$\boxed{\text{Ejercicio 1.2}}$

3- $\boxed{\text{Desigualdad triangular}}$ $\longrightarrow d_E(x,z) \leq d_E(x,y) + d_E(y,z) \quad \forall x,y,z \in \mathbb{R}^2$

Lo demostraremos en 3 pasos:

$\boxed{\text{Paso 1}} \to \boxed{\alpha^2+\beta^2 \geq 2\alpha\beta} \quad \forall \alpha,\beta \in \mathbb{R}$

En efecto:

$$(\alpha-\beta)^2 \geq 0 \Rightarrow \alpha^2-2\alpha\beta+\beta^2 \geq 0 \Rightarrow \alpha^2+\beta^2 \geq 2\alpha\beta$$

$\alpha = ad \quad \beta = cb$

$\boxed{\text{Paso 2}} \to \boxed{(ab+cd)^2 \leq (a^2+c^2)\cdot(b^2+d^2)} \quad \forall a,b,c,d \in \mathbb{R}.$

Aplicando "Paso 1": $a^2d^2+c^2b^2 \geq 2ad\cdot cb = 2abcd$

En efecto:

$$(ab+cd)^2 = a^2b^2+c^2d^2+2abcd \leq a^2b^2+c^2d^2 + a^2d^2+c^2b^2 =$$

[Sacamos factor común a^2 y c^2] $= a^2\cdot[b^2+d^2] + c^2\cdot[b^2+d^2] =$

[Sacamos factor común $[b^2+d^2]$] $= (a^2+c^2)\cdot(b^2+d^2)$

$\boxed{\text{Paso 3}} \to$ Llamemos: $a = |x_1-y_1| \quad b = |y_1-z_1| \quad c = |x_2-y_2| \quad d = |y_2-z_2|$

Tenemos que:

$$d_E(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = \sqrt{a^2+c^2} \qquad d_E(y,z) = \sqrt{(y_1-z_1)^2+(y_2-z_2)^2} = \sqrt{b^2+d^2}$$

Así:

Aplicamos Paso 2

$$[d_E(x,y)+d_E(y,z)]^2 = [\sqrt{a^2+c^2}+\sqrt{b^2+d^2}]^2 = a^2+c^2+b^2+d^2+2\sqrt{(a^2+c^2)\cdot(b^2+d^2)} \geq$$

$$\geq a^2+c^2+b^2+d^2+2\cdot\sqrt{(ab+cd)^2} =$$

$ab+cd \geq 0$

$$= a^2+b^2+c^2+d^2+2\cdot|ab+cd| = a^2+b^2+2ab + c^2+d^2+2cd =$$

$$= (a+b)^2+(c+d)^2 \geq (x_1-z_1)^2+(x_2-z_2)^2 = [d_E(x,z)]^2$$

$a+b = |x_1-y_1|+|y_1-z_1| \geq |x_1-z_1|$

$c+d = |x_2-y_2|+|y_2-z_2| \geq |x_2-z_2|$

Por tanto: $[d_E(x,z)]^2 \leq [d_E(x,y)+d_E(y,z)]^2$

y en consecuencia: $\boxed{d_E(x,z) \leq d_E(x,y)+d_E(y,z)}$