Rectangular maximum volume

Scoltech

5 GOSLINGS

Our Plan

- `https://github.com/ArkadiyAliev/Rectangular-maximum-volume`
- Problem statement
- Known algorithms
- Speed comparison
- Quality comparison
- Application in low-rank approximation
- Contribution of each team member

Problem statement

Volume of a matrix

Let $A \in \mathbb{C}^{m \times n}, m \geq n$. The volume of A is defined as

$$vol(A) = \sqrt{\det(A^* A)}$$

Problem statement

Let $A \in \mathbb{C}^{m \times n}, m \geq n, r \geq n$. Our goal is to find a submatrix $\hat{A} \in \mathbb{C}^{r \times n}$ with largest volume.

Dominant submatrix

Let $A \in \mathbb{C}^{m \times n}, m \geq n, r \geq n$ be of a full column rank. $\hat{A} \in \mathbb{C}^{r \times n}$ is a submatrix of A. $\hat{A}$ is called dominant submatrix of a matrix A, if the volume of $\hat{A}$ does not increase by swapping one of its rows with another row from A.

Algorithms: (A. Mikhalev, I.V. Oseledets, 2018)

In general, finding the maximum volume submatrix is a NP-complete problem.

maxvol

1. Input: $A \in \mathbb{R}^{M \times r}, M > r$
2. Output: $r \times r$ matrix — approximate solution of square maxvol problem.
3. Complexity: $O(Mr^2)$

rect-maxvol

1. Input:$A \in \mathbb{R}^{M \times r}, n, M \geq n > r$. r indices of rows of approximate solution of square maxvol problem.
2. Output: $n \times r$ matrix, — approximate solution of rectangular maxvol problem.
3. Complexity: $O(M(n^2 - r^2))$

Algorithms: (Osinsky A.I., 2019)

pre-maxvol

1. Input: $A \in \mathbb{R}^{M \times n}$, the required rank r.
2. Output: set of column indices $\mathcal{I}, |\mathcal{I}| = r$,corresponding to the submatrix, whose volume differs from the maximum by no more than $r!$ times.
3. Complexity: $O(Mnr)$

DominantC

1. Input: $A \in \mathbb{R}^{M \times r}$, set of indices $\mathcal{I}, |\mathcal{I}| = n$ to update.
2. Output: set of row indices $\mathcal{I}_1$, which corresponds to a dominant submatrix $n \times r$.
3. Complexity: $O(Mn \cdot iter)$ ($O(Mnr \cdot ln(r))$ after pre-maxvol). Here *iter* is number of switches.

Algorithms: (Osinsky A.I., 2019)

maxvol2

1. Input: $A \in \mathbb{R}^{M \times r}$, set of indices $\mathcal{I}, |\mathcal{I}| = r$ to update by adding rows, $n \geq r \in \mathbb{N}$.
2. Output: set of row indices $\mathcal{I}_1, |\mathcal{I}_1| = n$, which corresponds to a submatrix $n \times r$ with indices chosen greedily to maximize the volume.
3. Complexity: $O(M(n^2 - r^2))$

maxvol2-householder

1. Input: $A \in \mathbb{R}^{M \times r}$, set of indices $\mathcal{I}, |\mathcal{I}| = r$ to update by adding rows, $n \geq r \in \mathbb{N}$.
2. Output: set of row indices $\mathcal{I}_1, |\mathcal{I}_1| = n$, which corresponds to a submatrix $n \times r$ with indices chosen greedily to maximize the volume.
3. Complexity: $O(Mr(n - r))$

