# Online linear optimization with the log-determinant regularizer: Moridomi et al. (2017)

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# Outline

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# Collaborative filtering

- Making predictions about the preferences of a user for some items (e.g. based on ratings) based on the preferences of many others
- The information about the users' preferences increases over time.
  - At each t, we get new couples $(user_i, item_j)$ and we need to predict the preferences of some users for some items.
- This calls for an online learning approach.
- A standard framework: Follow-The-Leader (FTL)

# Follow-The-Leader

- Consider a convex function $f$
- Suppose we want to minimize the regret over a convex set $S$, where the regret is given by:

$$Regret(T,S) = \sum_{t=1}^{T} f_t(w_t) - \min_{u \in S} \sum_{t=1}^{T} f_t(u)$$

- At each t, we chose the vector which minimizes the cumulative loss:

**Follow-The-Leader (FTL)**

At each t:

$$w_t = \operatorname*{argmin}_{w} \sum_{s=1}^{t-1} f_s(w)$$

# Follow-The-Leader: success and failure

- Consider the following function $f_t(w) = \frac{1}{2}\|w - z_t\|_2^2$ for some $z_t$ and note $L = max_t\|z_t\|$. Then we can show that FTL achieves a low upper bound of regret of $4L^2(\log(T) + 1)$.
- We can show that FTL fails to solve irregular problems like the following:
  - Consider S = [-1, 1] and $f_t(w) = z_t w$, where:

$$z_t = \begin{cases} -0.5, & t = 1 \\ \ \ 1 \ \ , & t \ is \ even \\ -1 \ , t > 1 \ and \ t \ is \ odd \end{cases}$$

  - FTL will predict $w_t = 1$ if t is odd and $w_t = -1$ if t is even. The cumulative loss at $T$ will be $T$. Hence, comparing with the solution $u = 0$, we have a regret of T.

# Follow-The-Regularized-Leader

- To add stability to the algorithm, we add a regularizer to the FTL algorithm.

**Follow-The-Regularized-Leader (FTRL)**

At each t:

$$w_t = \underset{w}{\operatorname{argmin}} \; R(w) + \eta \sum_{s=1}^{t-1} f_s(w)$$

- Examples of regularizers:
  - $R(w) = \frac{1}{2}\|w\|_2^2$: euclidean regularization
  - $R(w) = -\sum_{i=1}^{d} w_i \log(w_i)$ entropic regularization
- Their matrix analogous:
  - $R(X) = \frac{1}{2}\|X\|_{Fr}^2$
  - $R(X) = Tr(X \log X - X)$

# Notations

In the following we define:

- For $\sigma_i$, the ith largest singular value of $X$ :
  - $\|X\|_{Tr} = \sum_i \sigma_i$
  - $\|X\|_{Sp} = \max_i \sigma_i$
  - $\|X\|_{Fr} = \sqrt{\sum_i \sigma_{i^2}}$
- E is the identity matrix
- $\forall X \in \mathbf{R}^{m \times n}$, $vec(X) = (X_{*1}^T, ..., X_{*m}^T)$ where $X_{*i}$ is the ith column of $X$
- For $m \times n$ matrices, $X \bullet L = \sum_{i,j}^{m,n} X_{ij} L_{ij} = vec(X)^T vec(L)$

# Outline

# Online SDP problems and regret bounds given by FTRL

In this section, we will:

- Derive a first bound of the regret is the FTRL in lemma 2.1
- Apply this bound to the Frobenius norm regularization in lemma 2.2
    - Show that the FTRL with the Frobenius norm regularization yields to the online gradient descent (OGD) algorithm seen in class (and go back to the vector case to show that)
    - Prove the lemma 2.1 in the vector case to obtain the same regret bound on the OGD as the one seen in class through a different proof
- Apply this bound to the log determinant regularization

# Online SDP problems

**Definition: Online SDP Problem**

An online SDP problem is defined by a pair $(\mathcal{K}, \mathcal{L})$ where $\mathcal{K} \subseteq \mathbb{S}_{+}^{N\times N}$ and $\mathcal{L} \subseteq \mathbb{S}^{N\times N}$. $\mathcal{K}$ is called the decision space and $\mathcal{L}$ is called the loss space. The online SDP problem is described by the following protocol:
At $t = 1, 2, ..., T$ the algorithm

- Chooses a matrix $X_t \in \mathcal{K}$
- Receives a loss matrix $L_t \in \mathcal{L}$
- Suffers the loss $X_t \bullet L_t$

The goal is to minimize the regret:

$$Reg(T, \mathcal{K}, \mathcal{L}) = \sum_{i=1}^{T} L_t \bullet X_t - \min_{U\in\mathcal{K}} \sum_{i=1}^{T} L_t \bullet U$$

# Follow the Regularized Leader

## Follow the Regularized Leader

For the online SDP problem $(\mathcal{K}, \mathcal{L})$, given a convex function $R : \mathcal{K} \to \mathbb{R}$, the FTRL with the regularizer $R$ suggests to choose a matrix $X_t \in \mathcal{K}$ as the decision at each round $t$ according to:

$$X_t = \operatorname*{argmin}_{X \in \mathcal{K}} \left( R(X) + \eta \sum_{s=1}^{t-1} L_s \bullet X \right)$$

where $\eta > 0$ is the learning rate.

# A first bound of the FTRL regret

**Lemma 2.1: A first bound of the FTRL regret (from S. Shalev-Schwartz)**

Assume that, for some real numbers $s, g > 0$ and a norm $\|.\|$, the following holds:

- R is s-strongly convex with respect to the norm $\|.\|$, ie $R(X) \geq R(Y) + \nabla R(X) \bullet (X - Y) + \frac{s}{2}\|X - Y\|^2$, or equivalently, if R is twice differentiable, for any $X \in \mathcal{K}$ and $W \in \mathbf{N} \times \mathbf{N}$, $vec(W)^T \nabla^2 R(X) vec(W) \geq s\|W\|^2$
- Any loss matrix $L \in \mathcal{L}$ satisfies $\|L\|_* \leq g$, where $\|.\|_*$ is the dual norm of $\|.\|$

Then the FTRL with regularizer $R$ achieves for an appropriate choice of $\eta$.:

$$Reg(T, \mathcal{K}, \mathcal{L}) \leq 2g \times \sqrt{\frac{max_{X,X' \in \mathcal{K}}(R(X) - R(X'))}{s} T}$$

# Application: FTRL with the Frobenius norm regularization

**Lemma 2.2: FTRL with the Frobenius norm regularization**

The FTRL with the Frobenius norm regularization $R(X) = \frac{1}{2}\|X\|_{Fr}^2$ achieves:

$$Reg(T, \mathcal{K}_2, \mathcal{L}_2) \leq \rho\gamma_2 \times \sqrt{2T}$$

where

- $\mathcal{K}_2 = \{X \in \mathbf{S}_+^{N \times N} : \|X\|_{Fr} \leq \rho\}$
- $\mathcal{L}_2 = \{L \in \mathbf{S}^{N \times N} : \|L\|_{Fr} \leq \gamma_2\}$

The FTRL with this regularizer yields the online gradient descent (OGD) algorithm seen in class.

