, \boldsymbol{U} = \begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} \\ 0 & u_{1,1} & u_{1,2} \\ 0 & 0 & u_{2,2} \end{bmatrix}$$

Note that we can simply combine them in one matrix, ie,

$$\begin{bmatrix} u_{0,0} & u_{0,1} & u_{0,2} \\ \boxed{\ell_{1,0}} & u_{1,1} & u_{1,2} \\ \boxed{\ell_{2,0}} & \boxed{\ell_{2,1}} & u_{2,2} \end{bmatrix}$$

This algorithm requires $u_{i,i}^{(i)} \neq 0$ at each step $i = 0, 1$ in order to obtain a unique solution.

LU Decomposition Methods: Doolittle's Method

Two important features of Doolittle's decomposition:

1. The matrix $\boldsymbol{U}$ is identical to the upper triangular matrix that results from Gauss elimination.
2. The off-diagonal elements of $\boldsymbol{L}$ are the pivot equation multipliers used during Gauss elimination; ie, ℓ_{ij} is the multiplier that eliminated a_{ij}.
3. $\det(\boldsymbol{A}) = \det(\boldsymbol{U}) = \prod u_{ii}$

LU Decomposition Methods: Doolittle's Method

1. Doolittle's factorization without permutation exists if $u^i_{i,i} \neq 0$, $i = 0, 1, \cdots, n-2$, where $u^0_{0,0} = u_{0,0} = a_{0,0}$.
2. Note that $u^{n-1}_{n-1,n-1}$ can be any number. If $u^{n-1}_{n-1,n-1} = 0$ then the last row of $\boldsymbol{U}$ is the null vector, ie, $\boldsymbol{u}_{n-1,\cdot} = \boldsymbol{0}$
3. For example: $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ for

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 7 & 7 \\ 4 & 9 & 5 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}; \boldsymbol{U} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

LU Decomposition Methods

Comments

1. Crout's decomposition: $\det(\boldsymbol{A}) = \det(\boldsymbol{L}) = \prod \ell_{ii}$
2. Cholesky's decomposition:
$\det(\boldsymbol{A}) = \det(\boldsymbol{U}^T)\det(\boldsymbol{U}) = \prod u_{ii}^2$

LU Decomposition

```python
def LU(A): # without permutation
  n = A.shape[0]
  L = np.eye(n)
  U = A.copy()
  for j in range(n-1):
    for i  in range(j+1, n):
      lij = U[i,j]/U[j,j]
      U[i,:] =  U[i,:] - lij*U[j,:]
      L[i,j] = lij
  return L, U
```

Note: $U[j,j] \neq 0$, $j = 0, 1, \cdots, n-2$, ie $U[n-1, n-1]$ could be zero.
See: scipy.linalg.lu, scipy.linalg.lu_factor, scipy.linalg.lu_solve

LU Decomposition

```
def Lu_compact(A): # without permutation
  n = A.shape[0]
  L = np.eye(n)
  U = A.copy()
  for j in range(n-1):
    for i  in range(j+1, n):
      lij = U[i,j]/U[j,j]
      U[i,:] =  U[i,:] - lij*U[j,:]
      U[i,j] = lij
  return U
```

Note: It only needs that $U[j,j] \neq 0, j = 0, 1, \cdots, n-2$

LU Decomposition with permutation: Example 1

$$\begin{array}{r} R_0 : \\ R_1 : \\ R_2 : \end{array} \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$

Permutation matrix $\boldsymbol{P} = \boldsymbol{I}$ or pivot index $\boldsymbol{p} = [0, 1, 2]$

$$\begin{array}{rc} & \boldsymbol{P} \\ \begin{array}{r} R_0 : \\ R_1 : \\ R_2 : \end{array} & \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4 & 4 & -4 \\ 0 & 0 & 1 & 1 & 3 & 1 \end{array}\right] \end{array}$$

LU Decomposition with permutation: Example 1

$$\begin{array}{c} R_0: \\ R_1: \\ R_2: \end{array} \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$

Pivot: row = 0, column = 0, maximum absolute value: 4 in row 1. Then, permute $R_0 \leftrightarrow R_1$, $\boldsymbol{P}_{01}$; then $\boldsymbol{P} = \boldsymbol{P}_{01}\boldsymbol{P}$ or $\boldsymbol{p} = [1, 0, 2]$

$$\begin{array}{c} \\ R_0: \\ R_1: \\ R_2: \end{array} \begin{array}{c} \boldsymbol{P} \\ \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 4 & 4 & -4 \\ 1 & 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 & 1 \end{array}\right] \end{array}$$

LU Decomposition with permutation: Example 1

$$
\begin{array}{r c}
 & \boldsymbol{P} \\
\begin{array}{r} R_0: \\ R_1: \\ R_2: \end{array} & \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 4 & 4 & -4 \\ 1 & 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 & 1 \end{array}\right]
\end{array}
$$

Forward elimination: pivot column 0, pivot row 0.

$$
\begin{aligned}
R_1 &\leftarrow R_1 - \frac{2}{4} R_0 = R_1 - \frac{1}{2} R_0 \\
R_2 &\leftarrow R_2 - \frac{1}{4} R_0
\end{aligned}
$$

LU Decomposition with permutation: Example 1

$$\begin{array}{r} \\ R_0 : \\ R_1 : \\ R_2 : \end{array} \begin{array}{c} \boldsymbol{P} \\ \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 4 & 4 & -4 \\ 1 & 0 & 0 & \boxed{0} & -1 & 7 \\ 0 & 0 & 1 & \boxed{0} & 2 & 2 \end{array}\right] \end{array}$$

we can use the new entries with zeroes to set the entries of matrix $\boldsymbol{L}$

$$\begin{array}{r} \\ R_0 : \\ R_1 : \\ R_2 : \end{array} \begin{array}{c} \boldsymbol{P} \\ \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 4 & 4 & -4 \\ 1 & 0 & 0 & \boxed{1/2} & -1 & 7 \\ 0 & 0 & 1 & \boxed{1/4} & 2 & 2 \end{array}\right] \end{array}$$

LU Decomposition with permutation: Example 1

Now we repeat the procedure

$$\begin{array}{c} \\ R_0 : \\ R_1 : \\ R_2 : \end{array} \begin{array}{c} \boldsymbol{P} \\ \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 4 & 4 & -4 \\ 1 & 0 & 0 & \boxed{1/2} & -1 & 7 \\ 0 & 0 & 1 & \boxed{1/4} & 2 & 2 \end{array}\right] \end{array}$$

Pivot: row = 1, column = 1.
Maximum absolute value: 2 in row 2. Then, permute $R_1 \leftrightarrow R_2$, $\boldsymbol{P}_{12}$, then $\boldsymbol{P} = \boldsymbol{P}_{12}\boldsymbol{P}$ or $\boldsymbol{p} = [1, 2, 0]$

$$\begin{array}{c} \\ R_0 : \\ R_1 : \\ R_2 : \end{array} \begin{array}{c} \boldsymbol{P} \\ \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 4 & 4 & -4 \\ 0 & 0 & 1 & \boxed{1/4} & 2 & 2 \\ 1 & 0 & 0 & \boxed{1/2} & -1 & 7 \end{array}\right] \end{array}$$