Our task is to calculate C and C' on the basis of C_0. The expression (9) shows that it is sufficient to find $\left(\hat{A}^*\hat{A}\right)^{-1}$. Since

$$\hat{A}^*\hat{A} = \hat{A}_0^*\hat{A}_0 + aa^* = \left(\hat{A}_0^*\hat{A}_0\right)\left(I + \left(\hat{A}_0^*\hat{A}_0\right)^{-1}aa^*\right),$$

then for the inverse

$$\begin{aligned}\left(\hat{A}^*\hat{A}\right)^{-1} &= \left(\hat{A}_0^*\hat{A}_0\right)^{-1}\left(I + \left(\hat{A}_0^*\hat{A}_0\right)^{-1}aa^*\right)^{-1} \\ &= \left(\hat{A}_0^*\hat{A}_0\right)^{-1}\left(I - \frac{\left(\hat{A}_0^*\hat{A}_0\right)^{-1}aa^*}{1 + a^*\left(\hat{A}_0^*\hat{A}_0\right)^{-1}a}\right).\end{aligned} \tag{10}$$

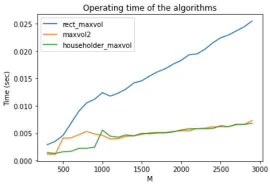
The expression in the brackets can be simplified by introducing the notation $c^* = C_{i,:}$. Indeed,

$$\begin{aligned}c^* &= C_{0i,:} = A_{i,:}\hat{A}_0^+ = a^*\left(\hat{A}_0^*\hat{A}_0\right)^{-1}\hat{A}_0^*, \\ c^*c &= a^*\left(\hat{A}_0^*\hat{A}_0\right)^{-1}\hat{A}_0^*\hat{A}_0\left(\hat{A}_0^*\hat{A}_0\right)^{-1}a = a^*\left(\hat{A}_0^*\hat{A}_0\right)^{-1}a.\end{aligned}$$

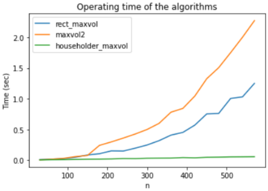
Also,

$$\left(\hat{A}_0^*\hat{A}_0\right)^{-1}aa^* = aa^*\left(\hat{A}_0^*\hat{A}_0\right)^{-1}.$$

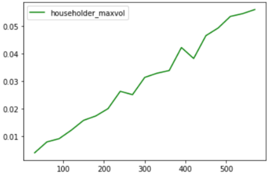
Input: $A \in \mathbb{R}^{M \times r}$, set of indices $\mathcal{I}$, $|\mathcal{I}| = r$ to update by adding rows, $n \geq r \in \mathbb{N}$.
Output: submatrix $n \times r$.



(a) Dependence on M, $r = 10$, $n = 20$



(b) Dependence on n, $r = 20$, $M = 600$



(c) Dependence on n, $r = 20$, $M = 600$

Figure: Speed comparison for rect-maxvol, maxvol2, maxvol2-householder

Quality comparison

Input: $A \in \mathbb{R}^{M \times r}$, $M > r$

Output: $r \times r$ matrix — approximate solution of square maxvol problem.

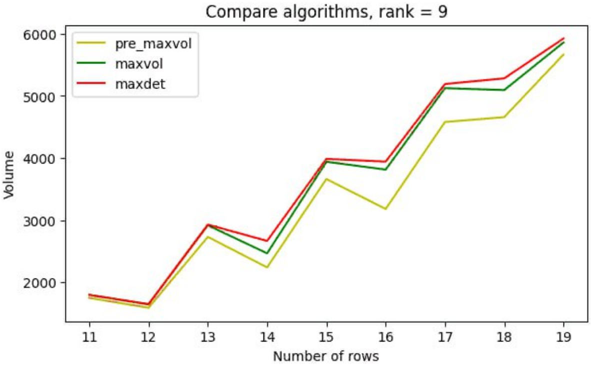


Figure: Quality comparison for premaxvol, maxvol and optimal algorithm ($r = 9$)