# Proof of the lemma

Let's first show that $R$ is 1-strongly with respect to $\|.\|_{Fr}$:

$$\forall W \in \mathbf{R}^{N\times N}, vec(W)^T \nabla^2 R(X) vec(W) = Tr(W)^2 \geq \|W\|_{Tr}^2$$

Now, using lemma 2.1, as the dual of $\|.\|_{Fr}$ is $\|.\|_{Fr}$:

$$\begin{aligned} Reg(T, \mathcal{K}_2, \mathcal{L}_2) &\leq 2\gamma_2 \times \sqrt{max_{X,X' \in \mathcal{K}}(R(X) - R(X'))T} \\ &= \gamma_2 \times \sqrt{4max_{X,X' \in \mathcal{K}}(R(X) - R(X'))T} \end{aligned}$$

Furthermore:

$$\begin{aligned} \forall X, X' \in \mathcal{K}, R(X) - R(X') &= \frac{1}{2}(\|X\|_{Fr^2} - \|X'\|_{Fr^2}) \leq \frac{1}{2}\|X - X'\|_{Fr}^2 \\ &\leq \frac{1}{2}\rho^2 \end{aligned}$$

Finally, we get: $Reg(T, \mathcal{K}_2, \mathcal{L}_2) \leq \rho\gamma_2 \times \sqrt{2T}$

# Link with the OGD algorithm in the vector case

All the results presented in the article for matrices can be used in particular for vectors by noticing that we can restrict $\mathcal{K}$ and $\mathcal{L}$ to diagonal matrices.
In particular:

$$\mathcal{K} = \{x \in \mathbb{R}_+^{N \times N}, \|diag(x)\|_{Fr} \leq \rho\} = \{x \in \mathbb{R}_+^{N \times N}, \|x\|_2 \leq \rho\}$$
$$\mathcal{L} = \{x \in \mathbb{R}^{N \times N}, \|diag(l)\|_{Fr} \leq \gamma_2\} = \{l \in \mathbb{R}^{N \times N}, \|l\|_2 \leq \gamma_2\}$$

We will now show that following the FTRL framework with the regularizer : $R(x) = \frac{1}{2}\|x\|_2^2$ is equivalent to running the Online Gradient Descent algorithm. Using the same hypotheses as in class, which amounts to chosing $\mathcal{K}$ and $\mathcal{L}$ defined as above, we derive the same bounds, but using a different proof.

# FTRL with the $l_2$ norm regularization and OGD in the vector case

**Online Gradient Descent**

Let $g_t : \mathcal{X} \to \mathbb{R}$ be a convex loss function. Let $\eta > 0$ and $x_1 = 0$.

- At t=1,..., T:

$$x_{t+1} = x_t - \eta . z_t, \text{ where } z_t \in \delta g_t(x_t)$$

In our case, $g_t(x) = x^\intercal l_t$. Hence the OGD algorithm writes:

- At t=1,..., T:

$$x_{t+1} = x_t - \eta . l_t$$

# FTRL with the $l_2$ norm regularization and OGD in the vector case

Suppose $R(x) = \frac{1}{2}\|x\|_2^2$

- Then in the FTRL framework, we chose $x_t$ at each iteration such that:

$$x_t = \operatorname{argmin} \eta \textstyle\sum_{i=1}^{t-1} x^\intercal l_i + \frac{1}{2}\|x\|_2^2$$

- Taking the FOC for the $t$-th and the $t+1$-th iterations, we have :

$$\eta \sum_{i=1}^{t} l_i + x_{t+1} = 0$$

$$\eta \sum_{i=1}^{t-1} l_i + x_t = 0$$

- Hence, at each iteration $t$, we chose $x_{t+1}$ s.t :

$$x_{t+1} = x_t - \eta l_t$$

# Bound on the FTRL with the $l_2$-norm regularizer in the vector case

To recover the $\rho\gamma_2\sqrt{2T}$ found in class, we propose to prove the following lemma, which is a particular case of lemma 2.1, in the vector case:

**Lemma: FTRL regret in the vector case for strongly convex regularizers**

Assume that :

- $R$ is s-strongly convex
- $\mathcal{K}$ and $\mathcal{L}$ are defined as above

Then, for an appropriate choice of $\eta$:

$$Reg(T, \mathcal{K}, \mathcal{L}) \leq 2\gamma_2 \times \sqrt{\frac{\max_{x,x' \in \mathcal{K}}(R(x) - R(x'))}{s} T}$$

In particular, for $R(x) = \frac{1}{2}\|x\|_2^2$:

$$Reg(T, \mathcal{K}, \mathcal{L}) \leq \rho\gamma\sqrt{2T}$$

# Proof of the lemma

We note: $g_t(x) = x^\intercal l_t$

- **Step 1**: We show that $\forall u \in \mathcal{K}$ :

$$\eta \sum_{t=1}^{T}(g_t(x_t) - g_t(u)) \leq R(u) - R(x_1) + \eta \sum_{t=1}^{T}(g_t(x_t) - g_t(x_{t+1}))$$

This amounts to proving that: $\eta \sum_{t=1}^{T} g_t(x_{t+1}) \leq R(u) - R(x_1) + \eta \sum_{t=1}^{T} g_t(u)$.
By induction, assume that

$$\forall u \in \mathcal{K},\ \eta \sum_{t=1}^{T-1} g_t(x_{t+1}) \leq R(u) - R(x_1) + \eta \sum_{t=1}^{T-1} g_t(u)$$

Adding $\eta . g_T(x_{T+1})$ to both sides and taking $u = x_{T+1}$ leads to:

$$\eta \sum_{t=1}^{T} g_t(x_{t+1}) \leq -R(x_1) + R(x_{T+1}) + \eta \sum_{t=1}^{T-1} g_t(x_{T+1})$$

And since $x_{T+1} = \operatorname{argmin} R(u) + \eta \sum_{t=1}^{T} g_t(u)$, we have $\forall u \in \mathcal{K}$:

$$\eta \sum_{t=1}^{T} g_t(x_{t+1}) \leq R(u) - R(x_1) + \eta \sum_{t=1}^{T} g_t(u)$$

which concludes the induction's proof.
Upper bounding $R(u) - R(x_1)$ by $\max_{x,x' \in \mathcal{K}}(R(x) - R(x'))$ leads to:

$$\sum_{t=1}^{T}(g_t(x_t) - g_t(u)) \leq \frac{1}{\eta} \max_{x,x' \in \mathcal{K}}(R(x) - R(x')) + \sum_{t=1}^{T}(g_t(x_t) - g_t(x_{t+1}))$$

