# Universality of Riemann Zeta Function value distribution on critical axis

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**Abstract**

We present a remarkable and striking universality property for the value distribution of the Riemann zeta function on the critical axis.

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**Mathematics Subject Classification (MSC)**: 11M06, 11-04.

## 1 Introduction

The universality relations and symmetries exhibited by a system are fundamental aspects of the system. In this work, we present a universality relation for the value distribution of the Riemann zeta function at Generalized Gram points.

Ref. [1] studied empirically the distribution of $Z(t)$ values at Gram points and showed that the distribution for even Gram points was the negative of the distribution for odd Gram points. Ref. [2]) extended the study to Generalized Gram points and showed that the value distribution of the Hardy $Z$ function at discrete points is anti-symmetrical for reflections around the mid-points of the Gram intervals (Eq. 3.3) and symmetrical for reflections around the Gram points(Eq. 3.4). Our new universality relation (Eq. 3.5) expresses the value distributions at different Generalized Gram points, in terms of three universal functions.

## 2 Notation for the Riemann zeta function

In this section we establish the required notation for the Riemann Zeta Function. For $\mathrm{Re}(s) > 1$ the Riemann Zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \in primes} \left(1 - p^{-s}\right)^{-1}. \tag{2.1}$$

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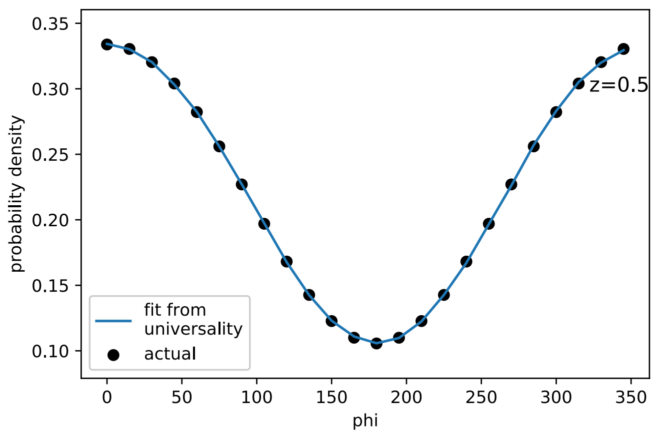


Figure 1: Test of universality. Comparison of probability density prediction from universality with actual values, for $z = 0.5$. The y axis is the probability density. The x axis is the angle characterizing the Generalized Gram point.

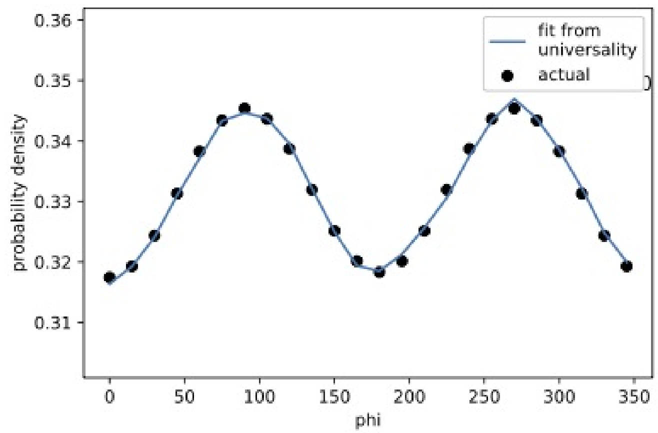


Figure 2: Test of universality. Comparison of probability density prediction from universality with actual values, for $z = 0.0$.

$\zeta(s)$ can be continued to the complex plane. Riemann's hypothesis, that the non-trivial zeros of $\zeta(s)$ lie on the critical axis $1/2 + it$, is probably the most famous unsolved problem in mathematics. The mean spacing $\delta$ of the zeros at large height $T$ is $\delta = 2\pi(\ln(T/2\pi))^{-1}$. For numerical studies of the Riemann hypothesis one defines Hardy's function

$$Z(t) = exp(i\theta(t))\zeta(1/2 + it) \tag{2.2}$$

where

$$\theta(t) = arg(\pi^{it/2}\Gamma(\frac{1}{4} + \frac{it}{2})). \tag{2.3}$$

The argument in Eq. 2.3 is defined by continuous variation of $t$ starting with the value 0 at $t = 0$. $Z(t)$ is real valued for real $t$, and we have $|Z(t)| = |\zeta(1/2+it)|$. Thus the zeros of $Z(t)$ are the imaginary part of the zeros of $\zeta(s)$ which lie on the critical line.