LU Decomposition with permutation: Example 1

$$
\begin{array}{c} \\ R_0 : \\ R_1 : \\ R_2 : \end{array}
\begin{array}{c} \boldsymbol{P} \\ \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 4 & 4 & -4 \\ 0 & 0 & 1 & \boxed{1/4} & 2 & 2 \\ 1 & 0 & 0 & \boxed{1/2} & -1 & 7 \end{array}\right] \end{array}
$$

Forward elimination: Pivot row = 1, column = 1. Then

$$
R_2 \quad \leftarrow \quad R_2 - \frac{-1}{2} R_1 = R_2 + \frac{1}{2} R_1
$$

LU Decomposition with permutation: Example 1

Finally, we get: $\boldsymbol{p} = [1, 2, 0]$

$$
\begin{array}{cc}
 & \begin{array}{ccc} \boldsymbol{P} & & \boldsymbol{L}\backslash\boldsymbol{U} \end{array} \\
\begin{array}{c} R_0 : \\ R_1 : \\ R_2 : \end{array} &
\left[\begin{array}{ccc|ccc}
0 & 1 & 0 & 4 & 4 & -4 \\
0 & 0 & 1 & \boxed{1/4} & 2 & 2 \\
1 & 0 & 0 & \boxed{1/2} & \boxed{-1/2} & 8
\end{array}\right]
\end{array}
$$

where $\boldsymbol{PA} = \boldsymbol{LU}$ with

$$
\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ \boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix}, \ \boldsymbol{U} = \begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}
$$

LU Decomposition with permutation: Example 1

$$\boldsymbol{PA} = \begin{bmatrix} 4 & 4 & -4 \\ 1 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix}$$

$$\boldsymbol{LU} = \begin{bmatrix} 4 & 4 & -4 \\ 1 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix}$$

LU Algorithm with permutation

```
def PLU(A): # LU with permutation
  n=A.shape[0]
  P,L,U=np.eye(n),np.eye(n),A.copy()
  for j in range(n-1):
    # Find arg max abs value, col 'j' and below row 'j-1'
    k=argmax(U,j)
    if k!=j: # permutes rows j and k
       U,P=permute(U,P,j,k)
    for i in range(j+1,n):
      lij=U[i,j]/U[j,j]
      U[i,:]=U[i,:]-lij*U[j,:]
      L[i,j]=lij
 return P,L,U
```

Note: It only needs that $U[j,j] \neq 0$, $j = 0, 1, \cdots, n-2$

LU Decomposition with permutation: Example 2

Permutation matrix $\boldsymbol{P} = \boldsymbol{I}$ or $\boldsymbol{p} = [0, 1, 2, 3, 4]$

$$\begin{array}{c} \\ R_0 : \\ R_1 : \\ R_2 : \\ R_3 : \\ R_4 : \end{array}
\begin{array}{c} \boldsymbol{P} \\ \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 3 & -8 & -7 & 6 & -4 \\ 0 & 1 & 0 & 0 & 0 & 6 & -18 & -12 & 12 & -6 \\ 0 & 0 & 1 & 0 & 0 & 2 & -5 & -5 & 5 & -8 \\ 0 & 0 & 0 & 1 & 0 & 3 & -7 & -10 & 8 & -9 \\ 0 & 0 & 0 & 0 & 1 & 2 & -5 & -3 & -1 & 4 \end{array}\right] \end{array}$$

LU Decomposition with permutation: Example 2

Permute $R_0 \leftrightarrow R_1$, $\boldsymbol{P}_{01}$; then $\boldsymbol{P} = \boldsymbol{P}_{01}\boldsymbol{P}$ or $\boldsymbol{p} = [1, 0, 2, 3, 4]$

$$
\begin{array}{r}
\\
R_0 : \\ R_1 : \\ R_2 : \\ R_3 : \\ R_4 :
\end{array}
\begin{array}{c}
\qquad \boldsymbol{P} \qquad\qquad\qquad\qquad\qquad \\
\left[\begin{array}{ccccc|ccccc}
0 & 1 & 0 & 0 & 0 & 6 & -18 & -12 & 12 & -6 \\
1 & 0 & 0 & 0 & 0 & 3 & -8 & -7 & 6 & -4 \\
0 & 0 & 1 & 0 & 0 & 2 & -5 & -5 & 5 & -8 \\
0 & 0 & 0 & 1 & 0 & 3 & -7 & -10 & 8 & -9 \\
0 & 0 & 0 & 0 & 1 & 2 & -5 & -3 & -1 & 4
\end{array}\right]
\end{array}
$$

LU Decomposition with permutation: Example 2

Forward elimination

$$
\begin{array}{r}
\\
R_0 : \\
R_1 : \\
R_2 : \\
R_3 : \\
R_4 :
\end{array}
\begin{array}{c}
\boldsymbol{P} \\
\left[\begin{array}{ccccc|ccccc}
0 & 1 & 0 & 0 & 0 & 6 & -18 & -12 & 12 & -6 \\
1 & 0 & 0 & 0 & 0 & \color{red}{1/2} & 1 & -1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & \color{red}{1/3} & 1 & -1 & 1 & -6 \\
0 & 0 & 0 & 1 & 0 & \color{red}{1/2} & 2 & -4 & 2 & -6 \\
0 & 0 & 0 & 0 & 1 & \color{red}{1/3} & 1 & 1 & -5 & 6
\end{array}\right]
\end{array}
$$

LU Decomposition with permutation: Example 2

Permute $R_1 \leftrightarrow R_3$, $\boldsymbol{P}_{13}$; then $\boldsymbol{P} = \boldsymbol{P}_{13}\boldsymbol{P}$ or $\boldsymbol{p} = [1, 3, 2, 1, 4]$

$$
\begin{array}{r}
\\
R_0 : \\ R_1 : \\ R_2 : \\ R_3 : \\ R_4 :
\end{array}
\begin{array}{c}
\boldsymbol{P} \\
\left[\begin{array}{ccccc|ccccc}
0 & 1 & 0 & 0 & 0 & 6 & -18 & -12 & 12 & -6 \\
0 & 0 & 0 & 1 & 0 & \color{red}{1/2} & 2 & -4 & 2 & -6 \\
0 & 0 & 1 & 0 & 0 & \color{red}{1/3} & 1 & -1 & 1 & -6 \\
1 & 0 & 0 & 0 & 0 & \color{red}{1/2} & 1 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & \color{red}{1/3} & 1 & 1 & -5 & 6
\end{array}\right]
\end{array}
$$

