Inverse Kinematics

Introduction to Robotics: Discussion 3
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20150915

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ADMINISTRIVA

- HW1 Self Grades: Thursday 17th @1700
- HW2 Due: Thursday 17th @1700

- Discussion sections: Tues 1000-1100, Wed 1100-1200
- Office hours: Mon, Thurs 1100-1200
- DSP Students: Letters of Accommodation required asap.

TERMINOLOGY

Forward Kinematics

– Given joint positions, find end effector coordinates

Inverse Kinematics

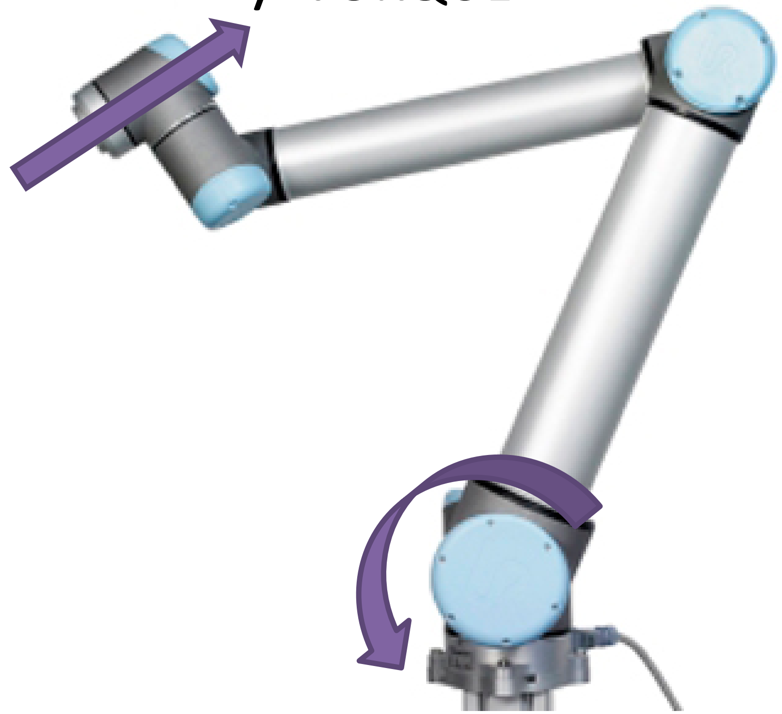
– Given end effector coordinates, find required joint positions

Forward Dynamics

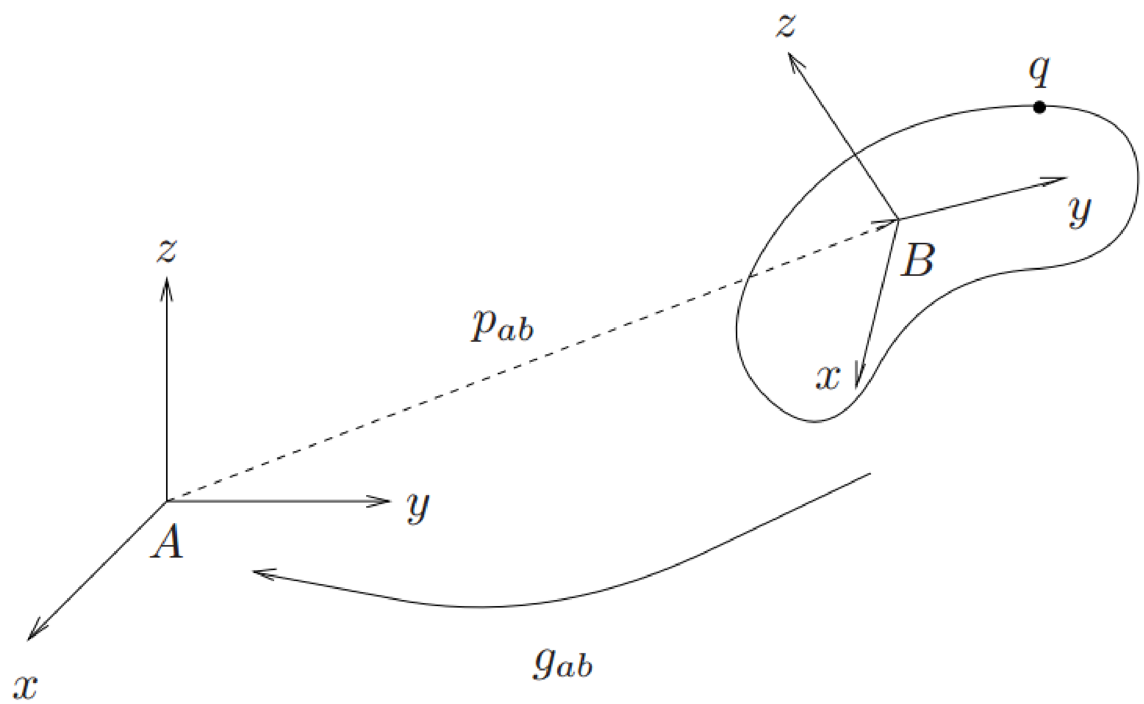
– Given joint torques, find end effector forces/torques

Inverse Dynamics

– Given a desired end effector force/torque, find required joint torques



RIGID BODY MOTION



$$\boldsymbol{q}_A = \boxed{\boldsymbol{p}_{AB}} + \boxed{R_{AB}\boldsymbol{q}_B}$$

Translation of Origin

Relative Rotation

HOMOGENEOUS COORDINATES

$$\overline{\boldsymbol{q}}_A = \begin{bmatrix} \boldsymbol{q}_A \\ 1 \end{bmatrix} = \begin{bmatrix} R_{AB} & \boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_B \\ 1 \end{bmatrix} = \overline{\boldsymbol{g}}_{AB} \overline{\boldsymbol{q}}_B$$

$$\overline{\boldsymbol{q}}_B = \begin{bmatrix} q_{B,1} \\ q_{B,2} \\ q_{B,3} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_B \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

Points

$$\overline{\boldsymbol{v}} = \begin{bmatrix} \boldsymbol{q}_B - \boldsymbol{q}_A \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} \in \mathbb{R}^4$$

Vectors

RIGID BODY MOTION

$$\boldsymbol{q}_A = \boldsymbol{p}_{AB} + R_{AB}\boldsymbol{q}_B$$

Homogeneous Coordinates:

$$\overline{\boldsymbol{q}}_A = \begin{bmatrix} \boldsymbol{q}_A \\ 1 \end{bmatrix} = \begin{bmatrix} R_{AB} & \boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_B \\ 1 \end{bmatrix} = \overline{\boldsymbol{g}}_{AB}\overline{\boldsymbol{q}}_B$$

Note: all configurations are **RELATIVE.**

FORWARD KINEMATICS

The ***Kinematics*** of a robotic manipulator describes the relationship between the ***motion of the joints*** and the ***motion of the rigid bodies*** that make up the manipulator.

$$\begin{bmatrix} \boldsymbol{q}_A \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_B \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{q}_B \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{BC} & \boldsymbol{p}_{BC} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_C \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{q}_A \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_{BC} & \boldsymbol{p}_{BC} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_C \\ 1 \end{bmatrix}$$

Note: all configurations are **RELATIVE.**

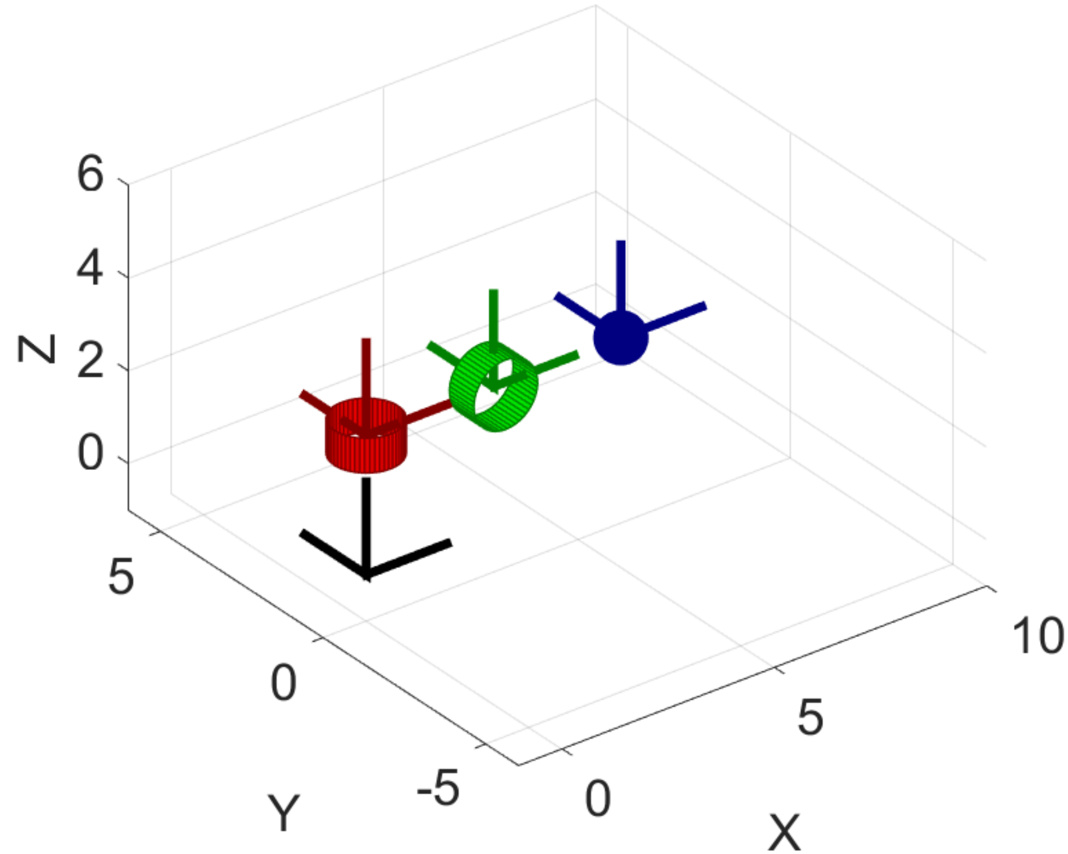
FORWARD KINEMATICS

$$\begin{bmatrix} \boldsymbol{q}_A \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_{BC} & \boldsymbol{p}_{BC} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_C \\ 1 \end{bmatrix}$$

$$\boldsymbol{R}_{AB} = \boldsymbol{R}_Z \qquad \boldsymbol{R}_{BC} = \boldsymbol{R}_Y$$

$$\boldsymbol{p}_{AB} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \qquad \boldsymbol{p}_{BC} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{q}_C = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$



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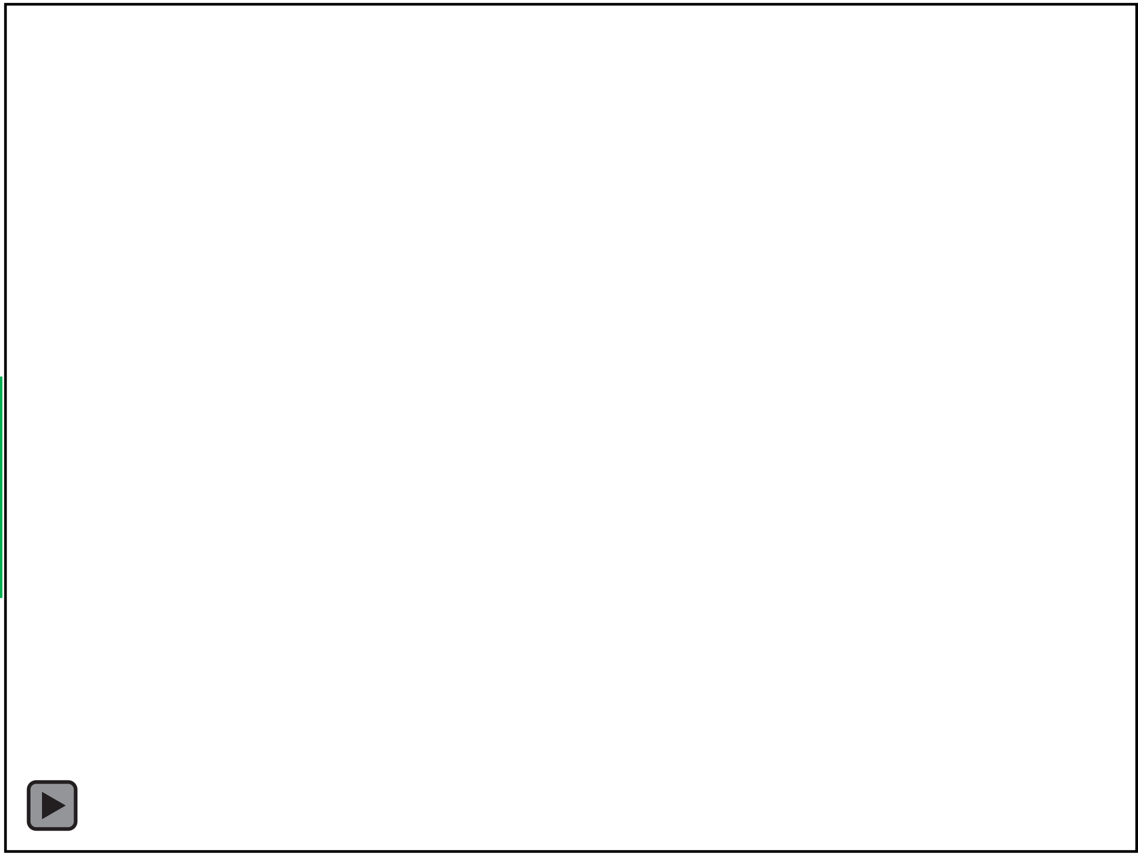


RELATIVE TRANSFORMS

Rigid body motion as:

coordinate transforms

$$\begin{bmatrix} \boldsymbol{q}_A \\ 1 \end{bmatrix} = \begin{bmatrix} R_{AB} & \boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_B \\ 1 \end{bmatrix}$$

EXPONENTIAL COORDINATES

Rigid body motion as:

coordinate transforms

$$\begin{bmatrix} \boldsymbol{q}_A \\ 1 \end{bmatrix} = \begin{bmatrix} R_{AB} & \boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_B \\ 1 \end{bmatrix}$$

solutions to differential equations

$$\bar{\boldsymbol{p}}(\theta) = e^{\widehat{\boldsymbol{\xi}\theta}} \bar{\boldsymbol{p}}(0)$$

$$\widehat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix}$$

FORWARD KINEMATICS

The effect of multiple RBMs can be found via the composition of multiple matrix exponents.

For any reference frame at a **zero configuration**, we can write:

$$g(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} g(0)$$

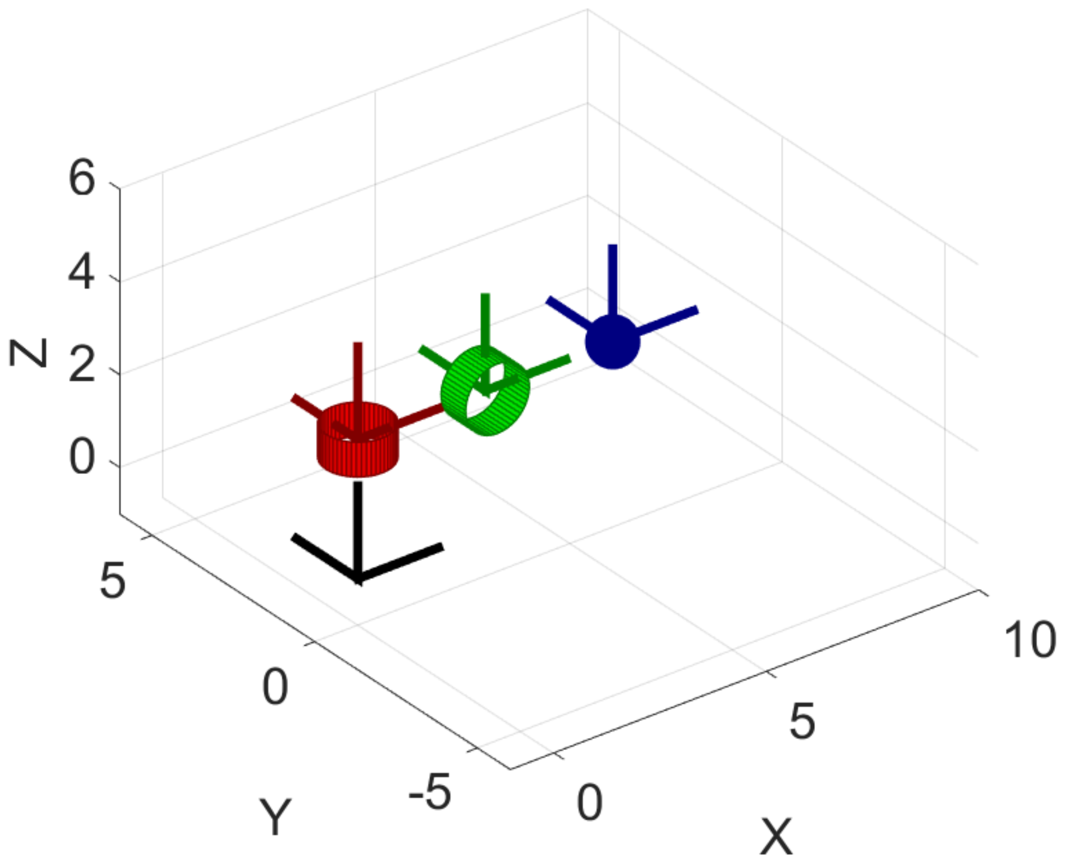
Note: all configurations are in **ABSOLUTE** coordinates.

FORWARD KINEMATICS

$$g(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} g(0)$$

$$\xi_1 = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$g(0) = \begin{bmatrix} [\mathbb{I}_3] & \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$



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INVERSE KINEMATICS

$$g(\theta) = e^{\hat{\xi}_1 \theta_1} g(0)$$

$$\xi_1 = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

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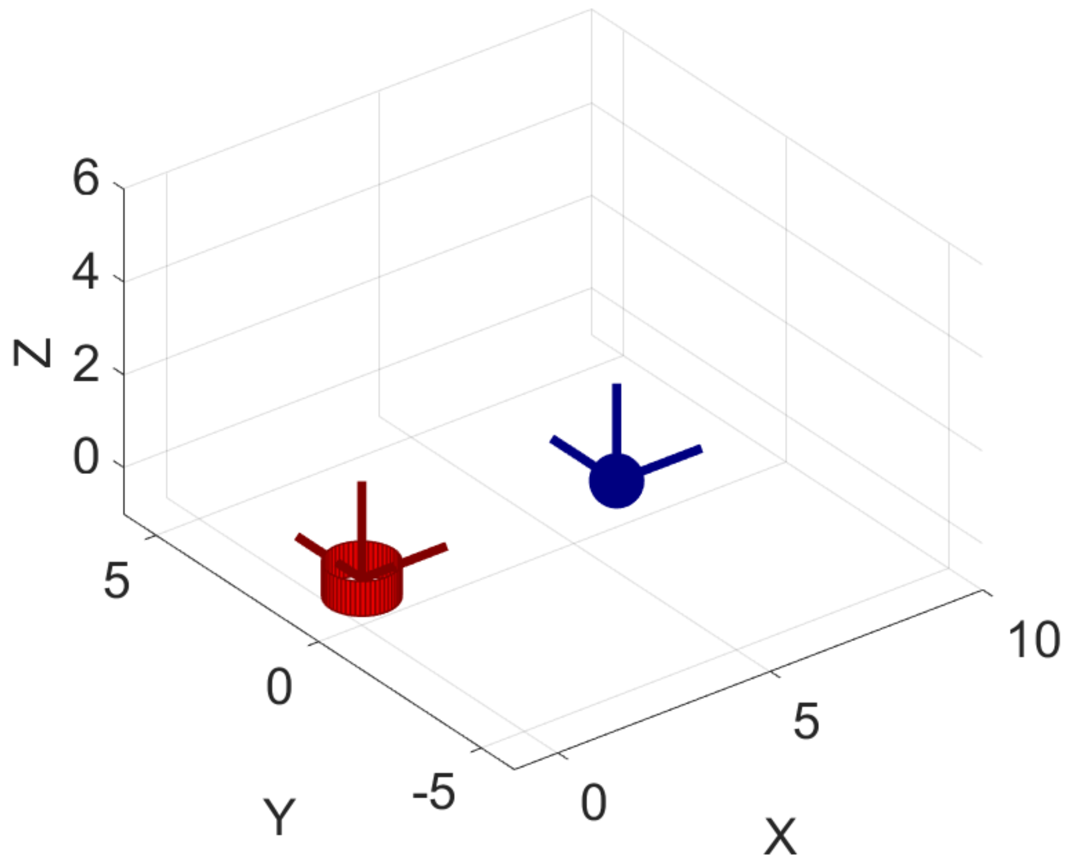

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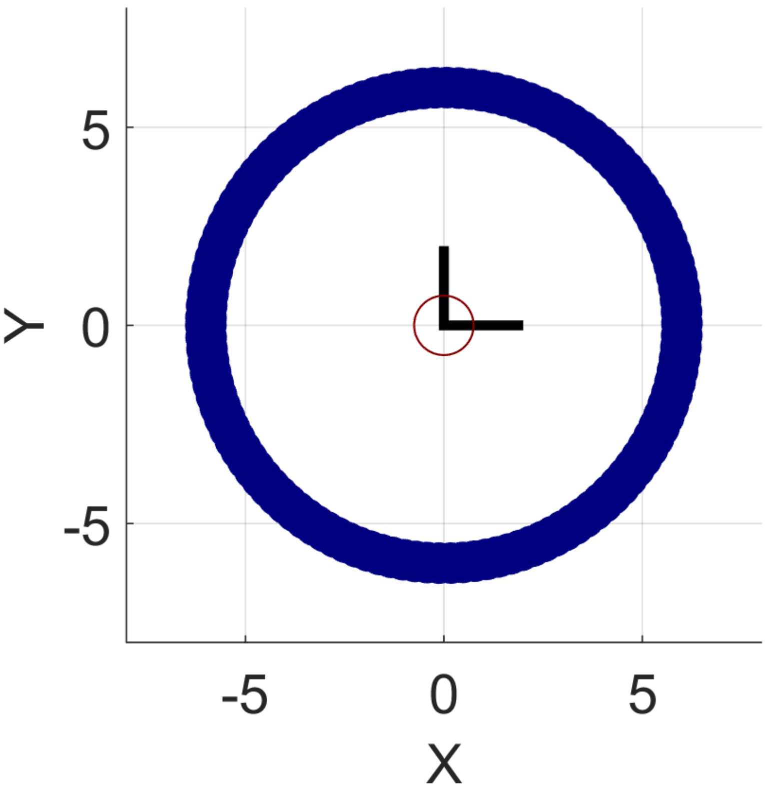
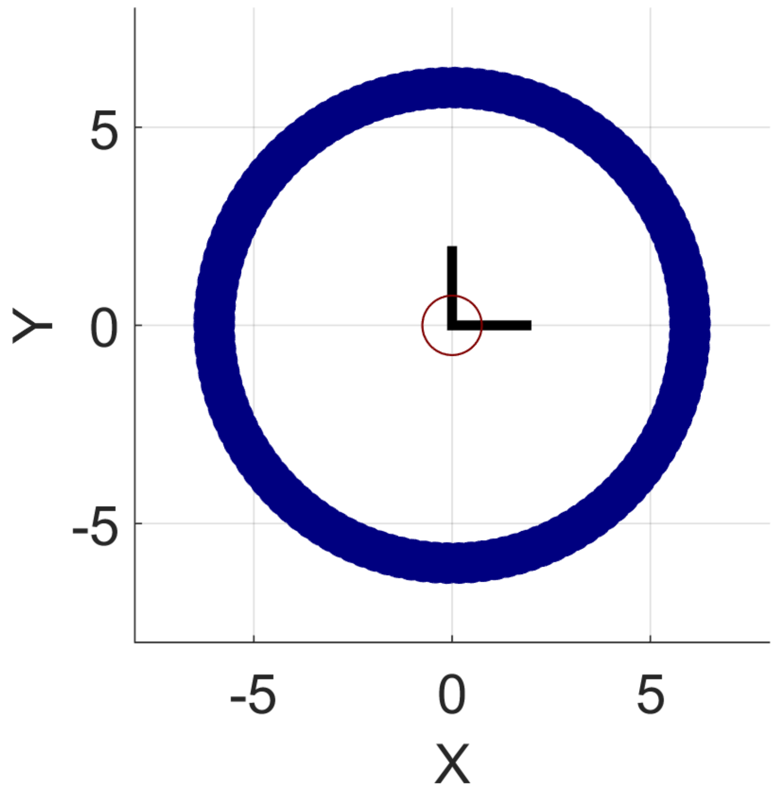


INVERSE KINEMATICS

Given a valid configuration in the workspace, find θ_1.



