The Discrete Fourier Transform (DFT) is a mathematical technique used to convert a finite sequence of equally spaced samples of a function into a sequence of coefficients of complex sinusoids, ordered by their frequencies. It is a specific case of the more general Fourier transform, applied to discrete data sets, and is widely used in digital signal processing, image analysis, and many other fields.

Definition

Given a sequence of N complex numbers $x_0, x_1, x_2, \ldots, x_{N-1}$, the DFT transforms this sequence into another sequence of N complex numbers $X_0, X_1, X_2, \ldots, X_{N-1}$ according to the following formula:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi i \frac{kn}{N}}$$

for $k = 0, 1, 2, \ldots, N-1$.

Inverse Discrete Fourier Transform (IDFT)

The inverse operation, which converts the frequency domain data back into the time domain, is given by:

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot e^{2\pi i \frac{kn}{N}}$$

for $n = 0, 1, 2, \ldots, N-1$.

Applications

1. **Signal Processing**: DFT is used to analyze the frequency content of signals, filter signals, and solve differential equations.
2. **Image Processing**: DFT is used in the processing and compression of images, such as in the JPEG standard.
3. **Audio Processing**: DFT helps in audio compression and feature extraction for tasks like speech recognition.
4. **Solving PDEs**: DFT is used in numerical methods to solve partial differential equations.

Example

Suppose we have a sequence $x = (x_0, x_1, x_2, x_3)$ of length $N = 4$. The DFT of this sequence is:

$$X_k = \sum_{n=0}^{3} x_n \cdot e^{-2\pi i \frac{kn}{4}}$$

This can be expanded into four separate sums for $k = 0, 1, 2, 3$:

$$\begin{aligned}
X_0 &= x_0 + x_1 + x_2 + x_3, \\
X_1 &= x_0 + x_1 \cdot e^{-2\pi i \frac{1}{4}} + x_2 \cdot e^{-2\pi i \frac{2}{4}} + x_3 \cdot e^{-2\pi i \frac{3}{4}}, \\
X_2 &= x_0 + x_1 \cdot e^{-2\pi i \frac{2}{4}} + x_2 \cdot e^{-2\pi i \frac{4}{4}} + x_3 \cdot e^{-2\pi i \frac{6}{4}}, \\
X_3 &= x_0 + x_1 \cdot e^{-2\pi i \frac{3}{4}} + x_2 \cdot e^{-2\pi i \frac{6}{4}} + x_3 \cdot e^{-2\pi i \frac{9}{4}}.
\end{aligned}$$

In summary, the Discrete Fourier Transform (DFT) is a fundamental tool in signal processing and other fields, providing a way to analyze the frequency components of discrete data sequences.