Lecture 7 - The Discrete Fourier Transform

7.1 The DFT

The Discrete Fourier Transform (DFT) is the equivalent of the continuous Fourier Transform for signals known only at N instants separated by sample times T (i.e. a finite sequence of data).
Let $f(t)$ be the continuous signal which is the source of the data. Let N samples be denoted $f[0], f[1], f[2], \ldots, f[k], \ldots, f[N-1]$.
The Fourier Transform of the original signal, $f(t)$, would be

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

We could regard each sample $f[k]$ as an *impulse* having area $f[k]$. Then, since the integrand exists only at the sample points:

$$\begin{aligned} F(j\omega) &= \int_{o}^{(N-1)T} f(t)e^{-j\omega t}dt \\ &= f[0]e^{-j0} + f[1]e^{-j\omega T} + \ldots + f[k]e^{-j\omega kT} + \ldots f(N-1)e^{-j\omega(N-1)T} \end{aligned}$$

$$\text{ie.} \qquad F(j\omega) = \sum_{k=0}^{N-1} f[k]e^{-j\omega kT}$$

We could in principle evaluate this for any ω, but with only N data points to start with, only N final outputs will be significant.

You may remember that the continuous Fourier transform could be evaluated over a finite interval (usually the fundamental period T_o) rather than from $-\infty$ to

$+\infty$ if the waveform was *periodic*. Similarly, since there are only a finite number of input data points, the DFT treats the data as if it were periodic (i.e. $f(N)$ to $f(2N-1)$ is the same as $f(0)$ to $f(N-1)$.)
Hence the sequence shown below in Fig. 7.1(a) is considered to be one period of the periodic sequence in plot (b).

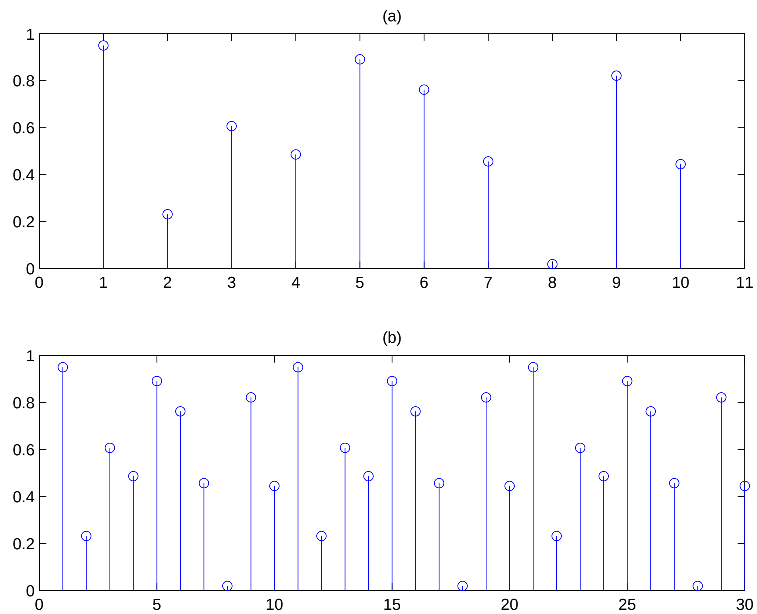

Figure 7.1: *(a) Sequence of $N = 10$ samples. (b) implicit periodicity in DFT.*

Since the operation treats the data as if it were periodic, we evaluate the DFT equation for the fundamental frequency (one cycle per sequence, $\frac{1}{NT}$Hz, $\frac{2\pi}{NT}$ rad/sec.) and its harmonics (not forgetting the d.c. component (or average) at $\omega = 0$).

$$\text{i.e.} \quad \text{set } \omega = 0, \frac{2\pi}{NT}, \frac{2\pi}{NT} \times 2, \ldots \frac{2\pi}{NT} \times n, \ldots \frac{2\pi}{NT} \times (N-1)$$

or, in general

$$F[n] = \sum_{k=0}^{N-1} f[k] e^{-j\frac{2\pi}{N}nk} \quad (n = 0 : N-1)$$

$F[n]$ is the Discrete Fourier Transform of the sequence $f[k]$.

We may write this equation in matrix form as:

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ \vdots \\ F[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{N-2} \\ 1 & W^3 & W^6 & W^9 & \dots & W^{N-3} \\ \vdots & & & & & \\ 1 & W^{N-1} & W^{N-2} & W^{N-3} & \dots & W \end{pmatrix} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{pmatrix}$$

where $W = \exp(-j2\pi/N)$ and $W = W^{2N}$ etc. $= 1$.