LU Decomposition with permutation: Example 2

Forward elimination

$$
\begin{array}{r}
 \\
R_0 : \\
R_1 : \\
R_2 : \\
R_3 : \\
R_4 :
\end{array}
\begin{array}{c}
\boldsymbol{P} \\
\left[\begin{array}{ccccc|ccccc}
0 & 1 & 0 & 0 & 0 & 6 & -18 & -12 & 12 & -6 \\
0 & 0 & 0 & 1 & 0 & {\color{red}1/2} & 2 & -4 & 2 & -6 \\
0 & 0 & 1 & 0 & 0 & {\color{red}1/3} & {\color{red}1/2} & 1 & 0 & -3 \\
1 & 0 & 0 & 0 & 0 & {\color{red}1/2} & {\color{red}1/2} & 1 & -1 & 2 \\
0 & 0 & 0 & 0 & 1 & {\color{red}1/3} & {\color{red}1/2} & 3 & -6 & 9
\end{array}\right]
\end{array}
$$

LU Decomposition with permutation: Example 2

Permute $R_2 \leftrightarrow R_4$, $\boldsymbol{P}_{24}$; then $\boldsymbol{P} = \boldsymbol{P}_{24}\boldsymbol{P}$ or $\boldsymbol{p} = [1, 3, 4, 1, 2]$

$$
\begin{array}{c}
 \\
R_0 : \\
R_1 : \\
R_2 : \\
R_3 : \\
R_4 :
\end{array}
\begin{array}{c}
\boldsymbol{P} \\
\left[\begin{array}{ccccc|ccccc}
0 & 1 & 0 & 0 & 0 & 6 & -18 & -12 & 12 & -6 \\
0 & 0 & 0 & 1 & 0 & \color{red}{1/2} & 2 & -4 & 2 & -6 \\
0 & 0 & 0 & 0 & 1 & \color{red}{1/3} & \color{red}{1/2} & 3 & -6 & 9 \\
1 & 0 & 0 & 0 & 0 & \color{red}{1/2} & \color{red}{1/2} & 1 & -1 & 2 \\
0 & 0 & 1 & 0 & 0 & \color{red}{1/3} & \color{red}{1/2} & 1 & 0 & -3
\end{array}\right]
\end{array}
$$

LU Decomposition with permutation: Example 2

Forward elimination

$$
\begin{array}{r@{\;}l}
 & \qquad\qquad\quad \boldsymbol{P} \\
\begin{array}{r} R_0 : \\ R_1 : \\ R_2 : \\ R_3 : \\ R_4 : \end{array} &
\left[\begin{array}{ccccc|ccccc}
0 & 1 & 0 & 0 & 0 & 6 & -18 & -12 & 12 & -6 \\
0 & 0 & 0 & 1 & 0 & \color{red}{1/2} & 2 & -4 & 2 & -6 \\
0 & 0 & 0 & 0 & 1 & \color{red}{1/3} & \color{red}{1/2} & 3 & -6 & 9 \\
1 & 0 & 0 & 0 & 0 & \color{red}{1/2} & \color{red}{1/2} & \color{red}{1/3} & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & \color{red}{1/3} & \color{red}{1/2} & \color{red}{1/3} & 2 & -6
\end{array}\right]
\end{array}
$$

LU Decomposition with permutation: Example 2

Permute $R_3 \leftrightarrow R_4$, $\boldsymbol{P}_{34}$; then $\boldsymbol{P} = \boldsymbol{P}_{34}\boldsymbol{P}$ or $\boldsymbol{p} = [1, 3, 4, 2, 1]$

$$\begin{array}{c} \\ R_0 : \\ R_1 : \\ R_2 : \\ R_3 : \\ R_4 : \end{array} \begin{array}{c} \boldsymbol{P} \\ \left[\begin{array}{ccccc|ccccc} 0 & 1 & 0 & 0 & 0 & 6 & -18 & -12 & 12 & -6 \\ 0 & 0 & 0 & 1 & 0 & {\color{red}1/2} & 2 & -4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & {\color{red}1/3} & {\color{red}1/2} & 3 & -6 & 9 \\ 0 & 0 & 1 & 0 & 0 & {\color{red}1/3} & {\color{red}1/2} & {\color{red}1/3} & 2 & -6 \\ 1 & 0 & 0 & 0 & 0 & {\color{red}1/2} & {\color{red}1/2} & {\color{red}1/3} & 1 & -1 \end{array}\right] \end{array}$$

LU Decomposition with permutation: Example 2

Forward elimination

$$
\begin{array}{r}
\\
R_0 : \\
R_1 : \\
R_2 : \\
R_3 : \\
R_4 :
\end{array}
\begin{array}{c}
\boldsymbol{P} \\
\left[\begin{array}{ccccc|ccccc}
0 & 1 & 0 & 0 & 0 & 6 & -18 & -12 & 12 & -6 \\
0 & 0 & 0 & 1 & 0 & \color{red}{1/2} & 2 & -4 & 2 & -6 \\
0 & 0 & 0 & 0 & 1 & \color{red}{1/3} & \color{red}{1/2} & 3 & -6 & 9 \\
0 & 0 & 1 & 0 & 0 & \color{red}{1/3} & \color{red}{1/2} & \color{red}{1/3} & 2 & -6 \\
1 & 0 & 0 & 0 & 0 & \color{red}{1/2} & \color{red}{1/2} & \color{red}{1/3} & \color{red}{1/2} & 2
\end{array}\right]
\end{array}
$$

LU Decomposition with permutation: Example 2

$\boldsymbol{PA} = \boldsymbol{LU}$ with

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 6 & -18 & -12 & 12 & -6 \\ 0 & 2 & -4 & 2 & -6 \\ 0 & 0 & 3 & -6 & 9 \\ 0 & 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} ; \; \boldsymbol{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \nicefrac{1}{2} & 1 & 0 & 0 & 0 \\ \nicefrac{1}{3} & \nicefrac{1}{2} & 1 & 0 & 0 \\ \nicefrac{1}{3} & \nicefrac{1}{2} & \nicefrac{1}{3} & 1 & 0 \\ \nicefrac{1}{2} & \nicefrac{1}{2} & \nicefrac{1}{3} & \nicefrac{1}{2} & 1 \end{bmatrix}$$

LU using Gaussian elimination: Let consider the following matrix

$$\boldsymbol{A}_j = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & a_{j+1,j} & & \\ & & \vdots & \ddots & \\ & & a_{n-1,j} & & 1 \end{bmatrix}$$

ie, it is the $n \times n$ identity matrix with its j-th column replaced by the vector $\begin{bmatrix} 0 & \cdots & 0 & 1 & a_{j+1,j} & \cdots & a_{n,j} \end{bmatrix}^T$.

