4. Relations and Functions

This section will reinforce some concepts that you have learnt in Chapter 2 and 3.

4.1 Relations

At the end of this section, students will be able to:

- identify and define a relation.
- describe a relation.
- identify and interpret the inverse of a relation and the composition of two relations
- identify properties of a relation and verify them
- define equivalence relations and classes.
- Identify the properties of equivalence classes.

4.1.1 Ordered pairs and the Cartesian product of two sets

Definition (ordered pair):

Suppose a, b are two objects. Then (a, b) is called an ordered pair and, a and b are the first and second elements of the ordered pair.

Note that $\{a, b\}= \{b, a\}$. But $(a, b) \neq (b,a)$.

Result 1: Suppose (a, b) and (c, d) are two ordered pairs. Then $(a, b) = (c, d)$ iff $a=c \wedge b=d$.

Definition (Cartesian product):

Suppose A and B are sets. Then their products AxB (read as A cross B) is defined as:

$$A \times B = \{ (x,y) \mid x \in A \wedge y \in B \}$$

Examples:

1) $A=\{1,5\}$ and $B=\{1,7,2\}$. Then $A \times B = \{(1,1), (1,7), (1,2), (5,1), (5,7), (5,2)\}$
2) $A=N^*$, $B=\{1,7\}$ and $N^*=\{1,2,3,\ldots\ldots\}$. Then

$$\begin{aligned} A \times B &= \{ (x,y) \mid x \in N^* \wedge y \in \{1,7\}\} \\ &= \{ (x,y) \mid x \in N^* \wedge (y =1 \vee y=7)\} \\ &= \{(1,1), (2,1), (3,1),\ldots\ldots (1,7), (2,7), (3,7),\ldots\ldots\ldots\} \end{aligned}$$

Results: Suppose A, B, C are three sets and Φ is the empty set. Then;

1) $A \times \Phi = \Phi \times A = \Phi$
2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
3) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Note: Results 2) and 3) are called distribution properties of x over ∪ and ∩ respectively.

4.1.2 Definition of a Relation

A relation can be thought of as a table that lists the relationship of elements to other elements. See the Table 1 below. Table 1 shows which students are taking which courses. For example, Kamal is taking Database and Software Engineering and Ajith is taking Mathematic. In the terminology of relations, we would say that Kamal is related to Database and Software Engineering and that Ajith is related to Mathematic.

Table 1: Relation of Students to Courses

Students	**Courses**
Kamal	Database
Ajith	Mathematic
Kamal	Software Engineering
Gamini	Database

We would also can see that this relation can be expressed as a set of ordered pairs if we represent the rows of the Table 1 as ordered pairs as in the Figure 1.

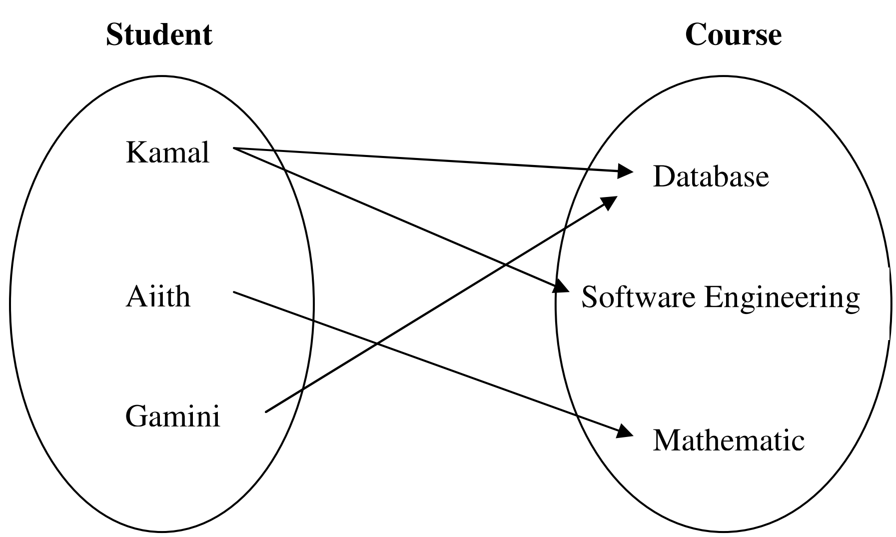


Figure 1: Relation of Students to Courses

Definition (Relation between two sets): A relation is a set of ordered pairs. For example, the relation in the above illustration, say ρ, can be given as

ρ = {(Kamal, Database), (Ajith, Mathematic), (Kamal, Software Engineering), (Gamini, Database)}

Note: the relation ρ is a subset of the Cartesian product between two sets Student and Course. i.e, $\rho \subseteq$ Student x Course.

