Quantum Fourier Transform (QFT) Overview

The Quantum Fourier Transform (QFT) is a quantum analogue of the discrete Fourier transform (DFT) applied to the amplitudes of a quantum state. It transforms a quantum state from the computational basis to the Fourier basis.

For a system with n qubits, the QFT acts on a basis state $|k\rangle$ (where k ranges from 0 to $2^n - 1$) as follows:

$$\mathrm{QFT}(|k\rangle) = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{2\pi i \frac{jk}{2^n}} |j\rangle$$

Applying QFT to Two Qubits in State |00⟩

1. **Initial State:** $|00\rangle$

For two qubits, the initial state $|00\rangle$ is expressed in the computational basis.

2. **QFT on |00⟩:**

The QFT for 2 qubits is given by:

$$\mathrm{QFT}(|00\rangle) = \frac{1}{2} \sum_{j=0}^{3} e^{2\pi i \frac{j \cdot 0}{4}} |j\rangle$$

Since $k = 0$ for $|00\rangle$:

$$\mathrm{QFT}(|00\rangle) = \frac{1}{2} \sum_{j=0}^{3} |j\rangle$$

Evaluating the sum:

$$\mathrm{QFT}(|00\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

This means that after applying the QFT, the state $|00\rangle$ transforms to an equal superposition of all possible basis states for two qubits.

Detailed Steps in Equations

1. **Initial state:**

$$|\psi_{\text{initial}}\rangle = |00\rangle$$

2. **QFT Transformation:**

$$\text{QFT}(|00\rangle) = \frac{1}{2} \sum_{j=0}^{3} e^{2\pi i \frac{0 \cdot j}{4}} |j\rangle$$

3. **Simplifying the sum:**

$$e^{2\pi i \frac{0 \cdot j}{4}} = e^0 = 1$$

4. **Resulting state:**

$$\text{QFT}(|00\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

So, the output state after applying the QFT to the input state $|00\rangle$ is:

$$|\psi_{\text{output}}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

This state is an equal superposition of all the possible states of two qubits.

QFT for |01⟩

1. **Initial State:** $|01\rangle$

2. **QFT on |01⟩:**

$$\text{QFT}(|01\rangle) = \frac{1}{2}\sum_{j=0}^{3} e^{2\pi i \frac{j \cdot 1}{4}} |j\rangle$$

3. **Evaluating the Exponentials:**

$$\text{QFT}(|01\rangle) = \frac{1}{2}(e^{2\pi i \frac{0}{4}}|00\rangle + e^{2\pi i \frac{1}{4}}|01\rangle + e^{2\pi i \frac{2}{4}}|10\rangle + e^{2\pi i \frac{3}{4}}|11\rangle)$$

   Simplifying the exponentials:

$$e^{2\pi i \frac{0}{4}} = 1, \quad e^{2\pi i \frac{1}{4}} = i, \quad e^{2\pi i \frac{2}{4}} = -1, \quad e^{2\pi i \frac{3}{4}} = -i$$

4. **Resulting State:**

$$\text{QFT}(|01\rangle) = \frac{1}{2}(|00\rangle + i|01\rangle - |10\rangle - i|11\rangle)$$

QFT for |10⟩

1. **Initial State:** $|10\rangle$

2. **QFT on |10⟩:**

$$\text{QFT}(|10\rangle) = \frac{1}{2}\sum_{j=0}^{3} e^{2\pi i \frac{j \cdot 2}{4}} |j\rangle$$

3. **Evaluating the Exponentials:**

$$\text{QFT}(|10\rangle) = \frac{1}{2}(e^{2\pi i \frac{0}{4}}|00\rangle + e^{2\pi i \frac{2}{4}}|01\rangle + e^{2\pi i \frac{4}{4}}|10\rangle + e^{2\pi i \frac{6}{4}}|11\rangle)$$

   Simplifying the exponentials:

$$e^{2\pi i \frac{0}{4}} = 1, \quad e^{2\pi i \frac{2}{4}} = -1, \quad e^{2\pi i \frac{4}{4}} = 1, \quad e^{2\pi i \frac{6}{4}} = -1$$

4. **Resulting State:**

$$\text{QFT}(|10\rangle) = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

QFT for |11⟩

1. **Initial State:** $|11\rangle$

2. **QFT on |11⟩:**

$$\text{QFT}(|11\rangle) = \frac{1}{2}\sum_{j=0}^{3} e^{2\pi i \frac{j\cdot 3}{4}} |j\rangle$$

3. **Evaluating the Exponentials:**

$$\text{QFT}(|11\rangle) = \frac{1}{2}(e^{2\pi i \frac{0}{4}}|00\rangle + e^{2\pi i \frac{3}{4}}|01\rangle + e^{2\pi i \frac{6}{4}}|10\rangle + e^{2\pi i \frac{9}{4}}|11\rangle)$$

   Simplifying the exponentials:

$$e^{2\pi i \frac{0}{4}} = 1, \quad e^{2\pi i \frac{3}{4}} = -i, \quad e^{2\pi i \frac{6}{4}} = -1, \quad e^{2\pi i \frac{9}{4}} = i$$

4. **Resulting State:**

$$\text{QFT}(|11\rangle) = \frac{1}{2}(|00\rangle - i|01\rangle - |10\rangle + i|11\rangle)$$

Summary

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- QFT(|00⟩):

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

- QFT(|01⟩):

$$\frac{1}{2}(|00\rangle + i|01\rangle - |10\rangle - i|11\rangle)$$

- QFT(|10⟩):

$$\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

- QFT(|11⟩):

$$\frac{1}{2}(|00\rangle - i|01\rangle - |10\rangle + i|11\rangle)$$