# Random Matrix Theory explanation for Riemann Zeta Value Distribution Symmetry

**O. Shanker**, Mountain View, CA 94041, U. S. A. email: oshanker@gmail.com

**Abstract**

We show that the value distribution of the characteristic polynomial of the Circular Unitary Ensemble (CUE) at discrete points can be expressed in terms of three functions which do not depend on the angle characterizing the Generalized Gram point. These relationships are very similar to those observed for the Riemann zeta function and Dirichlet $L$ functions.

**Keywords**: Circular Unitary Ensemble, Riemann zeta, Value Distribution, Symmetry **Mathematics Subject Classification (MSC)**: 11M06, 11-04.

## 1 Introduction

Inspired by striking new results found for the value distribution of the Riemann zeta function and Dirichlet $L$ functions, we searched for similar results for the Circular Unitary Ensemble (CUE) at discrete points. We did the study because of the well-known correspondence between the zeros of the Riemann zeta function and the eigenvalues of Random Matrix Theory (RMT). We show that the value distribution can be expressed in terms of three functions which do not depend on the angle characterizing the Generalized Gram point.

RMT has found applications [1, 2] in many areas of physics, mathematics, probability, statistics, and engineering. RMT is closely associated with the study of the zeroes of the Riemann zeta function and Generalized Zeta functions. The zeros are of interest to mathematicians and physicists. Mathematicians study the spacings because of its applications to analytic number theory, while physicists study it because of its relation to the theory of the spectra of random matrix theories and the spectra of classically chaotic quantum systems. Ref. [3] shows that the characteristic polynomial of the CUE has a limiting form when properly rescaled. See Ref. [4, 5, 6] for further references, examples and discussion. Ref. [1] shows that the CUE provides a good model for the moments and other properties of the $|\zeta(1/2 + it)|$.

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We were motivated to do the current study because of almost identical properties observed for the Riemann zeta function and Dirichlet $L$ functions. Ref. [7, 8] studied empirically the distribution of $Z(t)$ values at Gram points and showed that the value distribution of the Hardy $Z$ function at discrete points is anti-symmetrical for reflections around the mid-points of the Gram intervals (Eq. 2.5) and symmetrical for reflections around the Gram points(Eq. 2.6). Ref. [10] showed that the value distribution of the Riemann zeta function can be expressed in terms of three functions which do not depend on the angle characterizing the Generalized Gram point. Ref. [9] showed similar results for the Dirichlet $L$ functions.

## 2 Materials and Methods: the Riemann zeta function

In this section we establish the required notation for the Riemann Zeta Function. This will help us to define the corresponding quantities for the CUE characteristic polynomial. For $\mathrm{Re}(s) > 1$ the Riemann Zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \in primes} \left(1 - p^{-s}\right)^{-1}. \tag{2.1}$$

$\zeta(s)$ can be continued to the complex plane. Riemann's hypothesis, that the non-trivial zeros of $\zeta(s)$ lie on the critical axis $1/2 + it$, is probably the most famous unsolved problem in mathematics. The mean spacing $\delta$ of the zeros at large height $T$ is $\delta = 2\pi(\ln(T/2\pi))^{-1}$. One defines Hardy's function

$$Z(t) = exp(i\theta_{Riemann}(t))\zeta(1/2 + it) \tag{2.2}$$

where

$$\theta_{Riemann}(t) = arg(\pi^{it/2}\Gamma(\frac{1}{4} + \frac{it}{2})). \tag{2.3}$$

We use the subscript in $\theta_{Riemann}$ to differentiate this $\theta$ from the one which occurs in the study of the CUE matrices. The argument in Eq. 2.3 is defined by continuous variation of $t$ starting with the value 0 at $t = 0$. $Z(t)$ is real valued for real $t$, and we have $|Z(t)| = |\zeta(1/2 + it)|$. Thus the zeros of $Z(t)$ are the imaginary part of the zeros of $\zeta(s)$ which lie on the critical line.

Gram points play an important role in the the theory because many of the zeros are separated by them. When $t \geq 7$, the $\theta$ function Eq.(2.3) is monotonic increasing. For $n \geq -1$, the $n$-th Gram point $g_n$ is defined as the unique solution $> 7$ to $\theta_{Riemann}(g_n) = n\pi$. A Gram interval is the interval $G_n = [g_n, g_{n+1})$. In analogy with Gram points, we can associate an angle $\phi$ with a point $t$ on the critical axis as follows:

**Definition 2.1.** For $t \geq 7$, $t$ is said to be a generalized Gram point with value $\phi$ if $\theta_{Riemann}(t) = 2k\pi + \phi$, where $0 \leq \phi < 2\pi$.