We can easily compute the inverse of $\boldsymbol{A}_j$ by

$$\boldsymbol{A}_j^{-1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -a_{j+1,j} & & \\ & & \vdots & \ddots & \\ & & -a_{n-1,j} & & 1 \end{bmatrix}$$

we just need to change the sign of the entries $a_{k,j}$

Column stacking

Column stacking

$$\boldsymbol{A}_{0\to 1} := \boldsymbol{A}_0 \boldsymbol{A}_1 \quad = \begin{bmatrix} 1 & & & & & & \\ a_{10} & 1 & & & & & \\ a_{20} & a_{21} & 1 & & & & \\ \vdots & \vdots & 0 & \ddots & & & \\ \vdots & \vdots & \vdots & 1 & & & \\ \vdots & \vdots & \vdots & 0 & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ a_{n-1,0} & a_{n-1,1} & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Column stacking

$$
\begin{aligned}
\boldsymbol{A}_{0\to j} \quad &:= \quad \boldsymbol{A}_0 \boldsymbol{A}_1 \cdots \boldsymbol{A}_j;\ j = 0, \cdots, n-2 \\
&= \begin{bmatrix}
1 & & & & & & & \\
a_{10} & 1 & & & & & & \\
a_{20} & a_{21} & 1 & & & & & \\
a_{30} & a_{31} & a_{32} & & & & & \\
\vdots & \vdots & \vdots & \ddots & & & & \\
\vdots & \vdots & \vdots & & 1 & & & \\
\vdots & \vdots & \vdots & \cdots & a_{j+1,j} & 1 & & \\
\vdots & \vdots & \vdots & & \vdots & 0 & \ddots & \\
a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,j} & 0 & \cdots & 1
\end{bmatrix}
\end{aligned}
$$

Recall: $(\alpha \boldsymbol{e}_i + \beta \boldsymbol{e}_j)^T \boldsymbol{A} = \alpha \boldsymbol{a}_{i,.} + \beta \boldsymbol{a}_{j,.}$, ie, premultiplying $\boldsymbol{A}$ by

$$(\alpha \boldsymbol{e}_i + \beta \boldsymbol{e}_j)^T \quad = \quad [0, ..., \alpha, ..., \beta, ...0]$$

obtains a linear combination of rows i, j of matrix $\boldsymbol{A}$. In particular, if $i = 0$, $\alpha = -\lambda_j$ and $\beta = 1$

$$(-\lambda_j \boldsymbol{e}_0 + \boldsymbol{e}_j)^T \boldsymbol{A} \quad = \quad [-\lambda_j, ..., 1, ...0]\boldsymbol{A} = \boldsymbol{a}_{j,.} - \lambda_j \boldsymbol{a}_{0,.}$$

Note that this corresponds to the transformation

$$(E_j - \lambda_j E_0) \rightarrow (E_j)$$

with $\lambda_j = \frac{a_{ji}}{a_{ii}}$ in the Gaussian elimination process

Now we can stack the previous operations, $j = 1, 2, \cdots, n-1$, and at the top we use the row $\boldsymbol{e}_0^T$, obtaining

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -\lambda_1 & 1 & 0 & \cdots & 0 \\ -\lambda_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\lambda_{n-1} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -\lambda_1 & 1 & 0 & \cdots & 0 \\ -\lambda_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\lambda_{n-1} & 0 & 0 & \cdots & 1 \end{bmatrix} \boldsymbol{A} \qquad \begin{array}{l} E_0 \leftarrow E_0 \\ E_1 \leftarrow E_1 - \lambda_1 E_0 \\ E_2 \leftarrow E_2 - \lambda_2 E_0 \\ \cdots \\ E_{n-1} \leftarrow E_{n-1} - \lambda_{n-1} E_0 \end{array}$$

Note that, if

$$\lambda_j = \frac{a_{j,0}}{a_{0,0}}$$

then $\boldsymbol{M}\boldsymbol{A}$ corresponds to first step of the Gaussian elimination process and we call $\boldsymbol{M}$ a Gaussian elimination matrix

So, after the first iteration we obtain

$$\boldsymbol{M}\boldsymbol{A} \quad = \quad \begin{bmatrix} a_{0,0}^{(0)} & a_{0,1}^{(0)} & \cdots & \cdots & a_{0,n-1}^{(0)} \\ 0 & a_{1,1}^{(0)} & & \cdots & a_{1,n-1}^{(0)} \\ 0 & a_{2,1}^{(0)} & & \cdots & a_{2,n-1}^{(0)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n-1,1}^{(0)} & & \cdots & a_{n-1,n-1}^{(0)} \end{bmatrix}$$

with $a_{i,0}^{(0)} = a_{i,0}$. Now, we can repeat the previous idea in order to obtain the next steps of the Gaussian elimination.

Let us define

$$\hat{\boldsymbol{L}}^{(k)} = \begin{bmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\lambda_{k+1,k} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\lambda_{n-1,k} & 0 & \cdots & 1 \end{bmatrix}$$

with $k = 0, 1, \cdots, n-2$ and

$$\lambda_{j,k} = \frac{a_{j,k}^{(k)}}{a_{k,k}^{(k)}};\ j = k+1, k+2, \cdots, n-1$$

Then, the Gaussian elimination algorithm can be written as follows.

$$\begin{aligned} \boldsymbol{U}^{(0)} &= \boldsymbol{A} \\ \boldsymbol{U}^{(k+1)} &= \hat{\boldsymbol{L}}^{(k)}\boldsymbol{U}^{(k)} \end{aligned}$$

for $k = 0, 1, \cdots, n-2$. The last step is

$$\boldsymbol{U}^{(n-1)} = \hat{\boldsymbol{L}}^{(n-2)}\hat{\boldsymbol{L}}^{(n-3)} \cdots \hat{\boldsymbol{L}}^{(0)}\boldsymbol{A}$$

where $\boldsymbol{U}^{(n-1)}$ is an upper triangular matrix, which denoted by

$$\boldsymbol{U} := \boldsymbol{U}^{(n-1)}$$

In order to obtain the matrix factorization $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ we need to find $\boldsymbol{L}$

$$\boldsymbol{A} \;=\; \boldsymbol{L}\boldsymbol{U} = \boldsymbol{L}\hat{\boldsymbol{L}}^{(n-2)}\hat{\boldsymbol{L}}^{(n-3)}\cdots\hat{\boldsymbol{L}}^{(0)}\boldsymbol{A}$$

then we can simply define

$$\begin{aligned}\boldsymbol{L} &:= \left(\hat{\boldsymbol{L}}^{(n-2)}\hat{\boldsymbol{L}}^{(n-3)}\cdots\hat{\boldsymbol{L}}^{(0)}\right)^{-1}\\ &= \boldsymbol{L}^{(0)}\cdots\boldsymbol{L}^{(n-3)}\boldsymbol{L}^{(n-2)}\end{aligned}$$

with $\boldsymbol{L}^{(k)} := \left(\hat{\boldsymbol{L}}^{(k)}\right)^{-1}$, ie, $\boldsymbol{L}^{(k)}\hat{\boldsymbol{L}}^{(k)} = \boldsymbol{I}$