That is, a relation from set A to set B is a subset of A x B.

Definition (Relation on a set):

If ρ is a relation from A to itself, that is, if $\rho \subseteq A \times A$, then we say that R is a relation on A. For example, $\rho=\{ (x,y)|x,y\ \varepsilon R \wedge x \leq y\}$ is a relation on real number set R.

Definition (Domain and Range):

Suppose ρ is a relation. Then the domain and range of ρ written D(ρ) and R(ρ) respectively are defined as:

$$D(\rho)=\{x|\ \exists y, (x,y)\ \varepsilon\ \rho\}$$
$$R(\rho)=\{y|\ \exists x, (x,y)\ \varepsilon\ \rho\}$$

In the above relation between Student and Course,

D(ρ)={Kamal, Ajith, Gamini}
R(ρ)={Database, Mathematic, Software Engineering}

4.1.3 Inverse of a relation

Definition (inverse):

Suppose ρ is a relation. Then the inverse of ρ written ρ^{-1} is the relation defined as:

$$\rho^{-1} =\{(x,y)|(y,x)\ \varepsilon\ \rho\}$$

Example:

1. suppose $\rho = \{(1,4), (3,7),(6,6),(8,4)\}$ then
   $\rho^{-1} =\{(4,1), (7,3), (6.6), (4,8)\}$

2. suppose $\rho=\{ (x,y)|x,y\ \varepsilon R \wedge x \leq y\}$ then
   $\rho^{-1} = \{ (x,y)|x,y\ \varepsilon R \wedge y \leq x\}$

Result:

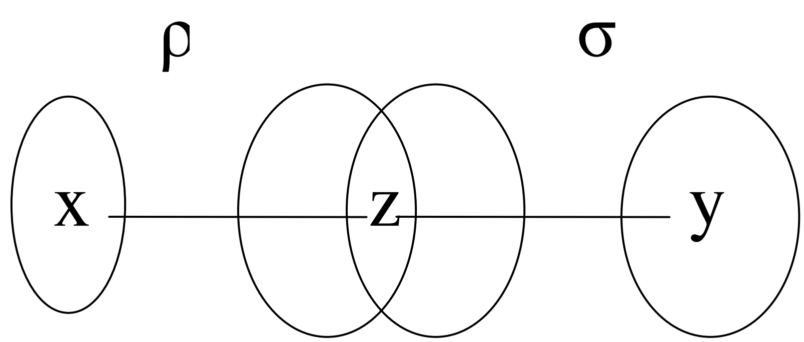
Suppose ρ is a relation. Then $(\rho^{-1})^{-1} = \rho$

4.1.4 Composition of Relations

Suppose ρ and σ are two relations. Then their composition written as σ o ρ is the relation defined by:

$\sigma \circ \rho = \{ (x,y) \mid \exists z\ (x,z)\ \varepsilon\ \rho \wedge (z,y)\ \varepsilon\ \sigma \}$

This can be graphically shown as in the following figure.



Note that z ε R(ρ) and z ε D(σ).

Example: Suppose ρ={(1,4), (2,3),(7,9),(1,8),(3,3),(5,2)} and
σ={(4,4),(3,10),(8,12),(8,5),(8,8),(15,2).(14,8)}. Then,
σ o ρ = {(1,4),(2,10),(1,12),(1,5),(1,8),(3,10)}
ρ o σ = {(8,2),(15,3)}

4.1.5 Directed Graph

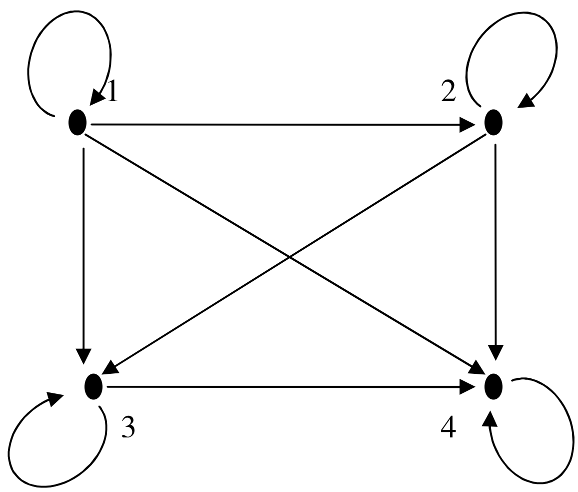
The directed graph is a way of picturing a relation on a set. To draw a directed graph of a relation on a set X,

1. draw dots or vertices to represents the elements of the set X.
2. if the element (x,y) is in the relation, draw an arrow from x to y. This arrow is called an edge.
3. an element of the form (x,x) in a relation corresponds to a directed edge from x to x. Such an edge is called a loop.