# Proof of the lemma

Hence, we have:

$$Reg(T, \mathcal{K}, \mathcal{L}) \leq \frac{1}{\eta} \max_{x,x' \in \mathcal{K}}(R(x) - R(x')) + \sum_{t=1}^{T}(g_t(x_t) - g_t(x_{t+1}))$$

- **Step 2:** Bounding : $\sum_{t=1}^{T}(g_t(x_t) - g_t(x_{t+1}))$

We note: $G_t(x) = \eta \sum_{i=1}^{t-1} g_i(x) + R(x)$.
Since $R$ is s-strongly convex, $G_t$ and $G_{t+1}$ are s-strongly convex too, and the following two inequalities hold:

$$G_t(x_{t+1}) \geq G_t(x_t) + \frac{s}{2}\| x_t - x_{t+1} \|_2^2$$

$$G_{t+1}(x_t) \geq G_{t+1}(x_{t+1}) + \frac{s}{2}\|x_t - x_{t+1}\|_2^2$$

Which leads to:

$$g_t(x_t) - g_t(x_{t+1}) \geq \frac{s}{\eta}\|x_t - x_{t+1}\|_2^2$$

Noticing that :
$g_t(x_t) - g_t(x_{t+1}) = (x_t - x_{t+1})^T l_t \leq \|x_t - x_{t+1}\|_2 \|l\|_2 \leq \gamma_2 \|x_t - x_{t+1}\|_2$, we have then:

$$\|x_t - x_{t+1}\|_2 \leq \frac{\gamma_2}{s}$$

# Proof of the lemma

Therefore, bounding $g_t(x_t) - g_t(x_{t+1})$ by $\eta\frac{\gamma_2^2}{s}$ and summing over $t = 0, ..., T-1$, we get:

$$\sum_{t=0}^{T-1}(g_t(x_t) - g_t(x_{t+1})) \leq \eta T\frac{\gamma_2^2}{s}$$

Therefore, we have the following bound that depends on $\eta$ :

$$Reg(T, \mathcal{K}, \mathcal{L}) \leq \frac{1}{\eta}\max_{x,x'\in\mathcal{K}}(R(x) - R(x')) + \eta T\frac{\gamma_2^2}{s}$$

The result follows by setting: $\eta = \frac{1}{g}\sqrt{\frac{s}{T}\max_{x,x'\in\mathcal{K}}(R(x) - R(x'))}$:

$$Reg(T, \mathcal{K}, \mathcal{L}) \leq 2\gamma_2\sqrt{\max_{x,x'\in\mathcal{K}}(R(x) - R(x'))\frac{T}{s}}$$

- **Step 4:** $R(x) = \frac{1}{2}\|x\|_2^2$

Since we have $\forall x, x' \in \mathcal{K}$, $\|x\|_2^2 - \|x'\|_2^2 \leq \|x - x'\|_2^2 \leq \rho^2$ and R is 1-strongly convex:

$$Reg(T, \mathcal{K}, \mathcal{L}) \leq \rho\gamma\sqrt{2T}$$

# Application: FTRL with the log-determinant regularization

**Theorem 2.1: FTRL with the log-determinant regularization**

The FTRL with the log-determinant norm regularization $R(X) = -\log\det(X + \epsilon E)$ with $\epsilon = \sigma$ achieves:

$$Reg(T, \mathcal{K}_\infty, \mathcal{L}_1) \leq 4\sigma\gamma_1 \times \sqrt{TN\log 2}$$

where

- $\mathcal{K}_\infty = \{X \in \mathbf{S}_+^{N\times N} : \|X\|_{Sp} \leq \sigma\}$
- $\mathcal{L}_1 = \{L \in \mathbf{S}^{N\times N} : \|L\|_{Tr} \leq \gamma_1\}$

# Application: FTRL with the log-determinant regularization

- This result is not very impressive: as $\mathcal{K}_\infty \subseteq \mathcal{K}_2$ with $\rho = \sqrt{N}\sigma$ and $\mathcal{L}_1 \subseteq \mathcal{L}_2$ with $\gamma_1 = \gamma_2$, the FTRL with the Frobenius norm has a slightly better regret bound for $(\mathcal{K}_\infty, \mathcal{L}_1)$
- In the fourth part, we will derive a tighter bound for the regret of the FTRL, and the log-determinant regularizer will prove to give the best bound among other regularizers

# Proof of the theorem

Before proving the regret bound, we prove some linear algebra formulas:

**Lemma 2.3: Gradient and Hessian of the log-determinant**

Consider the function $R(X) = -\log\det(X)$ for $X \in \mathbb{S}_+^{n\times n}$. Then:

$$\nabla R(X) = X^{-1}$$
$$\nabla^2 R(X) = X^{-1} \otimes X^{-1}$$

Where $\otimes$ denotes the Kronecker product between two matrices

**Proof of the lemma:**

- **The Gradient**: Consider matrices $Z, X \in \mathbb{S}_+^{n\times n}$ and define $\Delta X = Z - X$ where we suppose $\Delta X \approx 0$

# Proof of the lemma

$$\begin{aligned}\log\det(Z) &= \log\det(X+\Delta X) = \log\det(X^{\frac{1}{2}}(I + X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}})X^{\frac{1}{2}}) \\ &= \log\det(X) + \log\det(I + X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}}) \\ &= \log\det(X) + \sum_{i=1}^{n}\log(1+\lambda_i)\end{aligned}$$

where $(\lambda_i)_i$ are the eigenvalues of $X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}}$.
Since $\Delta X \approx 0$, the eigenvalues of $X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}}$ will be approximately 0.
Using the fact that: $\log(1+\lambda_i) \approx \lambda_i$:

$$\log\det(Z) = \log\det(X) + \sum_{i=1}^{n}\lambda_i = \log\det(X) + Tr(X^{-\frac{1}{2}}\Delta X X^{-\frac{1}{2}})$$

# Proof of the lemma

$$\log\det(Z) = \log\det(X) + Tr(X^{-1}\Delta X) = \log\det(X) + Tr((X^{-1})^T\Delta X)$$