Note that

$$\boldsymbol{L}^{(k)} = \begin{bmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \lambda_{k+1,k} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_{n-1,k} & 0 & \cdots & 1 \end{bmatrix}$$

with $k = 0, 1, \cdots, n-2$. Then,

$$\boldsymbol{L} = \boldsymbol{L}^{(0)} \boldsymbol{L}^{(1)} \cdots \boldsymbol{L}^{(n-3)} \boldsymbol{L}^{(n-2)} = \boldsymbol{L}_{0 \to n-2}$$

$$
\boldsymbol{L} = \begin{bmatrix}
1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\lambda_{1,0} & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\
\lambda_{k,0} & \cdots & \lambda_{k,k-1} & 1 & 0 & \cdots & & 0 \\
\lambda_{k+1,0} & \cdots & \lambda_{k+1,k-1} & \lambda_{k+1,k} & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & & \vdots & \vdots & \ddots & & \vdots \\
\lambda_{n-2,0} & \cdots & \lambda_{n-2,k-1} & \lambda_{n-2,k} & \lambda_{n-2,k+1} & \cdots & 1 & 0 \\
\lambda_{n-1,0} & \cdots & \lambda_{n-1,k-1} & \lambda_{n-1,k} & \lambda_{n-1,k+1} & \cdots & \lambda_{n-1,n-2} & 1
\end{bmatrix}
$$

Note that

$$
\begin{aligned}
\boldsymbol{L}\boldsymbol{U} &= \left(\boldsymbol{L}^{(0)}\boldsymbol{L}^{(1)}\cdots\boldsymbol{L}^{(n-3)}\boldsymbol{L}^{(n-2)}\right)\left(\hat{\boldsymbol{L}}^{(n-2)}\hat{\boldsymbol{L}}^{(n-3)}\cdots\hat{\boldsymbol{L}}^{(0)}\right)\boldsymbol{A} \\
&= \left(\boldsymbol{L}^{(0)}\boldsymbol{L}^{(1)}\cdots\boldsymbol{L}^{(n-3)}\right)\boldsymbol{L}^{(n-2)}\hat{\boldsymbol{L}}^{(n-2)}\left(\hat{\boldsymbol{L}}^{(n-3)}\cdots\hat{\boldsymbol{L}}^{(0)}\right)\boldsymbol{A} \\
&= \left(\boldsymbol{L}^{(0)}\boldsymbol{L}^{(1)}\cdots\boldsymbol{L}^{(n-3)}\right)\left(\hat{\boldsymbol{L}}^{(n-3)}\cdots\hat{\boldsymbol{L}}^{(1)}\hat{\boldsymbol{L}}^{(0)}\right)\boldsymbol{A} \\
&= \cdots \\
&= \boldsymbol{L}^{(0)}\hat{\boldsymbol{L}}^{(0)}\boldsymbol{A} \\
&= \boldsymbol{A}
\end{aligned}
$$

or simply

$$
\boldsymbol{L}\boldsymbol{U} = \left(\hat{\boldsymbol{L}}^{(n-2)}\hat{\boldsymbol{L}}^{(n-3)}\cdots\hat{\boldsymbol{L}}^{(0)}\right)^{-1}\left(\hat{\boldsymbol{L}}^{(n-2)}\hat{\boldsymbol{L}}^{(n-3)}\cdots\hat{\boldsymbol{L}}^{(0)}\right)\boldsymbol{A} = \boldsymbol{A}
$$

Summary:

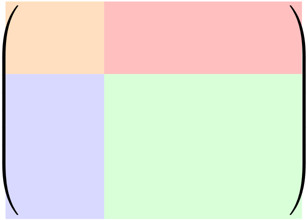
Doolittle's decomposition is obtained with a sequence of Gaussian elimination ($\boldsymbol{U}$) and a sequence of column stacking ($\boldsymbol{L}$)

$$
\begin{aligned}
\boldsymbol{U} &= \left(\hat{\boldsymbol{L}}^{(n-2)}\hat{\boldsymbol{L}}^{(n-3)}\cdots\hat{\boldsymbol{L}}^{(0)}\right)\boldsymbol{A} \\
\boldsymbol{L} &= \boldsymbol{L}^{(0)}\boldsymbol{L}^{(1)}\dots\boldsymbol{L}^{(n-3)}\boldsymbol{L}^{(n-2)} = \boldsymbol{L}_{0\to n-2}
\end{aligned}
$$

Preserving matrix form property

For matrices of the form $\boldsymbol{A}_{0\to k}$ and permutation matrix $\boldsymbol{P}_{ij}$

$$\boldsymbol{A}_{0\to k} =$$

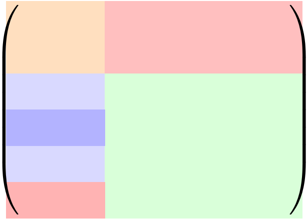


multiplying on both sides of $\boldsymbol{A}_{0\to k}$ by P_{ij} for $k+1 \leqslant i < j$ preserves the matrix form.

Preserving matrix form property

Multiplying on both sides of $\boldsymbol{A}_{0\to k}$ by P_{ij} for $k+1 \leqslant i < j$

$$\boldsymbol{P}_{ij}\boldsymbol{A}_{0\to k}\boldsymbol{P}_{ij} \quad =$$

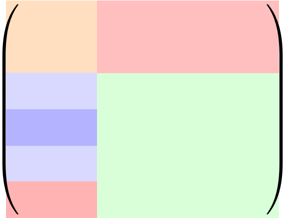


only permutes the subrows $\boldsymbol{a}_{i,0:k}$ and $\boldsymbol{a}_{j,0:k}$ of the matrix $\boldsymbol{A}_{0\to k}$, therefore the result has the same form as $\boldsymbol{A}_{0\to k}$ (matrix form preservation)

$$
\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 & 0 \\ 5 & 6 & 0 & 1 & 0 & 0 \\ 7 & 8 & 0 & 0 & 1 & 0 \\ 9 & 10 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

we can see that for $2 \leqslant i < j$

$$\boldsymbol{P}_{ij}\boldsymbol{A}\boldsymbol{P}_{ij} =$$



ie the matrix keeps the same structure and only permutes the subrows i, j in the submatrix in blue