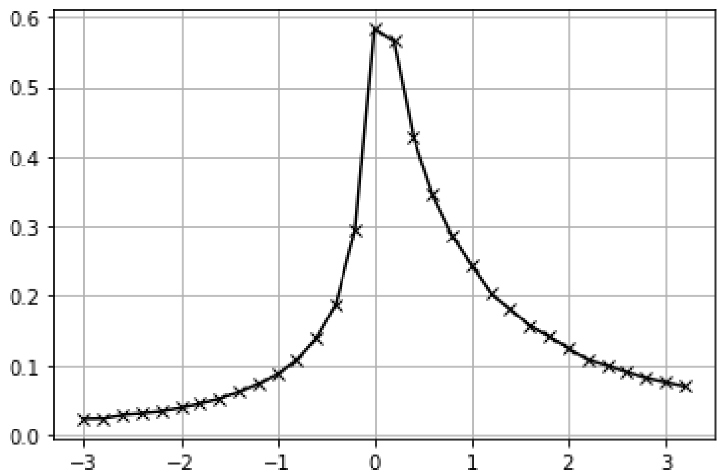


Figure 1: Probability density for $\phi = 0$.

We define the probability distribution function for $Z(t)$ at generalized Gram points $p_\phi(y)$:

**Definition 2.2.**

$$\int_a^b p_\phi(z)dz \tag{2.4}$$

is the probability that $a < Z(t) < b$ when we consider the values of $Z(t)$ for a large number of generalized Gram points in the sample space.

It is known (Ref. [8]) that the average value of the $Z$ distribution is $2cos(\phi)$ where $\phi$ is the angle characterizing the Generalized Gram point. We have also shown (Ref. [10]) that the probability distributions satisfy the anti-symmetry relation

$$p_\phi(z) = p_{\phi+\pi}(-z) \tag{2.5}$$

and the symmetry relation is

$$p_\phi(z) = p_{2\pi-\phi}(z). \tag{2.6}$$

Ref. [10] showed that the probability distributions at different $\phi$ can all be expressed in terms of three functions which are independent of $\phi$ and depend only on $z$. These are the observations which we wanted to model using the CUE. For generating the CUE samples for the study we extend the programs and results of Ref. [4].

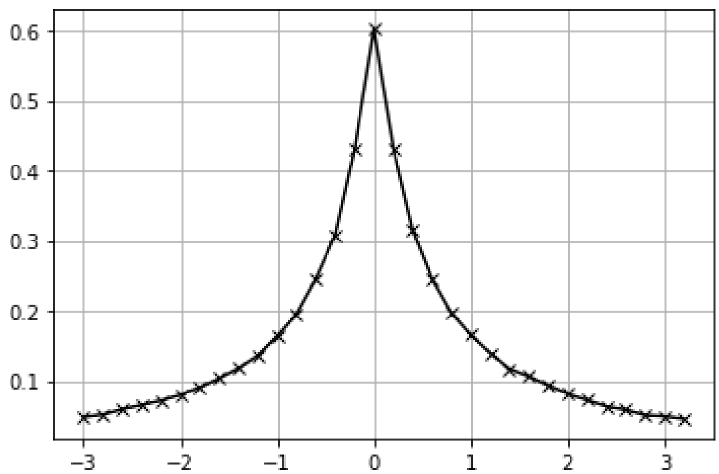


Figure 2: Probability density for $\phi = \pi/2$.

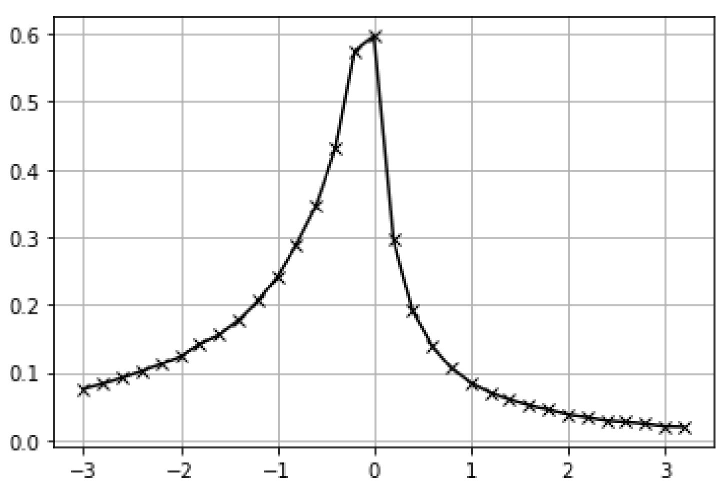


Figure 3: Probability density for $\phi = \pi$. Compare with Fig. 1 to observe the mirror symmetry.