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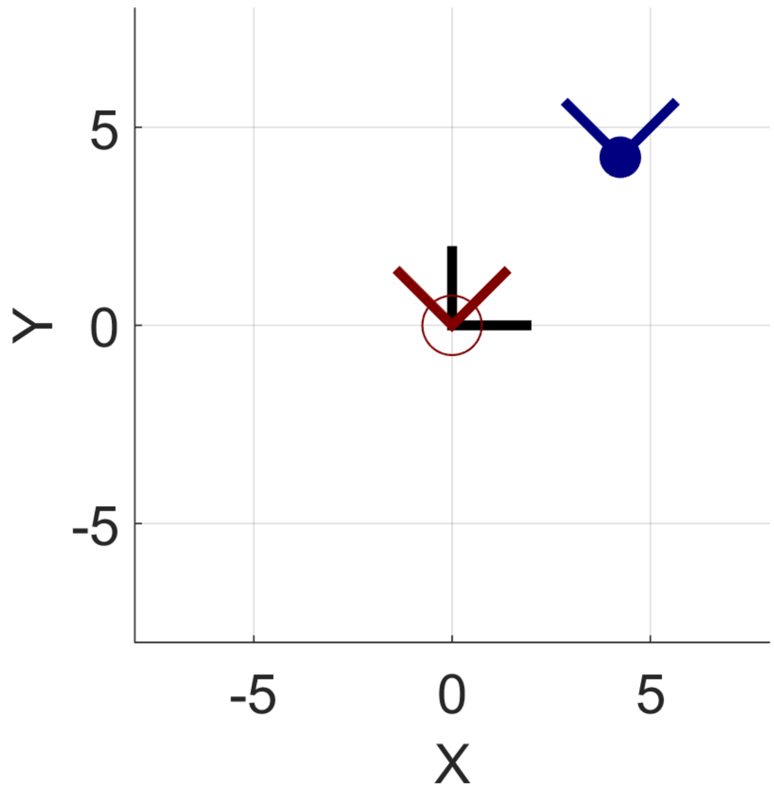
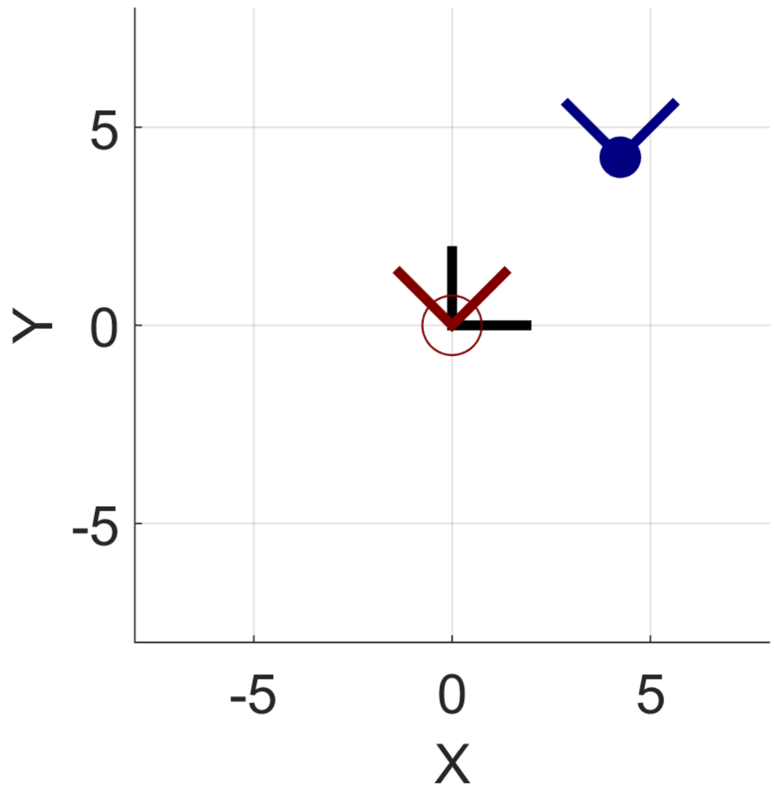


INVERSE KINEMATICS

Given a valid configuration in the workspace, find θ_1.

$$g_d(\theta_1) = \begin{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 3\sqrt{2} \\ 3\sqrt{2} \\ 0 \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$



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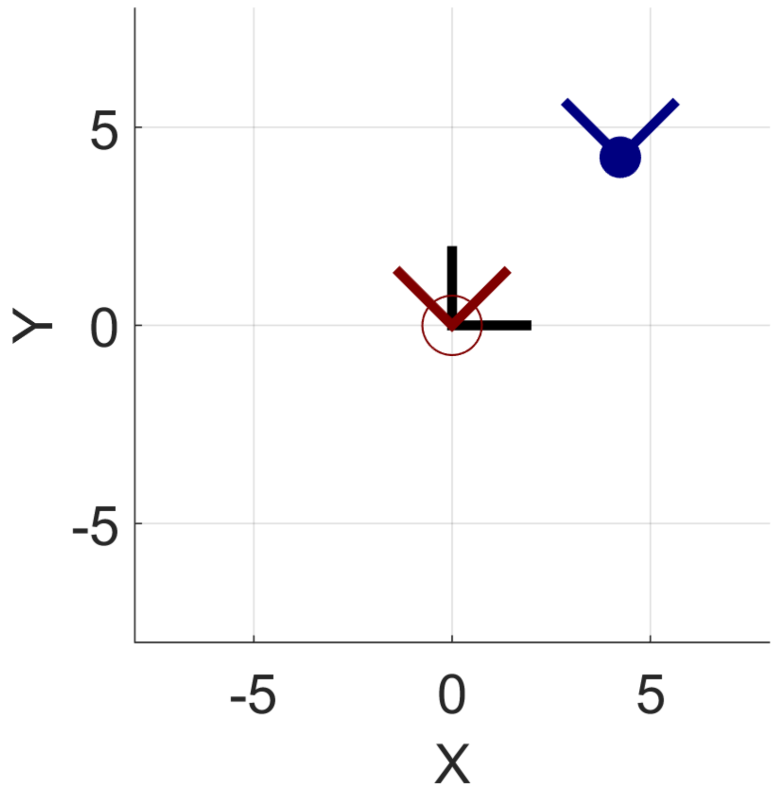
$$g(\theta) = e^{\hat{\xi}_1 \theta_1} g(0)$$



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$$g(\theta) = e^{\hat{\xi}_1 \theta_1} g(0)$$

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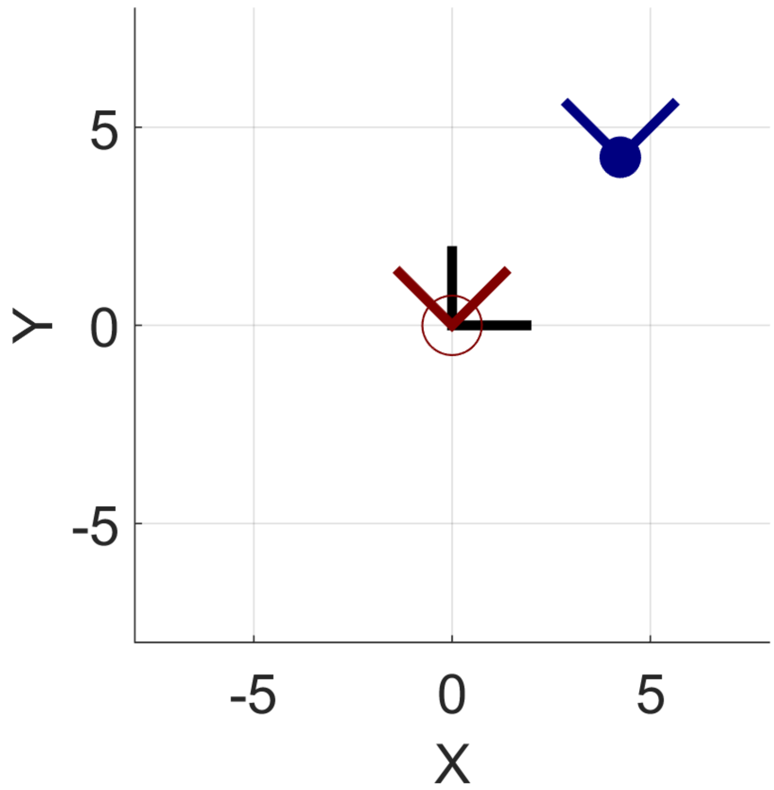

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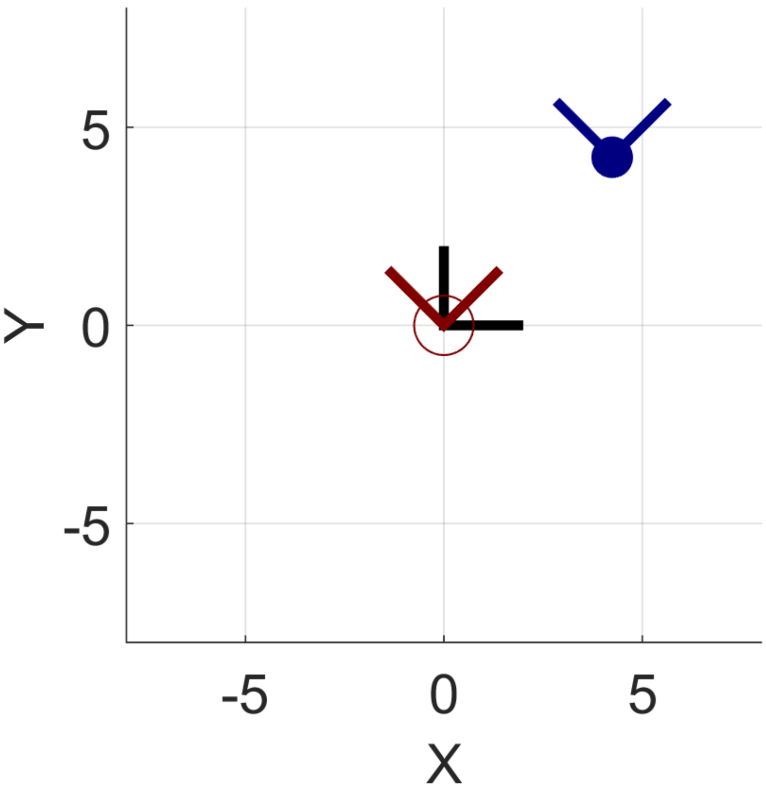
$$e^{\hat{\xi}_1 \theta_1} = g(\theta) g^{-1}(0)$$

$$e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$



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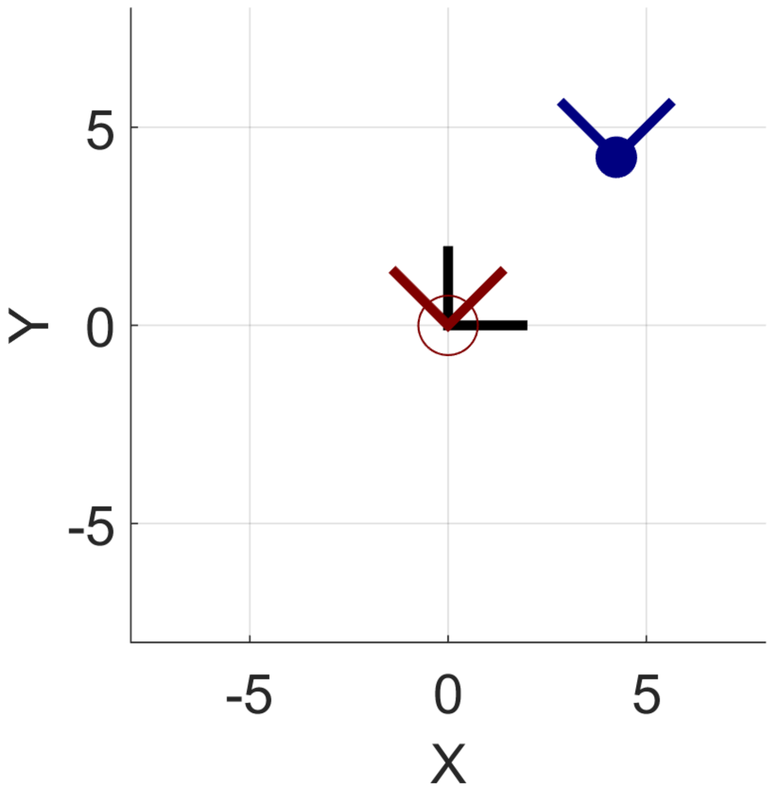
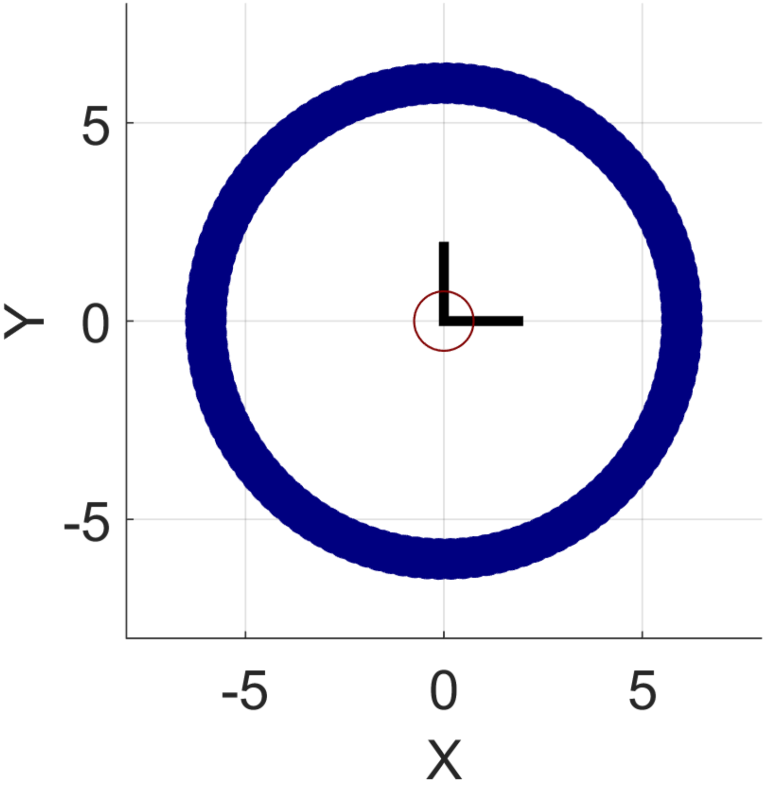
$$\hat{\xi}_1\theta_1 = \begin{bmatrix} \begin{bmatrix} 0 & -\pi/4 & 0 \\ \pi/4 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{0} & 0 \end{bmatrix}$$

$$\xi_1\theta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \pi/4 \end{bmatrix} \qquad \theta_1 = \pi/4$$



INVERSE KINEMATICS

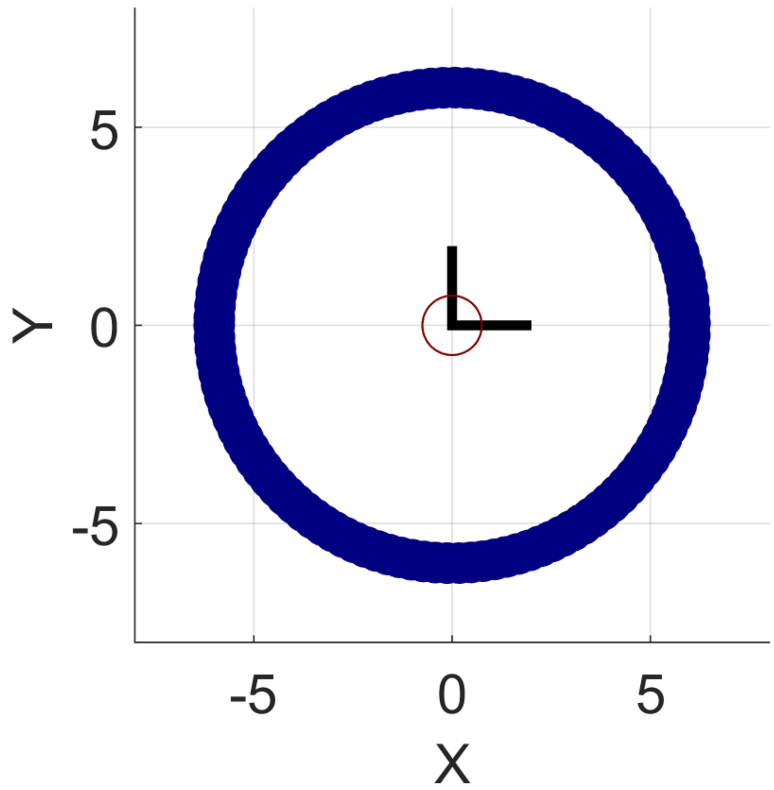
Given a valid configuration in the workspace, find θ_1.



INVERSE KINEMATICS

Given a valid configuration in the workspace, find θ_1.

If the point is not in the workspace it is infeasible, and no angle θ_1 can be found.

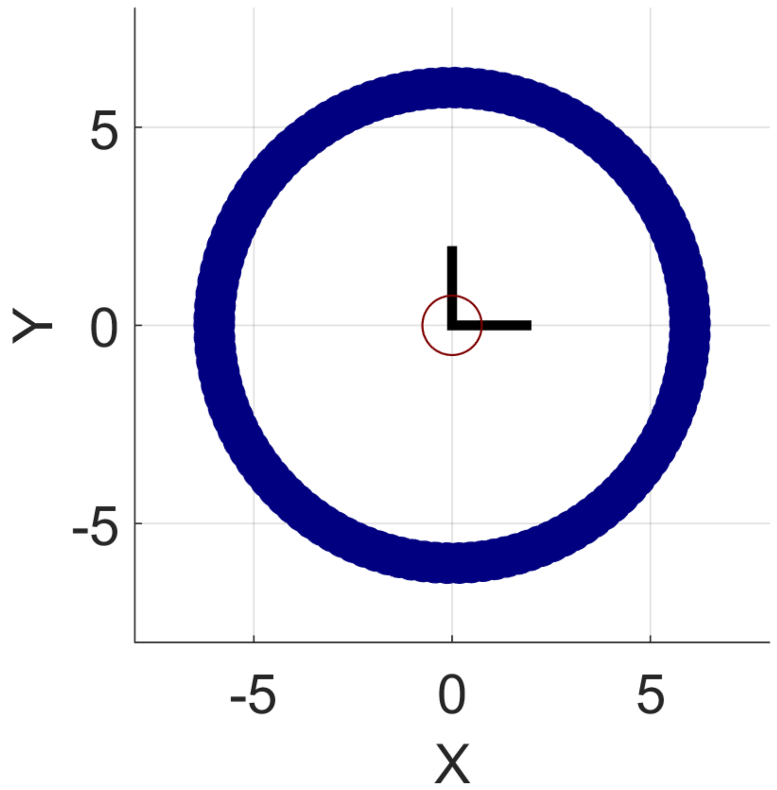


INVERSE KINEMATICS

Given a valid configuration in the workspace, find θ_1.

If the point is not in the workspace it is infeasible, and no angle θ_1 can be found.

The closest feasible point however can be found



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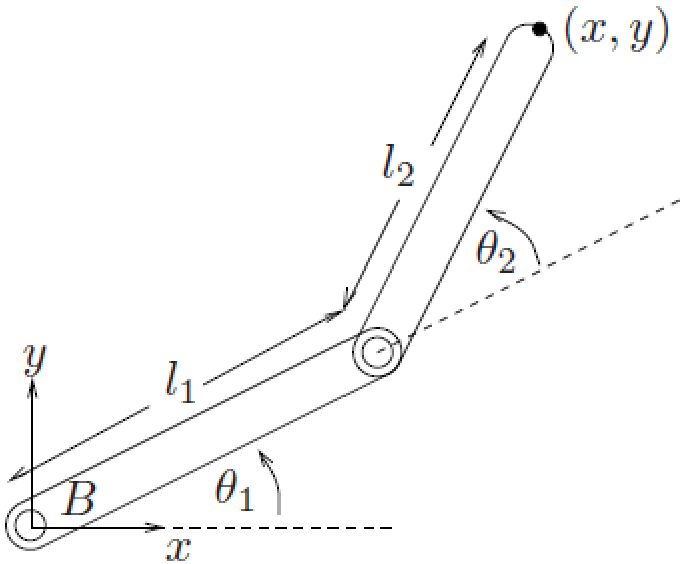
EXAMPLE I

Given a point (x, y), find the corresponding joint angles (θ_1, θ_2)



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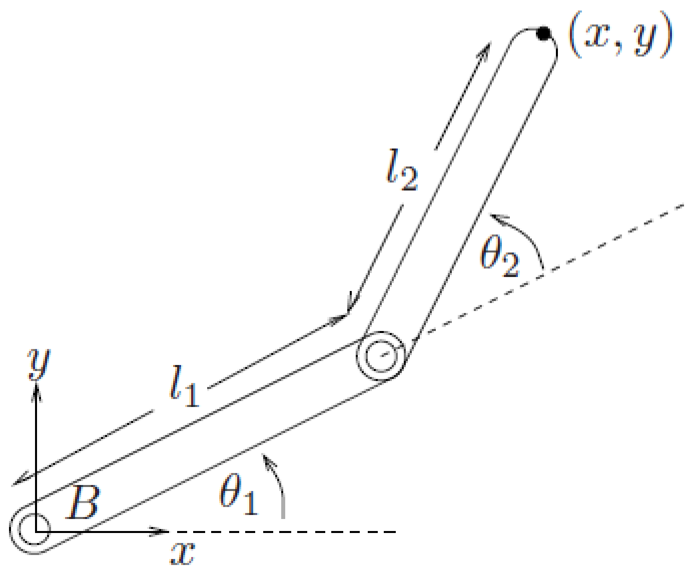
Given a point (x, y), find the corresponding joint angles (θ_1, θ_2)

Think of this problem in polar coordinates. Each target point (x, y) has a corresponding (r, ϕ)



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Given a point (x, y), find the corresponding joint angles (θ_1, θ_2)

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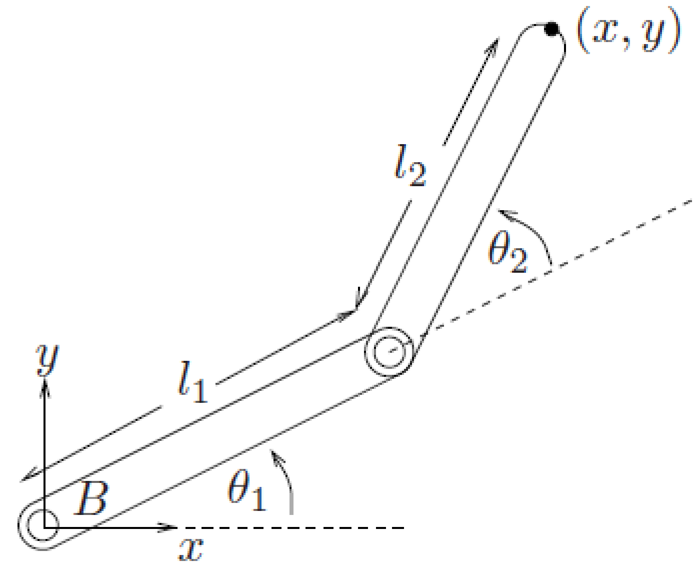
For (x, y) to be in the workspace, $r \leq l_1 + l_2$



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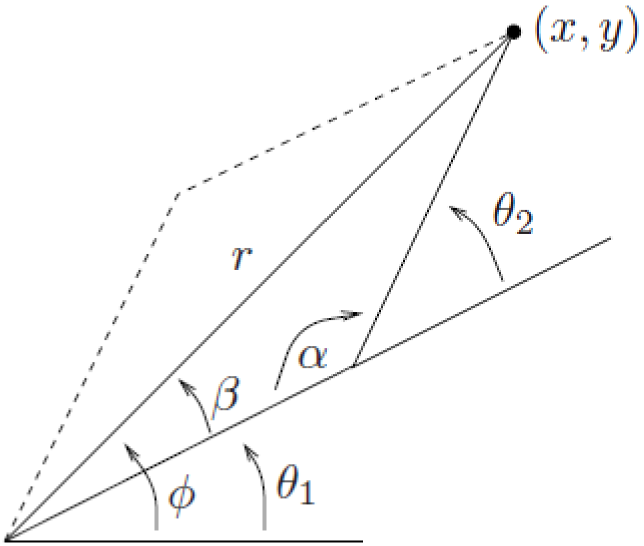
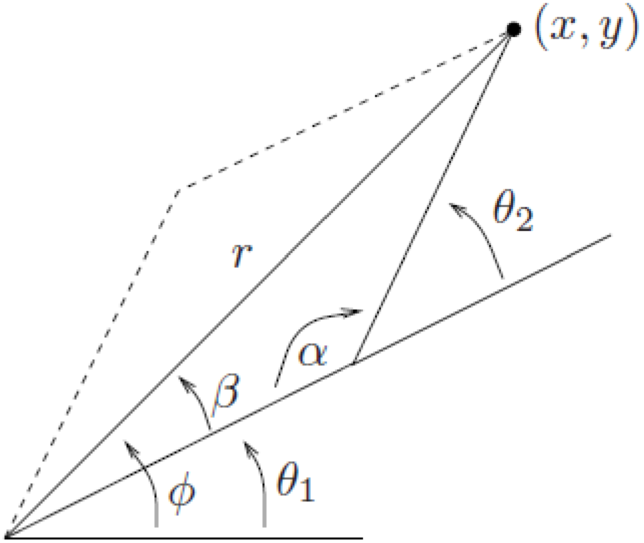


EXAMPLE I

Given a point (x, y), the lengths r, l_1, l_2 are known.