Quality comparison

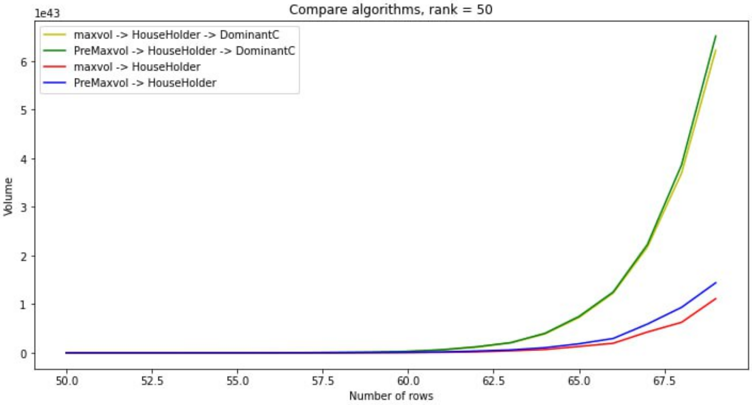


Figure: Input: Matrix $A \in \mathbb{R}^{200\times 50}$ and r - desired number of rows.

We see that adding DominantC improves the volume of output submatrix dramatically.

Quality comparison

$A \in \mathbb{R}^{N \times 50}$, desired number of rows 100.
Output: 100×50 matrix — approximate solution of maxvol problem.

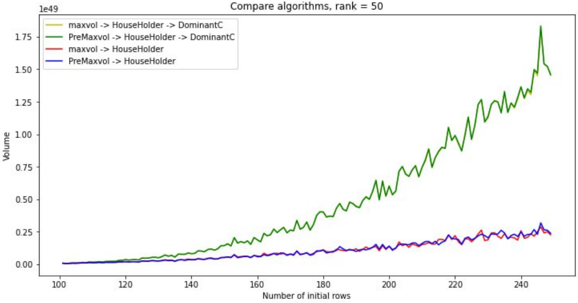
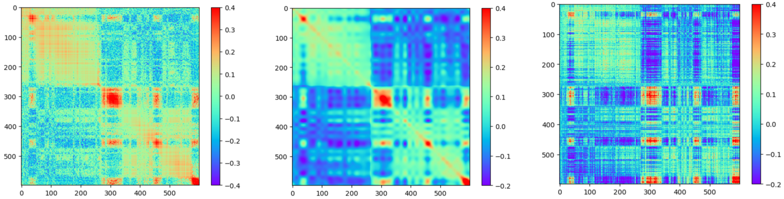


Figure: Quality comparison for different numbers of initial rows.

Application in low-rank approximation

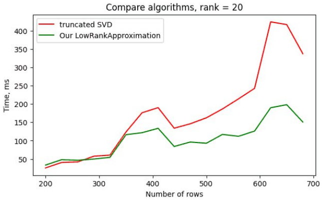


(a) Original (b) Truncated SVD (c) 2019

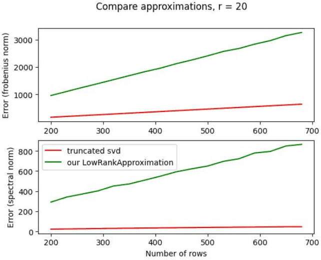
Figure: Truncated SVD vs (Osinsky A.I., 2019)

Application in low-rank approximation

Input: square matrix of size $N \times N$.
Output: low-rank approximation of this matrix by our algorithm.



(a) Speed comparison



(b) Error comparison

Contribution of each team member

- Arkadiy: LowRankApproximation, pre-maxvol, HouseHolder, DominantC implementation.
- Vsevolod: Correction of derivation of formulas, maxvol2, HouseHolder, DominantC implementation.
- Roman: maxvol, rect-maxvol implementation; numerical experiments for maxvol-rect vs maxvol2 vs HouseHolder
- Yuriy: numerical experiments for maxvol $\to$ HouseHolder $\to$ DominantC vs PreMaxvol $\to$ HouseHolder $\to$ DominantC vs maxvol$\to$HouseHolder vs PreMaxvol $\to$ HouseHolder
- Mikhail: suggested project idea, numerical experiments for LowRankApproximation vs Truncated SVD, premaxvol vs maxvol vs random r rows vs overkill

¿PREGUNTAS?