Gram points [3] play an important role in the the theory because many of the zeros are separated by them. When $t \geq 7$, the $\theta$ function Eq.(2.3) is monotonic increasing. For $n \geq -1$, the $n$-th Gram point $g_n$ is defined as the unique solution $> 7$ to $\theta(g_n) = n\pi$. A Gram interval is the interval $G_n = [g_n, g_{n+1})$. In analogy with Gram points, we can associate an angle $\phi$ with a point $t$ on the critical axis as follows:

**Definition 2.1.** For $t \geq 7$, $t$ is said to be a generalized Gram point with value $\phi$ if $\theta(t) = 2k\pi + \phi$, where $0 \leq \phi < 2\pi$.

| $z$ | $A$ | $B$ | $C$ | $R^2$ |
|---|---|---|---|---|
| −3.00 | −0.028 | 0.001 | 0.039 | 0.99982 |
| −2.50 | −0.036 | 0.001 | 0.051 | 0.99973 |
| −2.00 | −0.047 | 0.001 | 0.068 | 0.99988 |
| −1.50 | −0.063 | 0.001 | 0.095 | 0.99981 |
| −1.00 | −0.086 | −0.000 | 0.140 | 0.99991 |
| −0.50 | −0.115 | −0.004 | 0.223 | 0.99993 |
| 0.00 | −0.000 | −0.014 | 0.332 | 0.99403 |
| 0.50 | 0.114 | −0.004 | 0.223 | 0.99994 |
| 1.00 | 0.086 | 0.000 | 0.140 | 0.99992 |
| 1.50 | 0.062 | 0.001 | 0.095 | 0.99989 |
| 2.00 | 0.047 | 0.001 | 0.068 | 0.99989 |
| 2.50 | 0.036 | 0.001 | 0.051 | 0.99970 |
| 3.00 | 0.029 | 0.001 | 0.039 | 0.99982 |

Table 1: Values of the universal functions $A(z)$, $B(z)$ and $C(z)$ for $z$ in the range $-3.0$ to $3.0$, and the $R^2$ from the fit to actual values.

# 3 Probability Distribution

In this section we present the probability distribution function for the Riemann zeta function values at Generalized Gram points. We present a universality relation satisfied by these distributions.

The sample space for our study is the interval along the critical axis specified by $(T_1, T_2)$. While empirical studies necessarily use large but finite $T_1, T_2$, we are interested in the limit $T_1 \to \infty, T_2 \to \infty, T_2 - T_1 \to \infty$, however

$$T_2 - T_1 \ll T_1. \tag{3.1}$$

Because of Equation 3.1, we can consider $\ln(t)$ to be effectively constant over the interval. The latter condition is not essential but is convenient, in that it simplifies the numerical work. The notation $\ln(t)$ stands for the natural logarithm of $t$. We study the probability distribution function for $Z(t)$ at generalized Gram points, $p_\phi(y)$:

**Definition 3.1.**

$$\int_a^b p_\phi(z)dz \tag{3.2}$$

is the probability that $a < Z(t) < b$ when we consider the values of $Z(t)$ for a large number of generalized Gram points in the sample space.

The probability density $p_\phi(z)$ depends on the sample space (i.e., on the height $t$ and on the size of the sample space). In practice the densities are not sensitive to the choice of the sample space as long as the height $t$ is large enough and the length of the interval from which the sample is collected is large enough

| $\phi$ | 0 | | $\pi/6$ | | $\pi/4$ | | $\pi/3$ | | $\pi/2$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z$ | *actual* | *predict* | *actual* | *predict* | *actual* | *predict* | *actual* | *predict* | *actual* | *predict* |
| −3.00 | 0.012 | 0.012 | 0.016 | 0.015 | 0.019 | 0.019 | 0.025 | 0.024 | 0.038 | 0.038 |
| −2.50 | 0.016 | 0.016 | 0.020 | 0.020 | 0.025 | 0.025 | 0.032 | 0.032 | 0.050 | 0.050 |
| −2.00 | 0.023 | 0.022 | 0.028 | 0.028 | 0.035 | 0.035 | 0.044 | 0.044 | 0.067 | 0.067 |
| −1.50 | 0.033 | 0.033 | 0.041 | 0.041 | 0.050 | 0.051 | 0.063 | 0.063 | 0.094 | 0.094 |
| −1.00 | 0.054 | 0.054 | 0.065 | 0.066 | 0.079 | 0.080 | 0.098 | 0.097 | 0.140 | 0.140 |
| −0.50 | 0.105 | 0.105 | 0.122 | 0.122 | 0.143 | 0.142 | 0.167 | 0.168 | 0.227 | 0.227 |
| 0.00 | 0.316 | 0.317 | 0.324 | 0.324 | 0.331 | 0.331 | 0.337 | 0.338 | 0.345 | 0.345 |
| 0.50 | 0.334 | 0.334 | 0.320 | 0.320 | 0.303 | 0.304 | 0.282 | 0.282 | 0.227 | 0.227 |
| 1.00 | 0.227 | 0.226 | 0.213 | 0.214 | 0.200 | 0.201 | 0.183 | 0.183 | 0.140 | 0.140 |
| 1.50 | 0.157 | 0.158 | 0.149 | 0.150 | 0.140 | 0.139 | 0.127 | 0.126 | 0.093 | 0.095 |
| 2.00 | 0.115 | 0.115 | 0.109 | 0.109 | 0.101 | 0.101 | 0.091 | 0.091 | 0.068 | 0.067 |
| 2.50 | 0.088 | 0.088 | 0.083 | 0.083 | 0.077 | 0.076 | 0.068 | 0.069 | 0.050 | 0.050 |
| 3.00 | 0.069 | 0.069 | 0.065 | 0.064 | 0.059 | 0.060 | 0.054 | 0.053 | 0.039 | 0.039 |