DFT – example

Let the continuous signal be

$$f(t) = \underbrace{5}_{\text{dc}} + \underbrace{2\cos(2\pi t - 90^o)}_{\text{1Hz}} + \underbrace{3\cos 4\pi t}_{\text{2Hz}}$$

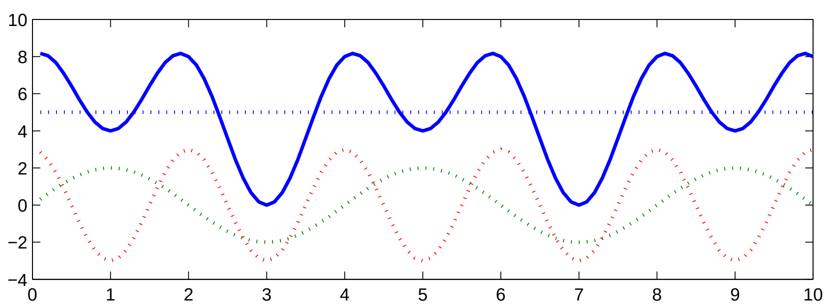

Figure 7.2: *Example signal for DFT.*

Let us sample $f(t)$ at 4 times per second (ie. f_s = 4Hz) from $t = 0$ to $t = \frac{3}{4}$. The values of the discrete samples are given by:

$$f[k] = 5 + 2cos(\tfrac{\pi}{2}k - 90^o) + 3cos\pi k \qquad \text{by putting } t = kT_s = \tfrac{k}{4}$$

i.e. $f[0]=8$, $f[1]=4$, $f[2]=8$, $f[3]=0$, $\qquad (N=4)$

$$\text{Therefore} \quad F[n] = \sum_{0}^{3} f[k]e^{-j\frac{\pi}{2}nk} = \sum_{k=0}^{3} f[k](-j)^{nk}$$

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \end{pmatrix} = \begin{pmatrix} 20 \\ -j4 \\ 12 \\ j4 \end{pmatrix}$$

The magnitude of the DFT coefficients is shown below in Fig. 7.3.

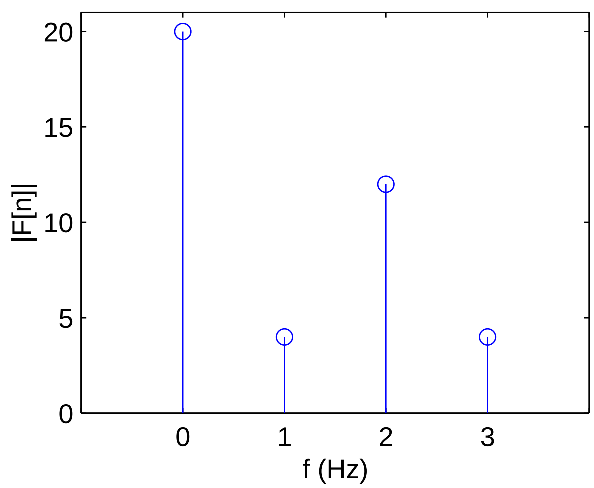

Figure 7.3: *DFT of four point sequence.*

Inverse Discrete Fourier Transform

The inverse transform of

$$F[n] = \sum_{k=0}^{N-1} f[k]e^{-j\frac{2\pi}{N}nk}$$

is

$$f[k] = \frac{1}{N}\sum_{n=0}^{N-1} F[n]e^{+j\frac{2\pi}{N}nk}$$

i.e. the inverse matrix is $\frac{1}{N}$ times the complex conjugate of the original (symmetric) matrix.

Note that the $F[n]$ coefficients are *complex*. We can assume that the $f[k]$ values are *real* (this is the simplest case; there are situations (e.g. radar) in which two inputs, at each k, are treated as a *complex pair*, since they are the outputs from $0\ ^{\text{o}}$ and $90\ ^{\text{o}}$ demodulators).

In the process of taking the inverse transform the terms $F[n]$ and $F[N-n]$ (remember that the spectrum is symmetrical about $\frac{N}{2}$) combine to produce 2 frequency components, only one of which is considered to be valid (the one at the *lower* of the two frequencies, $n \times \frac{2\pi}{T}$ Hz where $n \leq \frac{N}{2}$; the higher frequency component is at an "aliasing frequency" ($n > \frac{N}{2}$)).