Example:

Draw a directed graph for the relation ρ on a set X={1,2,3,4} defined by ρ={ (x,y)|x,y ε X ∧ x ≤ y}.

ρ={ (1,1),(1,2),(1,3),(1,4), (2,2),(2.3),(2,4),(3,3),(3,4),(4,4)}.



Definition (Reflexive):

A relation ρ on a set X is called reflexive if (x,x) ε ρ for every x. i.e., ∀x, x є D(ρ) => (x,x) є ρ.

For example, ρ={ (x,y)|x,y ε X ∧ x ≤ y}in above is reflexive. The directed graph of a reflexive relation has a loop at every vertex.

Definition (symmetric):

A relation ρ on a set X is called symmetric if for all (x,y) ε ρ, (y,x) ε ρ i.e., ∀x, ∀y, (x,y) є ρ => (y,x) є ρ.

For example, the relation ρ={ (1,1),(2.3),(3,2),(3,3)} on X={1,2,3}is a symmetric relation. The directed graph of a symmetric relation has the property that whenever there is a directed edge from u to v, there is also a directed edge from v to u.

Definition (transitive):

A relation ρ on a set X is called transitive if for all (x,y), (y,z) ε ρ, (x,z) ε ρ i.e., ∀x, ∀y, ∀z, (x,y) є ρ ∧ (y.z) є ρ => (x,z) є ρ

For example, the relation ρ={ (1,1),(2.3),(3,4),(2,4),(3,3)} on X={1,2,3,4}is a transitive relation. The directed graph of a transitive relation has the property that whenever there is a directed edge from u to v and v to s, there is also a directed edge from u to s.

4.1.6 Equivalence Relations

Definition(equivalence relation):

The relation ρ is said to be an equivalence relation if it satisfies the following three properties.

1) $\forall x, x \in D(\rho) \Rightarrow (x,x) \in \rho$ (reflexive)
2) $\forall x, \forall y, (x,y) \in \rho \Rightarrow (y,x) \in \rho$ (symmetric)
3) $\forall x, \forall y, \forall z, (x,y) \in \rho \wedge (y.z) \in \rho \Rightarrow (x,z) \in \rho$ (transitive)

Result:

If ρ is an equivalence relation, then $D(\rho) = R(\rho)$

Example:

The relation $\rho = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1), (4,5), (5,4) \}$ defined on $X=\{1,2,3,4,5,6\}$ is an equivalence relation as it is reflexive, symmetric and transitive.

Definition(equivalence classes):

Suppose ρ is an equivalence relation and $x \in D(\rho)$. Then the equivalence class of x denoted by $[x]_\rho$ is defined by

$$[x]_\rho = \{ y | (x,y) \in \rho \}$$

i.e., the equivalence class of x is the set of all elements that are related to x.

Example:

For the equivalence relation $\rho = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1), (4,5), (5,4) \}$ defined on $X=\{1,2,3,4,5,6\}$, Find $[1]_\rho$, $[2]_\rho$, $[3]_\rho$, $[4]_\rho$, $[5]_\rho$, $[6]_\rho$

$$[1]_\rho= [2]_\rho= [3]_\rho=\{1,2,3\}$$
$$[4]_\rho= [5]_\rho= \{4,5\}$$
$$[6]_\rho=\{6\}$$

Note that X can be written as the union of three disjoint sets. i.e., $X = \{1,2,3\}\cup\{4,5\}\cup\{6\}$

4.2 Functions

At the end of this section, students will be able to:

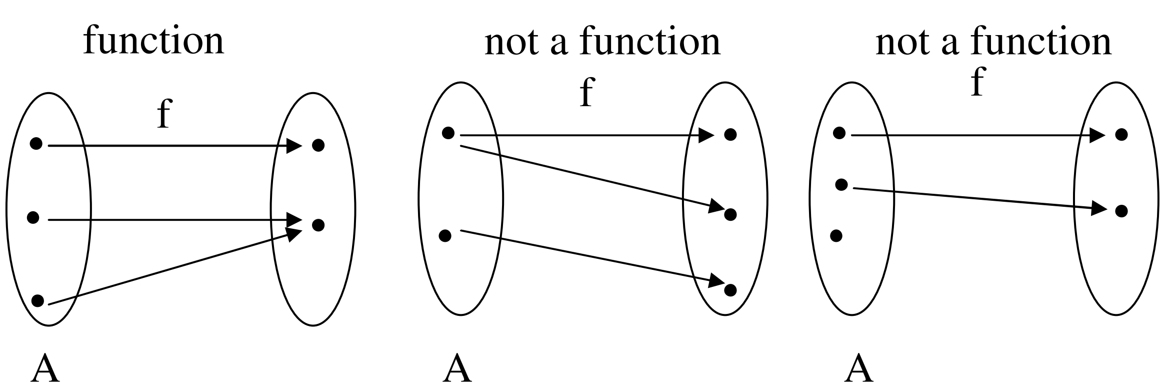
- distinguish between function and relation.
- interpret a function for its domain, range and characteristics
- determine if a given relation is a function.
- determine the degree of a function
- define the inverse of a function and prove some properties of it.
- define the composition of two functions and prove some properties of it.
- Define a function as mapping between two sets
- Distinguish between functions that map onto and into a set.

A function is a special kind of relation. Recall (see definition 4.1.2) that a relation ρ from X and Y is a subset of the Cartesian product X x Y.

Definition (function)**:** Suppose A is a non-empty set. Suppose for each element x in A, we assign an element denoted f(x) say. Then we have a function. Let's denote this function by f. A is said to be the domain of f. The domain of f is denoted by D(f).

e.g., D(f) = {1, 4, 7, 9}, f(1)=10, f(4)=0, f(7)=3, f(9)=6

Note:



Definition(graph): Suppose f is a function. Then the graph of f denoted by graph(f) is the set,

$$\text{graph}(f) = \{(x,y) \mid x \in D(f) \wedge y = f(x)\}$$

Definition(Range): Suppose f is a function. Then the range of f denoted by R(f) is the set,

$$R(f) = \{y \mid \exists x, x \in D(f) \wedge y = f(x)\}$$

Example:

D(f) = {1, 4, 7, 9}, f(1)=10, f(4)=0, f(7)=3, f(9)=6. Find graph(f) and R(f).

graph(f) = {(1,10), (4,0), (7,3), (9,6) }
R(f) = {10, 0, 3, 6) }

Definition(1-1): Suppose f is a function. Then f is said to be one to one (1-1 for short) if one of the following is satisfied.

1) $\sim(\exists x_1, \exists x_2 , x_1 \neq x_2 , f(x_1) = f(x_2))$
2) $(\forall x_1, \forall x_2\ x_1 \varepsilon D(f) \wedge x_2 \varepsilon D(f) \wedge x_1 \neq x_2)\Rightarrow f(x_1) \neq f(x_2))$
3) $(\forall x_1, \forall x_2 , f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$

Definition(inverse): Suppose f is a function and f is 1-1. Then the inverse of f denoted by $f^{-1}$is the function such that $D(f^{-1})$=R(f) and given any x ϵ $D(f^{-1})$, $f^{-1}(x)$=y where y is given by x=f(y).

Note: There exist unique y such that. $f^{-1}(x)$=y for x ϵ $D(f^{-1})$.

Result 1: Suppose f is a 1-1function. Then, $R(f^{-1}) = D(f)$.

Example: Show that f^{-1} is 1-1.

Suppose $f^{-1}(x_1) = f^{-1}(x_2)$.
Let $y = f^{-1}(x_1)$.
Then $x_1 = f(y)$ and $x_2 = f(y)$
Since f is a function, $x_1 = x_2$.
Therefore, $f^{-1}(x_1) = f^{-1}(x_2) \Rightarrow x_1 = x_2$.
i.e., f^{-1} is 1-1.

Example:

Suppose D(f) = {1,3,4,5} and f(1)=6, f(3)=7, f(4)=1 and f(5)=10. Find f^{-1}.

f is 1-1.
$D(f^{-1})$=R(f)={6,7,1,10}. $f^{-1}(6)=1$, $f^{-1}(7)=3$, $f^{-1}(1)=4$, $f^{-1}(10)=5$.