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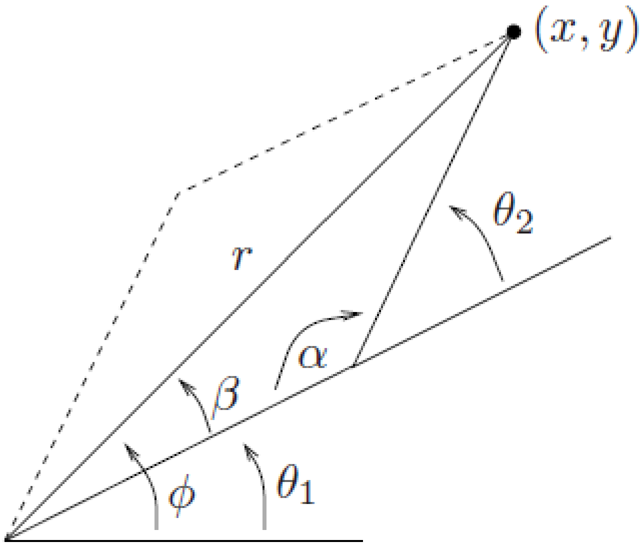
Given a point (x, y), the lengths r, l_1, l_2 are known.

Using the law of cosines:

$$\alpha = acos\left(\frac{l_1^2 + l_2^2 - r^2}{2l_1 l_2}\right)$$



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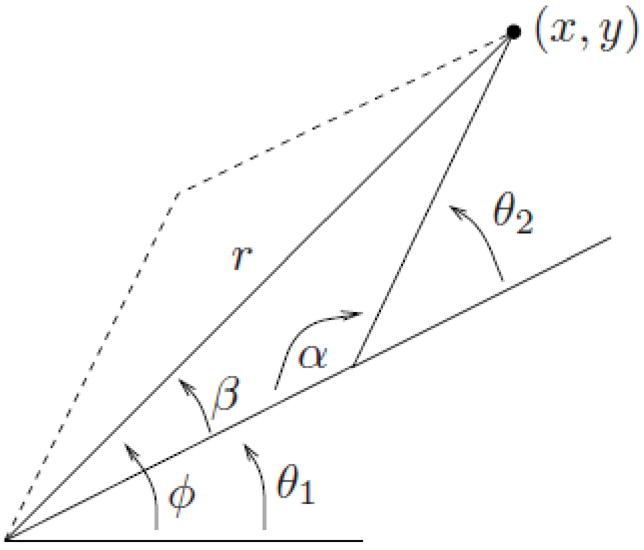
Therefore

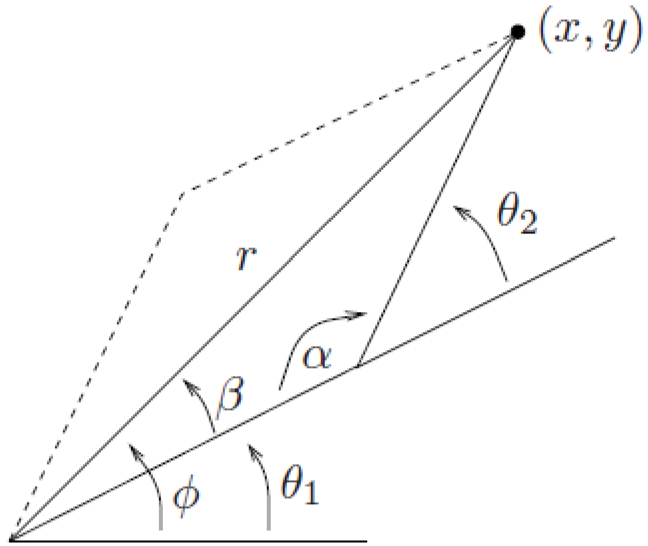
$$\theta_2 = \pi \pm \alpha$$



EXAMPLE I

Similarly:

$$\phi = atan2(y, x)$$



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EXAMPLE I

Similarly:

$$\phi = atan2(y, x)$$

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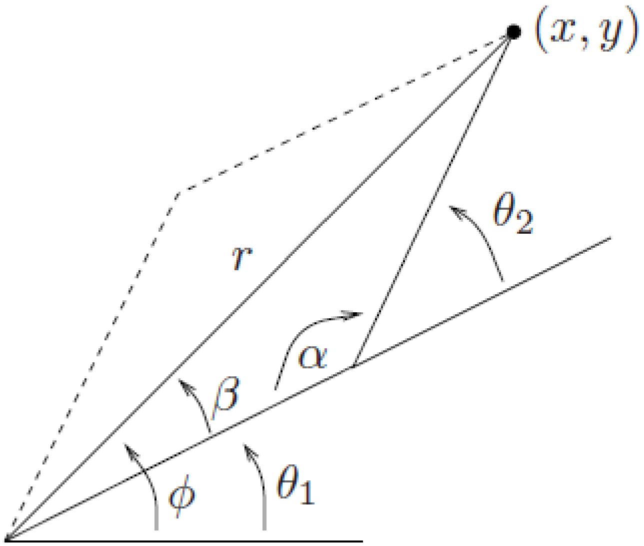
$$\beta = acos\left(\frac{r^2 + l_1^2 - l_2^2}{2l_1 r}\right)$$



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Similarly:

$$\phi = atan2(y, x)$$

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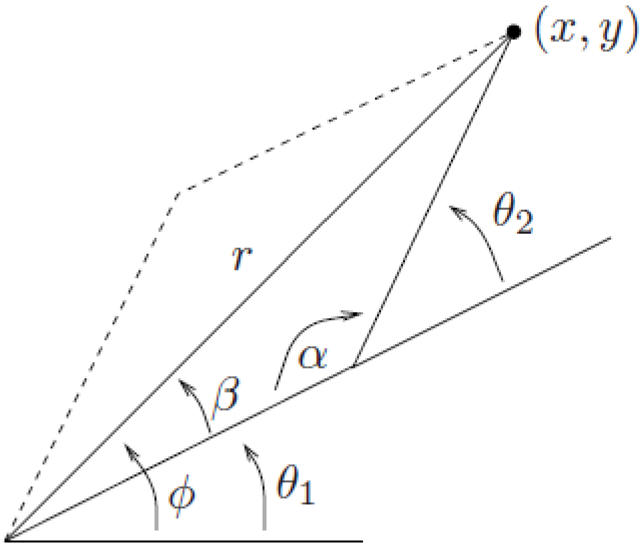
Therefore

$$\theta_1 = \phi \pm \beta$$



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Given a point (x, y), find the corresponding joint angles (θ_1, θ_2)

$$\theta_1 = \phi \pm \beta$$
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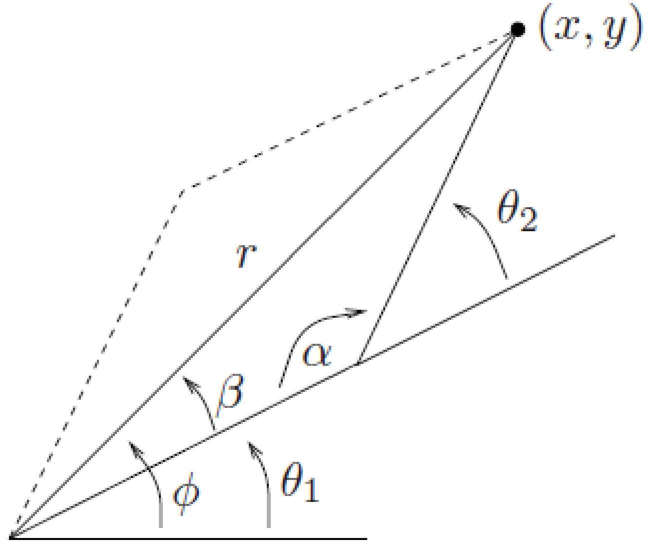

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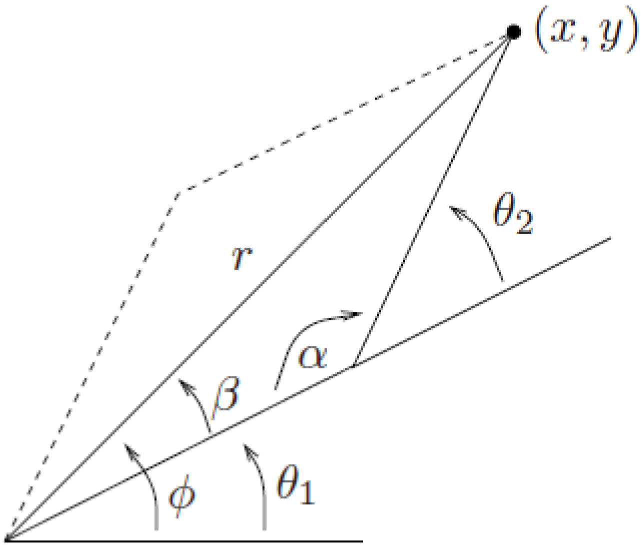

While (θ_1, θ_2) both determine the point (x, y), they separately control the radial position and distance

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Given a point (x, y), find the corresponding joint angles (θ_1, θ_2)

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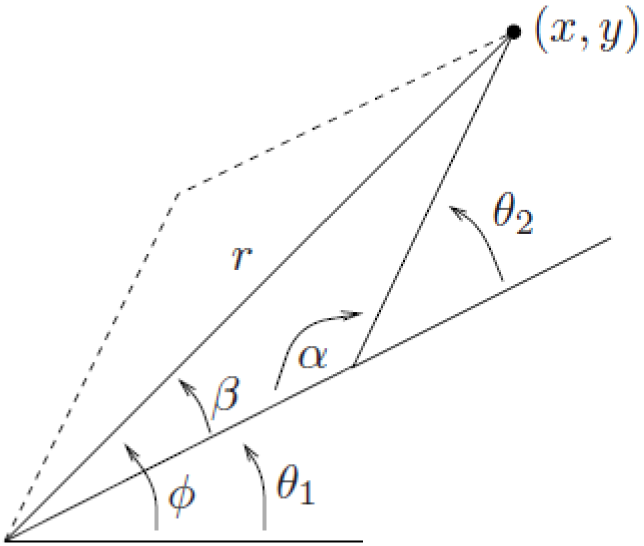
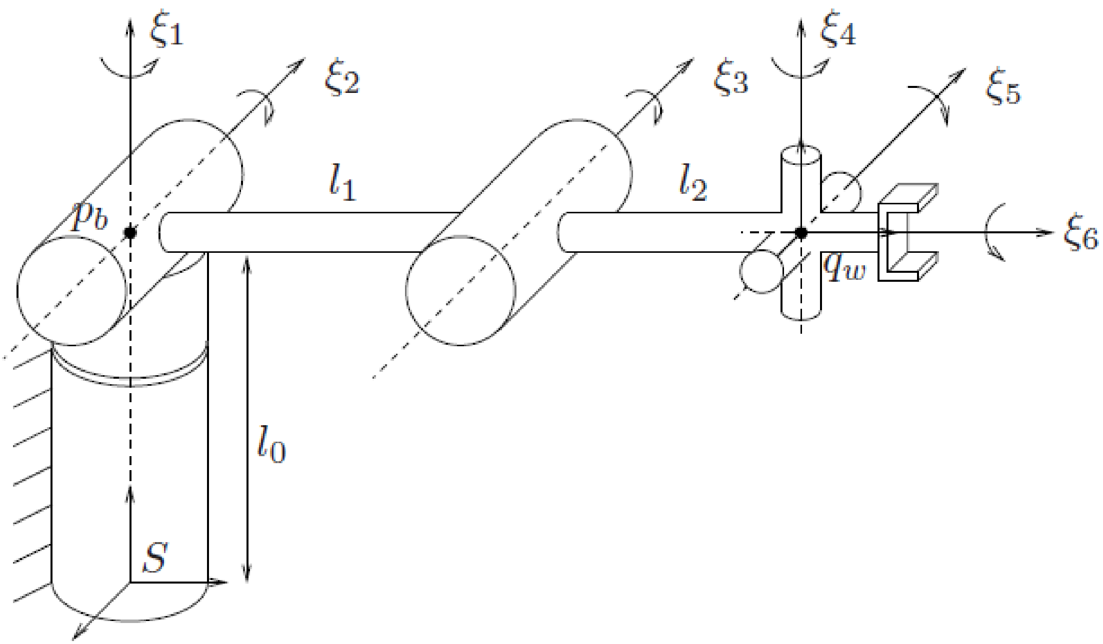


While (θ_1, θ_2) both determine the point (x, y), they separately control the radial position and distance

Separation is a useful IK strategy

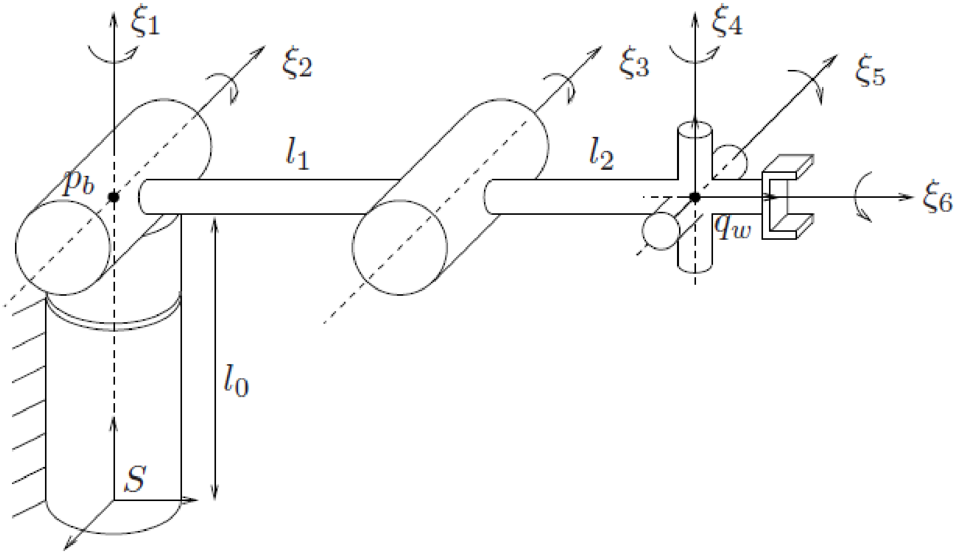
EXAMPLE II

Given a valid end effector configuration, find the corresponding joint angles $(\theta_1 - \theta_6)$



EXAMPLE II

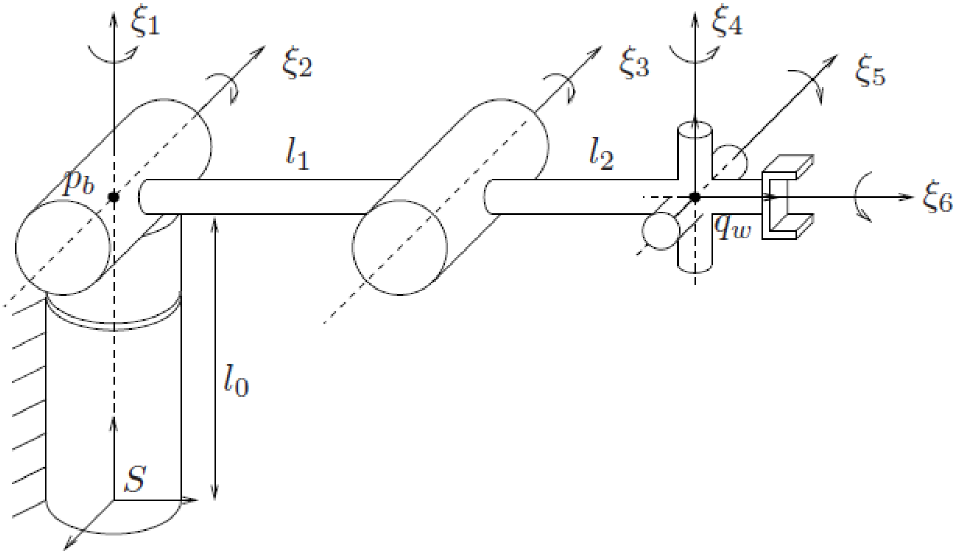
Separation of joints:



EXAMPLE II

Separation of joints:

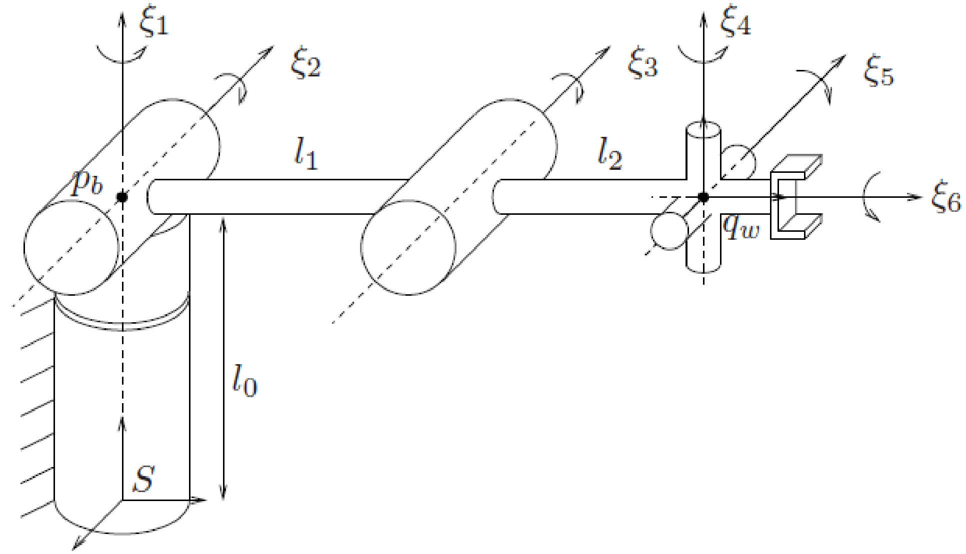
θ_3:



EXAMPLE II

Separation of joints:

θ_3: Distance $||q_w - p_b||$

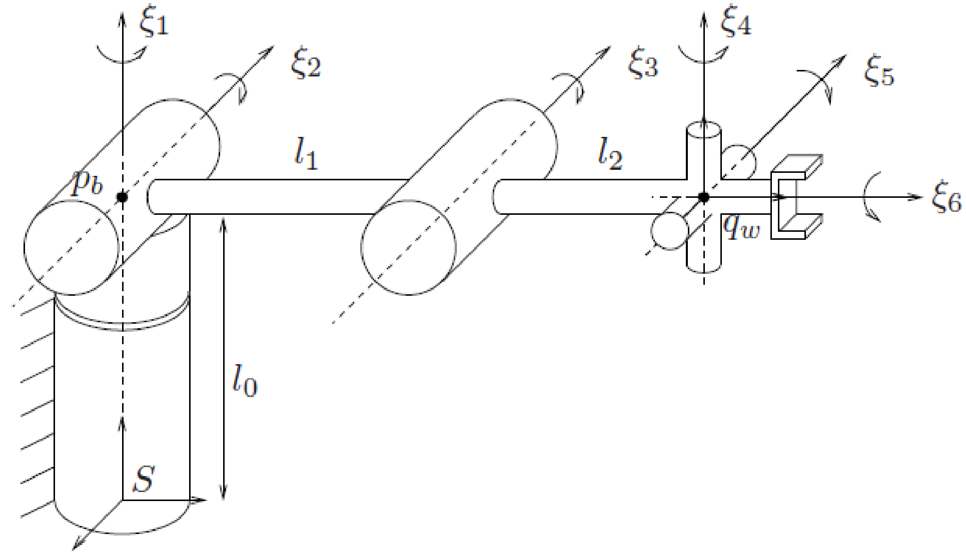


EXAMPLE II

Separation of joints:

θ_3: Distance $\|q_w - p_b\|$

$\theta_{1,2}$:

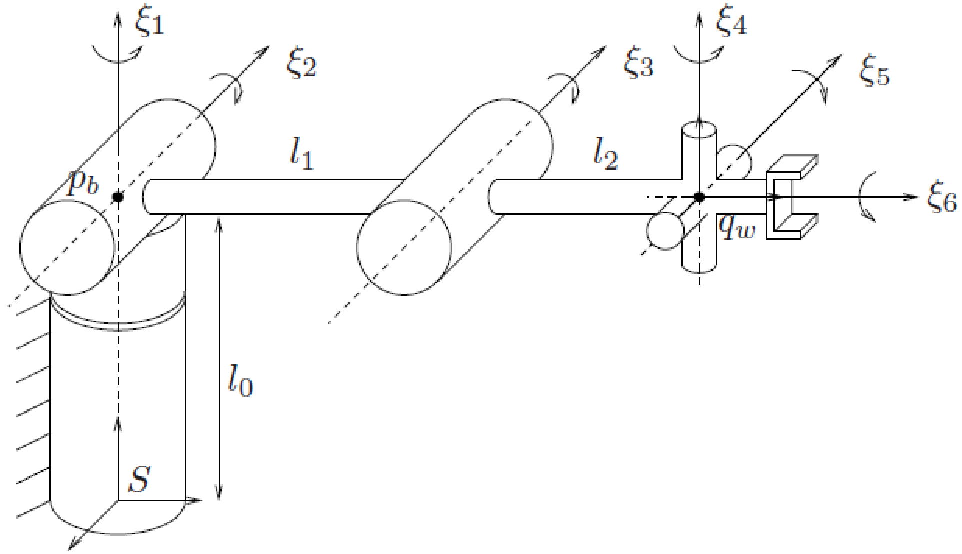


EXAMPLE II

Separation of joints:

θ_3: Distance $\|q_w - p_b\|$

$\theta_{1,2}$: Polar position of q_w



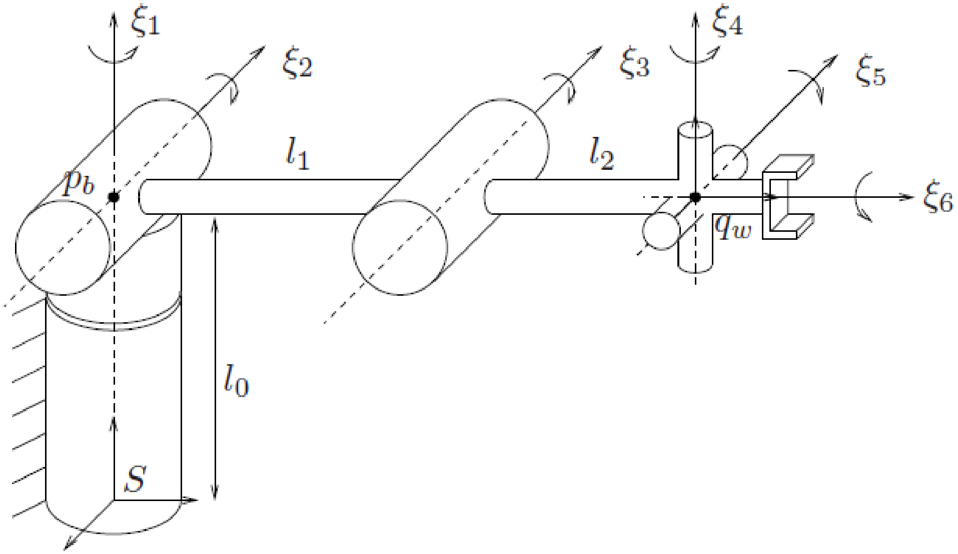
EXAMPLE II

Separation of joints:

θ_3: Distance $||q_w - p_b||$

$\theta_{1,2}$: Polar position of q_w

$\theta_{4,5,6}$:



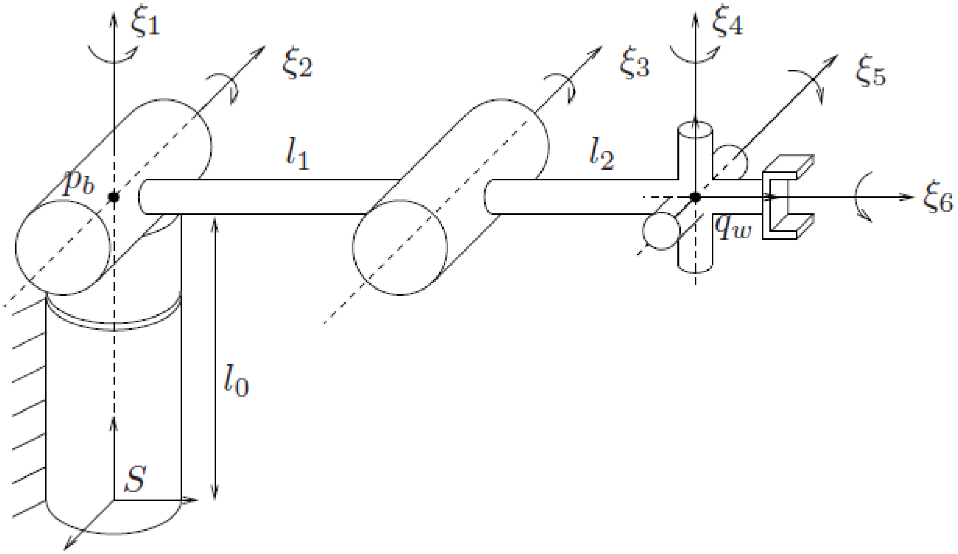
EXAMPLE II

Separation of joints:

θ_3: Distance $\|q_w - p_b\|$

$\theta_{1,2}$: Polar position of q_w

$\theta_{4,5,6}$: Orientation of the end effector



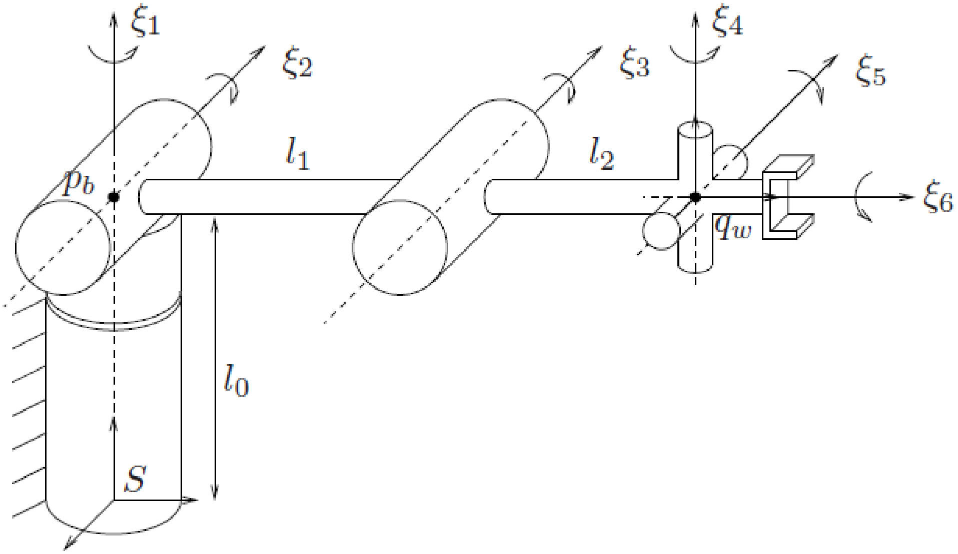
INVARIANT POINTS

The points q_w and p_b are special:

p_b does not change with rotations about $\theta_{1,2}$

q_w does not change with rotations about $\theta_{4,5,6}$

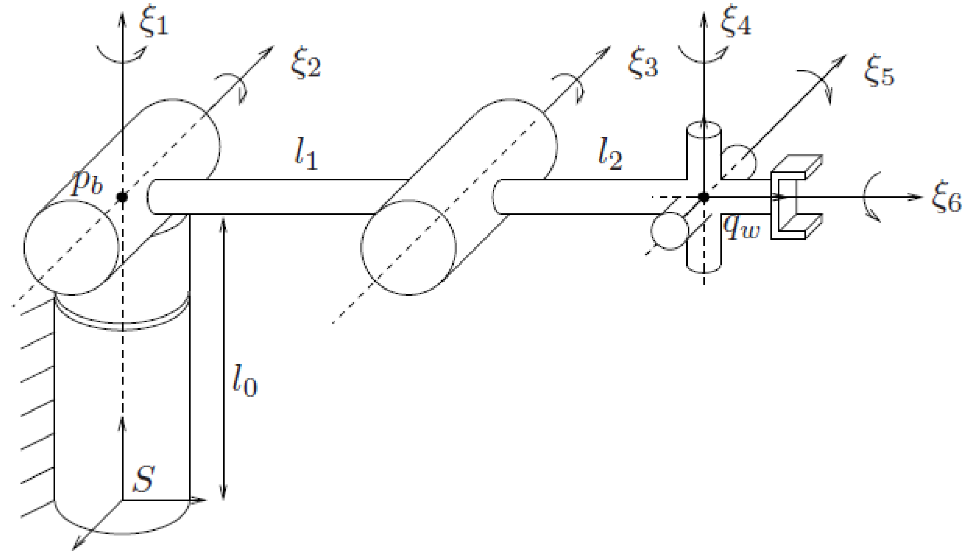
These points are said to be invariant



INVARIANT POINTS

Invariant points line on axes of rotation:

$$p = e^{\hat{\xi}_i \theta_i} p$$



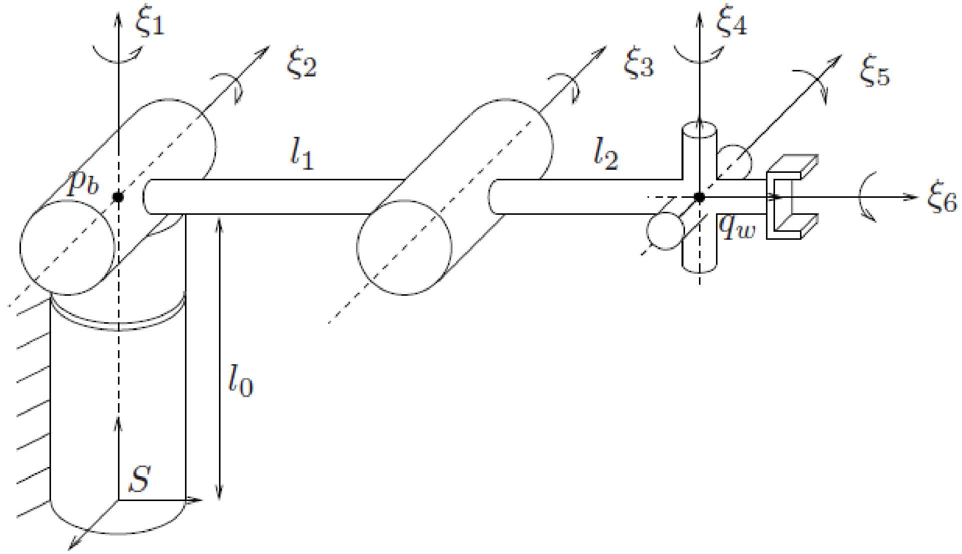
INVARIANT POINTS

Invariant points line on axes of rotation:

$$p = e^{\hat{\xi}_i \theta_i} p$$

ie:

$$p_b = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} p_b$$



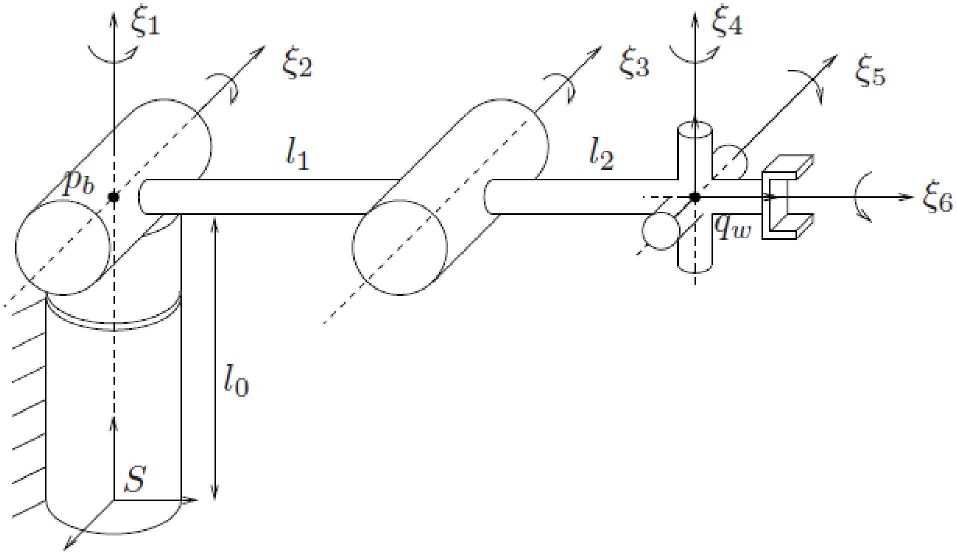
INVARIANT POINTS

Invariant points line on axes of rotation:

$$p = e^{\hat{\xi}_i \theta_i} p$$

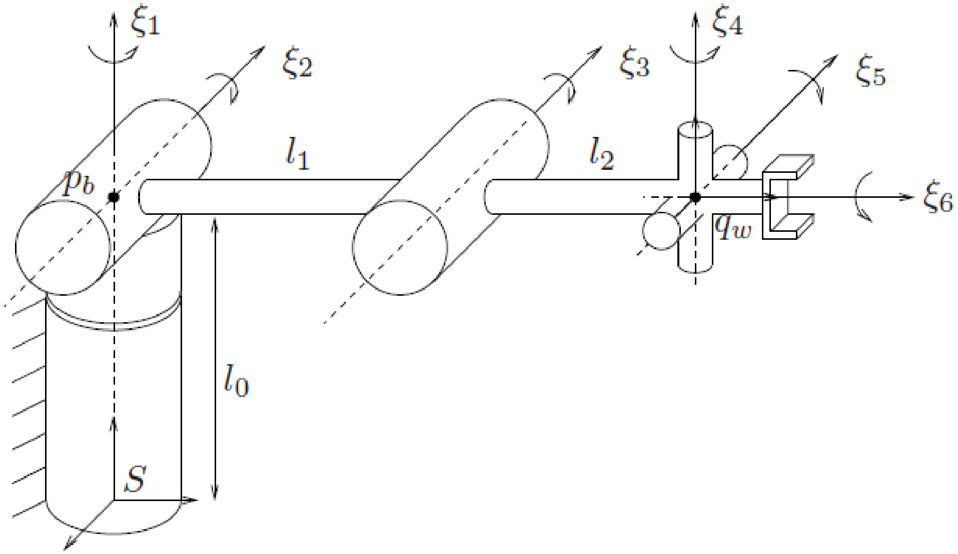
ie:

$$p_b = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} p_b$$
$$q_w = e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} e^{\hat{\xi}_6 \theta_6} q_w$$



EXAMPLE II

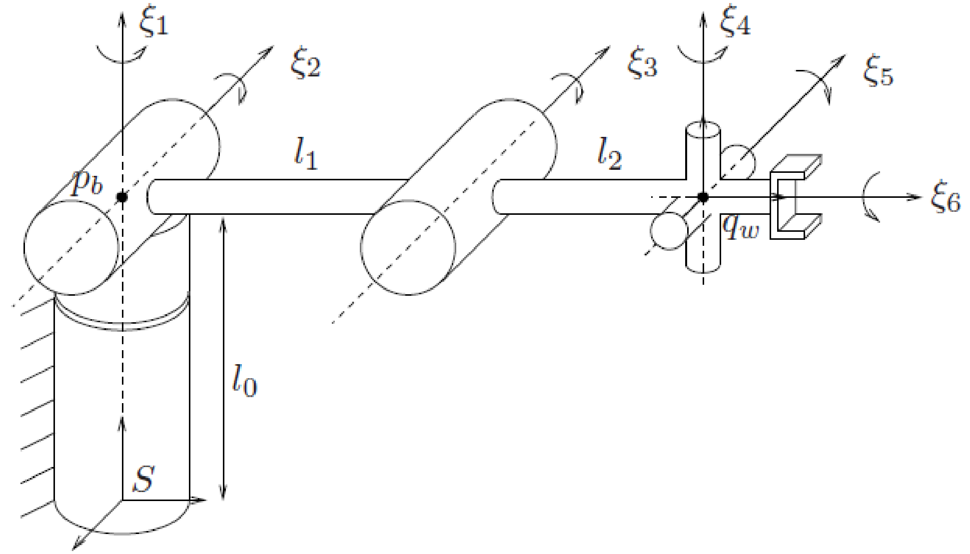
Given a desired end effector configuration g_d and an initial configuration g_0 find θ_{1-6}



EXAMPLE II

Given a desired end effector configuration g_d and an initial configuration g_0 find θ_{1-6}

$$g_d = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}g_0$$

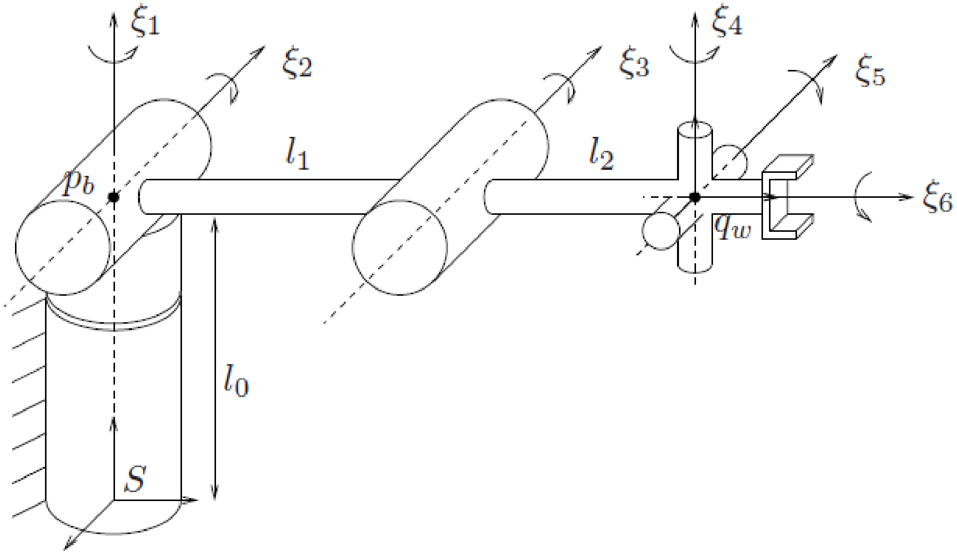


EXAMPLE II

Given a desired end effector configuration g_d and an initial configuration g_0 find θ_{1-6}

$$g_d = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}g_0$$

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$



PADEN-KAHAN SUBPROBLEMS

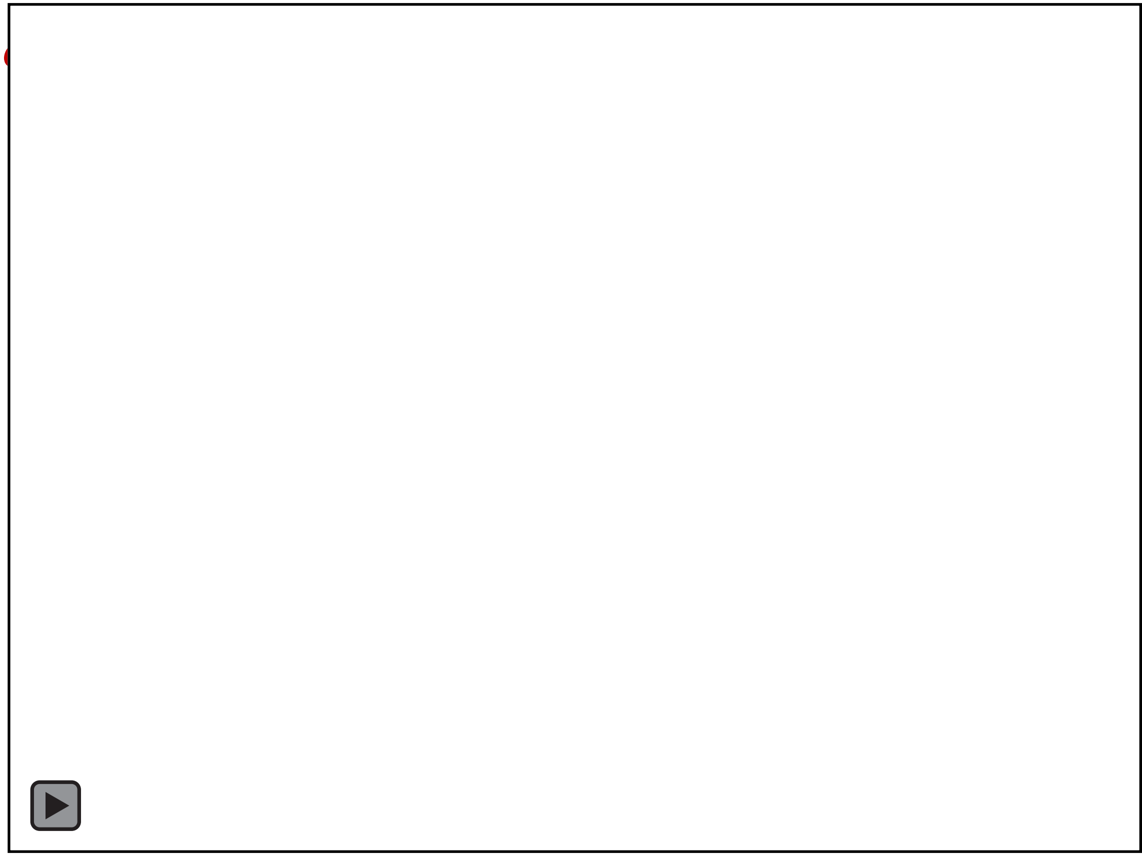
Analytical solutions for inverse kinematics:

Fast, efficient

Other methods (numerical) exist and will be explored in the labs.

PADEN-KAHAN SUBPROBLEM I

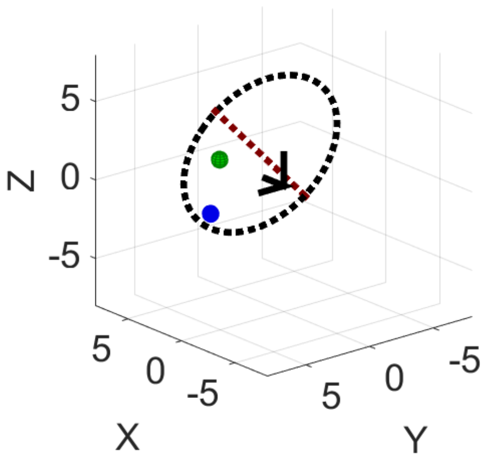
Rotations about a single axis

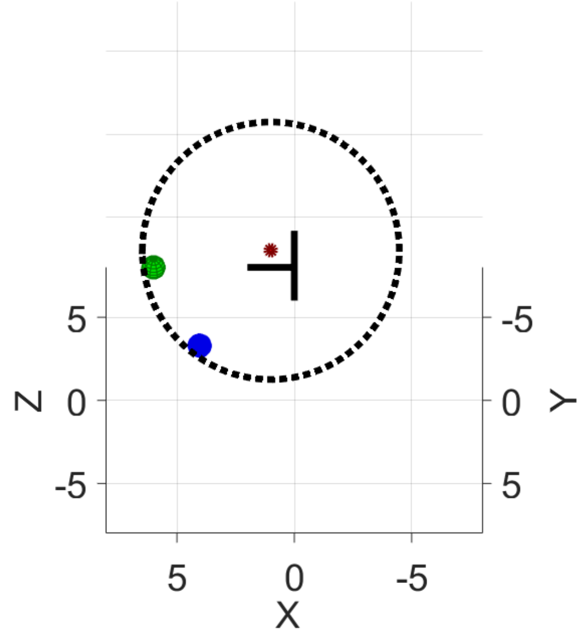


PADEN-KAHAN SUBPROBLEM I

Rotations about a single axis $\boldsymbol{\omega}$

Given two points $\boldsymbol{p}$ and $\boldsymbol{q}$ find the angle of rotation $\boldsymbol{\theta_1}$





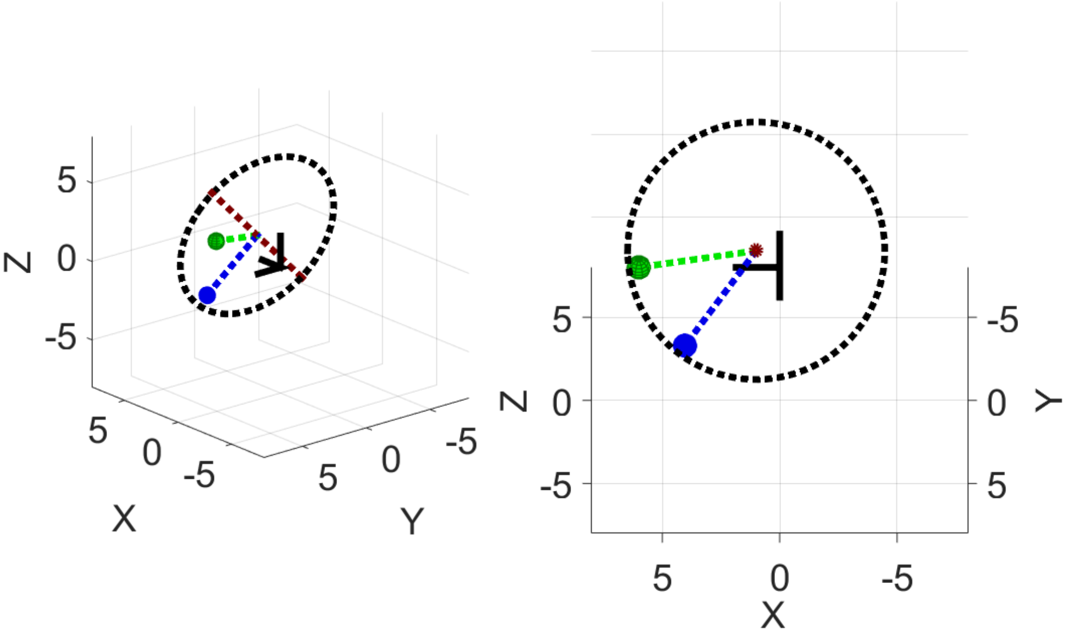
PADEN-KAHAN SUBPROBLEM I

Rotations about a single axis $\boldsymbol{\omega}$

Given two points $\boldsymbol{p}$ and $\boldsymbol{q}$ find the angle of rotation $\boldsymbol{\theta_1}$

Relative coordinates of $\boldsymbol{p}$ and $\boldsymbol{q}$ can be found from a point on the axis of rotation $\boldsymbol{r}$