From the inverse transform formula, the contribution to $f[k]$ of $F[n]$ and $F[N-n]$ is:

$$f_n[k] = \frac{1}{N}\{F[n]e^{j\frac{2\pi}{N}nk} + F[N-n]e^{j\frac{2\pi}{N}(N-n)k}\} \tag{7.2}$$

$$\text{For all} f[k] \text{ real,} \quad F[N-n] = \sum_{k=0}^{N-1} f[k]e^{-j\frac{2\pi}{N}(N-n)k}$$

$$\text{But} \quad e^{-j\frac{2\pi}{N}(N-n)k} = \underbrace{e^{-j2\pi k}}_{1 \text{ for all } k} e^{+j\frac{2\pi n}{N}k} = e^{+j\frac{2\pi}{N}nk}$$

i.e. $F[N-n] = F^*(n)$ (i.e. the complex conjugate)

Substituting into the Equation for $f_n[k]$ above gives,

$$f_n[k] = \frac{1}{N}\{F[n]e^{j\frac{2\pi}{N}nk} + F^*(n)e^{-j\frac{2\pi}{N}nk}\} \quad \text{since} \quad e^{j2\pi k} = 1$$

$$\text{ie.} \quad f_n[k] = \frac{2}{N}\{\mathsf{Re}\{F[n]\}\cos\frac{2\pi}{N}nk - \mathsf{Im}\{F[n]\}\sin\frac{2\pi}{N}nk\}$$

$$\text{or} \quad f_n[k] = \frac{2}{N}|F[n]|\cos\{(\frac{2\pi}{NT}n)kT + \arg(F[n])\}$$

i.e. a sampled sinewave at $\frac{2\pi n}{NT}$ Hz, of magnitude $\frac{2}{N}|F[n]|$.

For the special case of $n = 0$, $F[0] = \sum f[k]$ (i.e. sum of all samples) and the contribution of $F[0]$ to $f[k]$ is $f_0[k] = \frac{1}{N}F[0]$ = average of $f[k]$ = d.c. component.

Interpretation of example

1. $F[0] = 20$ implies a d.c. value of $\frac{1}{N}F[0] = \frac{20}{4} = 5$ (as expected)

2. $F[1] = -j4 = F^*[3]$ implies a fundamental component of peak amplitude $\frac{2}{N}|F[1]| = \frac{2}{4} \times 4 = 2$ with phase given by $\arg F[1] = -90^{\text{o}}$

$$\text{i.e.} \quad 2\cos(\frac{2\pi}{NT}kT - 90^{\text{o}}) = 2\cos(\frac{\pi}{2}k - 90^{\text{o}}) \quad \text{(as expected)}$$

3. $F[2] = 12 \quad (n = \frac{N}{2}$ – no other $N - n$ component here) and this implies a component

$$f_2[k] = \frac{1}{N}F[2]e^{j\frac{2\pi}{N}\cdot 2k} = \frac{1}{4}F[2]e^{j\pi k} = 3cos\pi k \quad \text{(as expected)}$$

since $\sin\pi k = 0$ for all k

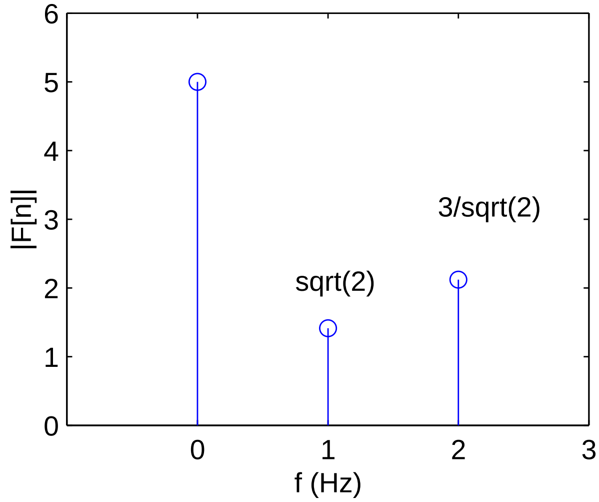

Figure 7.4: *DFT of four point signal.*

Thus, the conventional way of displaying a *spectrum* is not as shown in Fig. 7.3 but as shown in Fig. 7.4 (obviously, the information content is the same):
In typical applications, N is much greater than 4; for example, for $N = 1024$, $F[n]$ has 1024 components, but $513 - 1023$ are the complex conjugates of $511 - 1$, leaving $\frac{F[0]}{1024}$ as the d.c. component, $\frac{2}{1024}\frac{|F[1]|}{\sqrt{2}}$ to $\frac{2}{1024}\frac{|F[511]|}{\sqrt{2}}$ as complete a.c. components and $\frac{1}{1024}\frac{F[512]}{\sqrt{2}}$ as the cosine-only component at the highest distinguishable frequency $(n = \frac{N}{2})$.

Most computer programmes evaluate $\frac{|F[n]|}{N}$ (or $\frac{|F[n]|^2}{N}$ for the power spectral density) which gives the correct "shape" for the spectrum, except for the values at $n = 0$ and $\frac{N}{2}$.

7.2 Discrete Fourier Transform Errors

To what degree does the DFT approximate the Fourier transform of the function underlying the data? Clearly the DFT is only an approximation since it provides only for a finite set of frequencies. But how correct are these discrete values themselves? There are two main types of DFT errors: aliasing and "leakage":