Example

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 & 0 \\ 5 & 6 & 0 & 1 & 0 & 0 \\ 7 & 8 & 0 & 0 & 1 & 0 \\ 9 & 10 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{P}_{35}\boldsymbol{A}\boldsymbol{P}_{35} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 & 0 \\ 9 & 10 & 0 & 1 & 0 & 0 \\ 7 & 8 & 0 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{pmatrix}$$

ie permutes the subrows $i = 3$ and $j = 5$ in the submatrix in blue

The Gaussian elimination with permutation can be written as the following sequence of transformations (permutation and Gaussian elimination at each step)

$$\begin{aligned} \boldsymbol{U}^{(0)} &= \boldsymbol{A} \\ \boldsymbol{U}^{(k+1)} &= \hat{\boldsymbol{L}}^{(k)} \hat{\boldsymbol{P}}^{(k)} \boldsymbol{U}^{(k)} \end{aligned}$$

for $k = 0, 1, \cdots, n-2$, and $\hat{\boldsymbol{P}}^{(k)}$ is a two-row permutation or the identity matrix, ie, it has the form $\boldsymbol{P}_{ij}$, $\boldsymbol{P}_{ij}\boldsymbol{P}_{ij} = \boldsymbol{I}$. The last step is

$$\boldsymbol{U}^{(n-1)} = \left(\hat{\boldsymbol{L}}^{(n-2)} \hat{\boldsymbol{P}}^{(n-2)}\right) \left(\hat{\boldsymbol{L}}^{(n-3)} \hat{\boldsymbol{P}}^{(n-3)}\right) \cdots \left(\hat{\boldsymbol{L}}^{(0)} \hat{\boldsymbol{P}}^{(0)}\right) \boldsymbol{A}$$

where $\boldsymbol{U}^{(n-1)}$ is an upper triangular matrix, which denoted by

$$\boldsymbol{U} := \boldsymbol{U}^{(n-1)}$$

We need to find $\boldsymbol{P}$ and $\boldsymbol{L}$. Let us writte

$$\begin{aligned}\boldsymbol{L}^{-1}\boldsymbol{P} &= \left(\hat{\boldsymbol{L}}^{(n-2)}\hat{\boldsymbol{P}}^{(n-2)}\right)\left(\hat{\boldsymbol{L}}^{(n-3)}\hat{\boldsymbol{P}}^{(n-3)}\right)\cdots\left(\hat{\boldsymbol{L}}^{(0)}\hat{\boldsymbol{P}}^{(0)}\right)\\ \boldsymbol{L} &= \boldsymbol{P}\left[\left(\hat{\boldsymbol{L}}^{(n-2)}\hat{\boldsymbol{P}}^{(n-2)}\right)\left(\hat{\boldsymbol{L}}^{(n-3)}\hat{\boldsymbol{P}}^{(n-3)}\right)\cdots\left(\hat{\boldsymbol{L}}^{(0)}\hat{\boldsymbol{P}}^{(0)}\right)\right]^{-1}\end{aligned}$$

the problem is to find $\boldsymbol{L}$ (a lower triangular matrix) and $\boldsymbol{P}$ (a permutation matrix). Then

$$\begin{aligned}\boldsymbol{U} &= \boldsymbol{L}^{-1}\boldsymbol{P}\boldsymbol{A}\\ \boldsymbol{P}\boldsymbol{A} &= \boldsymbol{L}\boldsymbol{U}\end{aligned}$$

Let us define

$$\begin{aligned} \boldsymbol{B} &:= \boldsymbol{P}^T \boldsymbol{L} \\ &= \boldsymbol{P}^T \boldsymbol{P} \left[\left(\hat{\boldsymbol{L}}^{(n-2)} \hat{\boldsymbol{P}}^{(n-2)} \right) \left(\hat{\boldsymbol{L}}^{(n-3)} \hat{\boldsymbol{P}}^{(n-3)} \right) \cdots \left(\hat{\boldsymbol{L}}^{(0)} \hat{\boldsymbol{P}}^{(0)} \right) \right]^{-1} \end{aligned}$$

then

$$\begin{aligned} \boldsymbol{B} &= \left[\left(\hat{\boldsymbol{L}}^{(n-2)} \hat{\boldsymbol{P}}^{(n-2)} \right) \left(\hat{\boldsymbol{L}}^{(n-3)} \hat{\boldsymbol{P}}^{(n-3)} \right) \cdots \left(\hat{\boldsymbol{L}}^{(0)} \hat{\boldsymbol{P}}^{(0)} \right) \right]^{-1} \\ &= \boldsymbol{P}^{(0)} \boldsymbol{L}^{(0)} \cdots \boldsymbol{P}^{(n-2)} \boldsymbol{L}^{(n-2)} \end{aligned}$$

where $\boldsymbol{P}^{(k)} = \left(\hat{\boldsymbol{P}}^{(k)} \right)^{-1} = \left(\boldsymbol{P}^{(k)} \right)^{-1}$ ($\boldsymbol{P}_{ij} = \boldsymbol{P}_{ij}^{-1}$, $\boldsymbol{P}_{ij} \boldsymbol{P}_{ij} = \boldsymbol{I}$)
and $\boldsymbol{L}^{(k)} = \left(\hat{\boldsymbol{L}}^{(k)} \right)^{-1}$

Since $\boldsymbol{P}^{(1)}\boldsymbol{P}^{(1)} = \boldsymbol{I}$ then

$$
\begin{aligned}
\boldsymbol{B} &= \boldsymbol{P}^{(0)}\boldsymbol{L}^{(0)}\boldsymbol{P}^{(1)}\boldsymbol{L}^{(1)}\cdots\boldsymbol{P}^{(n-2)}\boldsymbol{L}^{(n-2)} \\
&= \boldsymbol{P}^{(0)}\boldsymbol{P}^{(1)}\left(\boldsymbol{P}^{(1)}\boldsymbol{L}^{(0)}\boldsymbol{P}^{(1)}\right)\boldsymbol{L}^{(1)}\cdots\boldsymbol{P}^{(n-2)}\boldsymbol{L}^{(n-2)}
\end{aligned}
$$

$\boldsymbol{L}_0 := \boldsymbol{P}^{(1)}\boldsymbol{L}^{(0)}\boldsymbol{P}^{(1)}$, where $\boldsymbol{L}_0$ and $\boldsymbol{L}^{(0)}$ have the same matrix form. Since the application of $\boldsymbol{P}^{(1)}$ on both sides of $\boldsymbol{L}^{(0)}$ only permutes subrows.

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{P}^{(0)}\boldsymbol{P}^{(1)}\boldsymbol{L}_0\boldsymbol{L}^{(1)}\boldsymbol{P}^{(2)}\boldsymbol{L}^{(2)}\cdots\boldsymbol{P}^{(n-2)}\boldsymbol{L}^{(n-2)} \\ &= \boldsymbol{P}^{(0)}\boldsymbol{P}^{(1)}\boldsymbol{P}^{(2)}\left(\boldsymbol{P}^{(2)}\boldsymbol{L}_{0\to1}\boldsymbol{P}^{(2)}\right)\boldsymbol{L}^{(2)}\cdots\boldsymbol{P}^{(n-2)}\boldsymbol{L}^{(n-2)} \end{aligned}$$

since $\boldsymbol{L}_0\boldsymbol{L}^{(1)}$ can be written as $\boldsymbol{L}_{0\to1}$.

