IT2103

Mathematics for Computing 1

Logic

Definition

Proposition

The basic units of mathematicalreasoning are **propositions** (or **statements**)

A **proposition** (or **statement**) is a sentence that is **either true or false**.

Examples

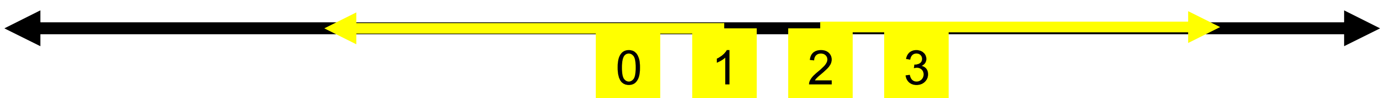
1) 2 is an integer.
2) 2 is not an integer.
3) Is 2 an integer?
4) $X^2 > 10$, Where x is an integer.
5) 5 + 3.

1) is a Proposition
2) is a Proposition
3) is a not a Proposition
4) is a not Proposition
5) is a not Proposition

Examples

The sentence $X^2 > 10$ is not a proposition.

Does the sentence $2 \leq x \leq 1$ a proposition ?



However no matter what number x happens to be the sentence $2 \leq x \leq 1$ is false. Therefore this sentence is a proposition. In sentences like this it is not important to know the value of the variables involved to determine whether such sentences are true of false.

Definition Truth value of a Proposition

A **proposition** (or **statement**) is a sentence that is **either true or false**.

Each proposition can be assigned with a **truth value**.

The truth value of a proposition is either "**True**" (or **T**) or "**False**" (or **F**) depending on whether the proposition is true or false respectively.

Examples

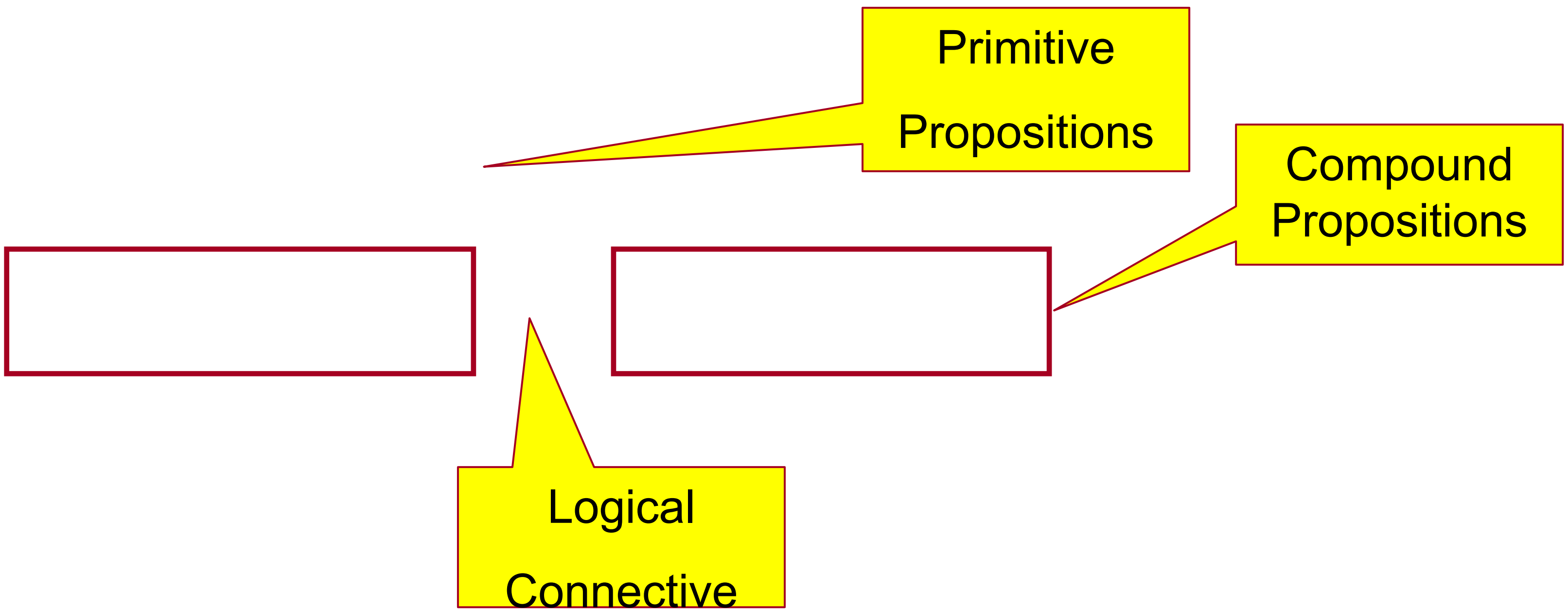
1) 2 is an integer.
2) 2 is not an integer
3) Is 2 an integer?
4) $X^2 > 10$
5) $5 + 3$

Can we assign Truth values for all the above sentences ?