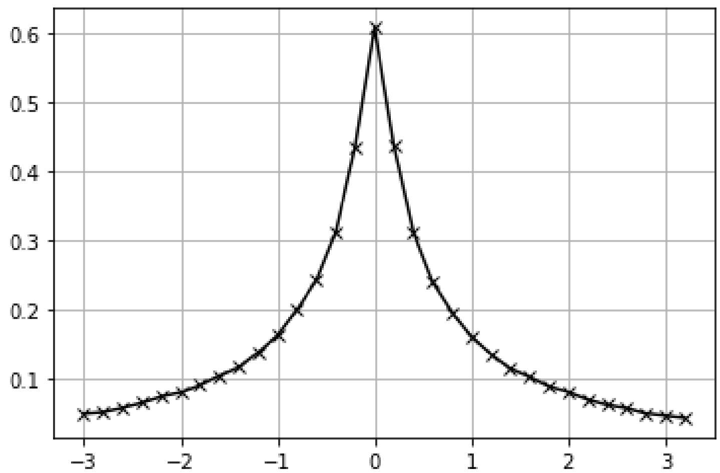


Figure 4: Probability density for $\phi = 3\pi/2$.

# 3 Results

## 3.1 Symmetry relations for CUE

The question of the values of the characteristic polynomial of a CUE matrix at a discrete sequence of points is quite interesting. In this section we present our numerical studies. The results provide very good evidence for two symmetry properties. Since the Riemann zeta studies were done at a height $T = 10^{28}$, we use the well-known correspondence $N \sim \ln(T/2\pi)$ where $N$ is the size of the unitary matrix we should consider. Thus, we chose $N = 63$ for our study. We studied 10000 sample $U$ to generate the distributions.

The characteristic polynomial can be written as

$$\Lambda(U,\theta) = det(I - U\exp^{-i\theta}) \tag{3.1}$$

$$= \prod_n (1 - \exp^{i(\theta_n - \theta)}). \tag{3.2}$$

This expression simplifies to

$$\Lambda(U,\theta) = 2^N \exp^{i(3N\pi/2 + \sum_n \theta_n/2 - N\theta/2)} \prod_n \sin((\theta_n - \theta)/2). \tag{3.3}$$

In analogy with the $Z$ in the Riemann zeta case, we can identify a $Z_{CUE}$ by defining

$$Z_{CUE}(\theta)) = \exp^{-i(3N\pi/2 + \sum_n \theta_n/2 - N\theta/2)} \Lambda(U,\theta) \tag{3.4}$$

$$= 2^N \prod_n \sin((\theta_n - \theta)/2). \tag{3.5}$$

| | | | |
|---|---|---|---|
| 0 | $\pi/2$ | $\pi$ | $3\pi/2$ |
| 2.03 | 0.00 | $-2.02$ | $-0.03$ |

Table 1: Mean $Z(t)$. Row 1: $\phi$, Row 2: mean $Z$

$Z_{CUE}(\theta)$) is real valued, and we have $|Z_{CUE}(\theta)| = |\Lambda(U,\theta)|$. The analogue of the Riemann zeta Gram points occurs when $\Lambda(U,\theta)$ is real, i.e., $(3N\pi/2 + \sum_n \theta_n/2 - N\theta/2) = m\pi$. We call $\theta$ a CUE generalized Gram point with value $\phi$ if $(3N\pi/2 + \sum_n \theta_n/2 - N\theta/2) = 2k\pi + \phi$, where $0 \leq \phi < 2\pi$. For a given $\phi$, we have $N$ distinct CUE generalized Gram points spaced uniformly around the unit circle. The definition for the probability distribution is analogous to the Riemann zeta definition.

The CUE probability distributions satisfy (see Fig. 1, 2, 3, 4) the anti-symmetry relation

$$p_\phi(z) = p_{\phi+\pi}(-z) \tag{3.6}$$

and the symmetry relation

$$p_\phi(z) = p_{2\pi-\phi}(z). \tag{3.7}$$

We also found (see Table 1) that the average value of the distribution is $2cos(\phi)$ where $\phi$ is the angle characterizing the CUE generalized Gram point, i.e.,

$$< z >_\phi = 2\cos(\phi). \tag{3.8}$$

Thus, we reproduce the behaviour for the Riemann zeta case.

### 3.2 Relations for distributions at different $\phi$

For the Riemann zeta and Dirchlet $L$ function probability distributions, we found [9, 10] that the probability distributions at different $\phi$ can all be expressed in terms of three functions which are independent of $\phi$ and depend only on $z$. We show that such a relation holds for the CUE distributions also. This is a very intriguing direction to probe more deeply.