Note $\boldsymbol{L}_1 := \boldsymbol{P}^{(2)}\boldsymbol{L}_{0\to1}\boldsymbol{P}^{(2)}$ and $\boldsymbol{L}_{0\to1}$ has the same matrix form, since the application of $\boldsymbol{P}^{(2)}$ on both sides of $\boldsymbol{L}_{0\to1}$ only permutes subrows.

$$
\begin{aligned}
\boldsymbol{B} &= \boldsymbol{P}^{(0)}\boldsymbol{P}^{(1)}\boldsymbol{P}^{(2)}\left(\boldsymbol{P}^{(3)}\boldsymbol{L}_1\boldsymbol{L}^{(2)}\boldsymbol{P}^{(3)}\right)\boldsymbol{L}^{(3)}\cdots\boldsymbol{P}^{(n-2)}\boldsymbol{L}^{(n-2)} \\
&= \boldsymbol{P}^{(0)}\boldsymbol{P}^{(1)}\boldsymbol{P}^{(2)}\left(\boldsymbol{P}^{(3)}\boldsymbol{L}_{0\to 2}\boldsymbol{P}^{(3)}\right)\boldsymbol{L}^{(3)}\cdots\boldsymbol{P}^{(n-2)}\boldsymbol{L}^{(n-2)}
\end{aligned}
$$

since $\boldsymbol{L}_1\boldsymbol{L}^{(2)}$ can be written as $\boldsymbol{L}_{0\to 2}$.

Note $\boldsymbol{L}_2 := \boldsymbol{P}^{(3)}\boldsymbol{L}_{0\to 2}\boldsymbol{P}^{(3)}$ and $\boldsymbol{L}_{0\to 2}$ has the same matrix form, since the application of $\boldsymbol{P}^{(3)}$ on both sides of $\boldsymbol{L}_{0\to 2}$ only permutes subrows.

We can repeat the previous procedure,

$$\boldsymbol{B} = \boldsymbol{P}^{(0)}\boldsymbol{P}^{(1)}\boldsymbol{P}^{(2)}\boldsymbol{P}^{(3)}\cdots\boldsymbol{P}^{(n-2)}\boldsymbol{L}_{n-3}\boldsymbol{L}^{(n-2)}$$

with $\boldsymbol{L}_{n-3} := \boldsymbol{P}^{(n-2)}\boldsymbol{L}_{0\to n-3}\boldsymbol{P}^{(n-2)}$. Since

$$\boldsymbol{B} = \boldsymbol{P}^T\boldsymbol{L}$$

we can define $\boldsymbol{P}^T := \boldsymbol{P}^{(0)}\boldsymbol{P}^{(1)}\boldsymbol{P}^{(2)}\boldsymbol{P}^{(3)}\cdots\boldsymbol{P}^{(n-2)}$ and $\boldsymbol{L} := \boldsymbol{L}_{n-3}\boldsymbol{L}^{(n-2)} = \boldsymbol{L}_{0\to n-2}$. Note that

$$\boldsymbol{P} = \boldsymbol{P}^{(n-2)}\boldsymbol{P}^{(n-3)}\cdots\boldsymbol{P}^{(0)}$$

Note that $\boldsymbol{P}$ is an orthogonal and $\boldsymbol{L}$ is a lower triangular matrix. We can use the recurrence

$$\begin{aligned}\boldsymbol{P} &= \boldsymbol{P}^{(0)}\\ \boldsymbol{P} &= \boldsymbol{P}^{(k)}\boldsymbol{P}\end{aligned}$$

to compute $\boldsymbol{P}$

Summary

Matrix form of LU with permutation

$$\begin{aligned}
\boldsymbol{U} &= \left(\hat{\boldsymbol{L}}^{(n-2)}\hat{\boldsymbol{P}}^{(n-2)}\right)\left(\hat{\boldsymbol{L}}^{(n-3)}\hat{\boldsymbol{P}}^{(n-3)}\right)\cdots\left(\hat{\boldsymbol{L}}^{(0)}\hat{\boldsymbol{P}}^{(0)}\right)\boldsymbol{A} \\
\boldsymbol{P} &= \boldsymbol{P}^{(n-2)}\boldsymbol{P}^{(n-3)}\dots\boldsymbol{P}^{(0)} \\
\boldsymbol{L} &= \boldsymbol{L}_{0\to n-2}
\end{aligned}$$

with the following sequence of subrows permutation and column stacking to compute $\boldsymbol{L}$

$$\begin{aligned}
\boldsymbol{L}_{0\to 0} &:= \boldsymbol{L}_0 \\
\boldsymbol{L}_{k-1} &:= \boldsymbol{P}^{(k)}\boldsymbol{L}_{0\to k-1}\boldsymbol{P}^{(k)} \text{ subrow permutation} \\
\boldsymbol{L}_{0\to k} &:= \boldsymbol{L}_{k-1}\boldsymbol{L}^{(k)} \text{ column stacking}
\end{aligned}$$

for $k = 1, 2, \cdots, n-2$

Example

$$\boldsymbol{A} = \begin{pmatrix} 3 & -8 & -6 & 6 \\ 6 & -18 & -12 & 12 \\ 2 & -5 & -5 & 5 \\ 3 & -7 & -10 & 8 \end{pmatrix}$$

Permutation

$$\boldsymbol{P}_{01}\boldsymbol{A} = \begin{pmatrix} 6 & -18 & -12 & 12 \\ 3 & -8 & -6 & 6 \\ 2 & -5 & -5 & 5 \\ 3 & -7 & -10 & 8 \end{pmatrix}$$

Example

Gaussian Elimination

$$\hat{\boldsymbol{L}}^0 \boldsymbol{P}_{01} \boldsymbol{A} = \begin{pmatrix} 6 & -18 & -12 & 12 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -4 & 2 \end{pmatrix}$$

Permutation

$$\boldsymbol{P}_{13} \hat{\boldsymbol{L}}^0 \boldsymbol{P}_{01} \boldsymbol{A} = \begin{pmatrix} 6 & -18 & -12 & 12 \\ 0 & 2 & -4 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Example

Gaussian Elimination

$$\hat{\boldsymbol{L}}^1 \boldsymbol{P}_{13} \hat{\boldsymbol{L}}^0 \boldsymbol{P}_{01} \boldsymbol{A} = \begin{pmatrix} 6 & -18 & -12 & 12 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$$

Permutation

$$\boldsymbol{P}_{23} \hat{\boldsymbol{L}}^1 \boldsymbol{P}_{13} \hat{\boldsymbol{L}}^0 \boldsymbol{P}_{01} \boldsymbol{A} = \begin{pmatrix} 6 & -18 & -12 & 12 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Example

Gaussian Elimination

$$\hat{\boldsymbol{L}}^2 \boldsymbol{P}_{23} \hat{\boldsymbol{L}}^1 \boldsymbol{P}_{13} \hat{\boldsymbol{L}}^0 \boldsymbol{P}_{01} \boldsymbol{A} = \begin{pmatrix} 6 & -18 & -12 & 12 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}$$

the we obtain $\boldsymbol{U} := \hat{\boldsymbol{L}}^2 \boldsymbol{P}_{23} \hat{\boldsymbol{L}}^1 \boldsymbol{P}_{13} \hat{\boldsymbol{L}}^0 \boldsymbol{P}_{01} \boldsymbol{A}$.