Hence, we have:

$$R(Z) = R(X) + < X^{-1}, Z - X >$$

with $Z \approx X$. Hence: $\nabla R(X) = X^{-1}$

- **The Hessian:**

We use the fact that: $I = XX^{-1}$ and take the derivative with respect to the components of $X$ in the two sides of the equation.
We find that (denoting $A' = \frac{\delta A}{\delta X_{ij}}$) : $0 = X'X^{-1} + X(X^{-1})'$.
Hence: $(X^{-1})' = X^{-1}J_{ij}X^{-1} = (X^{-1} \otimes X^{-1})_{ij}$

# Proof of the theorem

The eigenvalues of $\nabla^2 R(X)$ are a product of eigenvalues of $(X + \epsilon E)^{-1}$ and $(X + \epsilon E)^{-1}$. Hence, the lowest eigenvalue of $\nabla^2 R(X)$ is $(\|X\|_{Sp}^2 + \epsilon)^{-2}$
Hence, since $X \in \mathcal{K}_\infty$:

$$\begin{aligned} vec(W)^T \nabla^2 R(X) vec(W) &\geq (\|X\|_{Sp}^2 + \sigma)^{-2} vec(W)^T vec(W) \\ &= (\|X\|_{Sp}^2 + \sigma)^{-2} \|vec(W)\|_{Fr}^2 \\ &\geq (2\sigma)^{-2} \|vec(W)\|_{Sp}^2 \end{aligned}$$

Hence $R$ is $\frac{1}{4\sigma^2}$ - strongly convex.

- Moreover:

$$\begin{aligned} \det(X + \epsilon E) &= \prod_{i=1}^{N} (\lambda_i + \epsilon) \\ &\leq (\sigma + \epsilon)^N = (2\sigma)^N \end{aligned}$$

Hence:

$$R(X) \leq (2\sigma)^N \text{ . Therefore: } \max_{X, X' \in \mathcal{K}} R(X) - R(X') \leq N \log(2)$$

# Outline

# OMP and reduction to online SDP

In this section, we will:

- Present the general theory of OMP and show that well-known online learning issues such as online collaborative filtering and online max-cut can be formulated in the OMP framework
- Show that OMP problems can be reduced to SDP problems under some conditions, hence our interest for SDP problems
- Apply these results to the collaborative filtering example

# Examples of Online Matrix Prediction problems

## Online collaborative filtering

Consider a number $m$ of users and $n$ of items and an initial ratings matrix $W_0 \in \mathbf{R}^{m \times n}$. In each round $t = 1, 2, ..., T$, the algorithm:

- receives a couple (user, item): $(i_t, j_t)$
- predicts how much user $i_t$ likes item $j_t$: $\hat{y}_t = W_t(i_t, j_t)$
- suffers a loss $l_t(\hat{y}_t, y_t)$ when the true preference $y_t$ is revealed

## Online max-cut

Consider a graph of n nodes where one predicts if there is a vertice between two nodes. Consider an initial matrix $W_0 \in \{-1, 1\}^{m \ \times n}$. In each round $t = 1, 2, ..., T$, the algorithm:

- receives a pair of nodes: $(i_t, j_t)$
- predicts if there is an edge: $\hat{y}_t = W_t(i_t, j_t) \in \{-1, 1\}$
- suffers a loss $l_t(\hat{y}_t)$ when $y_t$ reveals whether there is an edge or not

## Definition 3.1: Online Matrix Prediction problem

Let $m, n \in \mathbb{N}^*$. An Online Matrix Prediction problem is defined by a couple $(\mathcal{W}, G)$, where $\mathcal{W} \subset [-1,1]^{m \times n}$ and $G \in \mathbb{R}_*^+$. In each round $t = 1, 2..., T$, the algorithm:

- receives a couple $(i_t, j_t) \in \{1, .., m\} \times \{1, ..., n\}$
- choses $W_t \in \mathcal{W}$ and predicts $\hat{y}_t = W_t(i_t, j_t)$
- receive a G-lipschitz loss function $l_t$
- suffers the loss $l_t(\hat{y}_t)$

The goal is then to minimize the regret at T defined by:

$$Reg_{OMP}(T, \mathcal{W}) = \sum_{t=1}^{T} l_t(W_t(i_t, j_t)) - \min_{U \in \mathcal{W}} \sum_{t=1}^{T} l_t(U(i_t, j_t))$$

# Reduction of OMP to online SDP

- We don't know how to tightly upper bound OMP problems' regret directly.
- But we can reduce OMP problem to online SDP problems if the OMP problem $\mathcal{W}$ is $(\beta, \tau)$-decomposable.

For a matrix $W \in \mathcal{W}$, let

$$sym(W) = \begin{bmatrix} 0 & W \\ W^T & 0 \end{bmatrix}$$

if $\mathcal{W}$ is not symmetric (ie some $W \in \mathcal{W}$ is not symmetric) and $sym(W) = W$ otherwise.
Let $p$ be the order of $sym(W)$, that is, $p = m + n$ if $\mathcal{W}$ is not symmetric, and $p = n$ otherwise.

# Reduction of OMP to online SDP

**Definition 3.2: $(\beta, \tau)$-decomposability**

$W$ is said to be $(\beta, \tau)$-decomposable if there exists $P, Q \in \mathbb{S}_{+}^{p \times p}$, where p is the order of $sym(W)$, such that:

- $sym(W) = P - Q$
- $Tr(P + Q) \leq \tau$
- $P_{ii}, Q_{ii} \leq \beta$ for any $i \in \{1, ..., p\}$

$\mathcal{W}$ is said to be $(\beta, \tau)$-decomposable if every $W \in \mathcal{W}$ is $(\beta, \tau)$ -decomposable.

# Reduction of OMP to online SDP

- The following theorem states that any OMP problem $(\mathcal{W}, G)$ can be reduced to a particular online SDP problem.

**Theorem 3.1: Reduction of OMP to online SDP**

Let ($\mathcal{W}$, G) be the OMP problem where $\mathcal{W} \subset [-1, 1]^{m+n}$ is $(\beta, \tau)$ - decomposable. Then this problem can be reduced to the online SDP problem $(\mathcal{K}, \mathcal{L})$, with $N = 2(m+n)$ if $\mathcal{W}$ is not symmetric and $N = m+n$ otherwise, defined by:

$$\mathcal{K} = \{X \in \mathbb{S}_+^{N \times N} : \|X\|_{Tr} \leq \tau, \forall i \in [N], X_{ii} \leq \beta\}$$
$$\mathcal{L} = \{L \in \mathbb{S}^{N \times N} : L_{ij} \leq G\ \forall (i,j) \in [N] \times [N], |(i,j) : L_{ij} \neq 0| = 4, L^2 \text{diagonal}\}$$

And we have the following upper bound on the OMP regret:

$$Reg_{OMP}(T, \mathcal{W}) \leq \frac{1}{2} Reg(T, \mathcal{K}, \mathcal{L})$$

# Proof of the reduction

We need to prove that for every $W \in \mathcal{W}$ and for every $G > 0$, we can find an online SDP problem whose regret tighthly upper bounds that of the OMP problem.
Consider $W \in \mathcal{W} \subset [-1,1]^{m \times n}$. Then $sym(W) \in [-1,1]^{p \times p}$, where $p = m + n$.

