Optimally Matched Wavelets

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Today's menu

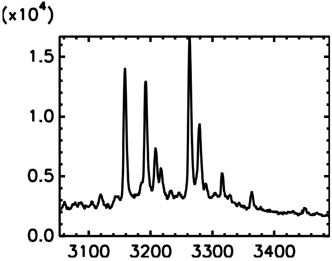
Problem

Signal contains the same pattern in different scales.

- Pattern matching - retrieve patterns
- De-noising - remove any non-matching structure

Applications

- Detection of component (ball-bearing) wear-out by observing the current of an engine
- Detection of pollution in rotor spinning machines
- Decomposition of (audio) signals into time-frequency atoms
- Extraction of peaks from a mass spectrogram



Obvious Method

Multi-scale correlation alias *Continuous Wavelet Transform*

Advantages:

- Multiple scales
- Translation invariant
- Fine frequency sampling
- Mathematically reversible
- Weak restrictions on pattern

Disadvantages:

- Not numerically reversible
- Slow

Alternative Method

Discrete wavelet transform

Advantages:

- Multiple scales
- Reversible
- Fast

Disadvantages:

- Strong restrictions on pattern
- Translation dependent
- Coarse frequency sampling

Discrete Wavelet Transform

The *Discrete Wavelet Transform* coincides with *Sub-band Coding*.
It is described by four filters $h, g, \widetilde{h}, \widetilde{g}$.

- Analysis: h, g
- Synthesis: $\widetilde{h}, \widetilde{g}$

Perfect reconstruction:

$$\begin{aligned} h*\widetilde{h} \; + g*\widetilde{g} \; &= \delta \\ h*\widetilde{h}_- + g*\widetilde{g}_- &= 0 \end{aligned}$$

where $(h_-)_k = (-1)^k \cdot h_k$.

Perfect reconstruction

Consequences:

- h and g are relatively prime.
- $$\widetilde{h} = (g \leftarrow 1)_-$$
$$\widetilde{g} = h_- \leftarrow 1$$

Dependencies:

- Choice of h is limited.
- Choice of h limits choice of g.
- Choice of h and g determines $\widetilde{h}$ and $\widetilde{g}$ and thus fixes the transform.

Continuous view on DWT

- Discrete Wavelet Transform = Filtering
- Alternative interpretation: Expansion into a wavelet base

- Filters $h, \widetilde{h}$ correspond to the *generator functions* $\varphi, \widetilde{\varphi}$, which are refinable.

$$\varphi = 2 \cdot (h * \varphi) \downarrow 2 \qquad \widetilde{\varphi} = 2 \cdot (\widetilde{h} * \widetilde{\varphi}) \downarrow 2$$

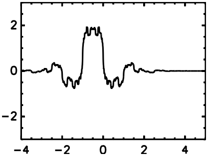
- Filters $g, \widetilde{g}$ correspond to the *wavelet functions* $\psi, \widetilde{\psi}$, which are finite linear combinations of translates of $\varphi, \widetilde{\varphi}$, respectively.

$$\psi = 2 \cdot (g * \varphi) \downarrow 2 \qquad \widetilde{\psi} = 2 \cdot (\widetilde{g} * \widetilde{\varphi}) \downarrow 2$$

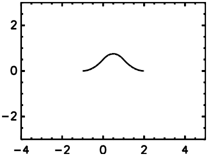
Wavelet basis functions

Basis of Cohen, Daubechies, Feauveau of order 3,5

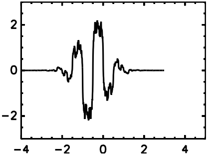
Generators: $h \sim \varphi$



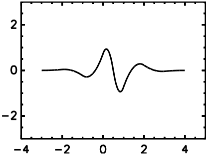
$\widetilde{h} \sim \widetilde{\varphi}$



Wavelets: $g \sim \psi$



$\widetilde{g} \sim \widetilde{\psi}$



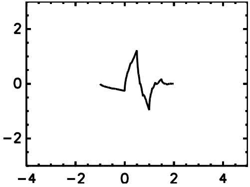
Standard wavelets

Standard wavelets are designed to fulfil mathematical properties like

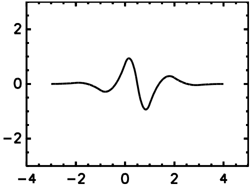
- Orthogonality
- Interpolation
- Symmetry
- Smoothness
- Vanishing moments
- Minimal mask length.

Standard wavelets

They look like



Daubechies-2



CDF-3,5

but never like a certain pattern you want.

Pattern meets Wavelet

Is it possible to create a discrete wavelet
similar to a given pattern?

Lifting

- Lifting step:
  Transform a perfect reconstruction filter bank into another such filter bank.
- If (h, g) constitutes a perfect reconstruction filter bank, then for any filter s
  $(h, g + h * (s \uparrow 2))$ allows for perfect reconstruction, too.
- Lifting both for wavelet transform and wavelet construction.