In order to find $\boldsymbol{L}$ and $\boldsymbol{P}$, let us choose $\boldsymbol{B}$ such that $\boldsymbol{B}\boldsymbol{U} = \boldsymbol{A}$

Example

Now, we are going to compute $\boldsymbol{P}$ and $\boldsymbol{L}$,

$$
\begin{aligned}
\boldsymbol{B} &:= (\hat{\boldsymbol{L}}^2 \boldsymbol{P}_{23} \hat{\boldsymbol{L}}^1 \boldsymbol{P}_{13} \hat{\boldsymbol{L}}^0 \boldsymbol{P}_{01})^{-1} \\
&= \boldsymbol{P}_{01} \boldsymbol{L}^0 \boldsymbol{P}_{13} \boldsymbol{L}^1 \boldsymbol{P}_{23} \boldsymbol{L}^2 \\
&= \boldsymbol{P}_{01} (\boldsymbol{P}_{13} \boldsymbol{P}_{13}) \boldsymbol{L}^0 \boldsymbol{P}_{13} \boldsymbol{L}^1 \boldsymbol{P}_{23} \boldsymbol{L}^2 \\
&= \boldsymbol{P}_{01} \boldsymbol{P}_{13} (\boldsymbol{P}_{13} \boldsymbol{L}^0 \boldsymbol{P}_{13}) \boldsymbol{L}^1 \boldsymbol{P}_{23} \boldsymbol{L}^2 \\
&= \boldsymbol{P}_{01} \boldsymbol{P}_{13} \boldsymbol{L}_0 \boldsymbol{L}^1 \boldsymbol{P}_{23} \boldsymbol{L}^2 \\
&= \boldsymbol{P}_{01} \boldsymbol{P}_{13} \boldsymbol{L}_{0\to 1} \boldsymbol{P}_{23} \boldsymbol{L}^2
\end{aligned}
$$

where $\boldsymbol{L}_0 = \boldsymbol{P}_{13} \boldsymbol{L}^0 \boldsymbol{P}_{13}$ and $\boldsymbol{L}_{0\to 1} = \boldsymbol{L}_0 \boldsymbol{L}^1$.

Example

$$
\begin{aligned}
\boldsymbol{B} &= \boldsymbol{P}_{01}\boldsymbol{P}_{13}(\boldsymbol{P}_{23}\boldsymbol{P}_{23})\boldsymbol{L}_{0\to1}\boldsymbol{P}_{23}\boldsymbol{L}^2 \\
&= \boldsymbol{P}_{01}\boldsymbol{P}_{13}\boldsymbol{P}_{23}(\boldsymbol{P}_{23}\boldsymbol{L}_{0\to1}\boldsymbol{P}_{23})\boldsymbol{L}^2 \\
&= \boldsymbol{P}_{01}\boldsymbol{P}_{13}\boldsymbol{P}_{23}\boldsymbol{L}_1\boldsymbol{L}^2 \\
&= \boldsymbol{P}^T\boldsymbol{L}_{0\to2} \\
&= \boldsymbol{P}^T\boldsymbol{L}
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{L}_1 &= \boldsymbol{P}_{23}\boldsymbol{L}_{0\to1}\boldsymbol{P}_{23} \\
\boldsymbol{P} &= \boldsymbol{P}_{23}\boldsymbol{P}_{13}\boldsymbol{P}_{01} \\
\boldsymbol{L}_{0\to2} &= \boldsymbol{L}_1\boldsymbol{L}^2
\end{aligned}
$$

since $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{U}$ then $\boldsymbol{P}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$

Example

Inverse of the first Gaussian elimination matrix

$$\boldsymbol{L}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix}$$

Preserving from property

$$\boldsymbol{L}_0 = \boldsymbol{P}_{13}\boldsymbol{L}^0\boldsymbol{P}_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ {\color{red}1/2} & 1 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \\ {\color{blue}1/2} & 0 & 0 & 1 \end{bmatrix}$$

Example

Inverse of the second Gaussian elimination matrix

$$L^1 \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix}$$

Column stacking of Gaussian elimination matrices

$$L_{0\to 1} := L_0 L^1 \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & 1/2 & 1 & 0 \\ 1/2 & 1/2 & 0 & 1 \end{bmatrix}$$

Example

Preserving from property

$$\boldsymbol{L}_1 = \boldsymbol{P}_{23}\boldsymbol{L}_{0\to1}\boldsymbol{P}_{23} \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ {\color{red}1/2} & {\color{red}1/2} & 1 & 0 \\ {\color{blue}1/3} & {\color{blue}1/2} & 0 & 1 \end{bmatrix}$$

Inverse of the third Gaussian elimination matrix

$$\boldsymbol{L}^2 \quad = \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \end{bmatrix}$$

Example

Column stacking of Gaussian elimination matrices

$$\boldsymbol{L}_1\boldsymbol{L}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 1/2 & 1 & 0 \\ 1/3 & 1/2 & 1/2 & 1 \end{bmatrix}$$

Matrix $\boldsymbol{L}$:

$$\boldsymbol{L} := \boldsymbol{L}_{0\to 2} = \boldsymbol{L}_1\boldsymbol{L}^2$$

Example

Permutation matrix $\boldsymbol{P}$

$$
\begin{aligned}
\boldsymbol{P}^T &= \boldsymbol{P}_{01}\boldsymbol{P}_{13}\boldsymbol{P}_{23} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
\boldsymbol{P} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

Example

Finally,

$$\begin{aligned} \boldsymbol{U} &= \hat{\boldsymbol{L}}_2 \boldsymbol{P}_{23} \hat{\boldsymbol{L}}_1 \boldsymbol{P}_{13} \hat{\boldsymbol{L}}_0 \boldsymbol{P}_{01} \boldsymbol{A} \\ \boldsymbol{P} &= \boldsymbol{P}_{23} \boldsymbol{P}_{13} \boldsymbol{P}_{01} \\ \boldsymbol{L} &= \boldsymbol{L}_{0\to 2} \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{L}_0 &= \boldsymbol{P}_{13} \boldsymbol{L}^0 \boldsymbol{P}_{13} \\ \boldsymbol{L}_{0\to 1} &= \boldsymbol{L}_0 \boldsymbol{L}^1 \\ \boldsymbol{L}_1 &= \boldsymbol{P}_{23} \boldsymbol{L}_{0\to 1} \boldsymbol{P}_{23} \\ \boldsymbol{L}_{0\to 2} &= \boldsymbol{L}_1 \boldsymbol{L}^2 \end{aligned}$$

Example

the PLU (permutation, lower-, and upper-triangular matrices) decomposition (factorization) is

$$\boldsymbol{PA} = \boldsymbol{LU}$$

with

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \; \boldsymbol{A} = \begin{bmatrix} 3 & -8 & -6 & 6 \\ 6 & -18 & -12 & 12 \\ 2 & -5 & -5 & 5 \\ 3 & -7 & -10 & 8 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 1/2 & 1 & 0 \\ 1/3 & 1/2 & 1/2 & 1 \end{bmatrix}; \; \boldsymbol{U} = \begin{bmatrix} 6 & -18 & -12 & 12 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$