Table 2: Comparison of actual and predicted probability density for some $\phi$

(but not too large on log scale). The emphasis in this work is on the empirical study of the distribution. There are important open theoretical questions about the distribution that we do not cover, and we mention those questions in the Appendix.

The anti-symmetry relation is

$$p_\phi(z) = p_{\phi+\pi}(-z). \tag{3.3}$$

The symmetry relation is

$$p_\phi(z) = p_{2\pi-\phi}(z). \tag{3.4}$$

The universality relation is

$$p_\phi(z) = A(z)\cos(\phi) + B(z)\cos(2\phi) + C(z), \tag{3.5}$$

where $A(z)$, $B(z)$ and $C(z)$ are universal functions, i.e., they do not depend on $\phi$.

We estimated $A(z)$, $B(z)$ and $C(z)$ by fitting Eq. 3.5 to the actual probability densities for several values of $z$ and $\phi$. Table 1 gives the fitted values of the universal functions. Fig 1 and Fig. 2 and Table 2 show the comparison of the predictions from the universality relation to the actual probability densities, for some values of the argument $z$.

From the symmetry relations Eq. 3.4 and Eq. 3.3 we find that the function $A(z)$ has to be anti-symmetric in $z$, and $B(z)$ and $C(z)$ have to be symmetric in $z$. Table 1 confirms these properties. The figures, and Table 2 show the excellent agreement of the actual probability densities with the universality relations prediction.

We briefly mention other studies of the value distribution of the Riemann zeta function and the closely related Hardy's function [4]. Ivić's monograph [5] has a comprehensive survey of the field. Selberg in unpublished work showed that at large $t$ $\log(\zeta(\frac{1}{2}+it))$ is approximately normally distributed with a standard deviation of order $\sqrt{log\ log\ t}$ (see Ref. [6]). He showed a similar result [7, 8] for $\log(|Z(t)|)$. Laurincikas [9] used probabilistic number theory to prove various results about the distribution of the Riemann zeta function.

Regarding the value distribution at specific points along the Gram interval, Titchmarsh [10] and Kalpokas and Steuding [11] present results pertaining to the mean value of the Riemann zeta function. Lester's [12] Ph. D. thesis also considers the distribution of $\log(|\zeta(\frac{1}{2}+it)|)$ for specific points along the Gram interval. The question of the values of Hardy's $Z$-function at a discrete sequence of points on the critical axis is quite interesting. The above references give some results in this direction that can be proved rigorously. At the same time, the present state of Riemann zeta function theory gives limited information about the value distribution of $Z(t)$ at discrete sequences of points. Possibly, these analogues could differ significantly from the continuous case. Our numerical studies of the value distribution of Hardy's $Z$-function at discrete points helps fill the gaps.

Hardy's function $Z(t)$ is evaluated using the Riemann−Siegel series

$$Z(t) = 2\sum_{n=1}^{m} \frac{\cos(\theta(t) - t\ln(n))}{\sqrt{n}} + R(t), \tag{3.6}$$

where $m$ is the integer part of $\sqrt{t/(2\pi)}$. $R(t)$ is a small remainder term which can be evaluated to the desired level of accuracy. We used the techniques in Refs. [13, 14, 15] to efficiently evaluate the zeta function at large $t$. To evaluate $Z(t)$ at several points in the Gram interval, we have to use band limited function interpolation [16]. We evaluate the coefficients in the series for band limited function interpolation at the Gram points, and use the series to evaluate $Z(t)$ at other points in the Gram interval. The most important source for loss of accuracy at large heights is the cancellation between large numbers that occur in the arguments of the cos terms in Eq. (3.6). We use a high precision module to evaluate the arguments. The rest of the calculation is done using regular double precision accuracy. The zeros from Ref [17] were used to check the accuracy of our zeta function calculations. Our evaluations of $Z(t)$ at $T = 10^{12}$ are accurate to better than $10^{-6}$.