- **Step 1:** $\mathcal{K}$
  - Since $W$ is $(\beta, \tau)$ - decomposable, $\exists P, Q \in \mathbb{S}_+^{p \times p}$ s.t:
    - $sym(W) = P - Q$
    - $Tr(P) + Tr(Q) \leq \tau$
    - $P_{ii}, Q_{ii} \leq \beta$ for all $i \in \{1, ..., p\}$
  - Noting $N = 2p$, we define:

$$\phi(W) = \begin{bmatrix} P & 0 \\ 0 & N \end{bmatrix} \in \mathbb{S}_+^{N \times N}$$

  - We can easily verify that $\phi(W) \in \mathcal{K}$

# Proof of the reduction

- **Step 2:** $\mathcal{L}$

  Given the loss functions $(l_t)_{t\in[T]}$ of the OMP problem $(\mathcal{W}, G)$, we need to find a loss space $\mathcal{L}$ such that:

  $$Reg_{OMP}(T, \mathcal{W}) \leq \frac{1}{2} Reg(T, \mathcal{K}, \mathcal{L})$$

  - At each iteration t, in the OMP problem, we need to predict: $W_t[i_t, j_t]$. Noting $q = 0$ if $W$ is symmetric and $q = m$ otherwise, we predict:

    $$\begin{aligned} \hat{y}_t = W_t[i_t, j_t] &= sym(W_t)[i_t, j_t + q] = P_t[i_t, j_t + q] - N_t[i_t, j_t + q] \\ &= \phi(W_t)[i_t, j_t + q] - \phi(W_t)[p + i_t, p + j_t + q] \end{aligned}$$

  - Let $g \in \frac{\delta l_t}{\delta y_t}(\hat{y}_t)$, we define for all $(i, j) \in [N] \times [N]$ :

$$L_t(i,j) = \begin{cases} g & \text{if } (i,j) = (i_t, j_t + q) \vee (i,j) = (j_t + q, i_t) \\ -g & \text{if } (i,j) = (p + i_t, p + j_t + q) \vee (i,j) = (p + j_t + q, p + i_t) \\ 0 & \text{otherwise} \end{cases} \qquad (\star)$$

# Proof of the reduction

Any matrix $L_t \in \mathbb{R}^{N \times N}$ which verifies the property $\star$ also verifies the following properties:

- $L_t$ has only 4 non-zero entries: $|(i,j) \in [N] \times [N] : L_{ij} \neq 0| = 4$
- $L_t^2$ is diagonal and also has only 4 non-zero entries all equalling $g^2$. Hence $Tr(L_t^2) = 4g^2$. Therefore, since $l_t$ is $G$ - lipschitz: $Tr(L_t^2) \leq 4G^2$

This justifies the following Lemma:

**Lemma 3.1**

We define:

$$\mathcal{L}^{'} = \{L \in \mathbb{S}^{N \times N} : L \text{ verifies } (\star)\}$$

Then: $\mathcal{L}^{'} \subseteq \mathcal{L}$, hence: $Reg(T, \tilde{\mathcal{K}}, \mathcal{L}^{'}) \leq Reg(T, \tilde{\mathcal{K}}, \mathcal{L})$

# Proof of the reduction

**Step3:** Proving the regret bound
Consider the following algorithm:

**Reduced OMP algorithm**

At each t=1,...,T:

- We receive a pair of indices $(i_t, j_t)$
- We predict $\hat{y}_t = X_t(i_t, j_t + q) - X_t(p + i_t, p + j_t + q)$
- We receive a loss function $l_t$ and pay $l_t(\hat{y}_t)$
- We take $g \in \frac{\delta l_t}{\delta y_t}(\hat{y}_t)$ and define $L_t$ as in $(\star)$
- We update $X_{t+1} = \text{argmin}_{X \in \tilde{\mathcal{K}}} \; R(X) + \eta \sum_{s=1}^{t} L_s \bullet X$

# Proof of the reduction

Let $(W_t)_{t\in[T]} \in \mathcal{W}$, $(L_t)_{t\in[T]}$ and $X_t \in \tilde{K}$ resutling from the algorithm. Then:

$$\phi(W) \bullet L_t = 2g(P[i_t, j_t] - N[i_t, j_t]) = 2gW(i_t, j_t)$$

$$\text{and } X_t \bullet L_t = 2g(X_t[i_t, j_t + q] - X_t[p + i_t, p + i_t + q]) = 2g\hat{y}_t$$

Hence: $X_t \bullet L_t - \phi(W) \bullet L_t = 2g(\hat{y}_t - W(i_t, j_t)) \geq 2l_t(\hat{y}_t) - l_t(W(i_t, j_t))$ by the convexity of $l_t$.
Summing over t=1,...,T:

$$\sum_{t=1}^{T} l_t(\hat{y}_t) - l_t(W(i_t, j_t)) \leq \frac{1}{2}(\sum_{t=1}^{T} X_t \bullet L_t - \phi(W) \bullet L_t) \leq \frac{1}{2} Reg(T, \tilde{\mathcal{K}}, \mathcal{L}')$$

Hence, using the previous lemma:

$$Reg_{OMP}(\mathcal{W}, T) \leq \frac{1}{2} Reg(T, \tilde{\mathcal{K}}, \tilde{\mathcal{L}})$$

# Application to online collaborative filtering

First, let's formulate online collaborative filtering as an OMP problem.

### OMP formulation of online collaborative filtering

The problem of online collaborative filtering can be formulated as the OMP problem $(\mathcal{W}_\tau, G)$ where:

$$\mathcal{W}_\tau = \{W \in [-1,1]^{m \times n} : \|W\|_{Tr}^2 \leq \tau\}$$

where:

- $m$ is the number of users and $n$ the number of items
- $\tau$ is a parameter controlling the rank of the matrices

**Remark**

We want two users with similar preferences for many items to also have the same preferences for other items, which means that we want many rows of our matrix to be similar, i.e. we want our matrix to have a low rank, hence the condition on the trace-norm.