Properties of lifting

- Complete
  Two arbitrary perfect reconstruction filter banks can always be transformed into each other by a sequence of lifting steps.
- Fast
  Applying the wavelet transform by the lifting scheme halves the computation time. This is possible because lifting can generate perfect reconstruction filter banks only.
- Reversible
  The reconstruction restores every bit exactly.

Lifting decomposition:
Transform a filter bank into lazy wavelet filter bank.

Lifting for CDF-$n, 0$; n even

$$
\begin{aligned}
a_0 &= \frac{1}{n} \\
a_m &= \frac{1}{(n - 2 \cdot m) \cdot (n + 2 \cdot m) \cdot a_{m-1}}
\end{aligned}
$$

$$
\begin{aligned}
x_{-1} &= (1, \mathbf{0}) \\
x_0 &= (\mathbf{1}) \\
x_{m+1} &= x_{m-1} + a_m \cdot (2 \cdot m + 1, \mathbf{0}, 2 \cdot m + 1) \leftarrow (-1)^m * x_m
\end{aligned}
$$

Lifting for CDF-$n, 1$; n odd

$$a_0 = \frac{1}{n}$$

$$a_m = \frac{1}{(n - 2 \cdot m + 1) \cdot (n + 2 \cdot m - 1) \cdot a_{m-1}}$$

$$x_{-1} = (1, \mathbf{0})$$

$$x_0 = (\mathbf{1})$$

$$x_1 = x_{-1} + x_0 * a_0 \cdot (\mathbf{1})$$

$$x_{m+1} = x_{m-1} + x_m * a_m \cdot (2 \cdot m - 1, \mathbf{0}, 2 \cdot m + 1) \leftarrow (-1)^m$$

Matching wavelets using Lifting

1. Choose a generator φ with mask h
   $\Rightarrow$ Determines smoothness of the wavelet, too.
2. Choose a wavelet ψ with mask g.
   Filters g and h must allow for perfect reconstruction.
3. Find an optimally matching lifting step.
4. $\Rightarrow$ Reduction to a simple least squares problem!
5. Structure of refinable functions allows for quick computation of normal equations.

Optimization target:

$$\operatorname*{argmin}_{c,s} \|\psi_{c,s} - f\|_2 \qquad \text{with} \qquad \psi_{c,s} = c \cdot \psi + s * \varphi$$

The set $\{\psi_{c,s} : c \in \mathbb{R}, s \in \ell_0(\mathbb{Z})\}$ is a linear space.

Example base

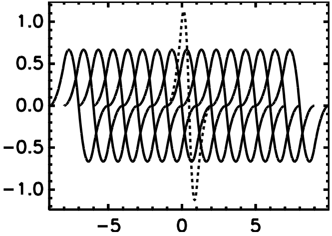
Most simple choice: The Binomial mask

$$h = (1, 1)^n$$

corresponds to B-Spline of order $n - 1$.

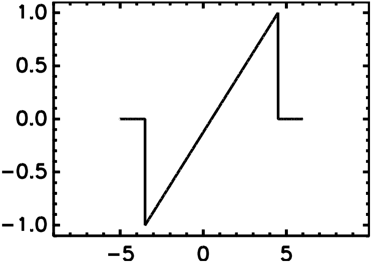
Example $n = 3$

Basis functions for
$\{c \cdot \psi + s * \varphi \ : \ c, s\}$

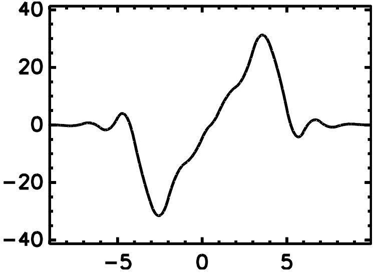


Nice approximation

Original

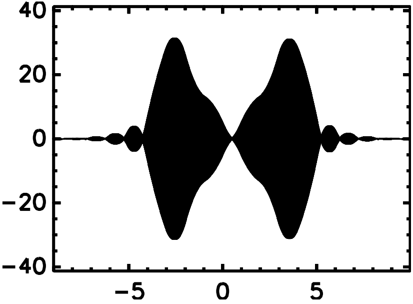


Approximation



Fractal dual generators

A nicely matched wavelet may lead to a rough dual generator.



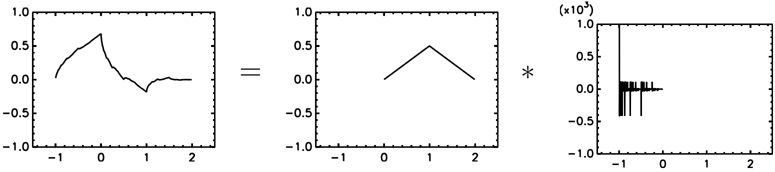
Absent smoothness means bad numerical properties.

Smoothness of a refinable function

If h_0 refines φ_0
and h_1 refines φ_1
then $h_0 * h_1$ refines $\varphi_0 * \varphi_1$

Divide refinable function into B-Spline and rest.

$$h_{\mathrm{daub2}} = \left(\frac{1}{2}, \frac{1}{2}\right)^2 * \left(\frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)$$



Smoothness: The VILLEMOES machine

- B-Spline is smooth
- the rest is usually fractal

The dual generator becomes smoother for

- big number of dual smoothness factors,
- small spectral radius of the *transition matrix* (roughly spoken).