- The truth value of proposition 1) is “True” (or T)
- The truth value of proposition 2) is “False” (or F)

Compound Propositions

- 2 is an integer.
- 2 is an integer and cat is a animal.



Definition Compound Propositions

A proposition is said to be **primitive,** if it cannot be broken down into simple propositions, otherwise the proposition is said to be **composite** (or **compound**).

This means that a composite proposition can always be broken down into a collection of primitive propositions.

Logical Connectives
(or Operations)

Logical connectives (operations) allow compound propositions to be built out of simple propositions. Each logical connective is denoted by a symbol.

Operation	Symbol	
not	~	**Negation**
and	$\wedge$	**Conjunction**
or	$\vee$	**Disjunction**

Examples:

compound propositions

1. $2+2=5$ and $5+3=8$.
2. Cat is a small animal or Dogs can fly.
3. It is not true that University of Colombo is in Matara.

These examples can be represented by using the symbols for logical connectives as below.

- $2+2=5 \wedge 5+3=8$.
- Cat is a small animal $\vee$ Dogs can fly.
- ~ (University of Colombo is in Matara).

Definition Proposition Variables

A variable that denotes a proposition is called a propositional variable.

Propositional variables can be replaced by propositions.

In logic the letters p,q,r, ……. Are used to denote propositional variables

Examples :

If p and q denotes two propositional variables

- $\sim p$ denotes the negation of p
- $p \wedge q$ denotes the conjunction of p and q
- $p \vee q$ denotes the disjunction of p and q

The proposition variables in a proposition can be replaced by any proposition.

For example p and q in $p \wedge q$ can be replaced by the two propositions "*A mouse is a small animal*" and "*Nimal is a boy*" resulting the proposition "*A mouse is a small animal* $\wedge$ *Nimal is a boy*"

Truth Tables

The sentences built out of propositions and logical connectives are also propositions. Therefore they must have truth values. The truth values of compound statements built out of the logical connectives are typically defined by using **truth tables**.

Definition of not(~)

p	~p
T	F
F	T

Truth table for ~p

Definition of and ($\wedge$)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth table of $p \wedge q$

Definition of or ($\vee$)

p	q	p $\vee$ q
T	T	T
T	F	T
F	T	T
F	F	F

Truth table of p $\vee$ q

Examples

Let

p : “It is hot” and

q: “It is sunny”.

Write each of the following sentences symbolically.

a) It is not hot but it is sunny.

b) It is not both hot and sunny.

c) It is neither hot nor sunny.

Examples

Let

p : “It is hot” and

q: “It is sunny”.

a) It is not hot **but** it is sunny.

The convention in logic is that the words *but* and *and* mean the same thing.

Solution :

$\sim p \wedge q$

Examples

Let

p : “It is hot” and

q: “It is sunny”.

b) It is not both hot and sunny.

It is not (hot and sunny)

Solution :

$\sim(p \wedge q)$

Examples

Let

p : “It is hot” and

q: “It is sunny”.

c) It is neither hot nor sunny.

It is not hot and It is not sunny

Solution :

$\sim p \wedge \sim q$

Construction of truth tables

- When constructing a truth table for a compound proposition composed of proposition variables, a column must be allocated for each proposition variable and for the final compound proposition.
- Columns may be allocated to record intermediate compound propositions, which are constructed only to help determination of truth values of the final proposition.

Construction of truth tables

- The table should have enough rows to represent all possible combinations of T and F for all proposition variables.

Number of rows in a Compound Proposition

For example if the compound proposition has two proposition variables there should be 4 (= 2^2) rows in the truth table. If the compound proposition has three proposition variables, there should be 8 (= 2^3) rows. Similarly if the compound proposition has n variables, there should be 2^n rows in the truth table.