# Decomposability of Online collaborative filtering (OCF)

We now prove the following proposition:

**Proposition 3.1**

$(\mathcal{W}_\tau, G)$ is $(\sqrt{m+n}, 2\tau)$ - decomposable

To do so, we use the following Lemma:

**Lemma 3.2**

Let $Y \in \mathbb{S}^{p \times p} \subseteq [-1,1]^{p \times p}$. Then we can write $Y = P - N$ where:

- $P, N \in \mathbb{S}_+^{p \times p}$
- $Tr(P) + Tr(N) = \|Y\|_{Tr}$
- $P_{ii}, N_{ii} \leq \sqrt{p}$ for all $i \in \{1, ..., p\}$

# Proof of the lemma

Let $Y \in \mathbb{S}^{p\times p}$. We can write the eigenvalue decomposition of Y:

$$Y = \sum_{i=1}^{p} \lambda_i v_i v_i^T = \sum_{i:\lambda_i>0} \lambda_i v_i v_i^T - \sum_{i:\lambda_i<0} -\lambda_i v_i v_i^T$$

We define: $P = \sum_{i:\lambda_i>0} \lambda_i v_i v_i^T$ and $N = \sum_{i:\lambda_i<0} -\lambda_i v_i v_i^T$. Then:

- $P, N \in \mathbb{S}_+^{p\times p}$
- $Tr(P) + Tr(N) = \sum_{i=1}^{p} |\lambda_i| = \|Y\|_{Tr}$
- $P_{ii}, N_{ii} \leq \sqrt{p}$ for all $i \in \{1, ..., p\}$

The last point needs some clarification. We define:

$$abs(Y) = P + N = \sum_{i=1}^{p} |\lambda_i| v_i v_i^T$$

# Proof of the lemma

Then: $abs(Y)^2 = Y^2$
Since $Y \subseteq [-1,1]^{p\times p}$, then: $\forall i,j \in [p] : y_{ij} \in [-1,1]$. Noting $Y^2 = (z_{ij})$:
$z_{ij} = \sum_{i=1}^{p} y_{ik}.y_{kj} \leq p$.
In particular, this is verified for the diagonal elements $z_{ii}$. Hence, by symmetry of $Y$: $\sum_{i=1}^{p} y_{ik}^2 \leq p$. Hence: $\forall i,k : y_{ij} \leq \sqrt{p}$.
Hence each entry of $abs(Y)$ is bounded by $\sqrt{p}$. In particular, its diagonal entries, $P_{ii} + N_{ii} \leq \sqrt{p}$

# Proof of the proposition on the decomposability of (OCF)

We now prove the proposition on the decomposability of (OCF).

- Let $W \in \mathcal{W} \subseteq [-1,1]^{p\times p}$. Then $sym(W) \in [-1,1]^{m\times n} \cap \mathbb{S}^{p\times p}$, where $p = m+n$. Hence we can use the previous lemma.
- $\exists P, N \in \mathbb{S}_+^{p\times p}$ s.t:
  - $P, N \in \mathbb{S}_+^{p\times p}$
  - $Tr(P) + Tr(N) = \sum_{i=1}^{p} |\lambda_i| = \|sym(W)\|_{Tr}$
  - $P(i,i), N(i,i) \leq \sqrt{p}$

The only thing left to prove is that: $\|sym(W)\|_{Tr} = 2\|W\|_{Tr}$
This is true since the eigenvalue decomposition of $sym(W)$ is given by:

$$\begin{pmatrix} \frac{1}{\sqrt{2}}U & \frac{1}{\sqrt{2}}U \\ \frac{1}{\sqrt{2}}V & -\frac{1}{\sqrt{2}}V \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}U^T & \frac{1}{\sqrt{2}}V^T \\ \frac{1}{\sqrt{2}}U^T & -\frac{1}{\sqrt{2}}V^T \end{pmatrix}$$

Where $W = U\Sigma V$ is SVD of $W$.

# Outline

# Online SDP with the log-determinant minimizer: Tightening the regret bound of FTRL

In this section, we will:

- Use a regret bound of the FTRL given by Hazan and improve it (in the main lemma) by introducing a new definition of strong convexity
- Show the strong convexity of the log-determinant regularizer
- Derive a new regret bound for the FTRL with the log-determinant regularizer using its strong convexity
- Go back to the OMP problem (through the reduction showed in part 3) and apply our results to the online collaborative filtering and the online max-cut examples

# Hypotheses

We consider the following SDP problem:

$$\tilde{\mathcal{K}} = \{X \in \mathbb{S}_+^{N\times N} : \|X\|_{Tr} \leq \tau, \forall i \in [N], X_{ii} \leq \beta\}$$
$$\tilde{\mathcal{L}} = \{L \in \mathbb{S}^{N\times N} : \|Vec(L)\|_1 \leq g_1\}$$

With $g_1 = 4G$, $\mathcal{K} = \tilde{\mathcal{K}}$ and $\mathcal{L} \subseteq \tilde{\mathcal{L}}$.

As a consequence, bounding $Reg(T, \tilde{\mathcal{K}}, \tilde{\mathcal{L}})$ implies bounding $Reg(T, \mathcal{K}, \mathcal{L})$, and therefore bounding $Reg(T, \mathcal{W})$ for an OMP problem where $\mathcal{W}$ is $(\beta, \tau)$-decomposable.

# A regret bound of the FTRL by Hazan

**Lemma 4.1: FTL-BTL (Follow The Leader Be The Leader)**

The FTRL with the regularizer $R : \mathcal{K} \to \mathbb{R}$ for an online SDP $(\mathcal{K}, \mathcal{L})$ achieves:

$$Reg(T, \mathcal{K}, \mathcal{L}) \leq \frac{H_0}{\eta} + \sum_{t=1}^{T} L_t \bullet (X_t - X_{t+1})$$

where $H_0 = max_{X, X' \in \mathcal{K}}(R(X) - R(X^{'})$

Our goal is to improve the bound above and to derive from it a tight bound for the FTRL with the regularizer $R(X) = -\log(\det(X))$.

To that effect, we will bound $H_0$ and $L_t \bullet (X_t - X_{t+1})$.

# Another definition of strong convexity

**Definition 4.1: Strong convexity**

For a decision space $\mathcal{K}$ and a real number $s > 0$, a regularizer $R : \mathcal{K} \to \mathbb{R}$ is said to be s-strongly convex with respect to a loss space $\mathcal{L}$ if for any $\alpha \in [0, 1]$, any $X, Y \in \mathcal{K}$, and any $L \in \mathcal{L}$:

$$R(\alpha X + (1-\alpha)Y) \leq \alpha R(X) + (1-\alpha)Y - \frac{1}{2}|L \bullet (X-Y)|^2$$

or equivalently:

$$R(X) \geq R(Y) + \nabla R(Y) \bullet (X-Y) + \frac{s}{2}|L \bullet (X-Y)|^2$$

This definition will enable us to give a tighter bound of the regret of the FTRL.

# A tighter bound of the regret of the FTRL

The lemma below gives a bound of the term $L_t \bullet (X_t - X_{t+1})$.