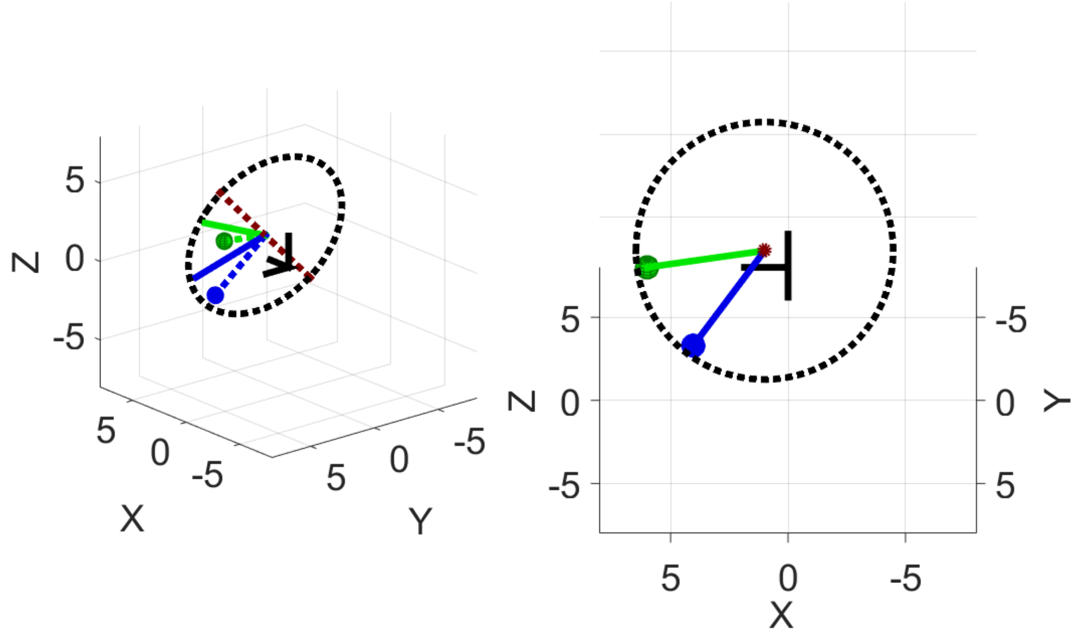
$$\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{r}$$
$$\boldsymbol{v} = \boldsymbol{q} - \boldsymbol{r}$$



PADEN-KAHAN SUBPROBLEM I

These relative points can then be projected on the circle of revolution:

$$u' = u - \omega\omega^T u$$
$$v' = v - \omega\omega^T v$$

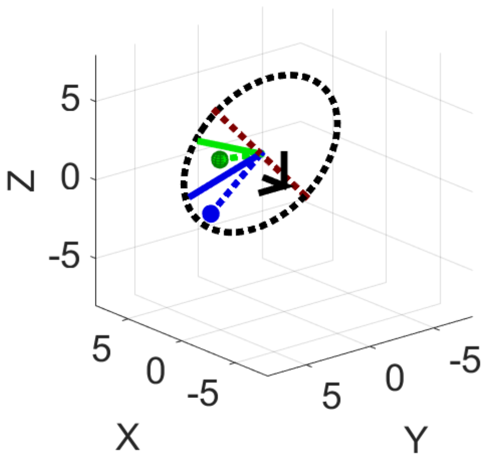


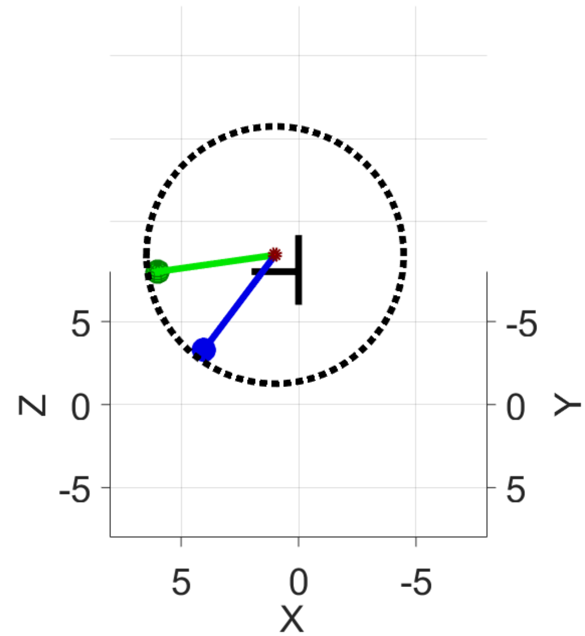
PADEN-KAHAN SUBPROBLEM I

These relative points can then be projected on the circle of revolution:

$$u' = u - \omega\omega^T u$$
$$v' = v - \omega\omega^T v$$

$$u' \times v' = \omega \sin(\theta_1)\|u'\|_2\|v'\|_2$$
$$u' \cdot v' = \cos(\theta_1)\|u'\|_2\|v'\|_2$$





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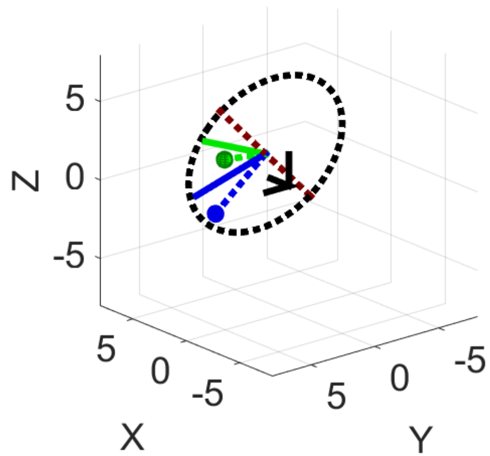
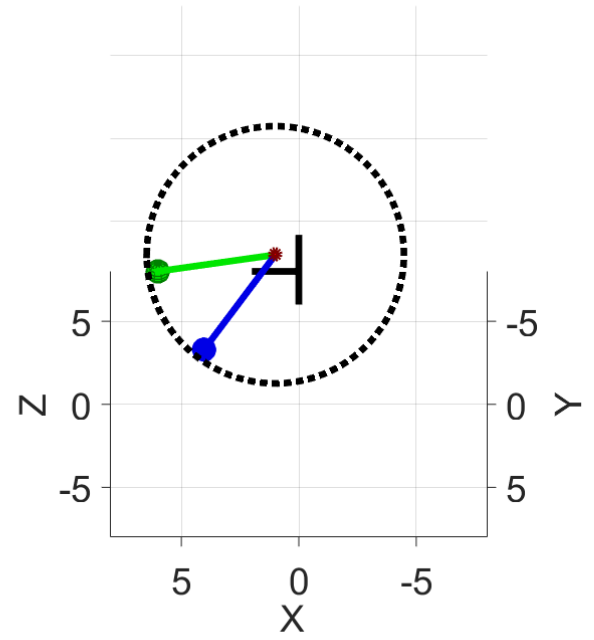
These relative points can then be projected on the circle of revolution:

$$u' = u - \omega\omega^T u$$
$$v' = v - \omega\omega^T v$$

$$u' \times v' = \omega sin(\theta_1)\|u'\|_2\|v'\|_2$$
$$u' \cdot v' = cos(\theta_1)\|u'\|_2\|v'\|_2$$

$$\theta_1 = atan2\left(\omega^T(u' \times v'), u'^T v'\right)$$





PADEN-KAHAN SUBPROBLEM II

Rotations about subsequent axes $\boldsymbol{\omega_1}$, $\boldsymbol{\omega_2}$.

$$g_{AB}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g_{AB}(0$$

PADEN-KAHAN SUBPROBLEM II

Rotations about subsequent axes $\boldsymbol{\omega_1}$, $\boldsymbol{\omega_2}$.

$$g_{AB}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g_{AB}(0$$

Given two points $\boldsymbol{p}$ and $\boldsymbol{q}$ find the angles θ_1, θ_2

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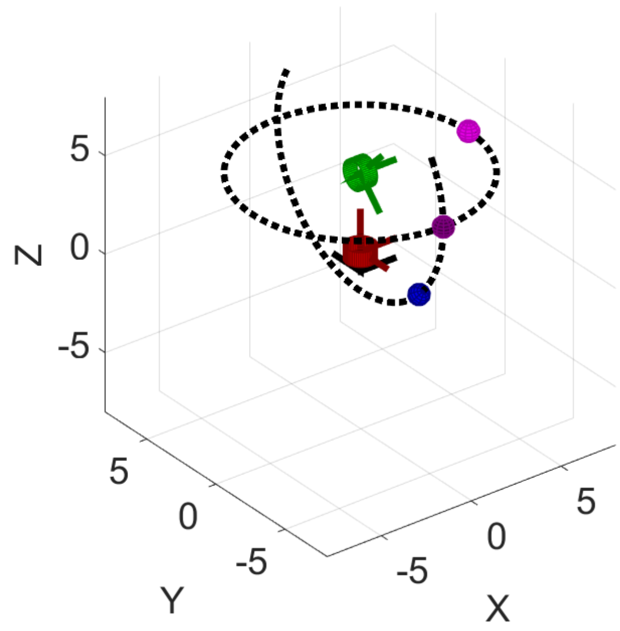
PADEN-KAHAN SUBPROBLEM II

Rotations about subsequent axes $\boldsymbol{\omega_1}$, $\boldsymbol{\omega_2}$.

$$g_{AB}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g_{AB}(0,0)$$

Given two points $\boldsymbol{p}$ and $\boldsymbol{q}$ find the angles θ_1, θ_2



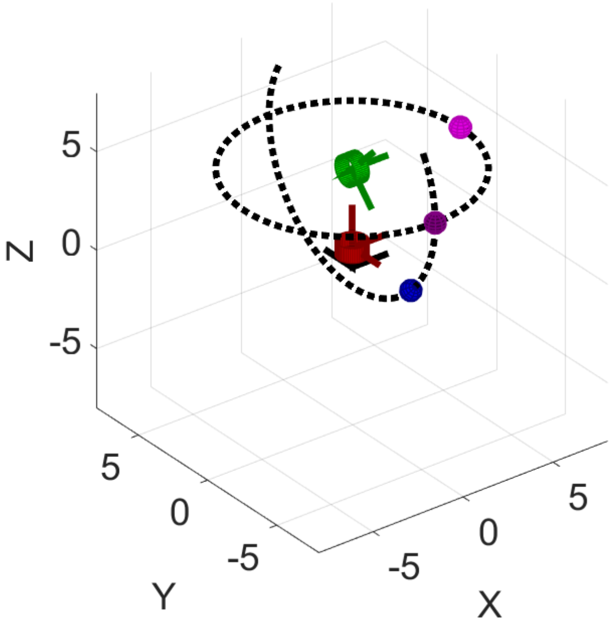
PADEN-KAHAN SUBPROBLEM II

Rotations about subsequent axes $\boldsymbol{\omega_1}$, $\boldsymbol{\omega_2}$.

$$g_{AB}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g_{AB}(0,0)$$

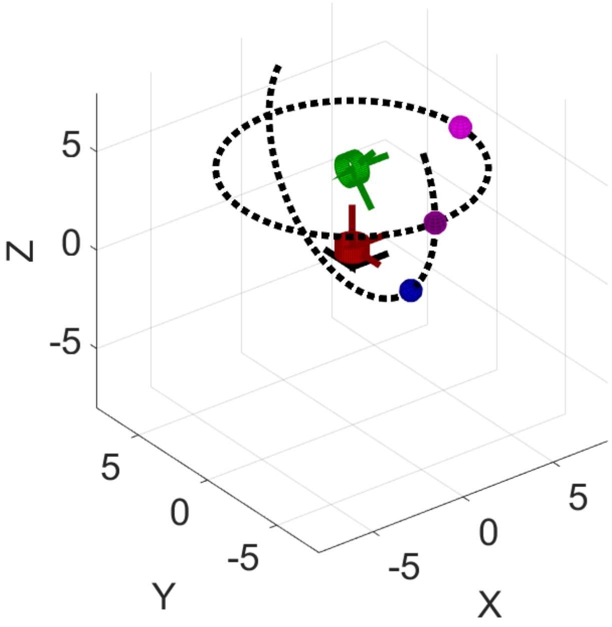
Given two points $\boldsymbol{p}$ and $\boldsymbol{q}$ find the angles θ_1, θ_2

Find intersection point $\boldsymbol{c}$.



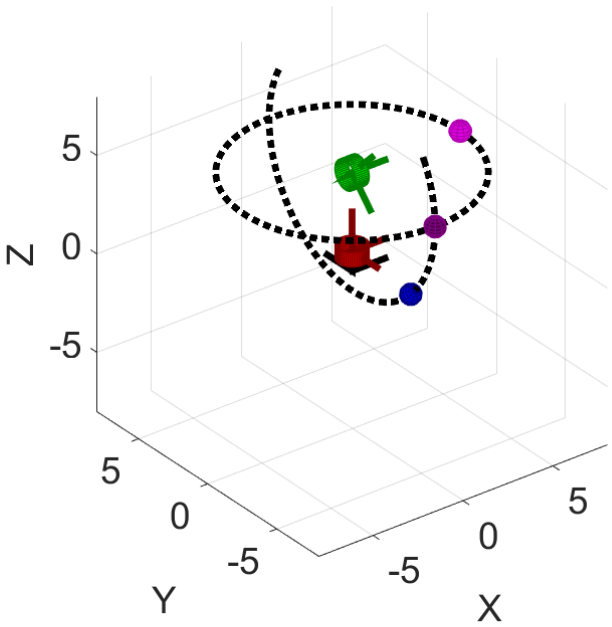
PADEN-KAHAN SUBPROBLEM II

$$q = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} p$$



PADEN-KAHAN SUBPROBLEM II

$$q = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p$$
$$e^{-\hat{\xi}_1\theta_1} q = e^{\hat{\xi}_2\theta_2} p$$

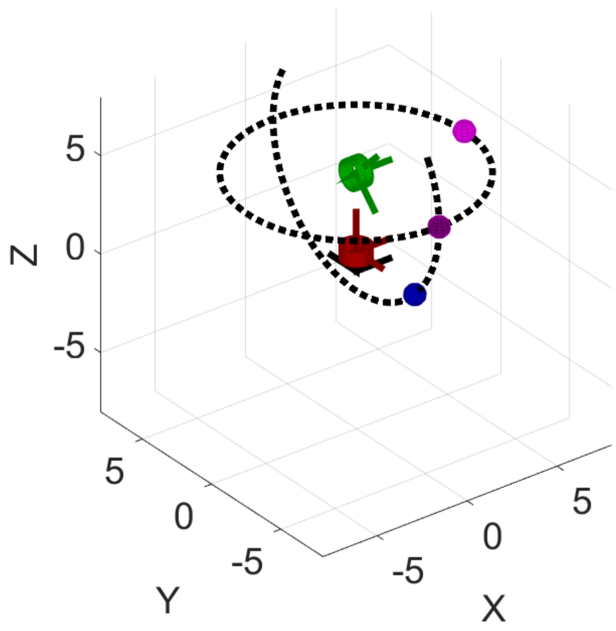


PADEN-KAHAN SUBPROBLEM II

$$q = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p$$

$$e^{-\hat{\xi}_1\theta_1} q = e^{\hat{\xi}_2\theta_2} p$$

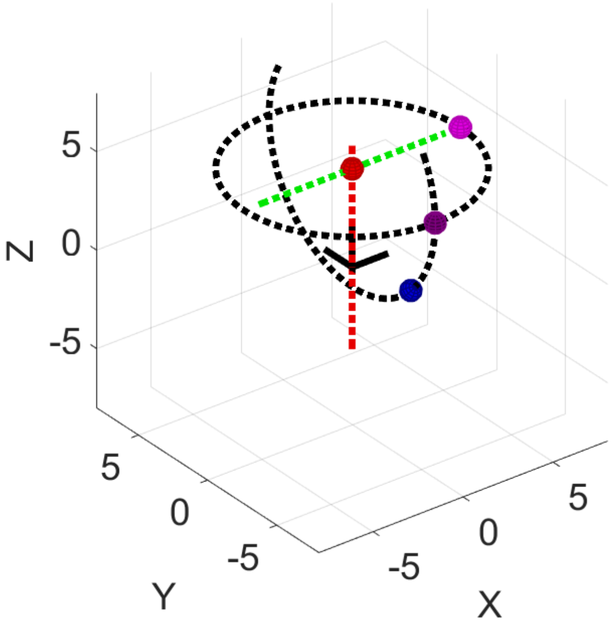
$$e^{-\hat{\xi}_1\theta_1} q = c = e^{\hat{\xi}_2\theta_2} p$$



PADEN-KAHAN SUBPROBLEM II

$$\boldsymbol{q} = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\boldsymbol{p}$$
$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{q} = e^{\hat{\xi}_2\theta_2}\boldsymbol{p}$$
$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{q} = \boldsymbol{c} = e^{\hat{\xi}_2\theta_2}\boldsymbol{p}$$

find a point common to both axes $\boldsymbol{r}$

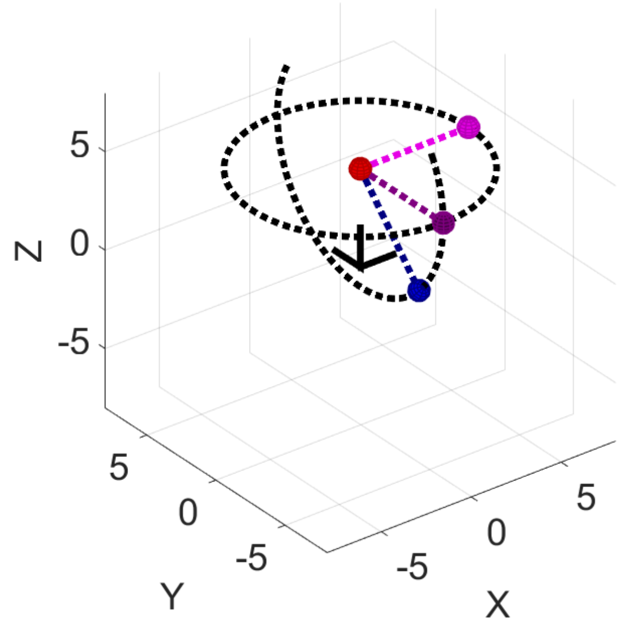


PADEN-KAHAN SUBPROBLEM II

$$q = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} p$$
$$e^{-\hat{\xi}_1 \theta_1} q = e^{\hat{\xi}_2 \theta_2} p$$
$$e^{-\hat{\xi}_1 \theta_1} q = c = e^{\hat{\xi}_2 \theta_2} p$$

find a point common to both axes r

$$u = p - r$$
$$z = c - r$$
$$v = q - r$$

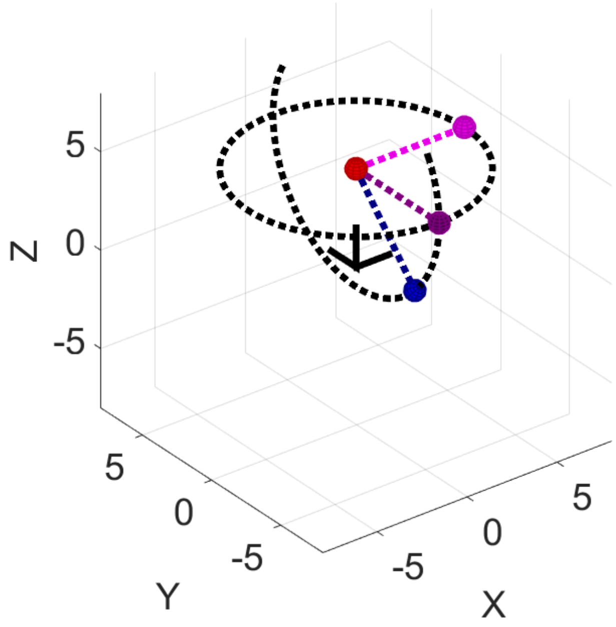


PADEN-KAHAN SUBPROBLEM II

$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{q} = \boldsymbol{c} = e^{\hat{\xi}_2\theta_2}\boldsymbol{p}$$

$$\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{r}$$
$$\boldsymbol{z} = \boldsymbol{c} - \boldsymbol{r}$$
$$\boldsymbol{v} = \boldsymbol{q} - \boldsymbol{r}$$

$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{v} = \boldsymbol{z} = e^{\hat{\xi}_2\theta_2}\boldsymbol{u}$$



PADEN-KAHAN SUBPROBLEM II

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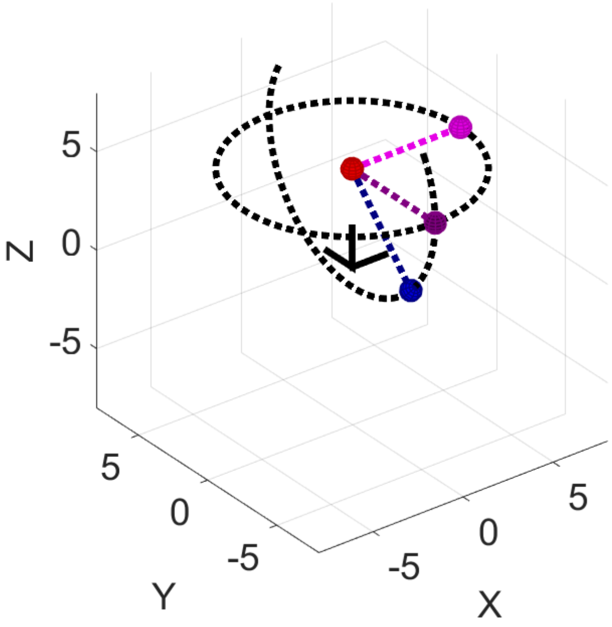
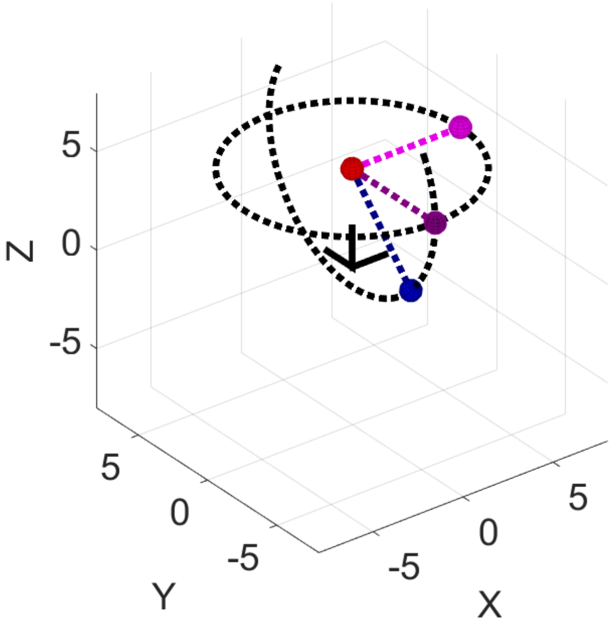


PADEN-KAHAN SUBPROBLEM II

$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{v} = \boldsymbol{z} = e^{\hat{\xi}_2\theta_2}\boldsymbol{u}$$

$\boldsymbol{\omega_1}$, $\boldsymbol{\omega_2}$ are linearly independent

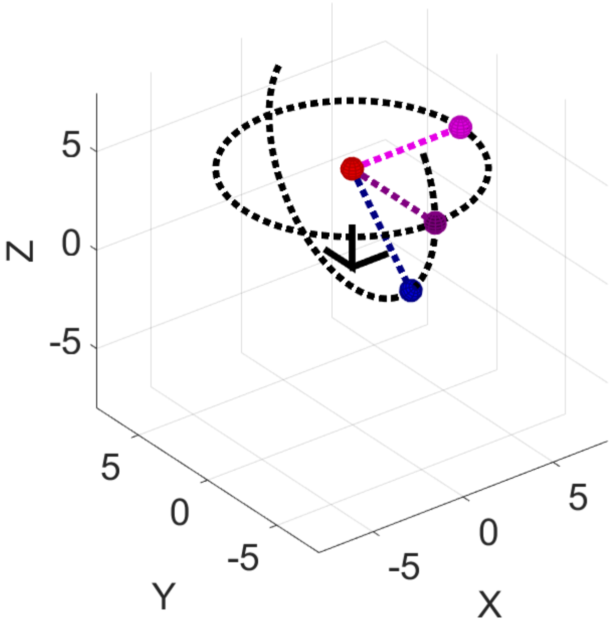


PADEN-KAHAN SUBPROBLEM II

$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{v} = \boldsymbol{z} = e^{\hat{\xi}_2\theta_2}\boldsymbol{u}$$

$\boldsymbol{\omega_1}$, $\boldsymbol{\omega_2}$ are linearly independent

$$\boldsymbol{z} = \boldsymbol{\alpha}\boldsymbol{\omega_1} + \boldsymbol{\beta}\boldsymbol{\omega_2} + \boldsymbol{\gamma}(\boldsymbol{\omega_1} \times \boldsymbol{\omega_2})$$



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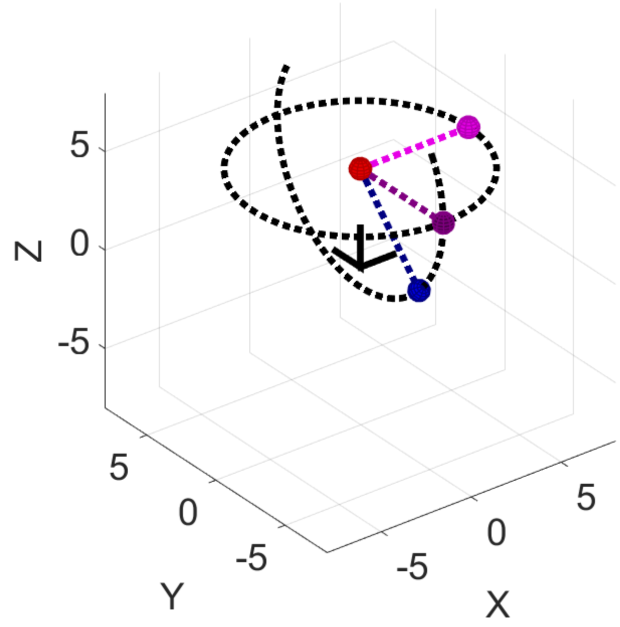
$$\boldsymbol{z} = \boldsymbol{\alpha}\boldsymbol{\omega_1} + \boldsymbol{\beta}\boldsymbol{\omega_2} + \boldsymbol{\gamma}(\boldsymbol{\omega_1} \times \boldsymbol{\omega_2})$$

$$\alpha = \frac{(\omega_1^T\omega_2)\omega_2^T u - \omega_1^T v}{(\omega_1^T\omega_2)^2 - 1}$$

$$\beta = \frac{(\omega_1^T\omega_2)\omega_1^T v - \omega_2^T u}{(\omega_1^T\omega_2)^2 - 1}$$

$$\gamma^2 = \frac{\|u\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\omega_1^T\omega_2}{\|\omega_1 \times \omega_2\|^2}$$



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PADEN-KAHAN SUBPROBLEM II

$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{v} = \boldsymbol{z} = e^{\hat{\xi}_2\theta_2}\boldsymbol{u}$$

$\boldsymbol{\omega_1}$, $\boldsymbol{\omega_2}$ are linearly independent

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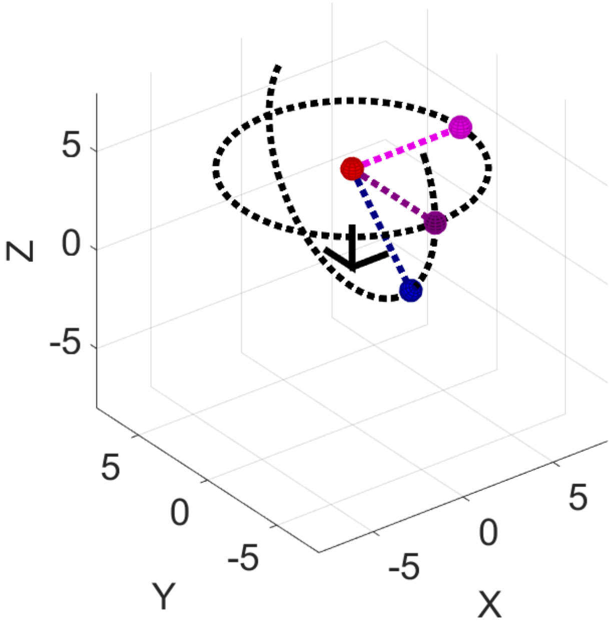
Two solutions!