Transfer matrix

$$(T_h)_{j,k} = h_{2\cdot j-k}$$

$$T_h = \begin{pmatrix} h_m & & & & & \\ h_{m+2} & h_{m+1} & h_m & & & \\ h_{m+4} & h_{m+3} & h_{m+2} & h_{m+1} & h_m & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & h_n & h_{n-1} & h_{n-2} & h_{n-3} & h_{n-4} \\ & & & h_n & h_{n-1} & h_{n-2} \\ & & & & & h_n \end{pmatrix}$$

- $2 \cdot T_{h*h^*}$ is the *transition matrix* of h
- if h contains no smoothness factor $(1,1)$: the higher the spectral radius, the less smooth is the refinable function

Regularized optimization target

$$\operatorname*{argmin}_{c,s} \left(\|\psi_{c,s} - f\|_2 + \lambda \cdot \widetilde{\varrho}\left(g + \frac{1}{c} \cdot s * h \right) \right)$$

Where $\widetilde{\varrho}(m)$ denotes an estimate of the spectral radius of the transfer matrix of the dual generator corresponding to the primal wavelet mask m, e.g.

- the spectral radius itself,
- the sum of the squares of the eigenvalues,
- the sum norm,
- the FROBENIUS norm.

The FROBENIUS norm was most successful.

Bounds for eigenvalues

Observation:

- To keep good matching only small reduction of the spectral radius is acceptable.
- If φ is refinable with respect to h and h contains no smoothness factor $(1, 1)$, then

$$\varphi \in H_2^s(\mathbb{R}) \quad \Rightarrow \quad s \leq \frac{1}{4} \cdot \log_2 \left(\frac{3}{2} \cdot \# h \right)$$

Vanishing moments

- Primal vanishing moments coupled with dual B-Spline order
- Matching procedure adapted for vanishing moment constraints

Wanted:

- High dual B-Spline order
- Only few primal vanishing moments

If the wavelet and the pattern have different number of vanishing moments they will not match very well.

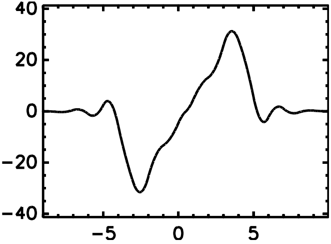
Non-matching vanishing moments

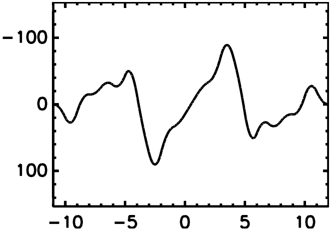
Example

- Pattern (ramp): 1 vanishing moment
- Wavelet:

  1 vanishing moment

  5 vanishing moments



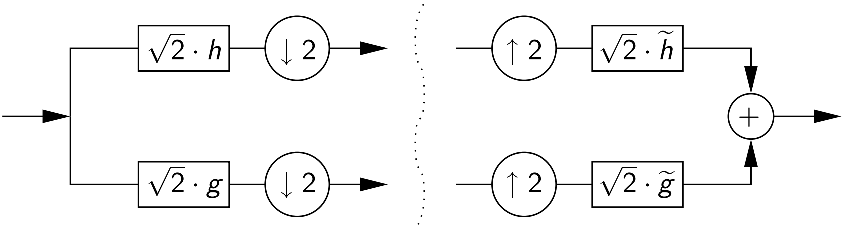


Modify the Discrete Wavelet Transform

- Assumption:
  Number of primal vanishing moments is smaller than number of dual smoothness factors.
- Idea:
  Modify DWT such that elimination of some moments is deferred to the reconstruction transform.
- Drawback:
  Double number of coefficients must be kept.

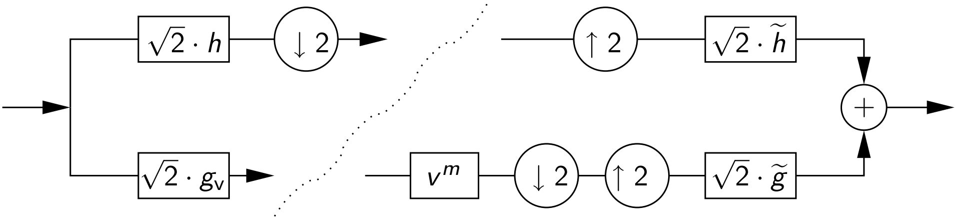
From critically sampled DWT . . .

Standard:



... to double density DWT

Modification: $g = g_v * v^m$ with $v = \frac{1}{2} \cdot (1, -1)$



Remaining problems

- Oscillations
  The synthesis wavelets of the double density transform oscillate heavily.
- Analysis and synthesis wavelet differ.
  What about orthogonal wavelets? Orthogonal wavelets are hard to control, they do not allow for a simple least squares approach. Orthogonal wavelets can not be symmetric. Can one benefit from additional degrees of design freedom: Translation invariant DWT, Multiwavelets?

Thank you for your patience!