**Lemma 4.2: A tight bound of the regret of the FTRL (Main lemma)**

Let $R : \mathcal{K} \to \mathbb{R}$ be s-strongly convex with respect to $\mathcal{L}$ for $\mathcal{K}$. Then, the FTRL with regularizer $R$ applied to $(\mathcal{K}, \mathcal{L})$ achieves:

$$Reg(T, \mathcal{K}, \mathcal{L}) \leq 2 \times \sqrt{\frac{H_0 \times T}{s}}$$

for an appropriate choice of $\eta$.

There is a small typo in the paper: Lemma 5.2 states that under the conditions above, $Reg(T, \mathcal{K}, \mathcal{L}) \leq 2 \times \sqrt{\frac{H_0}{s}}$

# A tighter bound of the regret of the FTRL

**An improvement over the lemma 2.1:**

- The main lemma 4.2 gives a more general method of deriving regret bounds. Indeed, if we assume that the first condition of the lemma 2.1 holds, the condition of the main lemma 4.2 also holds:

$$|L \bullet (X - Y)| \leq \|L\|_* \|X - Y\| \forall L \in \mathcal{L}, \forall X, Y \in \mathcal{K}$$

  and thus,

$$R(X) \geq R(Y) + \nabla R(X) \bullet (X - Y) + \frac{s}{2g^2} |L \bullet (X - Y)|^2$$

- The main lemma 4.2 does not require looking for appropriate norms to obtain good regret bounds.

# Proof of the main lemma

Let's first prove that $L_t \bullet (X_t - X_{t+1}) \leq \frac{\eta}{s}\ \forall t = 1...T$:

Any s-strongly convex function F with respect to $\mathcal{L}$ satisfies $\forall X \in \mathcal{K}$, $\forall L \in \mathcal{L}$, $\forall Y = \operatorname{argmin}_{X \in \mathcal{K}} F(X)$:

$$F(X) - F(Y) \geq \nabla F(Y)(X - Y) + \frac{s}{2}\|L \bullet (X - Y)\|^2 \geq \frac{s}{2}\|L \bullet (X - Y)\|^2 \quad (\star)$$

We recall the update rule of the FTRL: $X_{t+1} = \operatorname{argmin}_{X \in \mathcal{K}} F_t(X)$ where $F_t(X) = \sum_{i=1}^{t} \eta L_i \bullet X + R(X)$. $F_t$ is s-strongly convex with respect to $\mathcal{L}$ due to the linearity of $L_i \bullet X$. Applying $(\star)$ to $F_t$ and $F_{t-1}$ with $L = L_t$ :

$$F_t(X_t) \geq F_t(X_{t+1}) + \tfrac{s}{2}|L_t \bullet (X_t - X_{t+1})|^2$$
$$F_{t-1}(X_{t+1}) \geq F_{t-1}(X_t) + \tfrac{s}{2}|L_t \bullet (X_{t+1} - X_t)|^2$$

# Proof of the main lemma

Summing up:

$$F_t(X_t) + F_{t-1}(X_{t+1}) =$$

$$\sum_{i=1}^{t} \eta L_i \bullet X_t + \sum_{i=1}^{t-1} \eta L_i \bullet X_{t+1} + R(X_t) + R(X_{t+1}) \geq$$

$$\sum_{i=1}^{t} \eta L_i \bullet X_{t+1} + \sum_{i=1}^{t-1} \eta L_i \bullet X_t + R(X_{t+1}) + R(X_t) + s|L_t \bullet (X_t - X_{t+1})|^2$$

$$\Leftrightarrow \eta L_t \bullet (X_t - X_{t+1}) \geq s|L_t \bullet (X_t - X_{t+1})|^2$$

$$\Leftrightarrow L_t \bullet (X_t - X_{t+1}) \leq \frac{\eta}{s}$$

# Proof of the main lemma

Using the regret bound of the FTRL by Hazan, with $\eta = \sqrt{\frac{sH_0}{T}}$:

$$\begin{aligned} Reg(T, \mathcal{K}, \mathcal{L}) &\leq \frac{H_0}{\eta} + \sum_{t=1}^{T} L_t \bullet (X_{t+1} - X_t) \\ &\leq \frac{H_0}{\eta} + T \times \frac{\eta}{s} = H_0 \times \sqrt{\frac{T}{sH_0}} + T \times \sqrt{\frac{sH_0}{T}} \times \frac{1}{s} \\ &= 2 \times \sqrt{\frac{TH_0}{s}} \end{aligned}$$

# Strong convexity of the log-determinant regularizer

**Lemma 4.3: Sufficient condition from Christiano**

Let $X, Y \in \mathbb{S}_+^{N\times N}$ be such that:

$$\exists (i,j) \in [N]\times[N], |X_{i,j} - Y_{i,j}| \geq \delta(X_{i,i} + X_{j,j} + Y_{i,i} + Y_{j,j})$$

Then, the following inequality holds:

$$\begin{aligned}
&-\log\det(\alpha X + (1-\alpha)Y) \\
&\qquad \leq -\alpha \log\det(X) - (1-\alpha)\log\det(Y) - \frac{\alpha(1-\alpha)}{2}\frac{\delta^2}{72\times\sqrt{e}}
\end{aligned}$$

The proof of this lemma is given in the appendix of the article.

# Strong convexity of the log-determinant regularizer

The lemma below will enable us to prove the strong convexity of the log-determinant regularizer.

**Lemma 4.4: The condition of lemma 4.3 holds for our problem $(\tilde{\mathcal{K}}, \tilde{\mathcal{L}})$ for $\delta = \mathcal{O}(|L \bullet (X - Y)|)$**

Let $X, Y \in \mathbb{S}_+^{N \times N}$ be such that $X_{i,i} \leq \beta'$ and $Y_{i,i} \leq \beta'$ for every $i \in [N]$. Then, for any $L \in \tilde{\mathcal{L}}$, there exists $(i, j) \in [N] \times [N]$ such that:

$$|X_{i,j} - Y_{i,j}| \geq \frac{|L \bullet (X - Y)|}{4 g_1 \beta'} (X_{i,i} + X_{j,j} + Y_{i,i} + Y_{j,j})$$

# Proof of the lemma

By the Cauchy-Schwarz inequality:

$$|L \bullet (X - Y)| \leq \|Vec(L)\|_1 \|Vec(X - Y)\|_\infty \leq g_1 max_{i,j} |X_{i,j} - Y_{i,j}|$$

Furthermore, $X_{i,i} + X_{j,j} + Y_{i,i} + Y_{j,j} \leq 4\beta'$. Hence:

$$\frac{|L \bullet (X - Y)|}{g_1 \times 4\beta'} \times (X_{i,i} + X_{j,j} + Y_{i,i} + Y_{j,j}) \leq max_{i,j} |X_{i,j} - Y_{i,j}|$$

and so,

$$\exists (i,j) \in [N] \times [N] : \frac{|L \bullet (X - Y)|}{g_1 \times 4\beta'} \times (X_{i,i} + X_{j,j} + Y_{i,i} + Y_{j,j}) \leq |X_{i,j} - Y_{i,j}|$$

# Strong convexity of the log-determinant regularizer

We now come to the strong convexity of the log-determinant regularizer, which will allow us to apply the main lemma (lemma 4.2) to the log-determinant regularizer.