# 4 Conclusions

The most exciting new result is the discovery of a universality relation for the value distribution of the Hardy Z function at discrete points. Eq. 3.5 states the universality relations. Table 1 gives the fitted values of the universal functions. Fig 1 and Fig. 2 show the comparison of the predictions from the universality relation to the actual probability densities, for some values of the argument $z$. The figures, and Table 2 show the excellent agreement of the actual probability densities with the universality relations prediction.

# Appendix

A more precise definition for the function $p_\phi(y)$ is provided. We consider the interval along the critical axis specified by $(T_1, T_2)$. While empirical studies necessarily use large but finite $T_1, T_2$, we are interested in the limit $T_1 \to \infty, T_2 \to \infty, T_2 - T_1 \to \infty$, however

$$T_2 - T_1 \ll T_1. \tag{4.1}$$

Because of Equation 4.1, we can consider $\ln(t)$ to be effectively constant over the interval. A probability space $W$ is defined by a sample space of elementary events $\Omega$, a $\sigma$ algebra of all considered events $\mathcal{F}$, and a probability measure $P$. Our space of elementary events $\Omega$ is the set of all generalized Gram points with value phi in the interval. This is easily seen to be a discrete space. If we denote $\theta(T_1) = 2k_1\pi + \phi$, $\theta(T_2) = 2k_2\pi + \phi$, then we can index the set of generalized Gram points in the interval by the integer $k$, where $k$ lies in the interval $(k_1, k_2)$. Since the mean spacing $\delta$ of the zeros at height $t$ is $\delta = 2\pi(\ln(t/2\pi))^{-1}$, the cardinality of this set is $(T_2 - T_1) * (\ln(T_1/2\pi))/(2\pi)$. Since we are dealing with a discrete space, we follow standard practice and choose $\mathcal{F}$, the $\sigma$ algebra, to be the collection of all subsets of $\Omega$ . Finally, the random variable whose probability distribution we wish to study is the value of $Z(t)$ at the generalized Gram point $t$.

We can get some insight into the probability distribution $p_\phi(z)$ from the studies of Kalpokas and Steuding [11]. They show that for $\phi_1$ in the range $[0, \pi)$

$$\sum_{0<t<T, \zeta(\frac{1}{2}+it) \in e^{i\phi_1}\mathbb{R}} \zeta(\frac{1}{2} + it) = (2e^{i\phi_1} \cos(\phi_1)) \frac{T}{2\pi} \ln(\frac{T}{2\pi}) + O_\epsilon(T^{(\frac{1}{2}+\epsilon}). \tag{4.2}$$

From Equation 4.2 and the cardinality of our sample space, it follows that a finite mean can be defined for the value distribution in our sample space. This gives support to the existence of a limiting distribution. We now consider the possibility of defining higher order moments for the value distribution. Kalpokas and Steuding also show that

$$\sum_{0<t<T, \zeta(\frac{1}{2}+it) \in e^{i\phi_1}\mathbb{R}} |\zeta(\frac{1}{2}+it)|^2 = \frac{T}{2\pi}(\ln(\frac{T}{2\pi}))^2 + (2\gamma + 2\cos(2\phi_1)) \frac{T}{2\pi} \ln(\frac{T}{2\pi}) + \frac{T}{2\pi} + O_\epsilon(T^{(\frac{1}{2}+\epsilon}), \tag{4.3}$$

where $\gamma$ is Euler's constant. From Equation 4.3 and the cardinality of our sample space it seems that higher order moments of the value distribution diverge logarithmically. There is a lot of cancellation between positive and negative values of $Z(T)$ for odd moments, so odd moments will behave better than even moments. It seems likely that a limiting value distribution exists, it has a finite well-defined mean, and higher moments diverge logarithmically. Distributions which do not have all moments defined are used in applications, for example, the Cauchy-Lorentz distribution [18]. Equation 4.2 and 4.3 predict that the standard deviation should be independent of $\phi$. These equations also imply that the probability distribution will have a finite standard deviation if the argument is normalized by $\sqrt{\frac{T}{2\pi}}$.

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