Example : Construct the truth table for the compound proposition $\sim(p \wedge \sim q)$

– This proposition has 2 variables, p and q. Therefore the truth table should have $2^2 = 4$ rows.

Example : Construct the truth table for the compound proposition $\sim(p \wedge \sim q)$ …….

– You may allocate two additional columns to record the truth values of $\sim q$ and $(p \wedge \sim q)$ which will help to compute the final truth values easily.

Step 1: Fill in the columns p and q with all possible truth

values

p	q	~q	p∧~q	~(p∧~q)
T	T			
T	F			
F	T			
F	F			

Step 2: Fill in the columns ~q. To fill this column only the values of q is required.

p	q	~q	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F		
T	F	T		
F	T	F		
F	F	T		

Step 3: Now fill the column $p \wedge {\sim}q$ by using the columns for p and ~q.

p	q	${\sim}q$	$p \wedge {\sim}q$	${\sim}(p \wedge {\sim}q)$
T	T	F	F	
T	F	T	T	
F	T	F	F	
F	F	T	F	

Step 4: Now fill the final column $\sim(p \wedge \sim q)$ by using the column $p \wedge \sim q$.

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

It is better to remember the above truth tables in the following way;

(i) $p \wedge q$ is T only in <u>one</u> instance; i.e., when p-T and q-T.

(ii) $p \vee q$ is F only in <u>one</u> instance; i.e., when p-F and q-F.

Examples:

What are the logical meaning of the following

1) $y \leq 5$

2) $5 \leq z \leq 10$

Answers:

1) $y \leq 5$ means $(y < 5) \vee (y = 5)$
2) $5 \leq z \leq 10$ means $(5 \leq z) \cancel{\vee} \wedge (z \leq 10)$

Tautology

Consider the compound proposition $p \vee \sim p$. From the truth table it is clear that $p \vee \sim p$ is **always T**.

Such compound propositions are called **tautologies**.

p	$p \vee \sim p$
T	T
F	T

Contradictions

Consider the compound proposition $p \wedge \sim p$. From the truth table it is clear that $p \wedge \sim p$ is **always F**.

Such compound propositions are called **contradictions**.

p	$p \wedge \sim p$
T	F
F	F

Conditional statements

A mathematical statement of the form

“**if p then q**”

is called a **conditional** statement. In logic such a statement is denoted by

$$\mathbf{p \rightarrow q}$$

Truth table of the conditional statement

$$p \to q$$

$p \to q$ is false only when p is true and q is false.
In all the other cases $p \to q$ is true.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional Statements

A mathematical statement of the form

"**p if and only if q**"

is called a **biconditional** statement. In logic such a statement is denoted by

$$\mathbf{p \leftrightarrow q}$$

Truth table of the Biconditional statement

The biconditional statement $p \leftrightarrow q$ is true if both p and q have the **same truth value** and false if p and q have opposite truth values.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional statements ……

The biconditional "if and only if" sometimes abbreviated as **iff**

A proposition with propositional variables p, q, … can be denoted by **P(p,q,……)**

Example

$P(p,q) : p \wedge q$

This means the proposition $\mathbf{p \wedge q}$ is denoted by $P(p,q)$

Logical equivalence

Two propositions P(p,q,….) and Q(p,q,..) are said to be **logically equivalent** (or **equivalent** or **equal**), denoted by **P** $\equiv$ **Q**, if they have **identical truth tables**.

Example : Consider the following

two propositions.

$P(p,q) : p \rightarrow q$

$Q(p,q) : (\sim q) \rightarrow (\sim p)$

These two propositions are logically equivalent. This can be verified by constructing truth tables for P(p,q) and Q(p,q).

p	q	$p \rightarrow q$	$(\sim q) \rightarrow (\sim p)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

From this truth table it is obvious that $p \rightarrow q$ and $(\sim q) \rightarrow (\sim p)$ have identical truth tables. Hence these two propositions are logically equivalent.

Some more examples of equivalent propositions

$p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are equivalent.

$p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are equivalent.

$\sim(p \wedge q)$ and $(\sim p) \vee (\sim q)$ are equivalent.

$\sim(p \vee q)$ and $(\sim p) \wedge (\sim q)$ are equivalent.

$p \Rightarrow q$ and $(\sim p) \vee q$ are equivalent.

Exercise: Verify the above by using truth tables.