**Proposition 4.1: Strong convexity of the log-determinant regularizer**

The log-determinant regularizer $R(X) = -\log\det(X + \epsilon E)$ is s-strongly convex with respect to $\tilde{\mathcal{L}}$ for $\tilde{\mathcal{K}}$ with $s = \frac{1}{1152 \times \sqrt{e} g_1^2 (\beta+\epsilon)^2}$

# Proof of the proposition

We apply Lemma 4.4 to $X + \epsilon E$ and $Y + \epsilon E$ for $X, Y \in \tilde{\mathcal{K}}$ and $\beta' = \beta + \epsilon$. We obtain conditions to apply lemma 4.3 with $\delta = \frac{|L \bullet (X-Y)|}{4g_1(\beta+\epsilon)}$:

$$
\begin{aligned}
&-\log\det(\alpha(X+\epsilon E) + (1-\alpha)(Y+\epsilon E)) \\
&\quad \leq -\alpha \log\det(X+\epsilon E) - (1-\alpha)\log\det(Y+\epsilon E) - \frac{\alpha(1-\alpha)}{2} \frac{|L \bullet (X-Y)|^2}{1152 g_1^2 (\beta+\epsilon)^2 \times \sqrt{e}}
\end{aligned}
$$

Hence, $R(X) = -\log\det(X + \epsilon E)$ is s-strongly convex with $s = \frac{1}{1152 \times \sqrt{e} g_1^2 (\beta+\epsilon)^2}$

# Main theorem

**Theorem 4.1: A tight bound on the FTRL regret with the log-determinant regularizer (Main theorem)**

For the online SDP problem $(\tilde{\mathcal{K}}, \tilde{\mathcal{L}})$, the FTRL with the log-determinant regularizer $R(X) = -\log\det(X + \epsilon E)$ achieves:

$$Reg(T, \tilde{\mathcal{K}}, \tilde{\mathcal{L}}) \leq 175 g_1 \times \sqrt{\beta \tau T}$$

for an appropriate choice of $\eta$ and $\epsilon$.

# Proof of the theorem

We know that $Reg(T, \mathcal{K}, \mathcal{L}) \leq 2 \times \sqrt{\frac{H_0 \times T}{s}}$ from lemma 4.2. From proposition 4.1, $s = \frac{1}{1152 \times \sqrt{e} g_1^2 (\beta + \epsilon)^2}$. All we need to do is to bound $H_0$.

$$
\begin{aligned}
H_0 &= max_{X, X' \in \mathcal{K}} R(X) - R(X') = R(X_0) - R(X_1) \\
&= -\log\det(X_0 + \epsilon E) + \log\det(X_1 + \epsilon E) = \log \frac{\det(X_1 + \epsilon E)}{\det(X_0 + \epsilon E)} \\
&= \sum_{i=1}^{N} \log\left(\frac{\lambda_i(X_1) + \epsilon}{\lambda_i(X_0) + \epsilon}\right) \leq \sum_{i=1}^{N} \log\left(\frac{\lambda_i(X_1)}{\epsilon} + 1\right) \leq \sum_{i=1}^{N} \frac{\lambda_i(X_1)}{\epsilon} \\
&= \frac{Tr(X_1)}{\epsilon} = \frac{\|X_1\|_{Tr}}{\epsilon} \leq \frac{\tau}{\epsilon}
\end{aligned}
$$

where $X_0$ is the maximizer of $R$ in $\mathcal{K}$, $X_1$ is the minimizer of $R$ in $\mathcal{K}$, $\lambda_i$ is the ith eigenvalue of X.

## Proof of the theorem

Hence, with $\epsilon = \beta$:

$$
\begin{aligned}
Reg(T,\mathcal{K},\mathcal{L}) &\leq 2 \times \sqrt{\frac{\tau}{\beta}} \times \sqrt{1152 \times \sqrt{e} g_1^2 \times 4\beta^2 \times T} \\
&= g_1 \sqrt{\beta\tau} \times \sqrt{1152 \times \sqrt{e} \times 4} \times 2 \times \sqrt{T} \\
&\leq 175 g_1 \times \sqrt{\beta\tau T}
\end{aligned}
$$

# Back to the OMP problem

**Corollary 4.1: Regret bound on the OMP problem**

For the OMP problem $(\mathcal{W}, G)$ where $\mathcal{W} \subseteq [-1,1]^{m\times n}$ is $(\beta,\tau)$-decomposable, there exists an algorithm that achieves:

$$Reg_{OMP}(T,\mathcal{W}) = \mathcal{O}(G \times \sqrt{\beta\tau T})$$

**Improvement over the corollary 3.1:**

- Corollary 3.1 states that $Reg_{OMP}(T,\mathcal{W}) = \mathcal{O}(G \times \sqrt{\beta\tau T \log(n+m)})$
- The bound of Corollary 4.1 does not depend on the size $(n, m)$
- The bound is improved by a factor $\mathcal{O}(\sqrt{n+m})$

# Proof of the corollary

Since the OMP problem $(\mathcal{W}, G)$ is $(\beta, \tau)$-decomposable, it can be reduced to the online SDP problem $(\tilde{\mathcal{K}}, \tilde{\mathcal{L}})$ with $g_1 = 4G$. We then have:

$$Reg_{OMP}(T, \mathcal{W}) \leq \frac{1}{2} Reg(T, \tilde{\mathcal{K}}, \tilde{\mathcal{L}}) \leq \frac{1}{2} \times 175 \times 4G \times \sqrt{\beta \tau T}$$

# Examples

- **Online max-cut:** The problem is $(1, n)$-decomposable. Hence, $Reg_{OMP}(T, \mathcal{W}) = \mathcal{O}(G \times \sqrt{nT})$.
- **Collaborative filtering:** The problem is $(G \times \sqrt{\tau T \times \sqrt{n} \log n}, n)$-decomposable. Hence, $Reg_{OMP}(T, \mathcal{W}) = \mathcal{O}(G \times \sqrt{\tau T \times \sqrt{n}})$ for $n \geq m$.

# Outline

# Numerics

See notebook !

# Outline

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