PADEN-KAHAN SUBPROBLEM II

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$$\beta = \frac{(\omega_1^T\omega_2)\omega_1^T v - \omega_2^T u}{(\omega_1^T\omega_2)^2 - 1}$$

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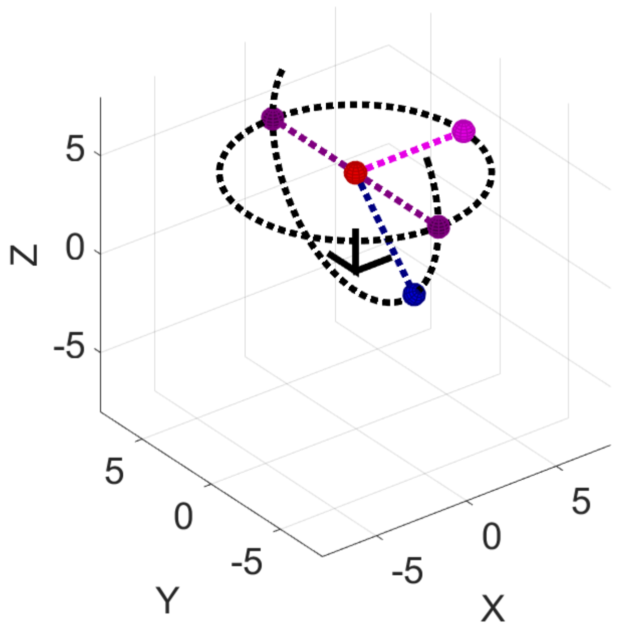
Two solutions!



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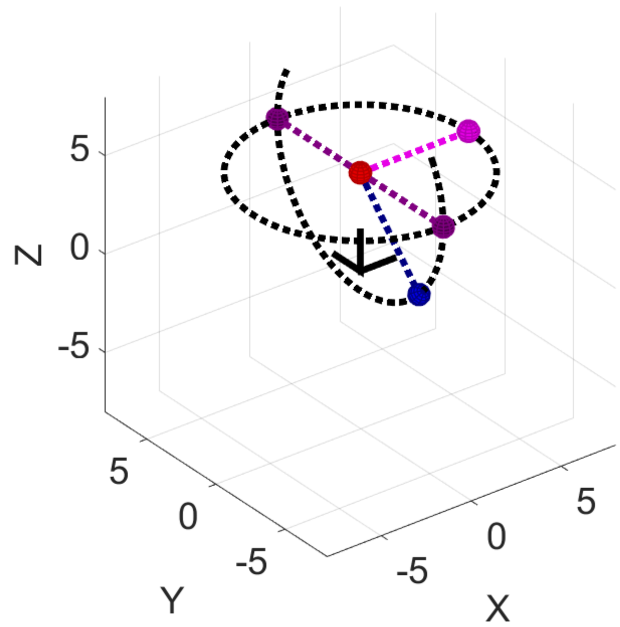
PADEN-KAHAN SUBPROBLEM II

$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{v} = \boldsymbol{z} = e^{\hat{\xi}_2\theta_2}\boldsymbol{u}$$



PADEN-KAHAN SUBPROBLEM II

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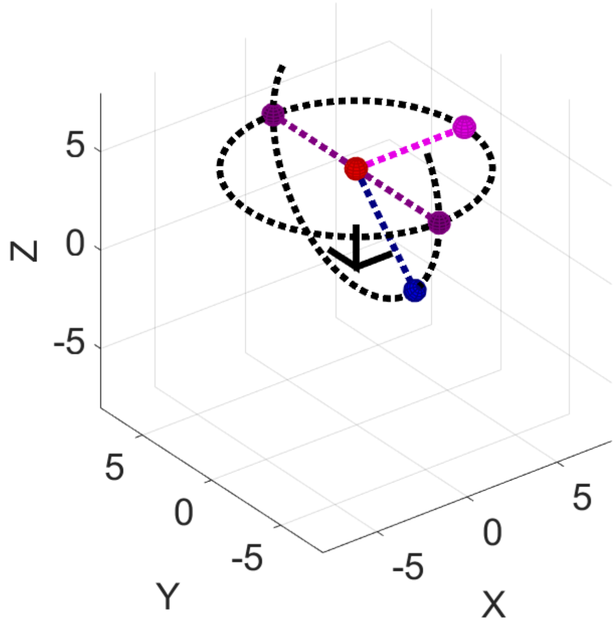
$$\boldsymbol{z}_1 = \boldsymbol{\alpha}\boldsymbol{\omega}_1 + \boldsymbol{\beta}\boldsymbol{\omega}_2 + \boldsymbol{\gamma}_1(\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2)$$

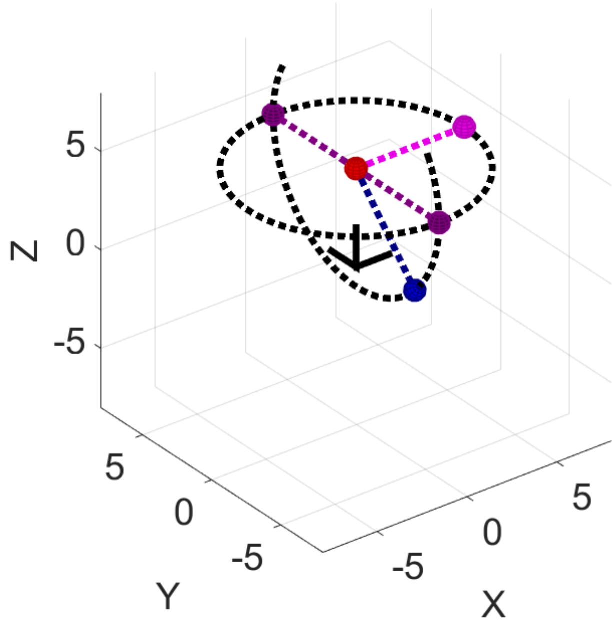


PADEN-KAHAN SUBPROBLEM II

$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{v} = \boldsymbol{z} = e^{\hat{\xi}_2\theta_2}\boldsymbol{u}$$

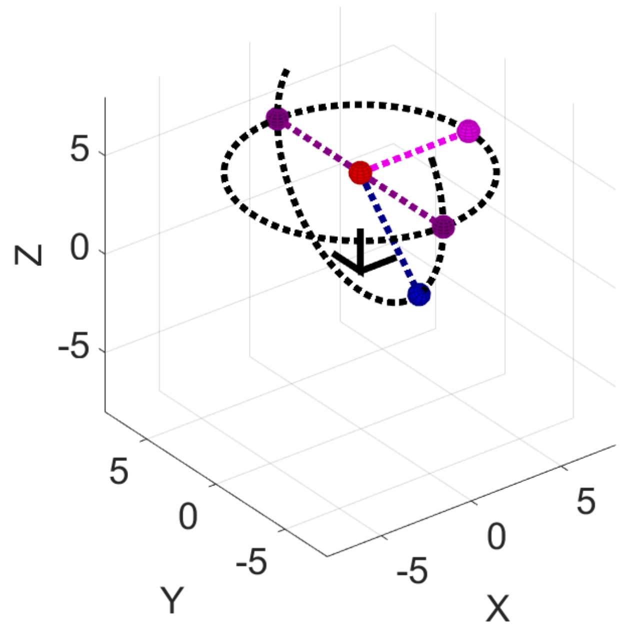
$$\boldsymbol{z_1} = \boldsymbol{\alpha\omega_1} + \boldsymbol{\beta\omega_2} + \boldsymbol{\gamma_1}(\boldsymbol{\omega_1} \times \boldsymbol{\omega_2})$$

$$\boldsymbol{z_2} = \boldsymbol{\alpha\omega_1} + \boldsymbol{\beta\omega_2} + \boldsymbol{\gamma_2}(\boldsymbol{\omega_1} \times \boldsymbol{\omega_2})$$



Paden-Kahan Subproblem II

$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{v} = \boldsymbol{z} = e^{\hat{\xi}_2\theta_2}\boldsymbol{u}$$
$$\boldsymbol{z_1} = \boldsymbol{\alpha}\boldsymbol{\omega_1} + \boldsymbol{\beta}\boldsymbol{\omega_2} + \boldsymbol{\gamma_1}(\boldsymbol{\omega_1} \times \boldsymbol{\omega_2})$$
$$\boldsymbol{z_2} = \boldsymbol{\alpha}\boldsymbol{\omega_1} + \boldsymbol{\beta}\boldsymbol{\omega_2} + \boldsymbol{\gamma_2}(\boldsymbol{\omega_1} \times \boldsymbol{\omega_2})$$



Solve as PK I:

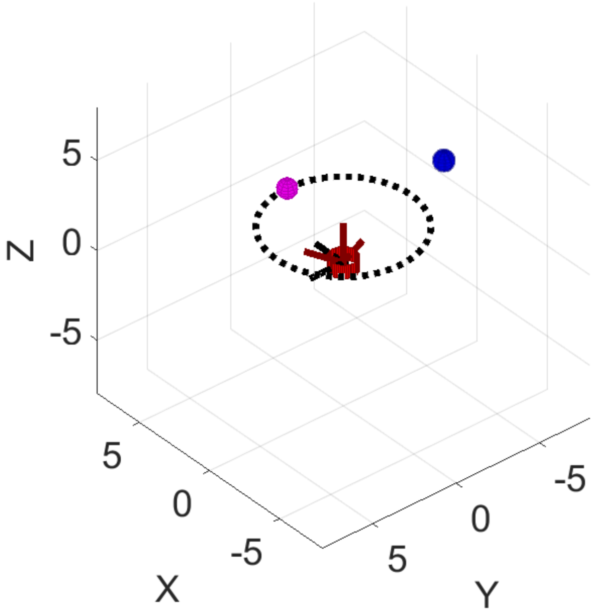
$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{v} = \boldsymbol{z_1} = e^{\hat{\xi}_2\theta_2}\boldsymbol{u}$$
$$e^{-\hat{\xi}_1\theta_1}\boldsymbol{v} = \boldsymbol{z_2} = e^{\hat{\xi}_2\theta_2}\boldsymbol{u}$$

PADEN-KAHAN SUBPROBLEM III

Rotations about axis $\boldsymbol{\omega_1}$ to a distance $\boldsymbol{\delta}$

$$\boldsymbol{\delta} = \left\| \boldsymbol{q} - e^{\hat{\xi}_1 \theta_1} \boldsymbol{p} \right\|$$

Given two points $\boldsymbol{p}$ and $\boldsymbol{q}$ find the angles θ_1



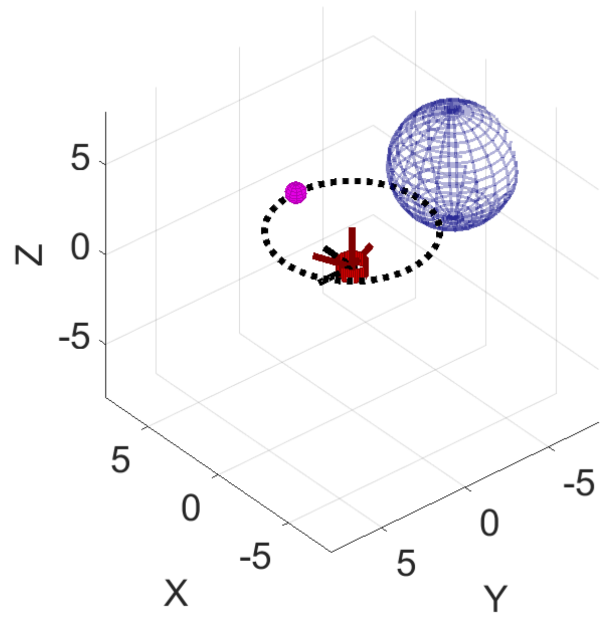
PADEN-KAHAN SUBPROBLEM III

Rotations about axis $\boldsymbol{\omega_1}$ to a distance $\boldsymbol{\delta}$

$$\boldsymbol{\delta} = \left\| \boldsymbol{q} - e^{\hat{\xi}_1 \theta_1} \boldsymbol{p} \right\|$$

Given two points $\boldsymbol{p}$ and $\boldsymbol{q}$ find the angles θ_1

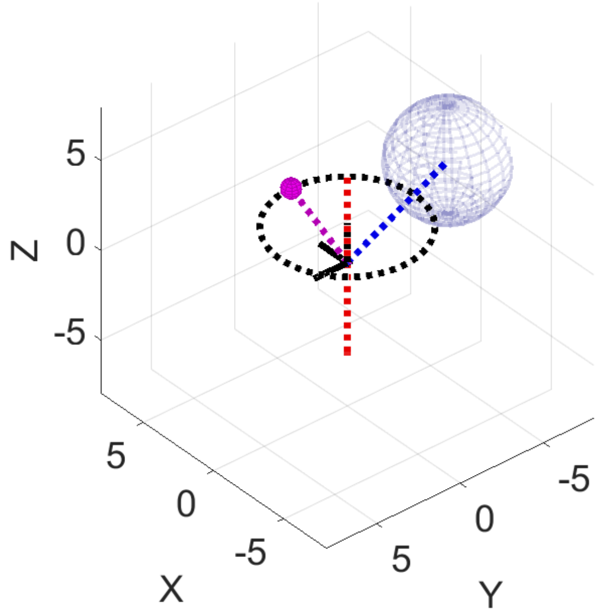
$\boldsymbol{\delta}$ can be thought of as a *ball* around $\boldsymbol{q}$



PADEN-KAHAN SUBPROBLEM III

Given a point $\boldsymbol{r}$ on the rotational axis, the relative coordinates can be found:

$$\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{r}$$
$$\boldsymbol{v} = \boldsymbol{q} - \boldsymbol{r}$$



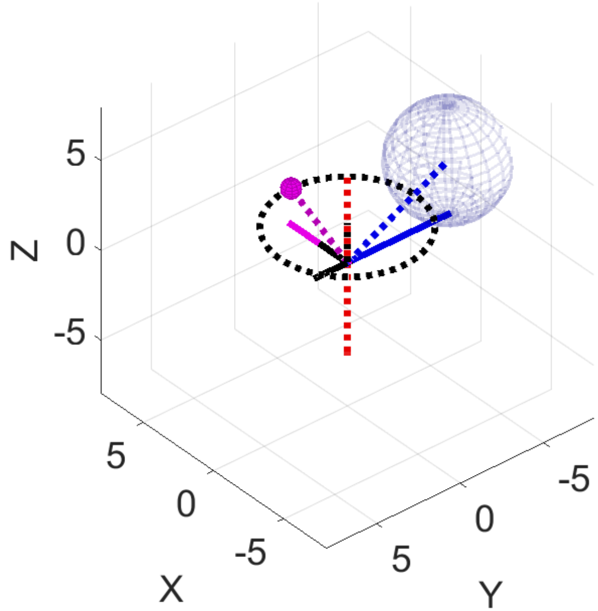
PADEN-KAHAN SUBPROBLEM III

Given a point $\boldsymbol{r}$ on the rotational axis, the relative coordinates can be found:

$$\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{r}$$
$$\boldsymbol{v} = \boldsymbol{q} - \boldsymbol{r}$$

The projection on the plane of rotation is given by:

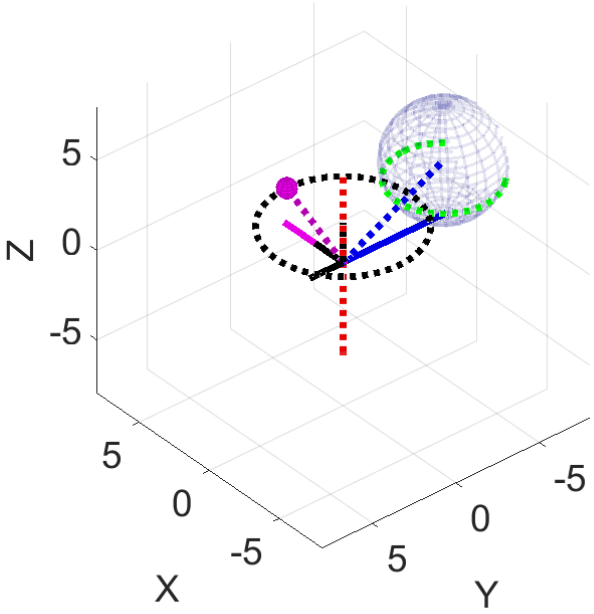
$$\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{u}$$
$$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}$$



PADEN-KAHAN SUBPROBLEM III

The projection of $\boldsymbol{\delta}$ on the rotational plane can also be found:

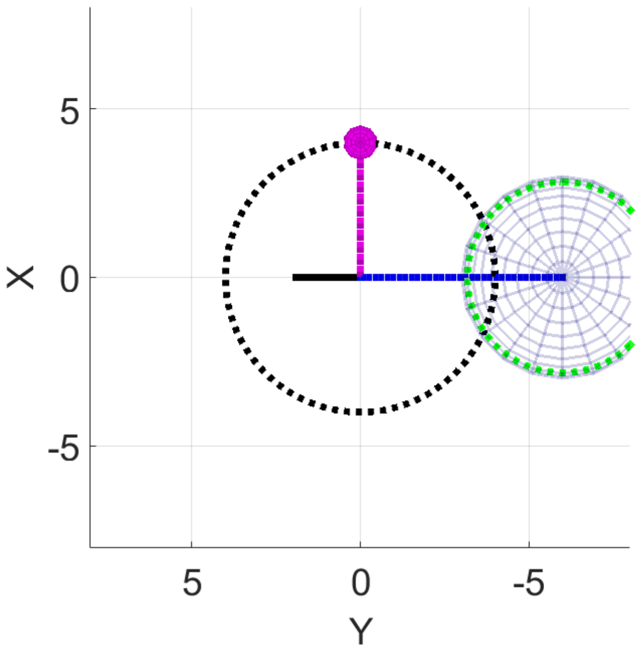
$$\boldsymbol{\delta}'^2 = \boldsymbol{\delta}^2 - |\boldsymbol{\omega}^T(\boldsymbol{p} - \boldsymbol{q})|^2$$



PADEN-KAHAN SUBPROBLEM III

The projection of $\boldsymbol{\delta}$ on the rotational plane can also be found:

$$\boldsymbol{\delta}'^2 = \boldsymbol{\delta}^2 - |\boldsymbol{\omega}^T(\boldsymbol{p} - \boldsymbol{q})|^2$$



PADEN-KAHAN SUBPROBLEM III

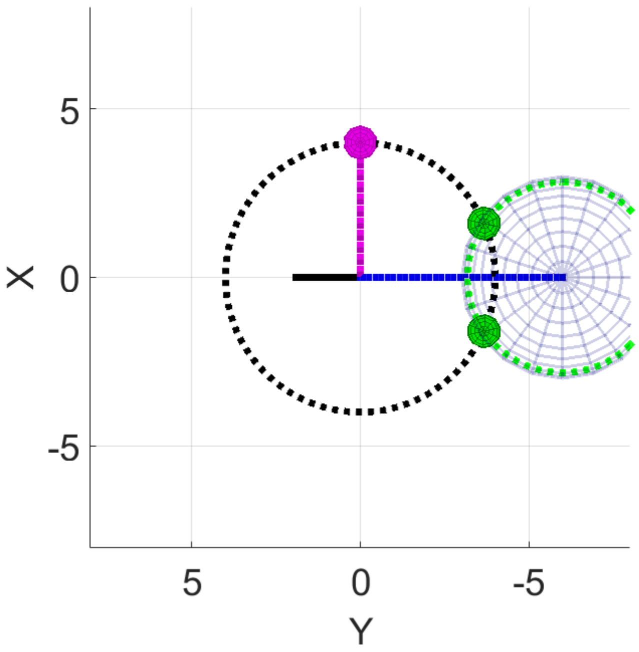
The projection of $\boldsymbol{\delta}$ on the rotational plane can also be found:

$$\boldsymbol{\delta}'^2 = \boldsymbol{\delta}^2 - |\boldsymbol{\omega}^T(\boldsymbol{p} - \boldsymbol{q})|^2$$

This defines two angles of rotation:

$$\boxed{\theta = \theta_0 \pm \theta_d}$$

Two solutions!