The universality relation is

$$p_\phi(z) = A(z) + B(z)\cos(\phi) + C(z)\cos(2\phi), \tag{3.9}$$

where $A(z)$, $B(z)$ and $C(z)$ are functions which do not depend on $\phi$. We estimated $A(z)$, $B(z)$ and $C(z)$ by fitting Eq. 3.9 to the actual probability densities for several values of $z$ and $\phi$. Fig 5 and Fig. 6 show the comparison of the predictions from the universality relation to the actual probability densities, for some values of the argument $z$. We find that the function $B(z)$ is anti-symmetric in $z$, and $A(z)$ and $C(z)$ are symmetric in $z$, which leads to Eq. 3.7 and Eq. 3.6. The figures show the excellent agreement of the actual probability densities with the universality relations prediction.

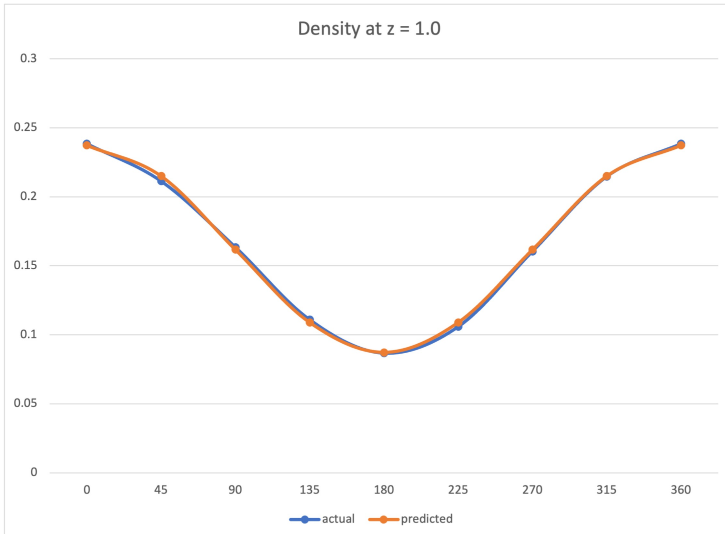


Figure 5: Test of universality. Comparison of probability density prediction from universality with actual values, for $z = 1.0$. The y axis is the probability density. The x axis is the angle characterizing the Generalized Gram point.

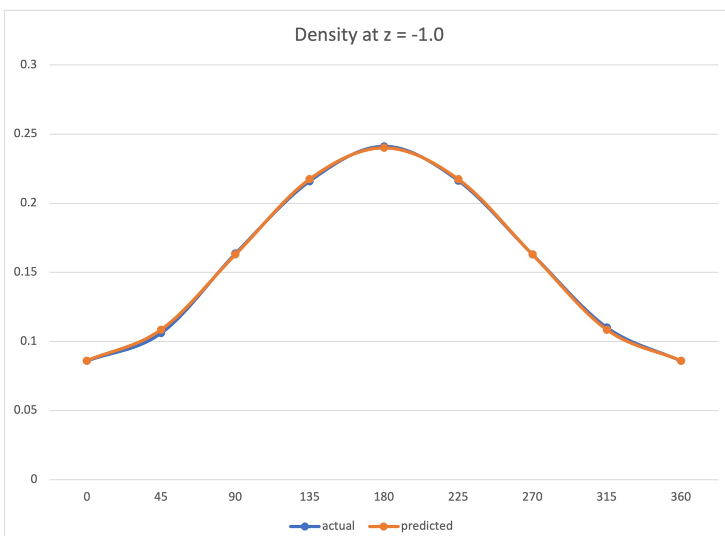


Figure 6: Test of universality. Comparison of probability density prediction from universality with actual values, for $z = -1.0$.

# 4 Conclusions

We relate anti-symmetry and symmetry properties for the value distribution of the CUE characteristic polynomial at discrete points to the observed properties for the Riemann zeta function. We presented a simple relation for the average value of the distribution. We showed that the value distribution can be expressed in terms of three functions which do not depend on the angle characterizing the Generalized Gram point. These relations are very similar to the relations observed for the Riemann zeta function.


# Acknowledgments and Funding Statement

The study was done as an independent researcher. There was no external funding.


# Ethical Compliance

No procedures were performed involving human participants in the study.

# Data Availability Statement

The computer programs used during the current study are available from the corresponding author on reasonable request.

# Conflict of Interest declaration

The authors declare that they have NO affiliations with or involvement in any organization or entity with any financial interest in the subject matter or materials discussed in this manuscript.

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