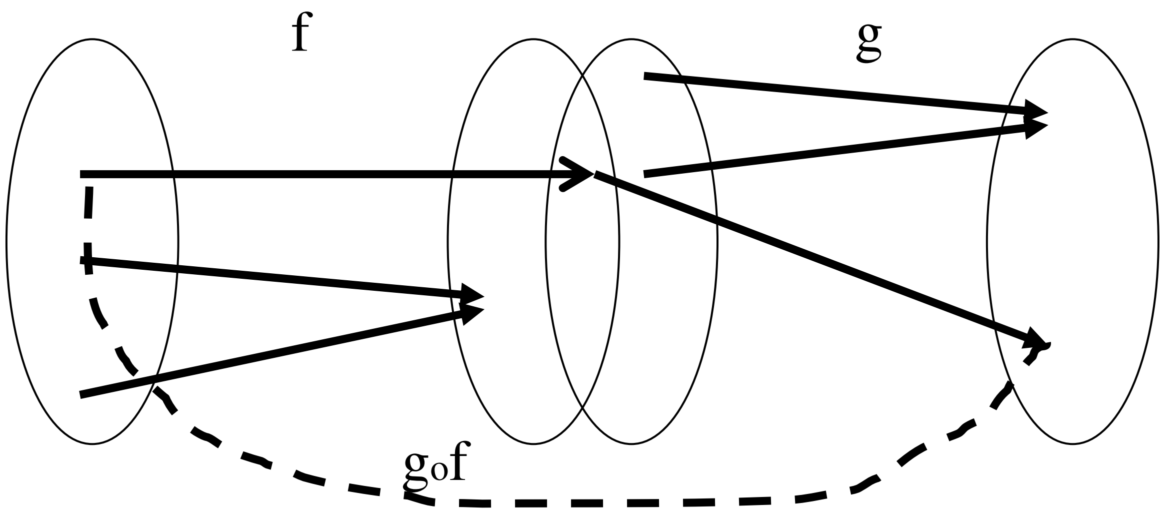
Example:

Suppose $D(f) = \{x | x \in R \wedge x \neq 2\}$ and f(x)=(x-1)/(x-2) for every $x \in D(f)$. Show that f is 1-1 and find f^{-1}.

Suppose $f(x_1) = f(x_2)$.
Then $(x_1-1)/(x_1-2) = (x_2-1)/(x_2-2)$
$x_1 = x_2$
Therefore, f is 1-1.
$D(f^{-1}) = \{ x | x \in R \wedge x \neq 1 \}$. $f^{-1}(x)=(2x-1)/(x-1)$.

Definition(Composition of functions): Suppose g, f are two functions. Then the composition denoted by gof is the function given by

$$D(gof) = \{x | x \in D(f) \wedge f(x) \in D(g) \} \text{ and } gof(x) = g(f(x))$$



Note: $D(gof) \neq \Phi$

Example:

D(f)={1,2,5,6,8}and f(1)=3, f(2)=4, f(5)=7, f(6)=9, f(8)=8
D(g)={1,5,8,3,12,14} and g(1)=1, g(5)=6, g(8)=7, g(3)=9, g(12)=1, g(14)=1
Find gof.

D(gof)={1,8}. gof(1)=9, gof(8)=7.

Definition (onto and into mapping):

Suppose A, B are two sets and f is a function s.t. D(f)=A and $R(f) \subseteq B$. Then we say that f is a mapping from A **into** B and we write $f: A \rightarrow B$. Further, if R(f)=B, we say f maps A **onto** B.

Note: onto means into. But into does not imply onto

Example:

Suppose f: A $\longrightarrow$ B, g: B$\longrightarrow$ C. Show that gof: A$\longrightarrow$C .
i.e., need to show that D(gof) = A and R(gof) $\subseteq$ C.

$x \in$ D(gof) iff $x \in$ D(f) $\wedge$ f(x) $\in$ D(g)
iff $x \in$ A $\wedge$ f(x) $\in$ B
iff $x \in$ A

Therefore, **D(gof) = A**

$y \in$ R(gof) => $x \in$ D(f) $\wedge$ y = gof (x)
=> $x \in$ D(f) $\wedge$ y = g(f (x))
=> $y \in$ R(g)
=> $y \in$ C   Since R(g) $\subseteq$ C

Therefore, **R(gof) $\subseteq$ C**

Definition (bijection): Suppose A, B are two sets and f is a function such that f is 1-1 and f maps A onto B. Then we say that f is a bijection from A to B.

Example:

Let A={1,2} and B={3,4}. Write all the bijection from A to B.

D(f_1)={1,2}and f_1(1)=1, f_1(2)=2
D(f_2)={1,2}and f_2(1)=2, f_2(2)=1