PADEN-KAHAN SUBPROBLEM III

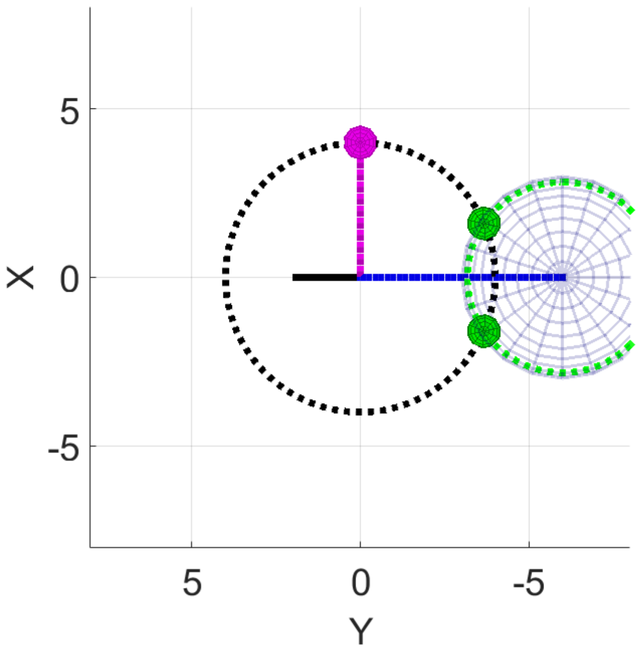
The two solutions have the form:

$$\theta = \theta_0 \pm \theta_d$$

where:

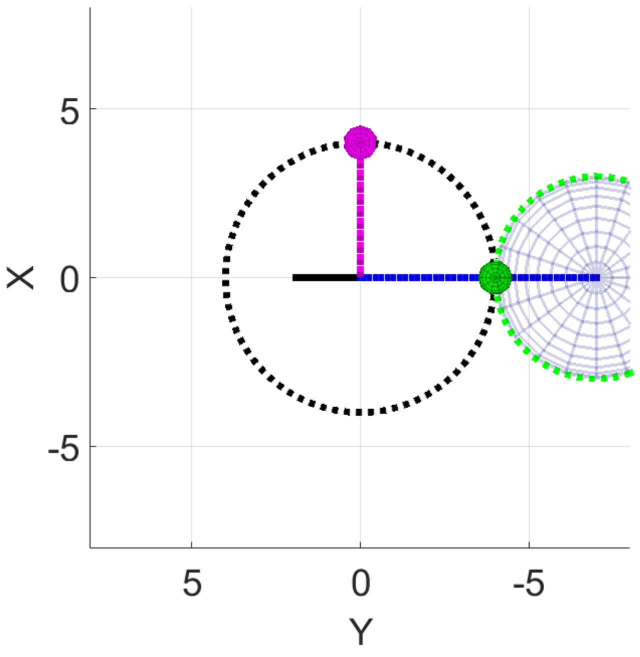
$$\theta_0 = atan2\left(\boldsymbol{\omega}^T(\boldsymbol{u}' \times \boldsymbol{v}'), \boldsymbol{u}'^T \boldsymbol{v}'\right)$$

$$\theta_d = \cos^{-1}\left(\frac{\|\boldsymbol{u}'\|^2 + \|\boldsymbol{v}'\|^2 - \boldsymbol{\delta}'^2}{2\|\boldsymbol{u}'\|\|\boldsymbol{v}'\|}\right)$$



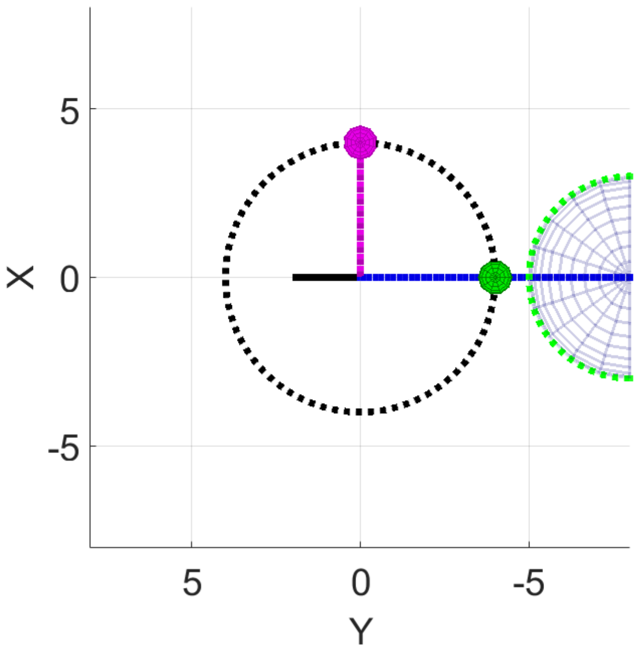
PADEN-KAHAN SUBPROBLEM III

If the desired point is on the edge of the ball there is only one solution



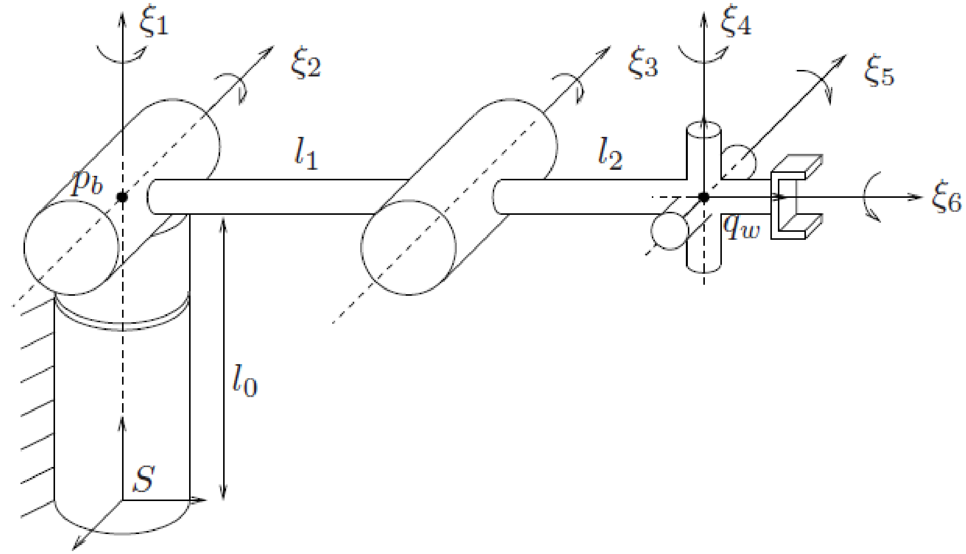
PADEN-KAHAN SUBPROBLEM III

If the desired point outside the edge of the ball there are no solutions



EXAMPLE II

Separation of joints:



θ_3: Distance $\|q_w - p_b\|$

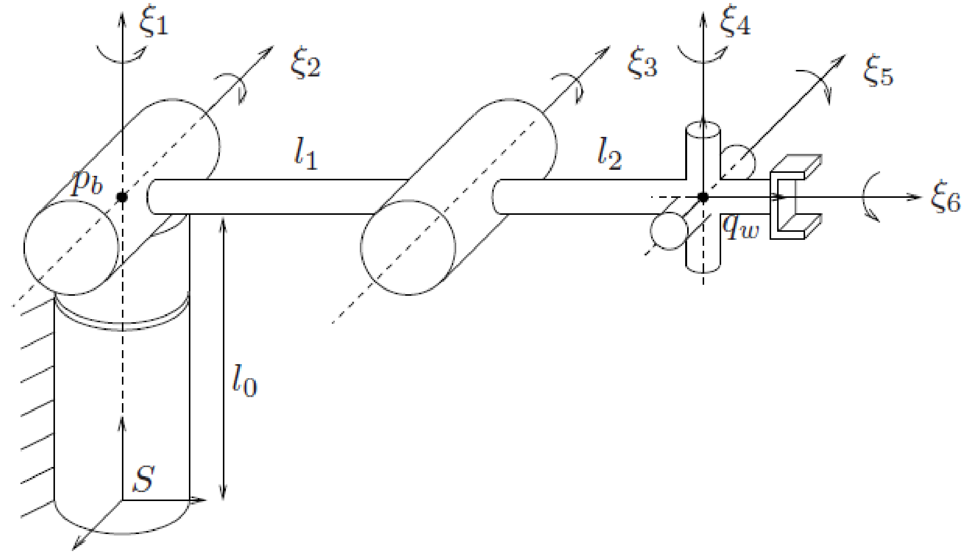
$\theta_{1,2}$: Polar position of q_w

$\theta_{4,5,6}$: Orientation of the end effector

EXAMPLE II

Given a desired end effector configuration g_d and an initial configuration g_0 find θ_{1-6}

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$



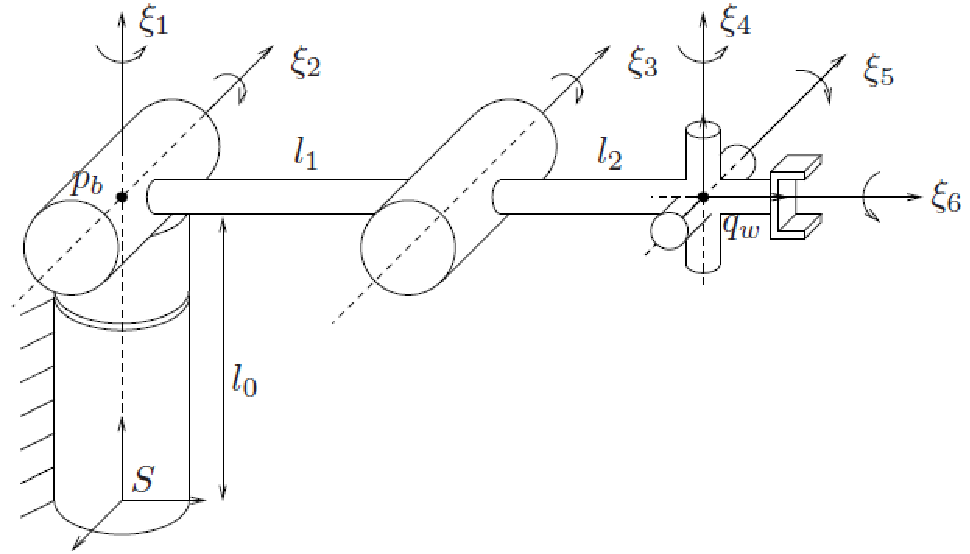
EXAMPLE II

Given a desired end effector configuration g_d and an initial configuration g_0 find θ_{1-6}

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$

Consider the invariant points p_b and q_w:

$$p_b = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p_b$$



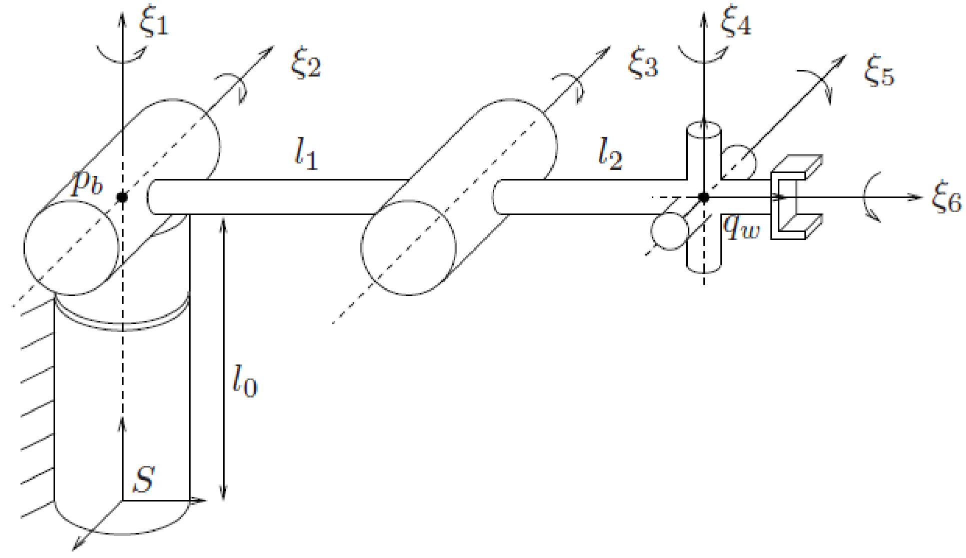
EXAMPLE II

Given a desired end effector configuration g_d and an initial configuration g_0 find θ_{1-6}

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$

Consider the invariant points p_b and q_w:

$$p_b = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p_b$$
$$q_w = e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}q_w$$

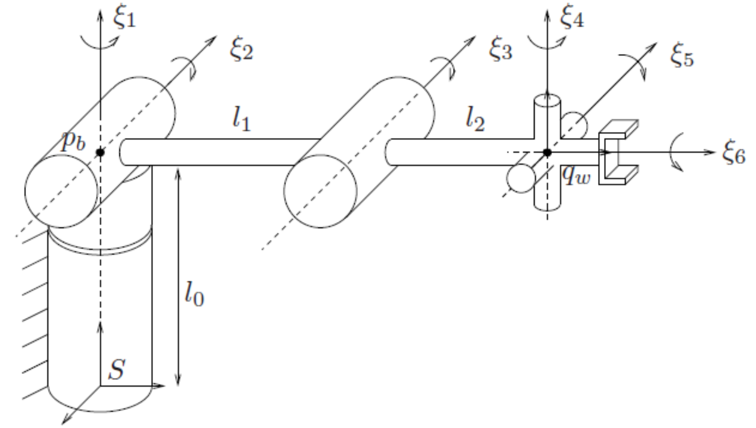


EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$

$$p_b = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p_b$$

$$q_w = e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}q_w$$



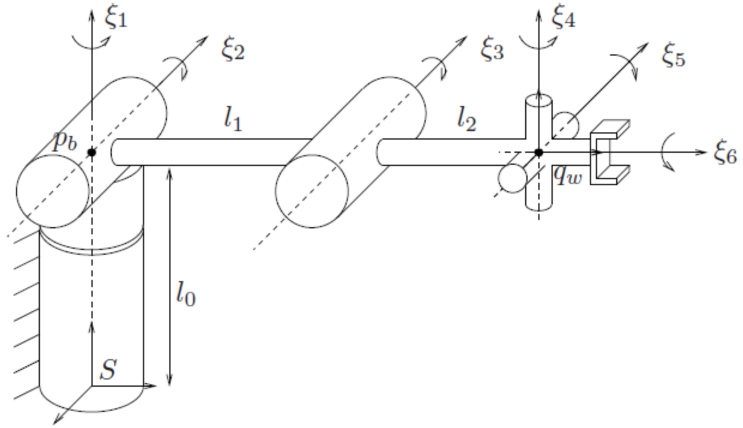
EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p_b$$

$$q_w = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w$$

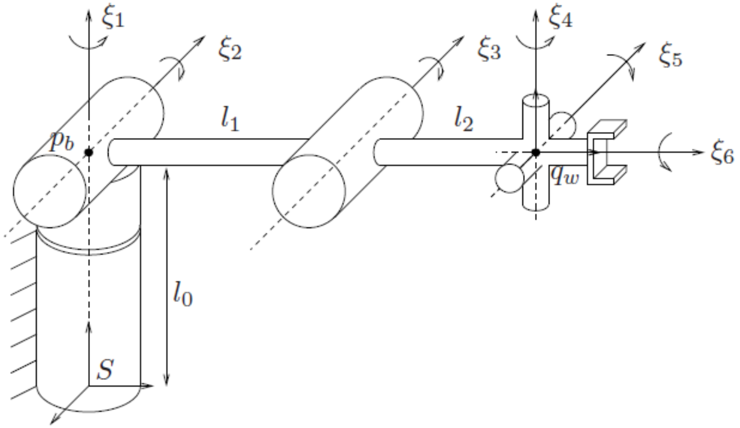
$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w$$



EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$
$$p_b = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p_b$$
$$q_w = e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}q_w$$

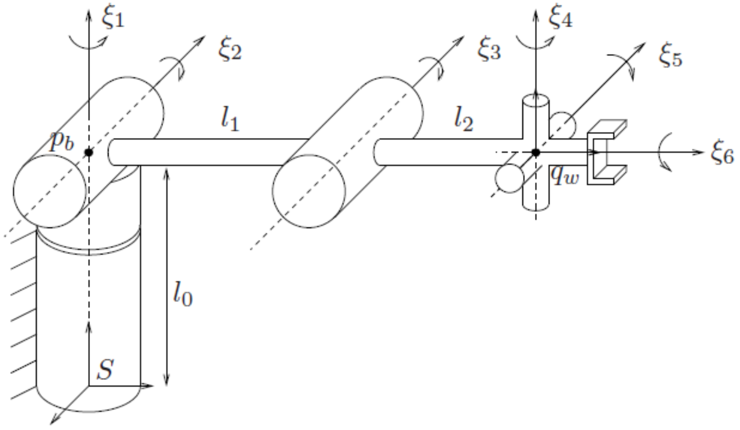
$$g_1 q_w = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}q_w$$
$$g_1 q_w = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_w$$



Example II

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$
$$p_b = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p_b$$
$$q_w = e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}q_w$$

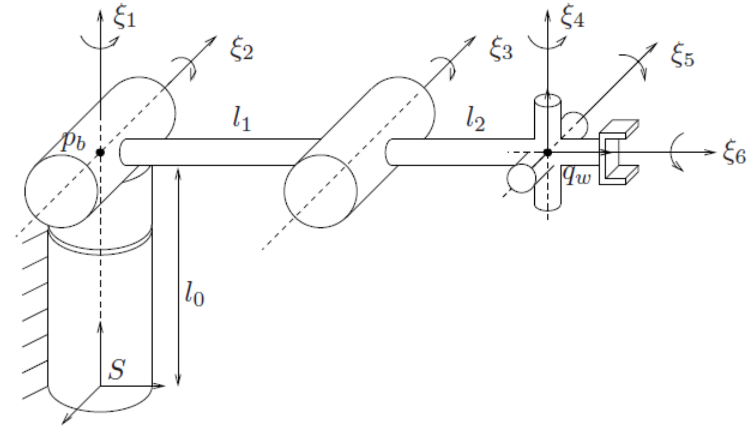
$$g_1 q_w = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}q_w$$
$$g_1 q_w = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_w$$
$$g_1 q_w - p_b = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_w - p_b$$



EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$
$$p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p_b$$
$$q_w = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w$$

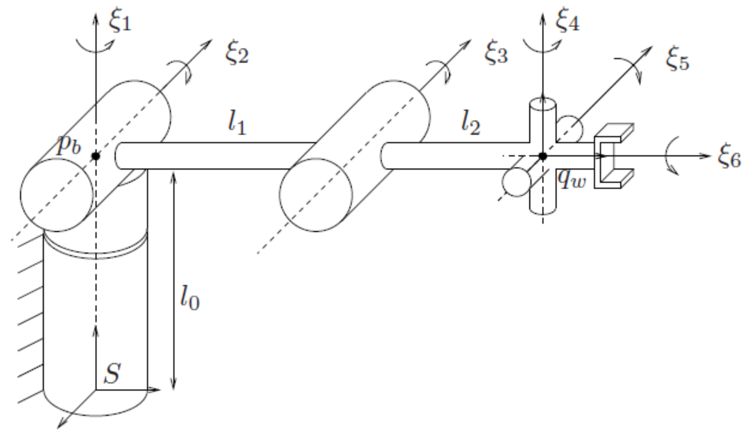
$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w$$
$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w$$
$$g_1 q_w - p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w - p_b$$
$$g_1 q_w - p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \left(e^{\hat{\xi}_3\theta_3} q_w - p_b \right)$$



EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$
$$p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p_b$$
$$q_w = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w$$

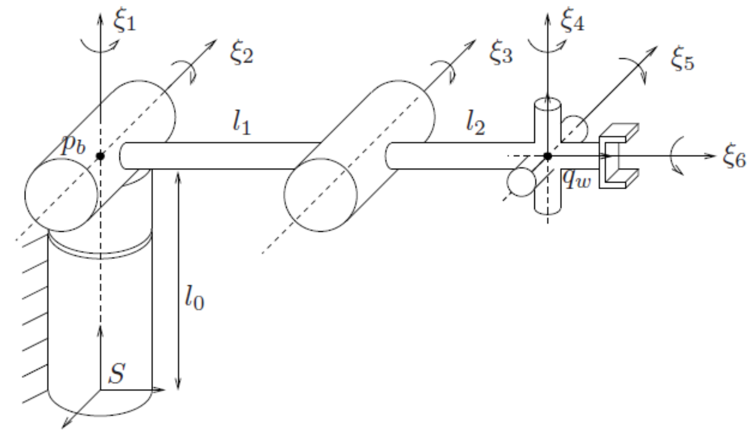
$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w$$
$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w$$
$$g_1 q_w - p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w - p_b$$
$$g_1 q_w - p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \left(e^{\hat{\xi}_3\theta_3} q_w - p_b \right)$$
$$\| g_1 q_w - p_b \| = \left\| e^{\hat{\xi}_3\theta_3} q_w - p_b \right\|$$



EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$
$$p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p_b$$
$$q_w = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w$$

$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w$$
$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w$$
$$g_1 q_w - p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w - p_b$$
$$g_1 q_w - p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \left(e^{\hat{\xi}_3\theta_3} q_w - p_b\right)$$
$$\|g_1 q_w - p_b\| = \left\|e^{\hat{\xi}_3\theta_3} q_w - p_b\right\|$$
$$\delta = \left\|e^{\hat{\xi}_3\theta_3} q_w - p_b\right\|$$



EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$

$$p_b = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p_b$$

$$q_w = e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}q_w$$

$$g_1 q_w = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}q_w$$

$$g_1 q_w = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_w$$

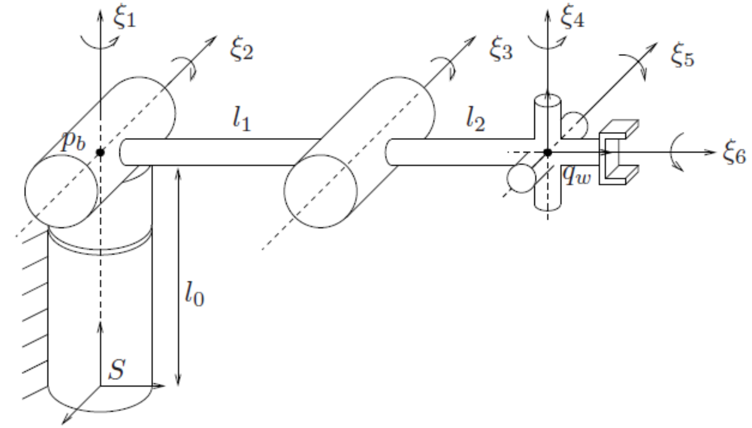
$$g_1 q_w - p_b = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_w - p_b$$

$$g_1 q_w - p_b = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\left(e^{\hat{\xi}_3\theta_3}q_w - p_b\right)$$

$$\|g_1 q_w - p_b\| = \left\|e^{\hat{\xi}_3\theta_3}q_w - p_b\right\|$$

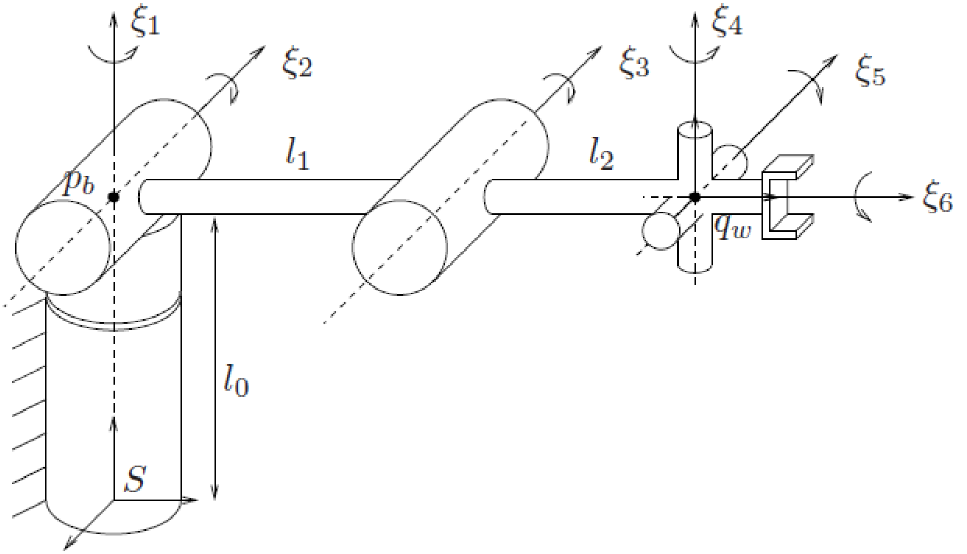
$$\delta = \left\|e^{\hat{\xi}_3\theta_3}q_w - p_b\right\|$$

PADEN KAHAN Subproblem III



EXAMPLE II

Separation of joints:



θ_3: Distance $\|q_w - p_b\|$

$\theta_{1,2}$: Polar position of q_w

$\theta_{4,5,6}$: Orientation of the end effector

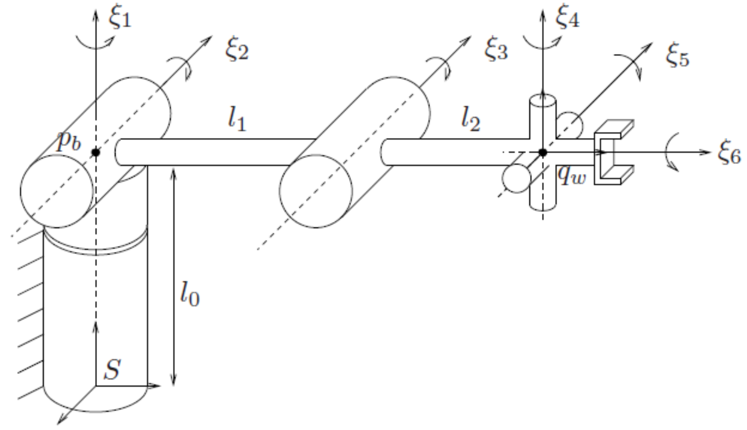
EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p_b$$

$$q_w = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w$$

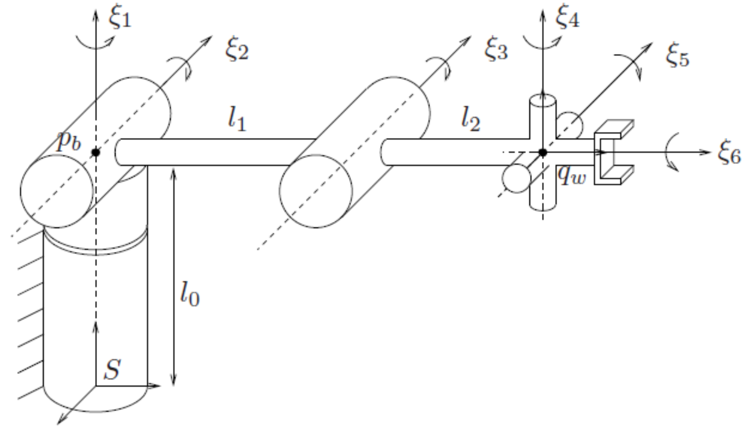
$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w$$



Example II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$
$$p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p_b$$
$$q_w = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w$$

$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w$$
$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \left(e^{\hat{\xi}_3\theta_3} q_w \right)$$

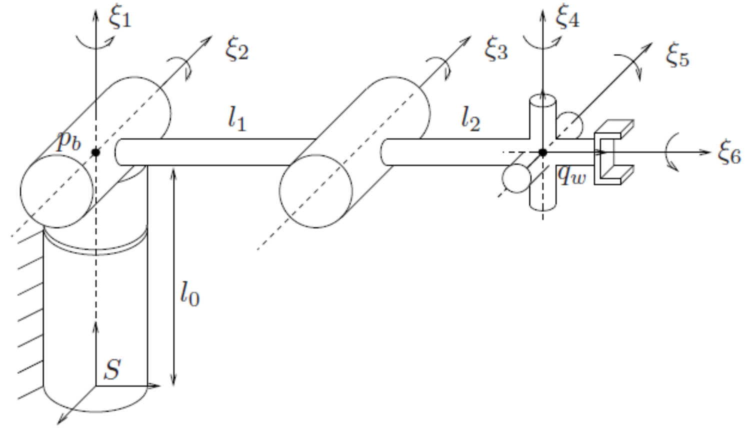


EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$p_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p_b$$

$$q_w = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w$$



$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w$$

$$g_1 q_w = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \left(e^{\hat{\xi}_3\theta_3} q_w \right)$$

PADEN KAHAN Subproblem II

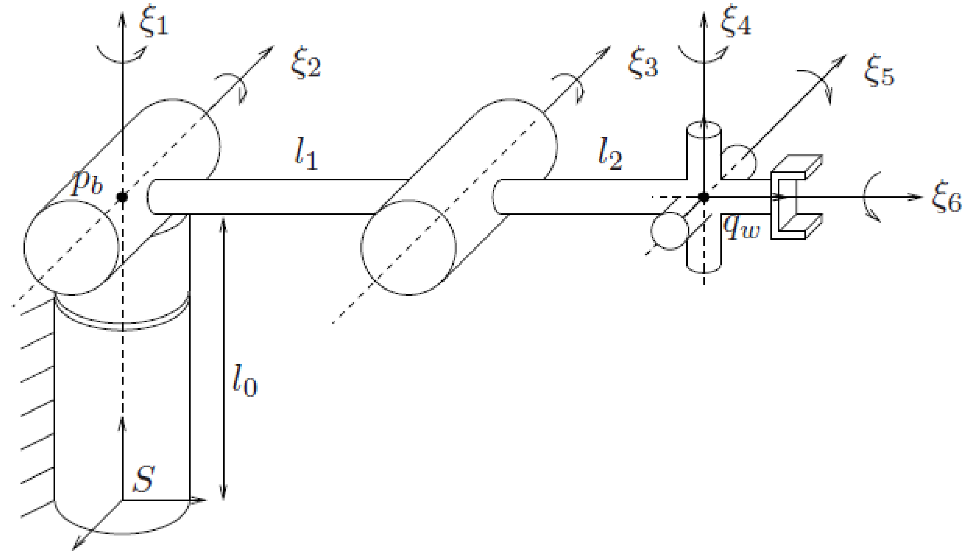
EXAMPLE II

Separation of joints:

θ_3: Distance $\|q_w - p_b\|$

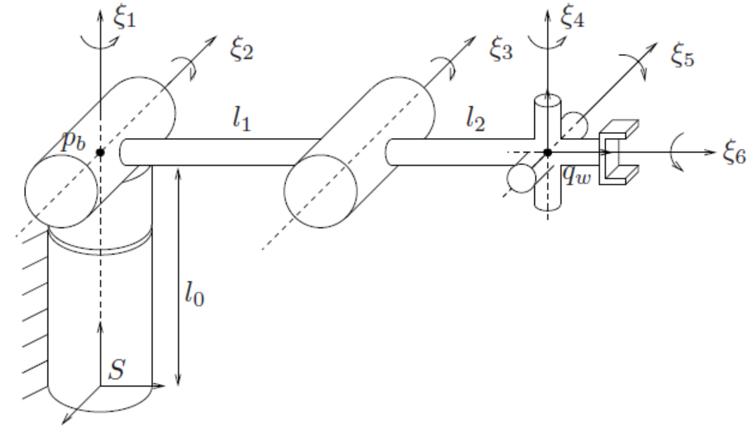
$\theta_{1,2}$: Polar position of q_w

$\theta_{4,5,6}$: Orientation of the end effector



EXAMPLE II

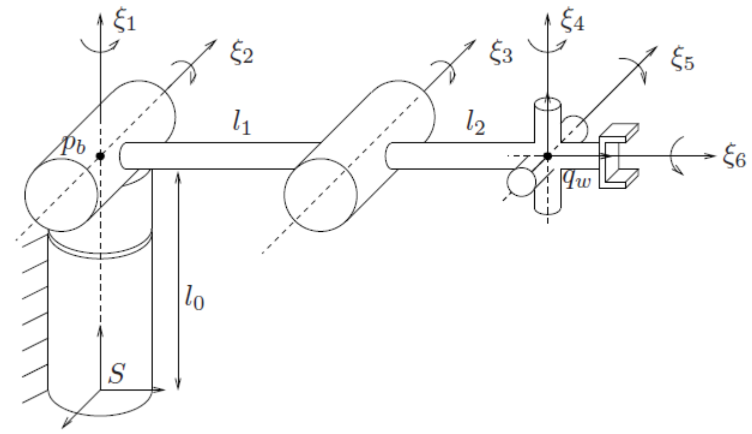
$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$



EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_1 = g_d g_0^{-1}$$

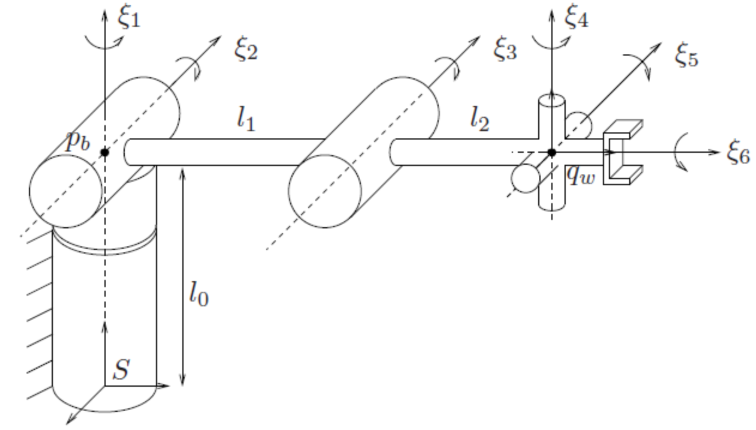


EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$

$$g_1 = g_d g_0^{-1}$$

$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3}e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1}$$



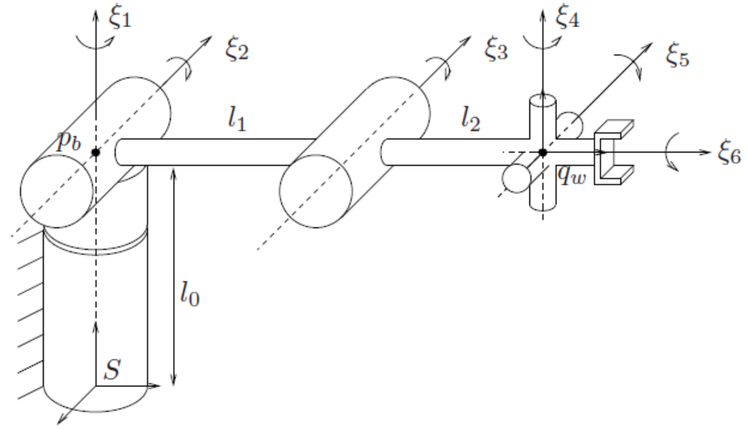
EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_1 = g_d g_0^{-1}$$

$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$



EXAMPLE II

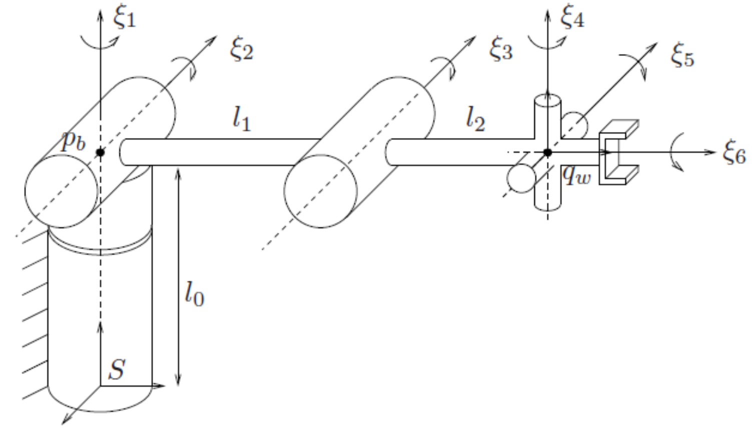
$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_1 = g_d g_0^{-1}$$

$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

Consider a point p_6 on the ξ_6 axis



EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

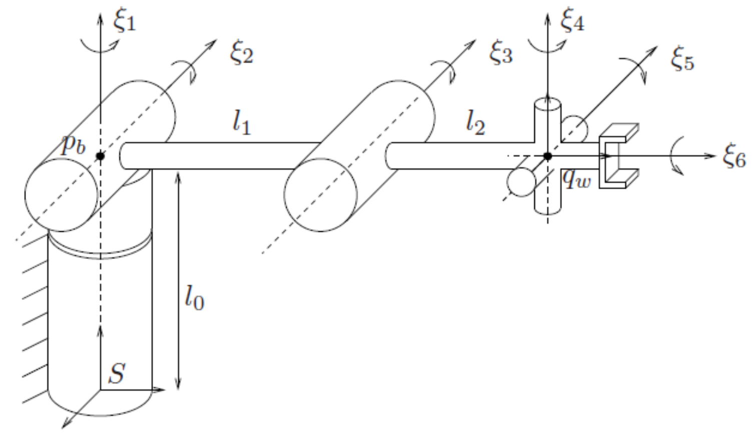
$$g_1 = g_d g_0^{-1}$$

$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

Consider a point p_6 on the ξ_6 axis

$$p_6 = e^{\hat{\xi}_6\theta_6} p_6$$



EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_1 = g_d g_0^{-1}$$

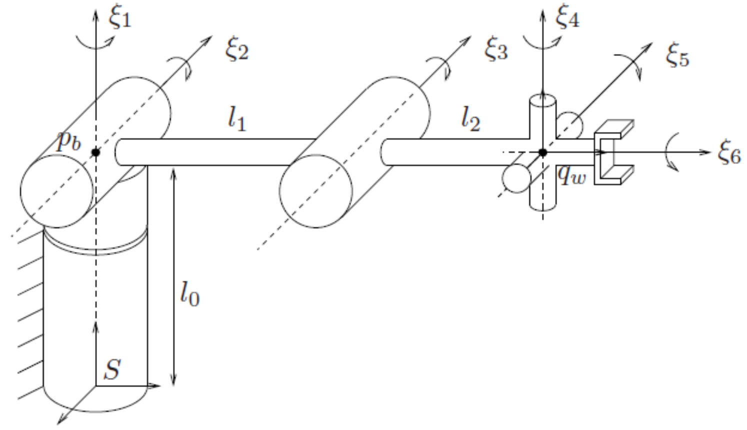
$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

Consider a point p_6 on the ξ_6 axis

$$p_6 = e^{\hat{\xi}_6\theta_6} p_6$$

$$g_2 p_6 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} p_6$$



Example II

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$

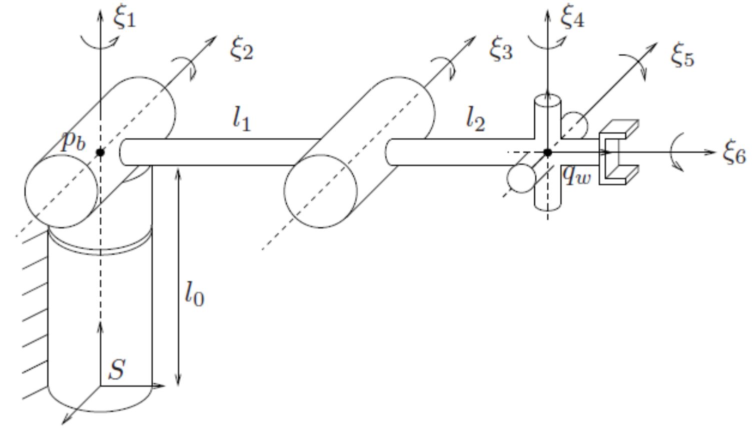
$$g_1 = g_d g_0^{-1}$$

$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3}e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$

Consider a point p_6 on the ξ_6 axis

$$p_6 = e^{\hat{\xi}_6\theta_6}p_6$$



$$g_2 p_6 = e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}p_6$$

$$g_2 p_6 = e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}p_6$$

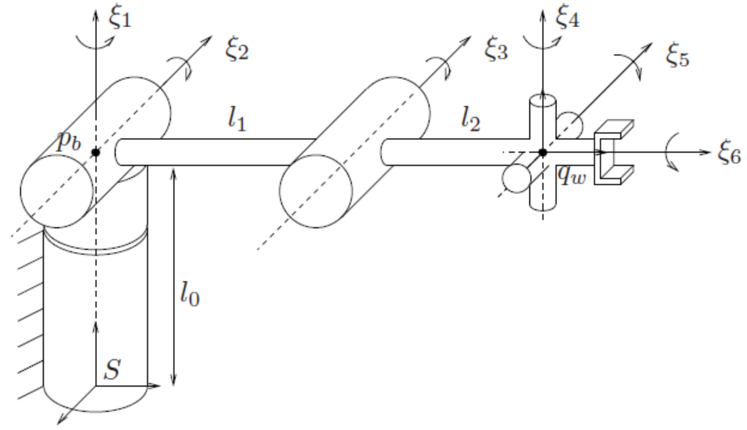
EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_1 = g_d g_0^{-1}$$

$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$



Consider a point p_6 on the ξ_6 axis

$$p_6 = e^{\hat{\xi}_6\theta_6} p_6$$

$$g_2 p_6 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} p_6$$

$$g_2 p_6 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} p_6$$

PADEN KAHAN Subproblem II

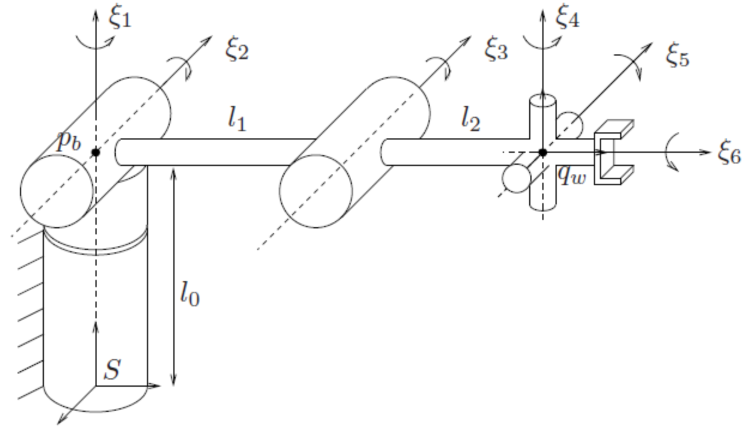
EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_1 = g_d g_0^{-1}$$

$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$



Example II

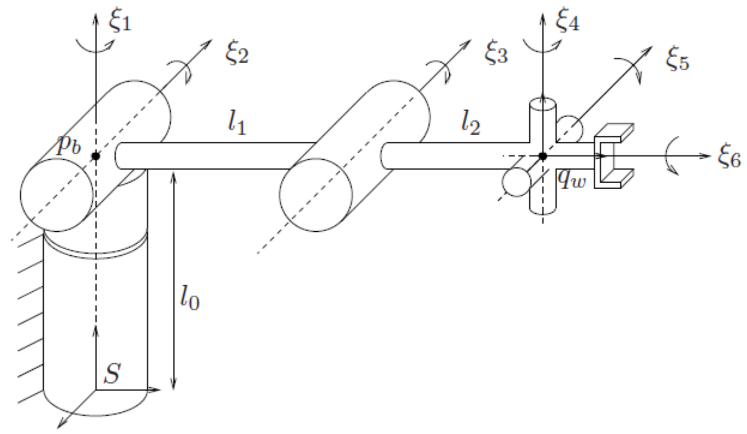
$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_1 = g_d g_0^{-1}$$

$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_3 = g_2 e^{-\hat{\xi}_5\theta_5} e^{-\hat{\xi}_4\theta_4}$$



EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$

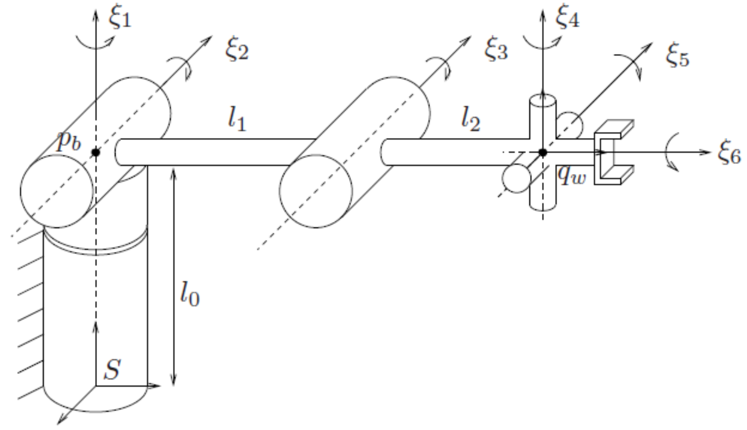
$$g_1 = g_d g_0^{-1}$$

$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3}e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}$$

$$g_3 = g_2 e^{-\hat{\xi}_5\theta_5}e^{-\hat{\xi}_4\theta_4}$$

$$g_3 = e^{\hat{\xi}_6\theta_6}$$



EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_1 = g_d g_0^{-1}$$

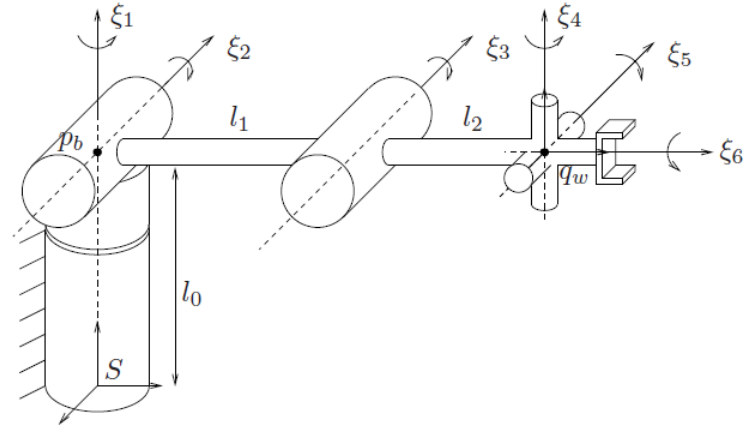
$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_3 = g_2 e^{-\hat{\xi}_5\theta_5} e^{-\hat{\xi}_4\theta_4}$$

$$g_3 = e^{\hat{\xi}_6\theta_6}$$

Consider a point p_E not on the ξ_6 axis



EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

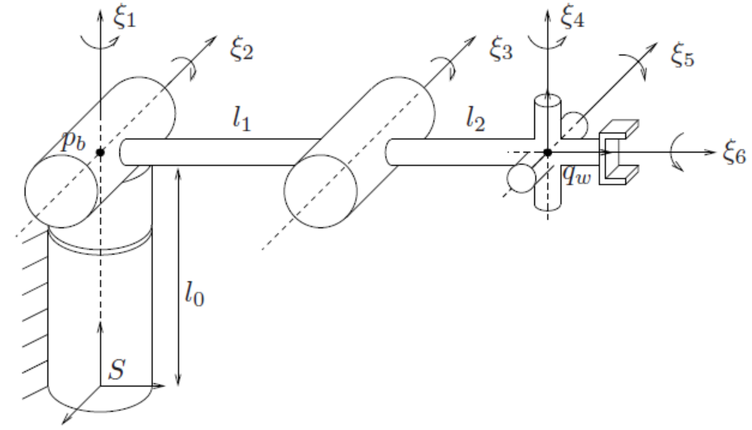
$$g_1 = g_d g_0^{-1}$$

$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_3 = g_2 e^{-\hat{\xi}_5\theta_5} e^{-\hat{\xi}_4\theta_4}$$

$$g_3 = e^{\hat{\xi}_6\theta_6}$$



Consider a point p_E not on the ξ_6 axis

$$g_3 p_E = e^{\hat{\xi}_6\theta_6} p_E$$

EXAMPLE II

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

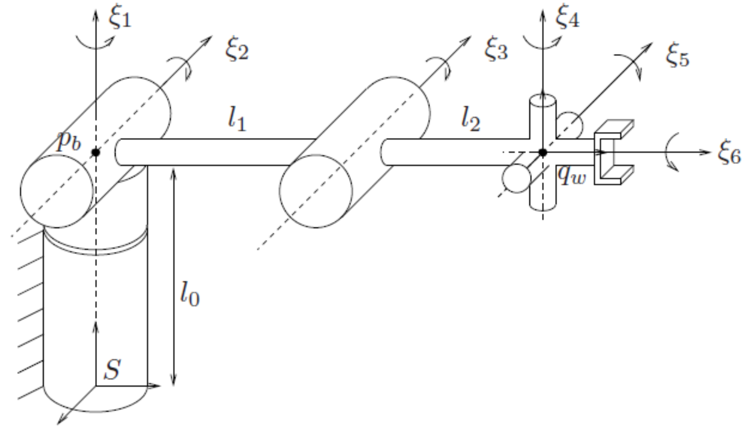
$$g_1 = g_d g_0^{-1}$$

$$g_2 = g_1 e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$$g_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$g_3 = g_2 e^{-\hat{\xi}_5\theta_5} e^{-\hat{\xi}_4\theta_4}$$

$$g_3 = e^{\hat{\xi}_6\theta_6}$$



Consider a point p_E not on the ξ_6 axis

$$g_3 p_E = e^{\hat{\xi}_6\theta_6} p_E$$

PADEN KAHAN Subproblem I